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MATHEMATICAL THEORY OF THE BUFFERNESS PHENOMENON IN OSCILLATORY SYSTEMS WITH DISTRIBUTED PARAMETERS

Abstract. A theoretical approach is presented to the investigation of the bufferness phenomenon in systems of differential equations. This phenomenon means the following: under an appropriate chŏice of parameters, the equation has any prescribed number of steady periodic solutions.

რეზიუმე. ნაშრომში გადმოცემულია თეორიული მიდგომა დიფერენციალურ განტოლებათა სისტემებში ბუფერულობის მოვლენისადმი. ეს მოვლენა მდგომარეობს შემდეგში: პარამეტრების სათანადოდ შერჩევისას განტოლებას აქვს ნებისმიერი წინასწარ დასახელებული რაოდენობა მდგრადი პერიოდული ამონახსნებისა.

The so-called “oscillatory systems with distributed parameters”, that is, the objects whose state depends on time and on spatial variables and whose variation of state at each point of the space takes place periodically, are widespread in physics, biology and modern engineering. Such are, for example, the well-known radio engineering devices generating periodic auto-oscillations. The work of these devices is simulated by means of partial differential equations (with certain boundary and initial conditions). To every real auto-oscillatory regime there corresponds a steady cycle (i.e., a periodic in time solution of the equation satisfying the boundary conditions).

A partial differential equation, being an adequate mathematical model of an oscillatory system with distributed parameters, may have (similar to the case of ordinary differential equations) one or several steady cycles. Naturally, a number of such cycles is, generally speaking, different for various values of parameters (which reflects quantitative physical characteristics of the object).

They say that the bufferness phenomenon takes place in an oscillatory system with distributed parameters if it possesses the following property: for any natural number N, one can choose physical characteristics of the

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system so that it has N auto-oscillatory regimes. In other words, the bufferness phenomenon consists in the fact that the mathematical model of the system, i.e., the partial differential equation, possesses under suitable values of parameters, an arbitrary, beforehand prescribed finite number of different steady cycles. Thus, by a suitable choice of parameters, an auto-oscillatory system with the bufferness property can be provided with an a priori fixed number of steady periodic regimes.

As an example of the typical oscillatory system with distributed parameters, we can mention the classical autogenerator with a long line. The problem on its auto-oscillations has been formulated by A. A. Witt [1] and studied in detail later on. Already in [1], Witt has emphasized that such an autogenerator may possess several steady cycles simultaneously; however the bufferness phenomenon, as such, has not been discovered yet.

Physicists were the first who observed experimentally the bufferness effect, the increase of the number of possible autooscillatory regimes under variation of the device parameters (see, e.g., [1]). However, mathematical investigation of the bufferness phenomenon begun on Yu. S. Kolesov's initiative who studied this phenomenon for parabolic reaction–diffusion type systems [3] by using numerical methods. Then he investigated it purely theoretically for hyperbolic equations [4] (see also [5]).

Note that the bufferness phenomenon may be of considerable interest in simulation of memory processes for the creation of memory cells [6].

Let us consider one mathematical result proving theoretically the existence of the bufferness phenomenon in the so-called LCRG autogenerator composed of a long line with a tunnel diode.

Omitting physical description of the autogenerator itself, we consider the boundary value problem (see [7]) which is its mathematical model (for a number of "probable" assumptions, in particular, for cubic approximation of the diode characteristics):

$$\begin{gathered} u_x = -Ri - Li_t, \quad i_x = -Gu - Cu_t, \\ i\big|_{x=1} = 0, \quad u\big|_{x=0} + k_0 u\big|_{x=1} + k_2 u^2\big|_{x=1} + k_3 u^3\big|_{x=1} = 0, \end{gathered} \tag{1}$$

where t is a time, x is the coordinate along the line, $u(t,x)$ and $i(t,x)$ are the tension and the current in the line, respectively; all the autogenerator parameters (such as the distributed resistance R, inductance L, conductivity G and capacity C) are constant. As is known, this problem is reduced to a boundary value problem for the so-called telegraph equation.

It is shown in [8] that the problem of finding auto-oscillations in this model is reduced (after technical transforms and change of variables) to the question of the existence and stability of periodic solutions of the nonlinear differential-difference equation

$$\begin{aligned} &\dot z - (1-\alpha\varepsilon)\dot z(t-h) + (1-\varepsilon)z(t) + (1-\alpha\varepsilon)(1+\varepsilon)z(t-h) = \\ &\quad = a[z(t) - (1-\alpha\varepsilon)z(t-h)]^2 - [z(t) - (1-\alpha\varepsilon)z(t-h)]^3; \end{aligned} \tag{2}$$

here $\varepsilon > 0$ is a small parameter, the parameter $\alpha > 0$ is of unity order, $h = const > 0$, $a = const$.

Denote by $\omega_1 < \omega_2 < \cdots$ all positive roots of the equation

$$\omega \operatorname{tg} \frac{\omega h}{2} = 1.$$

Lemma. *All the roots corresponding to zero equilibrium state of the equation* (2) *of the quasi-polynomial*

$$P(\lambda) = \lambda[1 - (1 - \alpha\varepsilon)e^{-\lambda h}] + (1 - \varepsilon) + (1 - \alpha\varepsilon)(1 + \varepsilon)e^{-\lambda h}$$

lie for $0 < \varepsilon \ll 1$ *and* $\alpha > 2(1 + \omega_1^2)^{-1}$ *in the half-plane* $\operatorname{Re}\lambda < 0$*; if we diminish the parameter* $\alpha > 0$*, then when passing through the values* $2(1 + \omega_n^2)^{-1}$*,* $n \geq 1$*, the roots of the quasi-polynomial* $P(\lambda)$ *transfer successively, one after another, to the half-plane* $\operatorname{Re}\lambda > 0$*.*

Theorem. *Let for some natural* $n \geq 1$ *the conditions*

$$0 < \alpha < \frac{2}{1 + 2\omega_n^2 - \omega_1^2}, \quad 1 + 10\omega_n^2 - 15\omega_n^4 \neq 0$$

be fulfilled. Then there exists $\varepsilon_n > 0$ *such that the equation* (2) *for every* $\varepsilon \in [0, \varepsilon_n]$ *has an exponentially* (*in metrics of the phase space* $W_2^1(-h, 0)$) *orbitally steady periodic solution* $z_n(\tau, \varepsilon)$*:*

$$z_n\Big(\tau + \frac{2\pi}{\omega_n}, \varepsilon\Big) = z_n(\tau, \varepsilon), \quad \tau = (1 + \delta_n(\varepsilon))t,$$

for which the asymptotic representation

$$z_n(\tau, \varepsilon) = \sqrt{\varepsilon}\, \frac{1 + \omega_n^2}{\sqrt{6}} \sqrt{\frac{2}{1 + \omega_n^2} - \alpha}\, \cos\omega_n\tau + O(\varepsilon), \quad \delta_n(\varepsilon) = O(\varepsilon),$$

is valid as $\varepsilon \to 0$.

It follows from the above Theorem (whose proof is rather nontrivial) that there takes place the following dynamics of the equation (2) under a suitable decrease of parameters ε and α.

Corollary. *Zero equilibrium state of the equation* (2) *for* $\alpha > 2(1 + \omega_1^2)^{-1}$ *is exponentially steady. When the parameter* α *passes through the value* $2(1+\omega_1^2)^{-1}$*, a steady cycle branches off this equilibrium state* (*the Andronov–Hopf bifurcation*)*. However, when the parameter* α *passes through each of the values* $2(1 + \omega_n^2)^{-1}$*,* $n \geq 2$*, there arises a nonsteady cycle* (*the second Andronov–Hopf bifurcation*) *which becomes steady for* $\alpha > 2(1+2\omega_n^2-\omega_1^2)^{-1}$*.*

Thus under suitable decrease of parameters ε and α, one may guarantee in the equation (2) the existence of any beforehand prescribed number of steady cycles. This in its turn means that the system (1) possesses the bufferness phenomenon, i.e., the autogenerator with a long line may possess, under a suitable choice of the values of its physical characteristics, an arbitrary number of steady periodic regimes. It should be noted that we have succeeded in illustrating the last conclusion experimentally.

To summarize the above-said, let us formulate specific features of the bufferness phenomenon from the physical point of view.

An oscillatory system with distributed parameters possessing the bufferness phenomenon has a set of steady periodic regimes; note that under a suitable choice of the parameters, this set may have arbitrarily many such regimes. For beforehand fixed values of the parameters, any potentially possible auto-oscillatory regime is realized depending on the initial conditions or external factors, but a pontaneous passage of the system to some other periodic movement is impossible.

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