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OSCILLATIONS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Consider the first order delay differential equation

$$x'(t) + p(t)x(t-\tau) = 0, \quad \tau > 0, \quad t \geq t_0, \tag{$*$}$$

and its discrete analogue

$$x_{n+1} - x_n + p_n x_{n-k} = 0, \quad k \in Z^+, \quad n = 0, 1, 2, \dots . \tag{$*$}'$$

Oscillation criteria are established for $(*)$ in the case where $0 < \liminf\limits_{t\to\infty} \int_{t-\tau}^{t} p\,(s) \leq \frac{1}{e}$ and $\limsup\limits_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < 1$, and for $(*)'$ when $\liminf\limits_{n\to\infty} \sum\limits_{i=n-k}^{n-1} p_i \leq \left(\frac{k}{k+1}\right)^{k+1}$ and $\limsup\limits_{n\to\infty} \sum\limits_{i=n-k}^{n} p_i < 1$.

რეზიუმე. ნაშრომში განხილულია პირველი რიგის დაგვიანებულარგუმენტიანი დიფერენციალური განტოლება $(*)$ და მისი დისკრეტული ანალოგი $(*)'$. დადგენილია რხევადობის პირობები $(*)$-ისთვის იმ შემთხვევაში, როცა $0 < \liminf\limits_{t\to\infty} \int_{t-\tau}^{t} p(s) \leq \frac{1}{e}$ და $\limsup\limits_{t\to\infty} \int_{t-\tau}^{t} p(s)ds < 1$ და $(*)'$-ისათვის იმ შემთხვევაში, როცა $\liminf\limits_{n\to\infty} \sum\limits_{i=n-k}^{n-1} p_i \leq \left(\frac{k}{k+1}\right)^{k+1}$ და $\limsup\limits_{n\to\infty} \sum\limits_{i=n-k}^{n} p_i < 1$.

1. Introduction

Consider the linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq T, \tag{1}$$

where p and τ are continuous functions defined on $[T, \infty), p(t) > 0, \tau(t) < t$ for $t \geq T$, $\tau(t)$ is nondecreasing and $\lim_{t\to\infty} \tau(t) = \infty$.

By a solution of the equation (1) we understand a continuously differentiable function defined on $[\tau(T_1), \infty)$ for some $T_1 \geq T$ such that (1) is satisfied for $t \geq T_1$. Such a solution is called oscillatory if it has arbitrarily large zeros. Otherwise it is called nonoscillatory.

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The first systematic study for the oscillation of all solutions of the equation (1) was undertaken by Myshkis. In 1950 [25], he proved that every solution of the equation (1) oscillates if

$$\limsup_{t\to\infty}[t-\tau(t)]<\infty, \quad \liminf_{t\to\infty}[t-\tau(t)]\cdot\liminf_{t\to\infty}p(t)>\frac{1}{e}. \qquad (\mathrm{C}_1)$$

In 1972, Ladas, Lakshmikantham and Papadakis [19] proved that the same conclusion holds if

$$\limsup_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>1. \qquad (\mathrm{C}_2)$$

In 1979 Ladas [18] and in 1982 Koplatadze and Chanturiya [14] improved (C_2) to

$$\liminf_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds>\frac{1}{e}. \qquad (\mathrm{C}_3)$$

Concerning the constant $\frac{1}{e}$ in (C_3), it is to be pointed out that if the inequality

$$\int_{\tau(t)}^{t}p(s)ds\le\frac{1}{e}$$

holds eventually, than, according to a result in [14], (1) has a non-oscillatory solution.

In 1982 Ladas, Sficas and Stavroulakis [21] and in 1984 Fukagai and Kusano [11] established oscillation criteria of the type of the conditions (C_2) and (C_3) for the equation (1) with an oscillating coefficient $p(t)$.

It is obvious that there is a gap between the conditions (C_2) and (C_3) when the limit

$$\lim_{t\to\infty}\int_{\tau(t)}^{t}p(s)ds$$

does not exist. How to fill this gap is an interesting problem which has been recently investigated by several authors.

Before the work of Erbe and Zhang [9] not much was known about the class of linear delay differential equations for which neither (C_2) nor (C_3) was satisfied. As far as we know, only the papers [4, 11, 13] contained results that could be applied also to some cases that were not covered by the above mentioned results. In 1988, Erbe and Zhang [9] developed new oscillation criteria by employing the upper bound of the ratio $x(\tau(t))/x(t)$

for possible nonoscillatory solutions $x(t)$ of the equation (1). Their result, when formulated in terms of the numbers m and L defined by

$$m = \liminf_{t\to\infty} \int\limits_{\tau(t)}^{t} p(s)ds \quad \text{and} \quad L = \limsup_{t\to\infty} \int\limits_{\tau(t)}^{t} p(s)ds,$$

says that all the solutions of the equation (1) are oscillatory if $0 < m \le \frac{1}{e}$ and

$$L > 1 - \frac{m^2}{4}. \tag{C$_4$}$$

Since then, several authors tried to obtain better results by improving the upper bound for $x(\tau(t))/x(t)$. In 1991 Jian Chao [2] derived the condition

$$L > 1 - \frac{m^2}{2(1-m)}, \tag{C$_5$}$$

while in 1992 Yu and Wang [28] and Yu, Wang, Zhang and Qian [29] obtained the condition

$$L > 1 - \frac{1 - m - \sqrt{1 - 2m - m^2}}{2}. \tag{C$_6$}$$

In 1990 Elbert and Stavroulakis [7] and in 1991 Kwong [17], using different techniques, improved (C_4) in the case where $0 < m \le \frac{1}{e}$ to the conditions

$$L > 1 - \Big(1 - \frac{1}{\sqrt{\lambda_1}}\Big)^2 \tag{C$_7$}$$

and

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1}, \tag{C$_8$}$$

respectively, where λ_1 is the smaller root of the equation $\lambda = e^{m\lambda}$.

In 1994 Koplatadze and Kvinikadze [15] improved (C_6), while in 1996 Philos and Sficas [26] derived the condition

$$L > 1 - \frac{m^2}{2(1-m)} - \frac{m^2}{2}\lambda_1. \tag{C$_9$}$$

Following this historical (and chronological) review, we also mention that in the case where

$$\int\limits_{\tau(t)}^{t} p(s)ds \ge \frac{1}{e} \quad \text{and} \quad \lim_{t\to\infty} \int\limits_{\tau(t)}^{t} p(s)ds = \frac{1}{e},$$

this problem has been studied in 1993 by Elbert and Stavroulakis [8] and in 1995 by Kozakiewicz [16], Li [23], [24] and by Domshlak and Stavroulakis

[5]. The methods previously used in [17] and [28] can be combined so that the conditions (C_2) and (C_4)-(C_9) may be weakened to

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - m - \sqrt{1 - 2m - m^2}}{2} \tag{C$_{10}$}$$

where λ_1 is the smaller root of the equation $\lambda = e^{m\lambda}$. It is to be noted that as $m \to 0$, then all conditions $(C_4) - (C_9)$ and also the condition (C_{10}) reduce to the condition (C_2). However the improvement is clear as $m \to \frac{1}{e}$. For illustrative purpose, we give the value of the lower bound in these conditions when $m = \frac{1}{e}$:

$$\begin{array}{ll}
(C_2): & 1.000000000 \\
(C_4): & 0.966166179 \\
(C_5): & 0.892951367 \\
(C_6): & 0.863457014 \\
(C_7): & 0.845181878 \\
(C_8): & 0.735758882 \\
(C_9): & 0.709011646 \\
(C_{10}): & 0.599215896
\end{array}$$

We see that the condition (C_{10}) essentially improves all the known results in the literature.

Consider next the delay difference equation

$$\Delta x_n + p_n x_{n-k} = 0, \quad n = 0, 1, 2, \ldots, \tag{1$'$}$$

where $\{p_n\}$ is a sequence of real numbers, k is a positive integer and Δ denotes the forward difference operator $\Delta x_n = x_{n+1} - x_n$. Note that the equation $(1)'$ is a discrete analogue of the equation (1).

By a solution of the equation $(1)'$ we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and which satisfies $(1)'$ for $n \geq 0$. A solution $\{x_n\}$ of the equation $(1)'$ is said to be oscillatory if the terms x_n of the solution are neither eventually all positive nor eventually negative. Otherwise, the solution is called non-oscillatory.

Erbe and Zhang [10] proved that if $p_n \geq 0$, then either one of the following conditions

$$\liminf_{n\to\infty} p_n > \frac{k^k}{(k+1)^{k+1}}, \tag{C$_0$}$$

or

$$\limsup_{n\to\infty} \sum_{i=n-k}^{n} p_i > 1 \tag{$C_2$$'$}$$

implies that all solutions of the equation $(1)'$ oscillate. Then Ladas, Philos and Sficas [20] proved that the same conclusion holds if $p_n \geq 0$ and

$$\liminf_{n\to\infty}\Big(\frac{1}{k}\sum_{i=n-k}^{n-1} p_i\Big) > \frac{k^k}{(k+1)^{k+1}}. \qquad (C_3)'$$

Therefore, they improved the condition (C_0) by replacing the p_n of (C_0) by the arithmetic mean of the terms $p_{n-k},\dots,p_{n-1}$ in $(C_3)'$. A further improvement of the above conditions is presented here as well as a sufficient condition under which all solutions of $(1)'$ oscillate without the assumption that $p_n \geq 0$ for all $n \geq 0$.

2. Main Results

We need the following lemmas which are also very interesting in their own right.

Lemma 1 ([12]). *Suppose that $m > 0$ and the equation (1) has an eventually positive solution $x(t)$. Then $m \leq 1/e$ and*

$$\lambda_1 \leq \liminf_{t\to\infty}\frac{x(\tau(t))}{x(t)} \leq \lambda_2,$$

where λ_1 and λ_2 are the roots of the equation $\lambda = e^{m\lambda}$.

Lemma 2 ([28]). *Let $0 < m \leq \frac{1}{e}$ and $x(t)$ be an eventually positive solution of the equation (1). Then*

$$\limsup_{t\to\infty}\frac{x(\tau(t))}{x(t)} \leq \frac{2}{1-m-\sqrt{1-2m-m^2}}.$$

Lemma 3 ([27]). *Assume that $\{p_n\}$ is a sequence of non-negative real numbers and that there exists $M > 0$ such that*

$$\lim_{n\to\infty}\inf\sum_{i=n-k}^{n-1} p_i > M.$$

If $\{x_n\}$ is an eventually positive solution of $(1)'$, then for every sufficiently large n there exists an integer n^* with $n-k \leq n^* \leq n-1$ such that

$$\frac{x_{n^*-k}}{x_{n^*}} \leq \Big(\frac{2}{M}\Big)^2.$$

Theorem 1 ([12]). *Let $0 < m \leq 1/e$ and let $x(t)$ be an eventually positive solution of the equation (1). Then*

$$L \leq \frac{1+\ln\lambda_1}{\lambda_1} - M,$$

where λ_1 is the smaller root of the equation $\lambda = e^{m\lambda}$ and $M = \liminf_{t\to\infty}\frac{x(t)}{x(\tau(t))}$.

Corollary 1. *Consider the differential equation* (1) *and assume that when* $L < 1$ *and* $0 < m \leq \frac{1}{e}$, *the following condition holds:*

$$L > \frac{\ln \lambda_1 + 1}{\lambda_1} - \frac{1 - m - \sqrt{1 - 2m - m^2}}{2}, \tag{C_{10}}$$

where λ_1 *is the smaller root of the equation* $\lambda = e^{m\lambda}$. *Then all solutions of the equation* (1) *oscillate.*

Example. Consider the delay differential equation

$$x'(t) + \frac{0.6}{\alpha\pi + \sqrt{2}}(2\alpha + \text{const})x(t - \frac{\pi}{2}) = 0,$$

where $\alpha = \frac{\sqrt{2}(0.6e+1)}{\pi(0.6e-1)}$. Then

$$\liminf_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} 0.6(2\alpha + \cos u)/(\alpha\pi + \sqrt{2})du = \frac{1}{e}$$

and

$$\limsup_{t\to\infty} \int_{t-\frac{\pi}{2}}^{t} 0.6(2\alpha + \cos u)/(\alpha\pi + \sqrt{2})du = 0.6.$$

Thus, according to Corollary 1, all solutions are oscillatory. Remark that none of the results mentioned in the introduction can be applied to this equation.

Theorem 2 ([27]). *Assume that there exists a sequence* $n_m \to \infty$ *such that* $p_n \geq 0$ *for* $n \in [n_m - (N+1)k, n_m]$ *and*

$$\sum_{i=n-k}^{n-1} p_i \geq c > \left(\frac{k}{k+1}\right)^{k+1} \quad \textit{for} \quad n \in [n_m - Nk, n_m], \quad m = 1, 2, \ldots,$$

where

$$N = 1 + \left[\frac{\log 4 - 2\log c}{\log c + (k+1)(\log(k+1) - \log k)}\right]$$

and $[\cdot]$ *denotes the greatest integer function. Then all solutions of the equation* $(1)'$ *oscillate.*

Theorem 3 ([27]). *Assume that* $\{p_n\}$ *is a non-negative sequence of real numbers and let* k *be a positive integer. Assume further that there exists* $M > 0$ *such that*

$$\liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > M \quad \textit{and} \quad \limsup_{n\to\infty} \sum_{i=n-k}^{n-1} p_i > 1 - \left(\frac{M}{2}\right)^2.$$

Then all solutions of the equation $(1)'$ *oscillate.*

Remark. The results concerning the equation (1) can be extended to advanced differential equations and inequalities (cf. [7]), to equations with positive and negative coefficients (cf. [29]), to neutral differential equations (cf. [3]) and also to higher order equations (cf. [6]) and essentially improve the existing results in the literature. While the results concerning $(1)'$ may be applied to the case when the sequence $\{p_n\}$ is not assumed to be non-negative everywhere and also when the conditions (C_0), $(C_2)'$ and $(C_3)'$ fail.

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