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ON THE QUESTION OF ENCLOSING SOLUTIONS OF LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

Abstract. A technique of constructing guaranteed a posteriori error bounds for approximate solutions of the Initial Value Problem and some Boundary Value Problems for linear functional differential equations is described. The main point of the consideration is construction of an approximation for the Cauchy matrix with the corresponding error bound. The described approach allows to develop a software oriented to solving some real problems in economic dynamics.

რეზიუმე. აღწერილია ტექნიკა, რომელიც საშუალებას იძლევა აიგოს ცდომილების გარანტირებული აპოსტეტიორული საზღვრები საწყისი და ზოგიერთი სასაზღვრო ამოცანის მიახლოებითი ამონახსნებისათვის. მთავარ მომენტს წარმოადგენს კოშის მატრიცის მიახლოების აგება ცდომილების საზღვრის მითითებით. აღწერილი მიდგომა საშუალებას იძლევა შეიქმნას პროგრამული პროდუქტი, რომელიც ორიენტირებულია ეკონომიკური დინამიკის ზოგიერთი რეალური ამოცანის ამოსახსნელად.

1. Introduction. The problem of enclosing solutions (estimating the accuracy, verifying, error estimating, validating) of ordinary differential equations or, more generally speaking, various classes of operator equations, indicates one of the "hottest" areas in current mathematical research. In support of this point of view, we point up the Proceedings of the International Symposium on Numerical Methods and Error Bounds [1], where the bulk of the contributions is devoted to different ways of obtaining error bounds for calculated results. Theoretical results in this area as well as the corresponding software for validated solutions give the biggest opportunity for solving real-world problems. As noted in [2], software development efforts have demonstrated its value to the tool builders working closely with scientists on appropriate problems. If they (the tool builders) develop the tools in a vacuum, they are in danger of spending much effort to develop the wrong tools.

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Here we consider a class of linear functional differential equations and describe a techniques of constructing a posteriori guaranteed error bounds for approximate solutions to the Initial Value Problem (IVP) and some Boundary Value Problems (BVP's). The main point of the consideration is the construction of an approximation for the Cauchy matrix [3, 4] with the corresponding error bound. Our approach has allowed to develop a software oriented to solving some problems in economic dynamics [5] that are being investigated at the Center of Analytical Researches "Prognosis", Perm, Russia. The application part of the work was made possible thanks to and jointly with the experts of the Center.

2. Preliminaries from the Theory of Functional Differential Equations. Consider the equation

$$\mathcal{L}x = f \tag{1}$$

with a linear bounded operator $\mathcal{L}$ acting from the space D^n of all absolutely continuous functions $x : [0,T] \to R^n$ into the space L^n of all summable functions $z : [0,T] \to R^n$. We assume, as usually, that $\|x\|_{D^n} = |x(0)|_n + \|\dot{x}\|_{L^n}$, $\|z\|_{L^n} = \int_0^T |z(t)|_n \, dt$, where $|\cdot|_n$ is a norm of R^n. Recall [3, 4] that any linear bounded operator $\mathcal{L} : D^n \to L^n$ is representable in the form

$$(\mathcal{L}x)(t) = (Q\dot{x})(t) - A(t)x(0), \quad t \in [0,T], \tag{2}$$

where the operator $Q : L^n \to L^n$ called the principal part of $\mathcal{L}$ is linear and bounded, the columns of $n \times n$-matrix A are elements of L^n. We shall restrict our consideration to the case the equation (1) with the operator Q of the form

$$(Qz)(t) = z(t) - \int_0^t K(t,s)z(s)\, ds, \tag{3}$$

where the elements $k^{ij}(t,s)$ of the kernel $K(t,s)$ are measurable on the set $0 \le s \le t \le T$ and satisfy the inequalities

$$|k^{ij}(t,s)| \le \mu(t), \quad \mu \in L^1. \tag{4}$$

Note (see [3, 4]) that the equation (1) includes the so called differential equations with concentrated delay

$$\dot{x}(t) - \sum_{i=1}^{\nu} P_i(t)x[h_i(t)] = v(t), \quad t \in [0,T], \tag{5}$$
$$x(\xi) = \varphi(\xi) \quad \text{as} \quad \xi \notin [0,T],$$

under the assumptions that the columns of the $n \times n$-matrix P_i belong to L^n, $h_i : [0,T] \to R^n$, $(h_i(t) \le t)$, $i = 1, \ldots, \nu$, are measurable functions, and $f = v + \sum_{i=1}^{\nu} P_i \varphi_i^h \in L^n$, where

$$\varphi_i^h(t) = \begin{cases} \varphi[h_i(t)] & \text{as} \quad h_i(t) \notin [0,T], \\ 0 & \text{as} \quad h_i(t) \in [0,T]. \end{cases}$$

In this case $K(t,s) = \sum_{i=0}^{\nu} \chi_{h_i}(t,s) P_i(t)$, where $\chi_{h_i}(t,s)$ is the characteristic function of the set $\{(t,s) \in [0,T] \times [0,T] : \ 0 \leq s \leq h_i(t) \leq T\}$. The equation with distributed delay

$$\dot{x}(t) - \int_0^t d_s \mathcal{T}(t,s) x(s) = f(t), \quad t \in [0,T],$$

where the elements $\tau^{ij}(t,s)$ of the $n \times n$-matrix $\mathcal{T}(t,s)$ are measurable on the set $0 \leq s \leq t \leq T$, $\tau^{ij}(\cdot, s) \in L^1$ for every $s \in [0,T]$,

$$\operatorname*{Var}_{s \in [0,t]} [\tau^{ij}(t,s)] = \rho^{ij}(t), \quad \rho^{ij} \in L^1$$

and $\mathcal{T}(t,t) = 0$, also can be written in the form (1) with $K(t,s) = -\mathcal{T}(t,s)$.

In the case under consideration, the operator $Q : L^n \to L^n$ is invertible, and Q^{-1} is the operator of the form

$$(Q^{-1}z)(t) = z(t) + \int_0^t R(t,s) z(s)\, ds, \tag{6}$$

where R(t,s) is the so called resolvent kernel corresponding to the kernel $K(t,s)$. Recall [3, 4] that the invertibility of Q is the criterion to the unique solvability of the Initial Value Problem

$$\mathcal{L}x = f \quad x(0) = \alpha \tag{7}$$

for every $f \in L^n$ and $\alpha \in R^n$. As this takes place, there exists the fundamental $n \times n$-matrix X of the homogeneous equation $\mathcal{L}x = 0$. This matrix is the matrix $X = (x_1, \ldots, x_n)$ with the columns X_i, $i = 1, \ldots, n$ such that X_i is a solution to the IVP

$$\mathcal{L}x = 0, \quad x(0) = e_i, \tag{8}$$

where e_i is the i-th column of the identity $n \times n$-matrix E. Thus for $\dot{X}(t)$ we have

$$\dot{X}(t) - \int_0^t K(t,s) \dot{X}(s)\, ds = A(t). \tag{9}$$

The solution of IVP (7) has the representation

$$x(t) = X(t)\alpha + \int_0^t C(t,s) f(s)\, ds, \tag{10}$$

where $C(t,s)$ is the Cauchy matrix [3, 4]. As is shown in [6],

$$\begin{aligned} C'_t(t,s) &= R(t,s), \quad 0 \le s \le t \le T, \\ C'_t(t,s) &= \int\limits_s^t C'_t(t,\tau)K(\tau,s)\,d\tau + K(t,s), \quad 0 \le s \le t \le T, \\ C(t,s) &= E + \int\limits_s^t C'_\tau(\tau,s)\,d\tau, \quad 0 \le s \le t \le T. \end{aligned} \tag{11}$$

Now consider the so called general linear Boundary Value Problem

$$\mathcal{L}x = f, \quad lx = \alpha, \tag{12}$$

where the operator $l : D^n \to R^n$ is linear and bounded. Recall [3, 4] that any l is representable in the form

$$lx = \Psi x(0) + \int\limits_0^T \Phi(s)\dot{x}(s)\,ds, \tag{13}$$

where Ψ is an $n \times n$-matrix, and the columns of an $n \times n$-matrix Φ belong to the space L^n_∞ of all measurable functions $z : [0,T] \to R^n$ such that

$$\|z\|_{L^n_\infty} \stackrel{def}{=} \operatorname*{vrai\,sup}_{s\in[0,T]} |z(s)|_n < +\infty.$$

The general boundary conditions $lx = \alpha$ include various classes of boundary conditions, among which are two- and multi-point conditions as well as integral ones. The criterion of the unique solvability of BVP (12) is the invertibility of the $n \times n$-matrix

$$lX \stackrel{def}{=} (lx_1, \ldots, lx_n).$$

As the condition

$$\det lX \ne 0 \tag{14}$$

takes place, the solution of BVP (12) has the representation

$$x(t) = X(t)(lX)^{-1}\alpha + \int\limits_0^T G(t,s)f(s)\,ds, \tag{15}$$

where $G(t,s)$ is the Green matrix [3, 4]. The Green matrix $G(t,s)$ and the Cauchy matrix are connected by the equality

$$G(t,s) = \chi(t,s)C(t,s) - U(t)V(s),$$

where

$$\chi(t,s)=\begin{cases}1, & 0\le s\le t\le T;\\ 0, & 0\le s\le t\le T;\end{cases}\quad U(t)=X(t)(lX)^{-1}, \tag{16}$$

$$V(s)=\Phi(s)+\int_s^T \Phi(\tau)C'_\tau(\tau,s)\,d\tau.$$

3. About a Posteriori Error Bounds. One of the most popular ways to obtain the mentioned error bounds is the use of the integral inequalities method. Too often this way has failed. Namely, let $\widetilde{x}$ be an approximate solution of IVP (7) such that $\widetilde{x}(0)=\alpha$. For $\widetilde{x}$ we have

$$\mathcal{L}\widetilde{x}=f+g,$$

where g is the corresponding residual function. A traditional scheme of obtaining the estimate to $z(t)=\dot{x}(t)-\dot{\widetilde{x}}(t)$ gives the following integral inequality

$$|z(t)|_n\le\int_0^t \|K(t,s)\|\,|z(s)|_n\,ds+|g(t)|_n$$

which implies the estimate

$$|z(t)|_n\le m(t)\int_0^t \exp\left(\int_s^t m(\tau)\,d\tau\right)|g(s)|_n\,ds+|g(t)|_n,\quad t\in[0,T],$$

where

$$m(t)=\left\|\{\mu^{ij}(t)\}\right\|,\quad \mu^{ij}(t)=\mu(t),\ \ i,j=1,\dots,n\ \ (\text{see}(4)).$$

The presence of an exponential factor in the right side produces even at medium values of n and T, say $n=10$, $T=5$, a very overstated estimate of real deviation of $\widetilde{x}$ from x.

We are going to use another way. First we construct an approximation $\widetilde{C}(t,s)$ of the Cauchy matrix $C(t,s)$ with a guaranteed quite high accuracy such that $\|C-\widetilde{C}\|_{L^n\to L^n_\infty}\le\Delta_C$, where $C,\ \widetilde{C}:L^n\to L^\infty$ are the linear integral Volterra operators with the kernels $C(t,s)$ and $\widetilde{C}(t,s)$ respectively. Next, taking into consideration that $\mathcal{L}(\widetilde{x}-x)=g$, $\widetilde{x}(0)-x(0)=0$, we obtain due to (10)

$$|\widetilde{x}(t)-x(t)|_n\le\int_0^t \|\widetilde{C}(t,s)\|\,|g(s)|_n\,ds+\Delta_C\int_0^T |g(s)|_n\,ds,\ \ t\in[0,T]. \tag{17}$$

As for the BVP, we can use the same way in principal points. Namely, let $\widetilde{x}$ be an approximate solution of BVP (12):

$$\mathcal{L}\widetilde{x}=f+g,\quad l\widetilde{x}=\alpha+\gamma.$$

Thus the difference $y(t) = \widetilde{x}(t) - x(t)$ is a solution of the BVP

$$\mathcal{L}y = g, \quad ly = \gamma.$$

By (15), this implies

$$|\widetilde{x}(t) - x(t)|_n \le \int_0^T \|G(t,s)\|\,|g(s)|_n\,ds + \|X(t)(lX)^{-1}\|\,|\gamma|_n.$$

The representation (16) ensures that an approximation

$$\widetilde{G}(t,s) = \chi(t,s)\widetilde{C}(t,s) - U(t)V(s) \tag{18}$$

as well as the estimate Δ_G,

$$\|G - \widetilde{G}\|_{L^n \to L^n_\infty} \le \Delta_G$$

can be calculated, where G, $\widetilde{G} : L^n \to L^n_\infty$ are the linear integral Fredholm operators with the kernels $G(t,s)$ and $\widetilde{G}(t,s)$, respectively. Thus we can get

$$\begin{aligned}|\widetilde{x}(t) - x(t)|_n \le \int_0^T \|\widetilde{G}(t,s)\|\,|g(s)|_n\,ds + \Delta_G \int_0^T |g(s)|_n\,ds + \\ + \|X(t)(lX)^{-1}\|\,|\gamma|_n, \quad t \in [0,T].\end{aligned} \tag{19}$$

4. An Approximation of the Cauchy Matrix and Its Error Bound. Let us split the interval [0,T] into $(N+1)$ equal parts by the points $0 \le t_0 < t_1 < \cdots < t_N < t_{N+1} = T$ and denote $t_{i+1} - t_i = h$.

Next, on every square

$$\Box_{ij} \stackrel{def}{=} (t_i, t_{i+1}) \times (t_{j-1}, t_j), \quad i = 1, \dots, N, \ \ j = 1, \dots, i$$

we replace the matrix $K(t,s)$ by a constant $n \times n$-matrix K_{ij} and assume that constant $n \times n$-matrices ΔK_{ij} are known such that

$$\|K(t,s) - K_{ij}\| \le \Delta K_{ij}, \quad (t,s) \in \Box_{ij}, \ \ i = 1, \dots, N, \ \ j = 1, \dots, i.$$

Here for a matrix $A = \{a^{ij}\}$, the symbol $\|A\|$ means the matrix $\{|a^{ij}|\}$.

Denote

$$\eta_i(t) = \begin{cases} 1, & t \in [t_i, t_{i+1}], \\ 0, & t \notin [t_i, t_i + 1], \end{cases} \qquad i = 0, 1, \dots, N,$$

$$H = \begin{pmatrix} E & 0 & 0 & \dots & 0 \\ -hK_{22} & E & 0 & \dots & 0 \\ -hK_{32} & -hK_{33} & E & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -hK_{N2} & -hK_{N3} & -hK_{N4} & \dots & E \end{pmatrix}, \quad H^{-1} = \{B_{ij}\}_{i,j=1,\dots,N},$$

$$\widetilde{K}(t,s) = K_{ij}, \quad (t,s) \in \Box_{ij}, \ \ i = 1, \dots, N, \ \ j = 1, \dots, i.$$

The resolvent kernel $\widetilde{R}(t,s)$ for the kernel $\widetilde{K}(t,s)$ can be found in the explicit form [7]:

$$\widetilde{R}(t,s) = \sum_{i=1}^{N} \eta_i(t) \sum_{k=1}^{i} R_{ik}\eta_{k-1}(s), \tag{20}$$

where

$$R_{ik} = \sum_{j=k}^{i} B_{ij}K_{jk}.$$

Now, we define

$$\widetilde{C}'_t(t,s) \overset{def}{=} \widetilde{R}(t,s), \quad \widetilde{C}(t,s) = E + \int_s^t \widetilde{R}(\tau,s)\,d\tau.$$

Also define linear operators K, $\widetilde{K}$, $\widetilde{R}$, $\Delta K : L^n \to L^n$ as the integral Volterra operators with the kernels $K(t,s)$, $\widetilde{K}(t,s)$, $\widetilde{R}(t,s)$, and $[\widetilde{K}(t,s) - K(t,s)]$, respectively.

Under the assumption that

$$\delta \overset{def}{=} \|\Delta K(I+\widetilde{R})\|_{L^n \to L^n} < 1, \tag{21}$$

we have, by means of the theorem on invertible operator (see, for instance, [8]), the following estimate

$$\|C - \widetilde{C}\|_{L^n \to L^n_\infty} \le \frac{\delta}{1-\delta}\,\|I+\widetilde{R}\|_{L^n \to L^n}. \tag{22}$$

Thus we can replace the constant Δ_C in (17) by the right side of the inequality (22).

As for an error bound of $\widetilde{G}$, we note that both (16) and (18) imply the equality

$$\|G - \widetilde{G}\|_{L^n \to L^n_\infty} = \|C - \widetilde{C}\|_{L^n \to L^n_\infty},$$

hence we can put in (18) $\Delta_G = \Delta_C$.

Finally note that estimates to both δ and $\|I+\widetilde{R}\|_{L^n \to L^n}$ can be calculated by a special computer program of calculation with rational numbers.

5. An illustrative example. Consider the IVP

$$\begin{gathered} \dot{x}(t) - p(t)x_h(t) = 1, \quad t \in [0,5], \\ x(0) = 0, \end{gathered} \tag{23}$$

where

$$x_h(t) = \begin{cases} x[h(t)], & \text{as} \quad h(t) \in [0,5], \\ 0 & \text{as} \quad h(t) \notin [0,5], \end{cases}$$

$$p(t) = \eta_1(t) - 2\eta_2(t) - 2\eta_3(t) + 3\eta_4(t) - \eta_6(t) - \eta_7(t) + 4\eta_8(t) + 4\eta_9(t),$$

$$h(t) = 0, \quad 4\eta_1(t) + 0.9\eta_2(t) + 0.1\eta_3(t) + 0.7\eta_4(t) - \eta_5(t) + \\ +0.2\eta_6(t) + \eta_7(t) + 2\eta_8(t) + 3\eta_9(t). \\ \eta_i(t) = \chi_{[0.5i, 0.5i+1]}(t), \quad i = 1, \dots, 9.$$

Let $\widetilde{x}(t)$ be an approximate solution of (23) such that the correspondig residual function $g(t)$ allows the estimate

$$|g(t)| \le \varepsilon, \quad t \in [0, 5].$$

The a posteriori error bound obtained in the traditional way (see, n. 3, this paper) is as follows:

$$|\widetilde{x}(t) - x(t)| \le \frac{5}{9}(e^9 - 1)\varepsilon, \quad t \in [0, 5].$$

Our approach gives the estimate

$$|\widetilde{x}(t) - x(t)| \le 165\varepsilon, \quad t \in [0, 5]$$

(here we have used the estimate $\|\widetilde{R}\|_{L^1 \to L^1} \le 10$ obtained as it was described above).

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