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Luis Adauto Medeiros and Juan Limaco Ferrel

ELLIPTIC REGULARIZATION AND NAVIER–STOKES SYSTEM

Abstract. Applications of the method of elliptic regularization to the Navier–Stokes system in a bounded domain of $\mathbf{R}^3$ with moving boundary are given. We follow the ideas of J. L. Lions [7] and Temam [13].

რეზიუმე. ნაშრომში მოყვანილია R^3-ში მოთავსებულ მოძრავსაზღვრიან შემოსაზღვრულ არეში ნავიე-სტოქსის სისტემისათვის ელიფსური რეგულარიზაციის მეთოდის ზოგიერთი გამოყენება. მასში გამოყენებულია ჟ. ლ. ლიონსის [7] და ტემამის [13] იდეები.

Introduction

There is an extensive literature about the existence of weak solutions for the Navier–Stokes system. In the references at the end of this article, one can find a small selection, far from completion, which can lead the reader indications to other references. Starting from the sixties, there appear many results on the Navier-Stokes system in noncylindrical domains, as, for example, Fujita–Sauer [5] and Lions [7]. Among the most employed methods, there was the so called penalty method, which consists in considering a perturbation, by means of a term with a parameter ε, transforming the problem into a cylindrical domain, and passing to limit as ε tends to zero. The objective of the present article is to study the Navier-Stokes system by a method idealized by Lions in 1983, cf. [7]. It consists in transforming a parabolic problem into a family of elliptic problems indexed by a parameter $\varepsilon > 0$. We solve these problems by elliptic methods and try to obtain the solution of the original parabolic problem as a limit as ε tends to zero. This method is also used, in a certain sense, by Salvi [11] in a different context. For the study of the Navier-Stokes system in the noncylindrical case by another method, we refer to M. Milla Miranda J. Limaco Ferrel [10]. The abstract aspect of the elliptic regularization can be found in Lions [9], and for the Navier–Stokes system in cylindrical domains see Temam [14], [15] or Tartar [13] and the references therein. We also use a techniques of Fujita-Sauer [5].

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1. Notation and Main Results

Let $T > 0$ be a number and $\{\Omega_t\}_{0\leq t\leq T}$ be a family of bounded open sets Ω_t of $\mathbf{R}^{n+1}$ with regular boundaries Γ_t. We consider the noncylindrical domain $\widehat{Q}$ in $\mathbf{R}^{n+1}$ defined by

$$\widehat{Q} = \underset{0<t<T}{\cup} \Omega_t \times \{t\},$$

with the lateral boundary

$$\widehat{\Sigma} = \underset{0<t<T}{\cup} \Gamma_t \times \{t\}.$$

We suppose this boundary $\widehat{\Sigma}$ to be regular.

We will consider the Navier-Stokes system in $\widehat{Q}$

$$\begin{cases} D_t u - \nu\Delta u + \sum_{j=1}^n y\frac{\partial u}{\partial x_j} = f - \nabla p \quad \text{on} \quad \widehat{Q}, \\ \operatorname{div} u = 0 \quad \text{on} \quad \widehat{Q}, \\ u = 0 \quad \text{on} \quad \widehat{\Sigma}, \\ u(x,0) = u_0(x) \quad \text{on} \quad \Omega_0. \end{cases} \tag{1}$$

Here x is a vector $(x_1, x_2, \ldots, x_n)$ of $\mathbf{R}^n$ and $u(x,t)$ is the velocity vector $(u_1(x,t), u_2(x,t), \ldots, u_n(x,t))$, $\Delta u = (\Delta u_1, \Delta u_2, \ldots, \Delta u_n)$, with Δ the Laplace operator in x, D_t is $\frac{\partial}{\partial t}$ and ∇=grad is the operator $(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n})$. Note that ν is the viscosity of the fluid and is supposed to be a positive constant.

By $\mathcal{D}(\Omega_t)$ we denote the space of all real C^∞-functions with a compact support in Ω_t, and

$$\mathcal{V}_t = \Big\{\varphi \in (\mathcal{D}(\Omega_t))^n; \;\; \operatorname{div}\varphi = 0 \;\text{ on }\; \Omega_t\Big\}.$$

We denote by $H^s(\Omega_t)$ the Sobolev space of the order $s > 0$ on Ω_t and by $V_s(\Omega_t)$ the closure of $\mathcal{V}_t$ in $(H^s(\Omega_t))^n$. The particular cases important for the present work are

$$V_1(\Omega_t) = V(\Omega_t) \quad \text{and} \quad V_0(\Omega_t) = H(\Omega_t)$$

with Hilbert structure given respectively by

$$(u,v)_{H(\Omega_t)} = \sum_{i=1}^n \int_{\Omega_t} u_i v_i \, dx \quad \text{and} \quad ((u,v))_{V(\Omega_t)} = \sum_{i,j=1}^n \int_{\Omega_t} \frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\, dx.$$

When $s > \frac{n}{2}$, we know by Sobolev theorem that $H^1(\Omega_t)$ is continuously embedded in $H_0^1(\Omega_t)$. Then, for $s > \frac{n}{2}$ we have

$$V(\Omega_t) = \Big\{u \in (H_0^1(\Omega_t))^n; \;\; \text{div } u = 0 \;\text{ on }\; \Omega_t\Big\}$$

(cf. Lions [9]). Note that $H(\Omega_t)$ is the closure of $\mathcal{V}_t$ in $(L^2(\Omega_t))^n$ and the embedding $V(\Omega_t) \subset H(\Omega_t)$ is continuous and dense.

We consider a trilinear form $b(u,v,w)$ defined for u, v, $w \in V(\Omega_t)$ by

$$b(u,v,w) = \sum_{i,j=1}^{n} \int_{\Omega_t} u_j \frac{\partial v_i}{\partial x_j} w_i \, dx.$$

We also consider the space

$$W = \Big\{ \varphi \in L^2(0,T;V(\Omega_t)); \quad D_t\varphi \in L^2(0,T;H(\Omega_t)) \\ \text{and} \quad \varphi_i(x,T) = 0 \ \text{ in } \ \Omega_t, \quad i = 1,2,\dots,n \Big\}$$

equipped with the Hilbert structure

$$(\varphi,\psi)_W = \int_0^T ((\varphi,\psi))_{V(\Omega_t)} \, dt + \int_0^T (D_t\varphi, D_t\psi)_{H(\Omega_t)} \, dt,$$

so

$$\|\varphi\|_W^2 = \int_0^T \|\varphi\|_{V(\Omega_t)}^2 \, dt + \int_0^T |D_t\varphi|_{H(\Omega_t)}^2 \, dt.$$

Let us consider $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$ in W. Multiply the i-component of the vector u in (1) by φ_i and integrate on $\widehat{Q}$. We obtain

$$\int_0^T \int_{\Omega_t} D_t u_i \cdot \varphi_i \, dx dt + \sum_{j=1}^{n} \nu \int_0^T \int_{\Omega_t} \frac{\partial u_i}{\partial x_j} \frac{\partial \varphi_i}{\partial x_j} \, dx + \\ + \sum_{j=1}^{n} \int_0^T \int_{\Omega_t} u_j \frac{\partial u_i}{\partial x_j} \varphi_i \, dx dt = \int_0^T \int_{\Omega_t} f_i \varphi_i \, dx dt. \qquad (2)$$

Remark 1. By the Gauss Lemma, we obtain

$$\sum_{i=1}^{n} \int_{\widehat{Q}} D_t(u_i \cdot \varphi_i) \, dx dt = \sum_{i=1}^{n} \int_{\partial \widehat{Q}} u_i \cdot \varphi_i \nu_t \, d(\partial \widehat{Q}) = - \sum_{i=1}^{n} \int_{\Omega_0} u_i(x,0) \varphi_i(x,0) \, dx$$

since we have $u = 0$ on $\widehat{\Sigma}$ by (1) and $\varphi_i(x,T) = 0$ on Ω_T by $\varphi \in W$. Note that $\partial \widehat{Q} = \widehat{\Sigma} \cup \Omega_0 \cup \Omega_T$ with $\nu = (0,0,\dots,-1)$ on Ω_0 and $\nu = (0,0,\dots,1)$ on Ω_T.

Summing in (2) with respect to $1 \le i \le n$, we have by Remark 1

$$\int_0^T \big((u(t),\varphi(t))\big)_{V(\Omega_t)}\,dt - \int_0^T (u(t),D_t\varphi(t))_{H(\Omega_t)}\,dt + \int_0^T b(u(t),u(t),\varphi(t))dt = \\ = \int_0^T (f(t),\varphi(t))_{H(\Omega_t)}\,dt + (u_0,\varphi(0))_{H(\Omega_0)}. \tag{3}$$

Definition 1. By a solution of the system (1) we mean a function

$$u \in L^2(0,T;V(\Omega_t))$$

satisfying (3) for all $\varphi \in W$.

Theorem 1. *Given*

$$u_0 \in H(\Omega_0) \quad \textit{and} \quad f \in L^2(0,T;H(\Omega_t)),$$

the system (1) is solvable for $n \le 3$.

The proof of Theorem 1 will be carried out by the elliptic regularization method as in Lions [7]. The method consists in considering a perturbation of (3) by the term $\varepsilon \int_0^T (D_t u, D_t\varphi)_{H(\Omega_t)}\,dt$ in order to obtain the scalar product of W in the right-hand side of (3), which permits to employ successfully the Galerkin method. We prove that the perturbed problem has a solution u^ε for each $\varepsilon > 0$, which converges to a solution u claimed in Theorem 1.

Proposition 1. *Let u_0 and f be as in Theorem* 1. *Then for each $\varepsilon > 0$ and $\varphi \in W$ there exists in W a solution u^ε of the equation*

$$\int_0^T \big((u^\varepsilon(t),\varphi(t))\big)_{V(\Omega_t)}\,dt + \varepsilon\int_0^T (D_t u^\varepsilon, D_t\varphi)_{H(\Omega_t)}\,dt + \\ + \int_0^T b\big(u^\varepsilon(t),u^\varepsilon(t),\varphi(t)\big)\,dt - \int_0^T (u^\varepsilon(t),D_t\varphi(t))_{H(\Omega_t)}\,dt = \\ = \int_0^T (f(t),\varphi(t))_{H(\Omega_t)}\,dt + (u_0,\varphi(0))_{H(\Omega_0)}. \tag{4}$$

Proof. Let $(w_k(x,t))$ be a Hilbert basis for W (cf. Brezis [1]). Denote by W_m the subspace of W generated by the first m vectors $w_k(x,t)$. If $u_m(x,t) \in W_m$, then

$$u_m(x,t) = \sum_{j=1}^m \xi_j\, w_j(x,t)$$

with real ξ_j, $1 \le j \le m$.

The approximate problem consists in finding a solution $u_m \in W_m$ of the system of algebraic equations

$$\int_0^T ((u_m(t), w_i))_{V(\Omega_t)}\,dt + \varepsilon \int_0^T (D_t u_m(t), D_t w_i)_{H(\Omega_t)}\,dt +$$
$$+ \int_0^T b\big(u_m(t), u_m(t), w_i\big)\,dt - \int_0^T (u_m(t), D_t w_i)_{H(\Omega_t)}\,dt =$$
$$= \int_0^T (f(t), w_i)_{H(\Omega_t)}\,dt + (u_0, \varphi(0))_{H(\Omega_t)}\,dt. \tag{5}$$

We need to prove that (5) has a solution u_m in W_m. To this end, we will apply the sharp angle lemma (cf. Lions [9], Visik [17] or Temam [14]). In fact, consider the vector $\eta = (\eta_i)_{1 \le i \le m}$ of $\mathbf{R}^m$ defined by

$$\eta_i = \int_0^T ((u_m, w_i))_{V(\Omega_t)}\,dt + \varepsilon \int_0^T (D_t u_m, D_t w_i)_{H(\Omega_t)}\,dt + \int_0^T b(u_m, u_m, w_i)\,dt -$$
$$- \int_0^T (u_m, D_t w_i)_{H(\Omega_t)}\,dt - \int_0^T (f, w_i)_{H(\Omega_t)}\,dt - (u_0, \varphi(0))_{H(\Omega_t)}.$$

Let $\xi = (\xi_i)_{1 \le i \le m}$ be the components of the vector u_m of W_m. The mapping $P : \mathbf{R}^n \to \mathbf{R}^n$ defined by $P\xi = \eta$ is continuous. If we prove that $(P\xi, \xi) \ge 0$ for $\|\xi\|_{\mathbf{R}^n} = \rho$, with an appropriate ρ, it will follow by the sharp angle lemma that there exists a ξ in the ball $B_\rho(0) \subset \mathbf{R}^m$ such that $P\xi = 0$. This implies the existence of a solution to (5). In fact, we have

$$(P\xi, \xi) = \sum_{i=1}^m \xi_i \eta_i = \int_0^T \|u_m\|^2_{V(\Omega_t)}\,dt + \varepsilon \int_0^T |D_t u_m|^2_{H(\Omega_t)}\,dt +$$
$$+ \frac{1}{2}|u_m(x,0)|^2_{H(\Omega_0)} - \int_0^T (f, u_m)_{H(\Omega_t)}\,dt - (u_0, u_m(0))_{H(\Omega_0)},$$

because $b(u, v, w) = 0$ if $v = w$. This implies that

$$(P\xi, \xi) \ge a\|u_m\|^2_W - b\|u_m\|_W - c,$$

where a, b, c are positive constants.

We can consider ρ to be so large that $(P\xi, \xi) \ge 0$ for $\|\xi\|_{\mathbf{R}^m} = \rho$. Then $P\xi = 0$ for some $\xi \in B\rho(0)$, which implies that the system (5) has a solution $u_m \in W_m$ corresponding to this ξ. Therefore, for any $\varepsilon > 0$ there exists a solution $u^\varepsilon_m \in W_m$ to (5). ■

ESTIMATES

Multiplying both sides of (5) by ξ_i and summing with respect to $1 \leq i \leq m$, we obtain

$$\int_0^T \|u_m^\varepsilon(t)\|_{V(\Omega_t)}^2\,dt + \varepsilon\int_0^T |D_t u_m^\varepsilon(t)|_{H(\Omega_t)}^2\,dt +$$

$$+\frac{1}{2}\,|u_m^\varepsilon(0)|_{H(\Omega_0)}^2 + \int_0^T (f, u_m^\varepsilon(t))_{H(\Omega_t)}\,dt + (u_0, u_m^\varepsilon(0))_{H(\Omega_0)}. \tag{6}$$

From (6), we get:

$$u_m^\varepsilon, \quad \text{as a function of} \quad \varepsilon > 0 \quad \text{and} \quad m, \quad \text{is bounded in}$$
$$L^2((0,T;V(\Omega_t)) \quad \text{as} \quad \varepsilon \to 0, \tag{7}$$
$$\sqrt{\varepsilon}D_t u_m^\varepsilon, \quad \text{as a function of} \varepsilon > 0 \quad \text{and} \quad m, \quad \text{is bounded in}$$
$$L^2(0,T;V(\Omega_t)) \quad \text{as} \quad \varepsilon \to 0. \tag{8}$$

Now, if we fix $\varepsilon > 0$, we can extract a subsequence (u_k^ε) of (u_m^ε) such that

$$u_k^\varepsilon \rightharpoonup u^\varepsilon \quad \text{weakly in} \quad L^2(0,T;V(\Omega_t)), \tag{9}$$
$$\sqrt{\varepsilon}D_t u_k^\varepsilon \rightharpoonup \sqrt{\varepsilon}D_t u^\varepsilon \quad \text{weakly in} \quad L^2(0,T;H(\Omega_t)), \tag{10}$$
$$u_m^\varepsilon(0) \rightharpoonup u^\varepsilon(0) \quad \text{in} \quad H(\Omega_0). \tag{11}$$

Analysis of the Nonlinear Term

Fixed $\varepsilon > 0$, from (9) and (10) we have

$$u_k^\varepsilon \quad \text{is bounded in} \quad L^2(0,T;V(\Omega_t)), \tag{12}$$
$$D_t u_k^\varepsilon \quad \text{is bounded in} \quad L^2(0,T;H(\Omega_t)). \tag{13}$$

Then

$$u_k^\varepsilon \to u^\varepsilon \quad \text{strongly in} \quad L^2(0,T;H(\Omega_t)) \tag{14}$$

since the embedding $V(\Omega_t) \subset H(\Omega_t)$ is compact.
There exists a subsequence such that

$$u_{ki}^\varepsilon \to u_i^\varepsilon \quad \text{a.e. on} \quad \widehat{Q}, \quad i = 1,2,3. \tag{15}$$

Then,

$$u_{k\ell}^\varepsilon u_{ks}^\varepsilon \to u_\ell^\varepsilon u_s^\varepsilon \quad \text{a.e. on} \quad \widehat{Q} \tag{16}$$

as $k \to \infty$ for $1 \leq \ell$, $s \leq 3$ and $\varepsilon > 0$ fixed.

By Sobolev's embedding theorem, $(u^\varepsilon_{kj})_{k\in\mathbf{N}}$, $1 \le j \le 3$, is bounded in $L^4(\widehat{Q})$, which yields

$$(u^\varepsilon_{k\ell}\, u_{ks})_{k\in\mathbf{N}}, \quad 1 \le \ell, s \le 3, \quad \text{is bounded in} \quad L^2(0,T;L^2(\Omega_t)). \tag{17}$$

By Lions [9], it follows from (16) and (17) that

$$u^\varepsilon_{k\ell}\, u^\varepsilon_{ks} \rightharpoonup u^\varepsilon_\ell\, u^\varepsilon_s \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega_t)). \tag{18}$$

We have from (18)

$$\int_0^T \int_{\Omega_t} u^\varepsilon_{k\ell}\, \frac{\partial w_s}{\partial x_\ell}\, u^\varepsilon_{ks}\, dxdt \to \int_0^T \int_{\Omega_t} u^\varepsilon_\ell\, \frac{\partial w_s}{\partial x_\ell}\, u^\varepsilon_s\, dxdt$$

as $k \to \infty$ for all $w_s \in W_m \subset W$, or

$$\int_0^T b(u^\varepsilon_k, u^\varepsilon_k, w_i)\, dt \to \int_0^T b(u^\varepsilon, u^\varepsilon, w_i)\, dt \tag{19}$$

for all $w_i \in W_m$.

If we insert $m = k$ in (5) and take the limits as $k \to \infty$ and $\varepsilon > 0$ is fixed, we obtain

$$u^\varepsilon \in L^2(0,T;V(\Omega_t)), \quad D_t u^\varepsilon \in L^2(0,T;H(\Omega_t))$$

and

$$\int_0^T ((u^\varepsilon,\varphi))_{V(\Omega_t)}\, dt + \varepsilon \int_0^T (D_t u^\varepsilon, D_t\varphi)_{H(\Omega_t)}\, dt + \int_0^T b(u^\varepsilon, u^\varepsilon, \varphi)\, dt -$$
$$- \int_0^T (u^\varepsilon, D_t\varphi)_{H(\Omega_t)}\, dt = \int_0^T (f,\varphi)_{H(\Omega_t)}\, dt - (u_0, \varphi(0))_{H(\Omega_0)} \tag{20}$$

for all $\varphi \in W$.

Proof of the Theorem 1. We must obtain estimates for $D_t u^\varepsilon$, where u^ε is a solution of (20), which would permit to pass to limit as $\varepsilon \to 0$. To this end, we apply an argument suggested by Lions [7] which consists in proving that $D_t u^\varepsilon$ is bounded in the L^2-norm in each cylinder contained in $\widehat{Q}$.

In fact, let $\mathcal{O}$ be an open bounded subset of $\mathbf{R}^n$. By $V(\mathcal{O})$ and $H(\mathcal{O})$ we denote the usual function spaces as defined in the case of Ω_t, (cf. §1). Identifying $H(\mathcal{O})$ with its dual $H(\mathcal{O})'$, we obtain the continuous and dense embeddings $V(\mathcal{O}) \subset H(\mathcal{O}) \subset V(\mathcal{O})'$. We denote by $\langle\ ,\ \rangle$ the scalar product of the objects from $V(\mathcal{O})'$ and $V(\mathcal{O})$. Consider an interval $]t_0, t_1[$ contained in $]0,T[$ and a cylinder $\mathcal{O}\times\,]t_0,t_1[$ such that

$$\overline{\mathcal{O}} \times [t_0, t_1] \subset \widehat{Q},$$

where $\overline{\mathcal{O}}$ is the closure of $\mathcal{O}$. For $\psi \in \mathcal{D}(]t_0, t_1[)$ and $v \in V(\mathcal{O})$, we consider the function $\varphi = \psi \otimes v$ defined by

$$\varphi(x,t) = \psi(t) \cdot v(x) \quad \text{for all} \quad (x,t) \in \mathcal{O} \times]t_0, t_1[\,.$$

Extend ψ by zero outside of its support and v by zero outside of $\mathcal{O}$, and still denote their product by φ, which belongs to W. Then, inserting φ in (20), we get

$$\int_{t_0}^{t_1} ((u^\varepsilon(t), v))_{V(\Omega_t)} \psi(t) dt + \varepsilon \int_{t_0}^{t_1} (D_t u^\varepsilon(t), v) D_t \psi dt - \int_{t_0}^{t_1} b(u^\varepsilon(t), v, u^\varepsilon(t)) \psi(t)\, dt -$$
$$- \int_{t_0}^{t_1} (u^\varepsilon(t), v) D_t \psi(t)\, dt = \int_{t_0}^{t_1} (f(t), v) \psi(t)\, dt. \tag{21}$$

We have $n = 1, 2, 3$, which implies that the linear form

$$v \to b(u^\varepsilon, v, u^\varepsilon)$$

is continuous on $L^2(t_0, t_1, V(\mathcal{O}))$. Then there exists $Bu^\varepsilon \in L^1(t_0, t_1, V(\mathcal{O})')$ such that $\langle Bu^\varepsilon, v\rangle = b(u^\varepsilon, v, u^\varepsilon)$. Note that $f \in L^2(t_0, t_1, H(\mathcal{O}))$. Then from (21) we obtain

$$-\varepsilon \int_{t_0}^{t_1} D_t^2 (u^\varepsilon, v) \psi\, dt + \int_{t_0}^{t_1} D_t (u^\varepsilon, v) \psi\, dt = \int_{t_0}^{t_1} \langle g, v\rangle \psi\, dt \tag{22}$$

for all $\psi \in D]t_0, t_1[$ extended by zero outside of its support and with $g \in L^1(t_0, t_1; V(\mathcal{O})')$.

From (22) and Temam [13], Ch.III, Lemma 1.1, it follows that

$$-\varepsilon D_t^2 u^\varepsilon + D_t u^\varepsilon = g \quad \text{in} \quad L^1(t_0, t_1; V(\mathcal{O})'). \tag{23}$$

Our next step is to obtain from (23) a uniform bound for $D_t u^\varepsilon$ in the space $L^1(\tau_0, \tau_1; V(\mathcal{O})')$, where $[\tau_0, \tau_1[\, \subset [t_0, t_1[\,$.

In fact, consider $\tau_0 = t_0$ and $\tau_0 < \tau_1 < t_1$ such that $\overline{\mathcal{O}} \times [t_0, t_1] \subset \widehat{Q}$. Let $\theta(t)$ be a smooth real function defined on $[t_0, t_1]$ and satisfying $0 \le \theta(t) \le 1$, $\theta(t) = 1$ near t_0 and $\theta(t) = 0$ near t_1.

If we put $v^\varepsilon = D_t u^\varepsilon$, we transform (23) into

$$-\varepsilon D_t(\theta v^\varepsilon) + \theta v^\varepsilon = h \quad \text{in} \quad L^1(t_0, t_1; V(\mathcal{O})') \tag{24}$$

with $h = \theta g - (D_t \theta) \varepsilon D_t u^\varepsilon$.

Note that h is bounded in $L^1(\tau_0, \tau_1; V(\mathcal{O})')$ because $\varepsilon D_t u^\varepsilon$ is bounded in $L^2(\tau_0, \tau_1; H(\mathcal{O}))$ for all $\varepsilon > 0$. Note that the uniform estimate for $D_t u^\varepsilon$ will be obtained for each cylinder $\mathcal{O} \times]\tau_0, \tau_1[$ whose closure is contained in $\widehat{Q}$.

Let

$$E_\varepsilon(t) = \left| \begin{array}{lll} \dfrac{1}{\varepsilon} & \exp\left(\dfrac{t}{\varepsilon}\right) & \text{if} \quad t < 0 \\ & 0 & \text{if} \quad t > 0 \end{array} \right.$$

for $\varepsilon > 0$. This function belongs to $L^1(-\infty, +\infty)$ and the solution of (24) is given by

$$\theta v^\varepsilon = E_\varepsilon * h, \tag{25}$$

where $*$ denotes the convolution of functions. We have $E_\varepsilon \in L^1(-\infty, +\infty; \mathbf{R})$ and $h \in L^1(\tau_0, \tau_1; V(\mathcal{O})')$. By the Young inequality it follows that

$$\|E_\varepsilon * h\|_{L^2(\tau_0,\tau_1;V(\mathcal{O})')} \leq \|E_\varepsilon\|_{L^1(-\infty,+\infty)} \|h\|_{L^1(\tau_0,\tau_1;V(\mathcal{O})')}.$$

But $\|E_\varepsilon\|_{L^1(-\infty,+\infty)} = 1$ and h is bounded in $L^1(\tau_0, \tau_1; V(\mathcal{O})')$. Then, by (25), we have

$$\int_{\tau_0}^{\tau_1} \|\theta v^\varepsilon\|^2 \, dt < c,$$

which implies

$$\int_{\tau_0}^{\tau_1} \|v^\varepsilon\|^2 \, dt = \int_{\tau_0}^{\tau_1} \|\theta v^\varepsilon\|^2 \, dt < c,$$

$\theta(t) = 1$ on $[\tau_0, \tau_1]$. Hence

$$D_t u^\varepsilon = v^\varepsilon \quad \text{is bounded in} \quad L^1(\tau_0, \tau_1; V(\mathcal{O})') \tag{26}$$

uniformly with respect to $\varepsilon > 0$ as $\varepsilon \to 0$.

Thus u^ε is bounded in $L^2(\tau_0, \tau_1; V(\mathcal{O}))$ and $D_t u^\varepsilon$ is bounded in $L^1(\tau_0, \tau_1; V(\mathcal{O})')$. We know that the embeddings $V(\mathcal{O}) \subset H(\mathcal{O}) \subset V(\mathcal{O})'$ are continuous and $V(\mathcal{O}) \subset H(\mathcal{O})$ is compact. Then, by the compactness theorem (cf. Lions [9], Temam [14], Simon [11]), we can extract a subsequence $u^{\varepsilon'}$ of u^ε such that

$$u_i^{\varepsilon'} \to u_i \quad \text{strongly in} \quad L^1(\tau_0, \tau_1; L^2(\mathcal{O})), \quad i = 1, 2, 3, \tag{27}$$

and a.e. on the cylinder $\mathcal{O} \times]\tau_0, \tau_1[$.

Consider $\varepsilon' = \frac{1}{n}$ with $n \in N$. Then we obtain a subsequence (u_n') such that

$$u_{ni} \to u_i \quad \text{strongly in} \quad L^1(\tau_0, \tau_1; L^2(\mathcal{O})) \;\; \text{for} \;\; i = 1, 2, 3 \tag{28}$$

and a.e. in the cylinder $\mathcal{O} \times]\tau_0, \tau_1[$. This argument is true for each cylinder $\mathcal{O} \times]\tau_0, \tau_1[$ such that its closure is contained in $\widehat{Q}$.

Our next step is to prove, employing the above result, that there exists a subsequence which we will still denote by (u_n), and $u \in L^1(\widehat{Q})$ such that

$$u_{ni} \to u_i \quad \text{strongly in} \quad L^1(\widehat{Q}) \;\; (i = 1, 2, 3)$$

a.e. on $\widehat{Q}$. The key points are a geometrical construction of Fujita and Sauer [5] and the diagonal process.

In fact, for $\delta > 0$ we consider a strip near the boundary $\widehat{\Sigma}$, but interior to $\widehat{Q}$. We consider first

$$W(\delta, t) = \{x \in \Omega_t;\ \ d(x, \Gamma_t) < \delta\},$$

where $d(x, \Gamma_t)$ is the distance from x, in Ω_t, to its boundary Γ_t. Then we define the strip

$$\widehat{W}(\delta) = \underset{0<t<T}{\cup} W(\delta, t) \times \{t\},$$

of length δ, inside of $\widehat{Q}$, near the boundary $\widehat{\Sigma}$.

Now let us construct, inside of $\widehat{Q}$, a domain not near the boundary $\widehat{\Sigma}$. Let ℓ be a natural number and

$$\Omega_t^\ell = \Big\{x \in \Omega_t;\ \ d(x, \Gamma_t) > \frac{1}{\ell}\Big\}.$$

Denoting by (t_j) a dense sequence in $[0, T]$, we define the cylinders contained in $\widehat{Q}$

$$G_{jk\ell} = \Omega_{t_j}^\ell \times]t_j, t_k[$$

such that their closure is inside of $\widehat{Q}$. Consider the set

$$S = \Big\{G_{jk\ell};\ \ j, k, \ell \in \mathbf{N} \ \text{ and the closure is in } \ \widehat{Q}\Big\},$$

which is countable.

Remark 2. Since for small $\delta > 0$ the closure of $\widehat{Q} - \widehat{W}(\delta)$ is compact in $\widehat{Q}$, there exists a finite covering $G_j \subset S$, $j = 1, 2, \ldots, N(\delta)$ such that

$$\widehat{Q} - \widehat{W}(\delta) \subset \overset{N(\delta)}{\underset{j=1}{\cup}} G_j,$$

or

$$\widehat{Q} \subset \Big(\overset{N(\delta)}{\underset{j=1}{\cup}} G_j\Big) \cup \widehat{W}(\delta).$$

Lemma 1. *If $\delta > 0$ is small, then for $v \in L^2(0, T; V(\Omega_t))$ we have*

$$|v|^2_{L^2(\widehat{W}(\delta))} \le c\delta^2 \int_0^T \|v(t)\|^2_{V(\Omega_t)}\, dt,$$

where c is a positive constant depending on $\widehat{\Sigma}$.

Proof. We will prove the lemma for the dense subset $\mathcal{V}=\{\varphi\in\mathcal{D}(\widehat{Q});\operatorname{div}\varphi=0\}$ of $L^2(0,T;V(\Omega_t))$. Suppose $v\in\mathcal{V}$ and let ν be the exterior unit normal to Γ_t at the point (η,τ). Since $v\in\mathcal{V}$, we have $v=0$ on Γ_t, that is, $v(\eta,\tau)=0$. Let us consider $\rho>0$ and a point $(\eta-\rho\nu,\tau)$ interior to Ω_t. Note that η is the space variable and τ the time variable, $0<\tau<T$. We have

$$v(\eta-\rho\nu,\tau)-v(\eta,\tau)=\int_0^\rho D_\lambda\, v(\eta-\lambda\nu,\tau)\,d\lambda.$$

Then, taking absolute values of both sides,

$$|v(\eta-\rho\nu,\tau)|\le\int_0^\rho|\nabla v(\eta-\lambda\nu,\tau)|\,d\lambda,$$

and applying the Schwarz inequality, we get

$$|v(\eta-\rho\nu,\tau)|^2\le\rho\int_0^\rho|\nabla v(\eta-\lambda\nu,\tau)|^2\,d\lambda, \tag{29}$$

where ρ is a parameter such that $0\le\rho\le\delta$.

Note that

$$\int_0^\delta\int_{\Gamma_t}|v(\eta-\rho\nu,\tau)|^2\,d\Gamma d\rho$$

is the integral on the strip of length δ, that is, on $W(t,\delta)$. If we integrate this integral with respect to τ over $]0,T[$, we obtain the integral on $\widehat{W}(\delta)$:

$$\int_0^T\int_0^\delta\int_{\Gamma_t}|v(\eta-\rho\nu,\tau)|^2\,d\Gamma d\rho d\tau=\int_{\widehat{W}(\delta)}|v(\eta-\rho\nu,\tau)|^2\,d\Gamma d\rho d\tau=|v|^2_{L^2(\widehat{W}(\delta))}.$$

If we integrate the right-hand side of the integral (29) over $]0,T[\times]0,\delta[$, we obtain

$$\int_0^T\Big(\int_0^\delta\rho\,d\rho\Big)\int_{\Gamma_i}\int_0^\delta\Big|\nabla v(\eta-\lambda\nu,\tau)\Big|^2\,d\lambda d\Gamma d\tau=$$

$$=\frac{\delta^2}{2}\int_0^T\int_{\Gamma_i}\int_0^\delta\Big|\nabla v(\eta-\lambda\nu,\tau)\Big|^2\,d\lambda d\Gamma d\tau=$$

$$=\frac{\delta^2}{2}\int_{\widehat{Q}}\Big|\nabla v(\eta,\tau)\Big|^2\,d\eta d\tau=\frac{\delta^2}{2}\int_0^T\Big\|v(t)\Big\|^2_{V(\Omega_t)}\,dt.$$

This proves the Lemma. ■

Proposition 2. *There exists a subsequence of* (u_n), *still denoted by* (u_n), *such that it converges strongly in* $L^1(G)$ *for all cylinders* G *of the family* S.

Proof. By (27), (u_n) converges strongly in $L^1(G_j)$ to u in a cylinder G_j, $j \in \mathbf{N}$. By the diagonal process we obtain a subsequence (u_n) that converges strongly in $L^1(G)$ for all $G \in S$. ■

Proposition 3. *There exists* $u \in L^1(\widehat{Q})$ *such that the subsequence of the Proposition* 1 *converges to* u *in the* $L^1(\widehat{Q})$*-norm.*

Proof. It is sufficient to prove that (u_n) is a Cauchy sequence in the $L^1(\widehat{Q})$-norm. In fact, given $\varepsilon > 0$, choose any δ such that $0 < \delta < \varepsilon$. For this $\delta > 0$ there exists $G_j \in S$, $j = 1, 2, \ldots, N(\delta)$, such that

$$\widehat{Q} \subset \Big(\bigcup_{j=1}^{N(\delta)} G_j \Big) \cup \widehat{W}(\delta).$$

We have, for all $m, n \in \mathbf{N}$,

$$|u_m - u_n|_{L^1(\widehat{Q})} \leq \sum_{j=1}^{N(\delta)} |u_m - u_n|_{L^1(G_j)} + |u_m - u_n|_{L^1(\widehat{W}(\delta))}. \tag{30}$$

Further,

$$|u_m - u_n|_{L^1(\widehat{W}(\delta))} \leq c|u_m - u_n|_{L^2(\widehat{W}(\delta))}, \tag{31}$$

and by Lemma 1 we obtain

$$|u_m - u_n|_{L^2(\widehat{W}(\delta))} \leq \left(c\delta^2 \int_0^T \|u_m - u_n\|^2_{V(\Omega_t)}\, dt \right)^{1/2}. \tag{32}$$

By (20), (u_n) is bounded in $L^2(0, T; V(\Omega_t))$ by a constant c. Then, by (31) and (32), we obtain

$$|u_m - u_n|_{L^1(\widehat{W}(\delta))} \leq c\delta < c\varepsilon. \tag{33}$$

Since (u_n) converges strongly in $L^1(G_j)$ for all cylinders $G_j \in S$, it follows that the sum

$$\sum_{j=1}^{N(\delta)} |u_m - u_n|_{L^1(G_j)} \to 0 \quad \text{as} \quad m, n \to \infty. \tag{34}$$

From (32) and (33) it follows that (u_n) is a Cauchy sequence in the $L^1(\widehat{Q})$-norm. Then it converges strongly in $L^1(\widehat{Q})$ or a.e. in $\widehat{Q}$. ■

After establishing the convergence, we can pass to limit in (20) for $\varepsilon = \frac{1}{n}$, as $n \to \infty$. Thus we obtain the solution claimed by Theorem 1. ■

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Authors' addresses:

Luis Adauto Medeiros
Instituto de Matemática UFRJ
CP 68530 - CEP 21944-970, Rio de Janeiro, RJ

Juan Limaco Ferrel
Instituto de Matemática UFF
Rua São Paulo s/n, Rio de Janeiro, RJ
CEP 24210-110 Niterói, RJ