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UNCERTAIN DYNAMIC SYSTEMS ON TIME SCALES

Abstract. Utilizing the framework of the theory of dynamic systems on time scales for measure chains, stability of moving invariant sets is discussed. These results include both continuous and discrete dynamic systems.

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1. Introduction

Nonlinear differential equations with uncertain parameters may cause change of equilibrium states. To investigate such situations, Siljak, Ikeda and Ohata [8] have introduced the notion of parametric stability and discussed its study which is interesting in itself.

A fundamental feedback control problem is that of obtaining some desired behavior from the given system which has uncertain information. Leitmann and associates [1, 2, 9] have dealt with such a problem in a series of papers. They have investigated continuous and discrete uncertain systems by means of Lyapunov functions.

Recently, a theory known as dynamic systems on time scales has been built which incorporates both continuous and discrete times, namely, time as an arbitrary closed set of reals, and permit us to handle both systems simultaneously [6]. This theory allows one to get some insight into and better understanding of the subtle differences between discrete and continuous systems.

To study uncertain systems, a different idea is employed recently [5], which exhibits moving invariant sets as the parameter changes. By reducing the problem to a simpler comparison problem, the stability of moving invariant sets is discussed employing comparison method. The derivative of the Lyapunov function involved is estimated from opposite directions relative to suitable sets in phase space that depend on the moving parameter.

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In this paper, utilizing the framework of the theory of dynamic systems on time scale, we will investigate uncertain dynamic systems on time scale relative to stability of moving invariant sets. As an application of our results, we will consider the control of uncertain dynamic system on time scales and obtain the desired stability behavior of moving invariant sets.

2. Preliminaries

Let $\mathbb{T}$ be a time scale (any subset of R with order and topological structure defined in a canonical way) with $t_0 \geq 0$ as a minimal element and no maximal element. Since a time scale $\mathbb{T}$ may or may not be connected, we need the following concept of jump operators.

Definition 2.1. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) = inf[s \in \mathbb{T} : s > t] \quad \text{and} \quad \rho(t) = sup[s \in \mathbb{T} : s < t]$$

are called the jump operators.

Definition 2.2. A nonmaximal element $t \in \mathbb{T}$ is called right-dense (rd) if $\sigma(t) = t$, right-scattered (rs) if $\sigma(t) > t$, left-dense (ld) if $\rho(t) = t$, left-scattered (ls) if $\rho(t) < t$. In the case $\mathbb{T} = R$, we have $\sigma(t) = t$, and if $\mathbb{T} = hZ$, then $\sigma(t) = t + h$.

Definition 2.3. The mapping $\mu^* : \mathbb{T} \to R_+$ defined by $\mu^*(t) = \sigma(t) - t$ is called graininess. If $\mathbb{T} = R$, then $\mu^*(t) = 0$, and when $\mathbb{T} = Z$, we have $\mu^*(t) = 1$.

Definition 2.4. The mapping $u : \mathbb{T} \to X$, where X is a Banach space is called rd-continuous if it is continuous at each right-dense $t \in \mathbb{T}$, and at each left-dense t, the left-sided limit $u(t^-)$ exists.

Let $C_{rd}[\mathbb{T}, X]$ denote the set of rd-continuous mappings from $\mathbb{T}$ to X. It is clear that a continuous mapping is rd-continuous. However, if $\mathbb{T}$ contains left-dense and right scattered points, then rd-continuity does not imply continuity. But on a discrete time scale the two notions coincide.

Definition 2.5. A mapping $u : \mathbb{T} \to X$ is said to be differentiable at $t \in \mathbb{T}$, if there exists an $\alpha \in X$ such that for any $\epsilon > 0$ there exists a neighborhood N of t satisfying

$$|u(\sigma(t)) - u(s) - \alpha(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \text{for all} \quad s \in N.$$

Let $u^\Delta(t)$ denote the derivative of u. Note that if $\mathbb{T} = R$, then $\alpha = u^\Delta = \frac{du(t)}{dt}$ and if $\mathbb{T} = Z$, then $\alpha = u^\Delta = u(t+1) - u(t)$. It is easy to see that if u is differentiable at t, then it is continuous at t, if u is continuous at t and t is right-scattered, then u is differentiable and

$$u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

Definition 2.6. For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then we define the generalized derivative (or Dini derivative), $D^+u^\Delta(t)$, to mean that, given $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\mu^*(t,s)} < D^+u^\Delta(t) + \epsilon \quad \text{for} \quad s \in N_\epsilon, \quad s > t, \quad \text{where} \quad \mu(t,s) = \sigma(t) - s.$$

In case t is rs and u is continuous at t, we have, as in the case of the derivative,

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

Definition 2.7. Let h be a mapping from $\mathbb{T}$ to X. The mapping $g : \mathbb{T} \to X$ is called the antiderivative of h on $\mathbb{T}$ if it is differentiable on $\mathbb{T}$ and satisfies $g^\Delta(t) = h(t)$ for $t \in \mathbb{T}$.

Following Definition 2.6, define $D^+V^\Delta(t, x(t))$ for $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$ to mean that, given $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such that

$$\frac{1}{\mu(t,s)}[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - \mu(t,s)f(t, x(t)))] <$$
$$< D^+V^\Delta(t, x(t)) + \epsilon$$

for each $s \in N_\epsilon$, $s > t$. As before, if t is rs and $V(t, x(t))$ is continuous at t, this reduces to

$$D^+V^\Delta(t, x(t)) = \frac{V(\sigma(t), x(\sigma(t))) - V(t, x(t))}{\mu^*(t)}.$$

We need the following comparison results in terms of Lyapunov-like functions. See [6].

Theorem 2.1. *Let $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, $V(t,x)$ be locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd, and let*

$$D^+V^\Delta(t,x) \leq g(t, V(t,x)),$$

where $g \in C_{rd}[\mathbb{T} \times R_+, R]$, $g(t,u)\mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and $r(t) = r(t, t_0, u_0)$ is the maximal solution of $u^\Delta = g(t,u)$, $u(t_0) = u_0 \geq 0$, existing on $\mathbb{T}$. Then, $V(t_0, x_0) \leq u_0$ implies that $V(t, x(t)) \leq r(t, t_0, u_0)$, $t \in \mathbb{T}$, $t \geq t_0$.

A result giving the lower estimate is also true.

Theorem 2.2. *Let $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, $V(t,x)$ be locally Lipschitzian in x for each $t \in \mathbb{T}$ which is rd, and let*

$$D^+V^\Delta(t,x) \geq g(t, V(t,x)),$$

where $g \in C_{rd}[\mathbb{T} \times R_+, R]$, $g(t,u)\ \mu^*(t) + u$ is nondecreasing in u for each $t \in \mathbb{T}$, and $\rho(t) = \rho(t, t_0, u_0)$ is the minimal solution of $u^\Delta = g(t,u)$, $u(t_0) =$

$u_0 \geq 0$, existing on $\mathbb{T}$. Then, $V(t_0, x_0) \geq u_0$ implies that $V(t, x(t)) \geq \rho(t)$, $t \in \mathbb{T}$, $t \geq t_0$.

We need both comparison results in our discussion below.

3. Main Results

Consider the dynamic system on time scales

$$x^{\Delta} = f(t, x, \lambda), \quad x(t_0) = x_0, \quad t_0 \in \mathbb{T}, \tag{3.1}$$

where $f \in C_{rd}[\mathbb{T} \times R^n \times R^d, R^n]$, $\lambda \in R^d$ is an uncertain parameter and $\mathbb{T}$ is a time scale. Consider also the comparison dynamic equation

$$u^{\Delta} = g(t, u, \mu), \quad u(t_0) = u_0 \geq 0, \tag{3.2}$$

where $g \in C_{rd}[\mathbb{T} \times R_+^2, R]$ and $\mu = \mu(\lambda) \geq 0$ is a parameter depending on λ.

Let $\rho_0 \leq r_0 \leq r \leq \rho$ be depending on λ. Then we will say that the set $B = [x \in R^n : \rho_0 \leq |x| \leq \rho]$ is conditionally invariant with respect to $A = [x \in R^n : r_0 \leq |x| \leq r]$ and is uniformly asymptotically stable (UAS) relative to (2.1) if

- (I) $r_0 \leq |x_0| \leq r$ implies $\rho_0 \leq |x(t)| \leq \rho$, $t \in \mathbb{T}$, $t \geq t_0$;
- (ii) given $\epsilon > 0$ and $t_0 \in \mathbb{T}$,
  - (a) there exists a $\delta = \delta(\epsilon) > 0$ such that $r_0 - \delta \leq |x_0| \leq r + \delta$ implies $\rho_0 - \epsilon < |x(t)| < \rho + \epsilon$, $t \geq t_0$, $t \in \mathbb{T}$;
  - (b) there exist a $\delta_0 > 0$ and a $T = T(\epsilon) > 0$ such that $r_0 - \delta_0 \leq |x_0| \leq r + \delta_0$ implies $\rho_0 - \epsilon < |x(t)| < \rho + \epsilon$, $t \geq t_0 + T$, $t \in \mathbb{T}$;

where $x(t) = x(t, t_0, x_0)$ is any solution of (3.1).

Relative to the comparison equation (3.2), we will say that $\Omega = [u \in R_+ : R_0 \leq u \leq R]$ is invariant and is UAS relative to (3.2) if

- (I) $R_0 \leq u_0 \leq R$ implies $R_0 \leq u(t) \leq R$, $t \geq t_0$, $t \in \mathbb{T}$;
- (ii) given $\epsilon > 0$ and $t_0 \in \mathbb{T}$,
  - (a) there exists a $\delta = \delta(\epsilon) > 0$ such that $R_0 - \delta \leq u_0 \leq R + \delta$ implies $R_0 - \epsilon < u(t) < R + \epsilon$, $t \geq t_0$, $t \in \mathbb{T}$;
  - (b) there exists a $\delta_0 > 0$ and a $T = T(\epsilon) > 0$ such that $R_0 - \delta_0 \leq u_0 \leq R + \delta_0$ implies $R_0 - \epsilon < u(t) < R + \epsilon$, $t \geq t_0 + T$, $t \in \mathbb{T}$,

where $u(t) = u(t, t_0, u_0)$ is any solution of (3.2).

Let us define the usual class K of functions by $K = [a \in C[R_+, R_+] : a(u)$ is strictly increasing in u with $a(0) = 0$ and $a(u) \to \infty$ as $u \to \infty]$.

We can now prove the following result on UAS of the conditionally invariant set B with respect to A, relative to the system (3.1). Let us define the sets Ω_r, Ω_{r_0} by $\Omega_r = [x \in R^n : x \in A$ and $|x| \geq r]$, $\Omega_{r_0} = [x \in R^n : x \in A$ and $|x| \leq r_0]$.

Theorem 3.1. *Assume that*

(A_0) *for each* $\lambda \in R^d$, *there exist* $r = r(\lambda)$, $r_0 = r_0(\lambda)$, $r_0 \leq r$ *satisfying* $r \to 0$ *as* $|\lambda| \to 0$ *and* $r_0 \to \infty$ *as* $|\lambda| \to \infty$;

(A_1) *there exists* $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$ *such that* $V(t,x)$ *is locally Lipschitzian in* x *for each right dense* $t \in \mathbb{T}$ *and for* a_i, $b_i \in K$, $I = 1, 2$,

$$b_1(|x|) \leq V(t,x) \leq a_1(|x|) \quad if \ x \in \Omega_r,$$
$$b_2(|x|) \leq V(t,x) \leq a_2(|x|) \quad if \ x \in \Omega_{r_0};$$

(A_2) *if* $x \in \Omega_r$, $D^+V^\Delta(t,x) \leq g(t, V(t,x), r)$, *and if* $x \in \Omega_{r_0}$, $D^+V^\Delta(t,x) \geq g(t, V(t,x), r_0)$, *where* $g \in C_{rd}[\mathbb{T} \times R_+^2, R]$, $g(t,u,\mu)\mu^*(t) + u$ *is nondecreasing in* u *for each* (t,u);

(A_3) *for each* $r_0 \leq r$, *there exists* $R_0 \leq R$ *such that* $R = a_1(r) = b_1(\rho)$ *and* $R_0 = b_2(r_0) = a_2(\rho_0)$, *where* $\rho_0 \leq r_0 \leq r \leq \rho$ *and* $R \to 0$ *as* $r \to 0$, $R_0 \to \infty$ *as* $r_0 \to \infty$;

(A_4) *the set* Ω *is invariant and is UAS with respect to* (3.2).

Then the set B *is conditionally invariant with respect to* A *and is UAS relative to the system* (3.1).

Proof. We will first prove that B is conditionally invariant with respect to A and (3.1). If not, there would exist a solution $x(t) = x(t, t_0, x_0)$ of (3.1) with $r_0 \leq |x_0| \leq r$ and $t_0 < t_2$ such that either

(i) $|x(t_2)| > \rho$ and $r_0 \leq |x(t)|$, $t \in [t_0, t_2] \cap \mathbb{T}$,
or
(ii) $|x(t_2)| < \rho_0$ and $|x(t)| \leq r$, $t \in [t_0, t_2] \cap \mathbb{T}$.

Because of (A_2), using comparison Theorems 2.2, 2.3, we get either

$$V(t, x(t)) \leq r(t, t_0, V(t_0, x_0)),$$

or

$$V(t, x(t)) \geq \rho(t, t_0, V(t_0, x_0)),$$

for $t \in [t_0, t_2] \cap \mathbb{T}$, where $r(t, t_0, u_0)$ and $\rho(t, t_0, u_0)$ are the maximal and minimal solutions of (3.2). Hence using (A_3) and (A_4), in the case (i) we have

$$\begin{aligned} b_1(\rho) < b_1(|x(t_2)|) &\leq V(t_2, x(t_2)) \leq r(t_2, t_0, V(t_0, x_0)) \leq \\ &\leq r(t_2, t_0, a_1(|x_0|)) \leq r(t_2, t_0, a_1(r)) \leq a_1(r) = b_1(\rho), \end{aligned}$$

or, in the case (ii), we get

$$\begin{aligned} a_2(\rho_0) > a_2(|x(t_2)|) &\geq V(t_2, x(t_2)) \geq \rho(t_2, t_0, V(t_0, x_0)) \geq \\ &\geq \rho(t_2, t_0, b_2(|x_0|)) \geq \rho(t_2, t_0, b_2(r_0)) \geq b_2(r_0) = a_2(\rho_0). \end{aligned}$$

Thus, we have a contradiction in both cases and hence B is conditionally invariant with respect to A and (3.1).

Let $0 < \epsilon < \rho_0$ and $t_0 \in \mathbb{T}$ be given. Since (A_4) holds and $a_1(r) = b_1(\rho) = R$, $R_0 = b_2(r_0) = a_2(\rho_0)$, given $a_2(\rho_0 - \epsilon)$, $b_1(\rho + \epsilon)$, there exist ϵ_1, δ_1, $\delta > 0$ such that

$$R_0 + \delta_1 = a_1(r + \delta) < b_1(\rho + \epsilon) = R + \epsilon_1,$$

and

$$R_0 - \epsilon = a_2(\rho_0 - \epsilon) < b_2(r_0 - \delta) = R_0 - \delta_1,$$

satisfying

$$R_0 - \delta_1 < u_0 < R + \delta_1 \text{ implies } R_0 - \epsilon_1 < u(t) < R + \epsilon_1, \ \ t \geq t_0, \ \ t \in \mathbb{T},$$

where $u(t) = u(t, t_0, u_0)$ is any solution of (3.2). We claim that with this $\delta > 0$, the set B is US relative to A, that is,

$$r_0 - \delta < |x_0| < r + \delta \text{ implies } \rho_0 - \epsilon < |x(t)| < \rho + \epsilon, \ \ t \geq t_0, \ \ t \in \mathbb{T}.$$

If this is not true, there would exist a solution $x(t)$ of (3.1) with $r_0 - \delta < |x_0| < r + \delta$ and a $t_2 > t_0$ such that either

(a) $|x(t_2)| \geq \rho + \epsilon$ and $|x(t)| \geq r_0$, $[t_0, t_2] \cap \mathbb{T}$,
or
(b) $|x(t_2)| \leq \rho_0 - \epsilon$ and $|x(t)| \leq r$, $[t_0, t_2] \cap \mathbb{T}$.

Consider (a). As before, we obtain

$$V(t, x(t)) \leq r(t, t_0, V(t_0, x_0)), \quad [t_0, t_2] \cap \mathbb{T},$$

and therefore, we arrive at the contradiction

$$\begin{aligned} b_1(\rho + \epsilon) &\leq b_1(|x(t_2)|) \leq V(t_2, x(t_2)) \leq r(t_2, t_0, V(t_0, x_0)) \leq \\ &\leq r(t_2, t_0, a_1(|x_0|)) \leq r(t_2, t_0, a_1(r + \delta)) < b_1(\rho + \epsilon). \end{aligned}$$

Similarly, in case (b), we first get

$$V(t, x(t)) \geq \rho(t, t_0, V(t_0, x_0)), \ \ [t_0, t_2] \cap \mathbb{T},$$

and then

$$\begin{aligned} a_2(\rho_0 - \epsilon) &> a_2(|x(t_2)|) \geq V(t_2, x(t_2)) \geq \rho(t_2, t_0, V(t_0, x_0)) \geq \\ &\geq \rho(t_2, t_0, b_2(|x_0|)) \geq \rho(t_2, t_0, b_2(r_0 - \delta)) \geq b_2(r_0) = a_2(\rho_0 - \epsilon), \end{aligned}$$

which is again a contradiction. Hence the set B is US relative to A.

To prove UAS of the set B relative to A, let us fix $\epsilon = \rho_0$ and designate $\delta_0 = \delta(\rho_0)$ so that we have

$$r_0 - \delta_0 < |x_0| < r + \delta_0 \text{ implies } 0 < |x(t)| < \rho + \rho_0, \ \ t \geq t_0, \ \ t \cap \mathbb{T}.$$

Let $0 < \epsilon < \rho_0$ and $t_0 \in \mathbb{T}$. Since Ω is UAS, given $a_2(\rho_0 - \epsilon)$, $b_1(\rho + \epsilon)$ there exists a $T = T(\epsilon) > 0$, with $t_0 + T \in \mathbb{T}$ such that

$$b_2(r_0 - \delta_0) < u_0 < a_1(r + \delta_0) \text{ implies } a_2(\rho_0 - \epsilon) < u(t) < b_1(\rho + \epsilon), \ \ t \geq t_0 + \mathbb{T}.$$

We claim that whenever $r_0 - \delta_0 < |x_0| < r + \delta_0$, we have

$$\rho_0 - \epsilon < |x(t)| < \rho + \epsilon, t \geq t_0 + \mathbb{T}, \quad t \in \mathbb{T}.$$

If this is not true, there would exist a solution $x(t)$ of (3.1) such that

(a) $|x(t_2)| \geq \rho + \epsilon$, $t_2 \geq t_0 + \mathbb{T}$, $t_2 \in \mathbb{T}$,

(b) $|x(t_2)| \leq \rho_0 - \epsilon$, $t_2 \geq t_0 + \mathbb{T}$, $t_2 \in \mathbb{T}$, where $r_0 - \delta_0 < |x_0| < r + \delta_0$. As before, using (A_2) and (A_3), we get successively

$$b_1(\rho + \epsilon) \leq V(t_2, x(t_2)) \geq r(t_2, t_0, a_1(r + \delta_0)) < b_1(\rho + \epsilon),$$

and

$$a_2(\rho_0 - \epsilon) \geq V(t_2, x(t_2)) \geq \rho(t_2, t_0, b_2(r_0 - \delta_0)) > a_2(\rho_0 - \epsilon),$$

which are contradictions. Hence we have B is UAS with respect to A relative to the system (3.1) and the proof is complete. ■

Remarks. If $\mathbb{T} = R$, then (3.1), (3.2) reduce to the continuous differential systems. Since, in this case, $\mu^*(t) = 0$, the results of Theorem 3.1 reduce to those in [4]. Note that the conditions (A_1) and (A_2), which are sufficient to prove UAS, are then weaker. If, on the other hand, $\mathbb{T} = Z$, so that $\mu^*(t) = 1$, (3.1) and (3.2) reduce to difference equations, and consequently, one needs stronger conditions (A_1), (A_2). Theorem 3.1 offers results in this special case.

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