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INTERIOR ESTIMATES IN THE THEORY OF PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATION TO INITIAL VALUE PROBLEMS

Abstract. The present paper is aimed at solving initial value problems by the contraction-mapping principle in case an interior estimate is true in the function space under consideration.

რეზიუმე. ნაშრომის მიზანია საწყისი ამოცანის კუმშვითი ასახვის პრინციპით ამოხსნა იმ შემთხვევაში, როცა განსახილველ სივრცეში სამართლიანია პირველი რიგის წარმოებულების შეფასება განსაზღვრის არის შიგნით მოთავსებულ არეებზე.

1. Introduction

Consider an initial value problem of type

$$\partial_t u = \mathcal{F}u, \quad u(0,x) = u_0(x)$$

in its integral rewriting

$$u(t,x) = u_0(x) + \int\limits_0^t \mathcal{F}u(\tau,x)d\tau, \tag{1}$$

where $\mathcal{F}$ is a first order differential operator acting with respect to the variable $x = (x_1, \dots, x_n)$.

An interior estimate is an estimate of the (first order) derivatives of a function which is true in a subset of the domain of definition having a positive distance from the boundary. Originally interior estimates have been introduced for solving boundary value problems for elliptic equations (Schauder's technique of a-priori estimates).

Later on, M. Nagumo [8] found a functional-analytic approach to initial value problems of Cauchy-Kovalevskaya type being based on an integral rewriting of that problem. Again, the solvability of the corresponding

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integro-differential equation follows from an interior estimate of the derivative of a holomorphic function.

Using I. N. VEKUA's theory of generalized analytic functions [14], one gets analogous interior estimates for generalized analytic functions finally leading to the solution of initial value problems with generalized analytic initial functions (see [11],[13]). Moreover, using B. BOJARSKI's theory [1] of generalized analytic vectors, A. CRODEL solved initial value problems with generalized analytic vectors as initial data in his Thesis [3], cf. also A. CRODEL's paper [4].

Interior estimates for generalized analytic functions and generalized analytic vectors resp. can be applied for solving initial value problems not only within the framework of scales of BANACH spaces but also when using a weighted norm introduced by W. WALTER in his elementary proof [15] of the (classical) CAUCHY-KOVALEVSKAYA theorem. This approach to initial value problems with a generalized analytic initial function is used in [12].

The present paper develops an abstract version of this approach to initial value problems.

2. Interior Estimates

Let $u = u(x)$ be defined in the bounded domain $\Omega \subset R^n$. Let, further, Ω' be a subdomain having positive distance dist $(\Omega', \partial\Omega)$ from the boundary $\partial\Omega$ of Ω. In order to solve the integro-differential equation (1) by the contraction-mapping principle, one needs an interior estimate of the type

$$\|\partial_{x_j} u\|_{\Omega'} \leq \frac{C}{\text{dist } (\Omega', \partial\Omega)} \|u\|_{\Omega}, \tag{2}$$

where $\|\cdot\|_{\Omega}$ and $\|\cdot\|_{\Omega'}$ denote suitable norms with respect to Ω and Ω' resp. and C is independent of the choice of $u = u(x)$, $j = 1, \ldots, n$.

3. Weighted Banach Spaces in Conical Domains

Using the interior estimate (2), we are going to estimate the integro-differential operator in (1). In order to apply (2), the given domain Ω will be exhausted by a family of subdomains Ω_s, $0 < s < s_0$, satisfying the following conditions:

(1) Every point $x \in \Omega$ (with the only exception of some fixed point x_0 in Ω) belongs to the boundary of a uniquely determined domain $\Omega_{s(x)}$ of the exhaustion.

(2) The distance of a smaller subdomain from the boundary of a larger subdomain can be estimated uniformly from below by the difference of the indices characterizing the two subdomains under consideration, i.e., there exists a constant c_0 such that

$$\text{dist}(\Omega_{s'}, \partial\Omega_s) \geq c_0(s - s') \tag{3}$$

for any pair s, s' with $0 < s' < s < s_0$.

The initial function may have singularities at the boundary $\partial\Omega$ of Ω because it is defined in Ω only. Therefore, the nearer a point x to the boundary $\partial\Omega$, the smaller the t-interval in which the solution is expected to exist. In other words, the solution will exist in a conical domain in the x, t-space, in general (conical evolution). Since $s_0 - s(x)$ can be used as a measure for the distance of $x \in \Omega$ from the boundary $\partial\Omega$, this conical domain can be defined by

$$M = \{(t, x) : x \in \Omega, 0 \leq t < \eta(s_0 - s(x))\} .$$

The height of M depends on η which will be fixed later. The intersection of M with a plane $t = \tilde{t}, \quad 0 < \tilde{t} < \eta s_0$, is $\Omega_{\tilde{s}}$, where $\tilde{s}$ is defined by

$$\tilde{t} = \eta(s_0 - \tilde{s}). \tag{4}$$

Finally, define the pseudo-distance $d(t, x) = s_0 - s(x) - \frac{t}{\eta}$ measuring the distance of $(t, x) \in M$ from the lateral surface of M.

Now take any BANACH space $\mathcal{R}$ with the norm $\|\cdot\|$ such as the space of continuous functions in $\overline{\Omega}$, the space of HÖLDER continuous functions in Ω, or the L_p-space. Denote the space of the restrictions of its elements to Ω_s by $\mathcal{R}_s$ and the corresponding norm by $\|\cdot\|_s$. Now consider (real-, complex- or vector-valued) functions defined in M such that $u(\tilde{t}, x)$ belongs to $\mathcal{R}_s$ in case $\tilde{t}$ and $\tilde{s}$ are connected by (4). Introduce, further, the functional

$$\|u\|_* = \sup_M \|u\|_{s(x)} d^\sigma(t, x),$$

where σ is a fixed chosen positive number. Define, finally, the space $\mathcal{R}_*(M)$ consisting of all functions $u = u(t, x)$ for which $\|u\|_*$ is finite. Immediately the definition of the *-norm implies for every $u(t, x) \in \mathcal{R}_*(M)$ and for every point $(t, x) \in M$ the following estimate of the s-norm:

$$\|u\|_{s(x)} \leq \frac{\|u\|_*}{d^\sigma(t, x)}. \tag{5}$$

In compact subsets of M in which $d(t, x) \geq \delta > 0$, the s-norm of $u(t, x)$ (for fixed t) can be estimated by the *-norm: $\|u\|_s \leq \frac{1}{\delta^\sigma} \|u\|_*$. Taking into account the completeness of $\mathcal{R}_s$, this estimate shows that $\mathcal{R}_*(M)$ is complete, too.

4. An axiomatic system for solving initial value problems

Suppose the following assumptions are satisfied:

- There exists a subspace $\mathcal{S}$ of $\mathcal{R}$ such that $\mathcal{F}(t, u)$ belongs to $\mathcal{S}$ for each t and for each $u \in \mathcal{S}$
- $\mathcal{S}$ is complete, and an interior estimate of the type (1) is true in $\mathcal{S}$

Then the following theorem holds:

Theorem. *Provided the initial function u_0 belongs to $\mathcal{S}$, the initial value problem*

$$\partial_t u = \mathcal{F}(t,u), \quad u(0,x) = u_0(x)$$

can be solved by the contraction-mapping principle in conical domains with the height sufficiently small.

Proof. Suppose (t,x) belonging to M, i.e., $d(t,x) = s_0 - s(x) - \frac{t}{\eta} > 0$. Define $r = \frac{1}{\sigma+1} d(t,x)$ and $\tilde{s} = s(x) + r$. Since $d(t,x) \le s_0 - s(x)$, one has

$$\tilde{s} \le s(x) + \frac{1}{\sigma+1}(s_0 - s(x)) = \frac{\sigma}{\sigma+1} s(x) + \frac{1}{\sigma+1} s_0 < s_0.$$

Consider an arbitrary point $\tilde{x}$ with $s(\tilde{x}) = \tilde{s}$, i.e., $\tilde{x} \in \partial\Omega_{\tilde{s}}$ and, therefore, $d(t,\tilde{x}) = d(t,x) - r$. Applying the estimate (5) to the point $(t,\tilde{x})$, one obtains

$$\|u\|_{\tilde{s}} \le \frac{\|u\|_*}{d^\sigma(t,\tilde{x})} = \frac{\|u\|_*}{(d(t,x)-r)^\sigma}. \tag{6}$$

Taking into account the estimates (3) and (6), the interior estimate (2) yields

$$\left\|\partial_{x_j} u\right\|_{s(x)} \le \frac{C}{c_0} \frac{1}{\tilde{s} - s(x)} \frac{\|u\|_*}{(d(t,x)-r)^\sigma}.$$

Substituting the above defined values of r and $\tilde{s}$, one gets

$$\left\|\partial_{x_j} u\right\|_{s(x)} \le \frac{C}{c_0}(\sigma+1)\left(1+\frac{1}{\sigma}\right)^\sigma \frac{1}{d^{\sigma+1}(\tau,x)} \|u\|_* .$$

Obviously, an analogous estimate is true for $\|u\|_{s(x)}$ because $\tilde{s} - s(x) < s_0$ and, therefore, $1 \le \frac{s_0}{\tilde{s}-s(x)}$. To sum up, one gets an estimate of the form

$$\|\mathcal{F}u\|_{s(x)} \le \frac{CC_{\mathcal{F}}}{c_0}(\sigma+1)\left(1+\frac{1}{\sigma}\right)^\sigma \frac{1}{d^{\sigma+1}(t,x)} \|u\|_* \tag{7}$$

in case $\mathcal{F}u$ is a linear combination of u and its first order derivatives whose coefficients are bounded in a suitably chosen metric (where $C_{\mathcal{F}}$ depends on the coefficients of $\mathcal{F}$).

An elementary calculation shows that

$$\int_0^t \frac{d\tau}{d^{\sigma+1}(\tau,x)} < \frac{\eta}{\sigma} \frac{1}{d^\sigma(t,x)}.$$

Taking into consideration that (7) holds for every τ with $0 \le \tau \le t$ instead of t, one gets

$$\left\| \int_0^t \mathcal{F}u(\tau,\cdot)d\tau \right\|_{s(x)} \le \eta \frac{C_{\mathcal{F}}C}{c_0}\left(1+\frac{1}{\sigma}\right)^{\sigma+1} \frac{1}{d^{\sigma+1}(t,x)} \|u\|_* .$$

Hence

$$\left\| \int_0^t \mathcal{F}u(\tau,\cdot)d\tau \right\|_* \leq \eta \frac{C_{\mathcal{F}}C}{c_0}\left(1+\frac{1}{\sigma}\right)^{\sigma+1} \|u\|_* .$$

Thus the integro-differential operator in (1) is contractive in case η is small enough. This completes the proof of the theorem. ■

5. Interior Estimates in Associated Spaces

Suppose $\mathcal{F}$ is associated to a given differential operator $\mathcal{G}$, i.e., $\mathcal{G}u = 0$ implies $\mathcal{G}(\mathcal{F}u) = 0$. Then the subspace $\mathcal{S}$ contains all solutions of $\mathcal{G}u = 0$ belonging to $\mathcal{R}$. Since the complex differentiation transforms spaces of holomorphic functions into itself, associated operators for generalized analytic functions are constructed in [10], [11], [13]. The necessary interior estimates can be obtained from integral representations using the Cauchy kernel*.

Interior estimates for general elliptic systems are obtained in the paper [5] of A. Douglis and L. Nirenberg. Generalizing J. Schauder's results of 1934, this paper contains estimates of the Hölder norm of solutions. Similar interior estimates of the L_p-norm are given, for instance, in the papers [2] of F.E. Browder and [9] of L. Nirenberg†.

Since the present paper is aimed at new applications of interior estimates, we are going to conclude it by hinting to a general way of getting interior estimates using integral representation with fundamental solutions. This method will be explained by a very simple but typical example:

Let $\mathcal{G}u = 0$ be the partial differential equation $\Delta^2 u = C(x_1)u$ in the $z = x_1 + ix_2$-plane. Then $\mathcal{F} = \partial/\partial x_2$ and $\mathcal{G}$ are associated. Moreover, each solution $u = u(z)$ of $\mathcal{G}u = 0$ in Ω can be represented in the form

$$u(z) = u_0(z) - \frac{1}{8\pi}\iint_{\Omega} |\zeta - z|^2 \log|\zeta - z|\, u(\zeta) d\xi d\eta, \quad \zeta = \xi + i\eta,$$

where u_0 is a suitably chosen solution of the biharmonic equation $\Delta^2 u_0 = 0$. Using both this representation formula and an interior estimate for the biharmonic equation (see, for instance, Example 3 in [7] which is based on the results in A. Douglis' and L. Nirenberg's paper [5]), one gets an interior estimate of the type (2) for $\mathcal{G}u = 0$.

6. Concluding Remarks

First, the above axioms for solving initial value problems by the contraction-mapping principle show in which direction interior estimates have to be developed when having in mind to apply them to initial value problems.

*Concerning generalized analytic vectors, see A. Crodel's Thesis [3] and his paper [4]

†In the quoted papers [2], [5], [9], one can find further references to the literature and historical remarks.

Second, using a family of complete subspaces $\mathcal{S}_j$ of $\mathcal{R}$, a decomposition theorem in the style of [6] can be formulated within the framework of this system of axioms, too.

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