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COMPLEX ANALYSIS METHODS IN THE THEORY OF INFINITESIMAL BENDINGS OF SURFACES WITH A FLAT POINT

Abstract. Using I. Vekua's analytic methods, the problem of one-to-one correspondence between infinitesimal bendings of surfuces with a flat point is studied.

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1. Introduction. In the paper the objects of study are two surfaces S_0 and S given in a rectangular Cartesian coordinate system $Ox_1y_1z_1$ by the equations $S_0 : z_1 = (x_1^2 + y_1^2)^{n/2}$, $S : z_1 = (x_1^2 + y_1^2)^{n/2} f(x_1, y_1)$. It is assumed that S_0 and S are defined in a domain G_1, $(0,0) \in G_1$, $f(x_1, y_1) \in C^3(G_1)$, $f(0,0) > 0$, n ($n > 2$) is any real number, and for all points of G_1 other than $(0,0)$, the gaussian curvature of S is positive.

It is clear that the point $(0,0)$ is a flat one on the surfaces S_0 and S. At it not only the Gaussian curvature but also all coefficients of the second quadratic forms of S_0 and S vanish. At this point the surfaces have with their tangent planes a contact order greater than 1. We call the surface S_0 *model* with respect to S as it is a particular case of S and can be obtained from S under the condition $f(x_1, y_1) = 1$.

The aim of the paper is to establish the following result.

Theorem 1. *There exists a one-to-one correspondence between the sets of continuous infinitesimal bendings of the surfaces S_0 and S.*

2. An equivalent analytic problem [1]. We extend the I.Vekua analytic methods on investigating infinitesimal bendings of the above surfaces [2]. On S_0 and S, we introduce a conjugate isometric parametrization $z = x + iy, i^2 = -1$. Then infinitesimal bendings of these surfaces will be characterized by the functions $\Phi(z) = z^2 K_0^{1/4}(z)(\delta M_0 + i\delta L_0)$, $w(z) = z^2 K^{1/4}(z)(\delta M + i\delta L)$, where $K_0(z)$ and $K(z)$ are the Gaussian

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curvatures of S_0 and S, and $\delta M_0, \delta L_0$ and $\delta M, \delta L$ are variations of the coefficients of the second quadratic forms of S_0 and S, respectively.

In the domain G which is the image of the domain G_1 by the mapping $z = z(x_1, y_1)$, these functions satisfy the following generalized Cauchy–Riemann systems:

$$2\overline{z}\partial_{\overline{z}}\Phi - b(0)\overline{\Phi} = 0, \tag{1}$$

$$2\overline{z}\partial_{\overline{z}}w - b(z)\overline{w} = 0. \tag{2}$$

Here the singular point $z=0$ belongs to the domain G, $b(0)=(n-2)/2\sqrt{n-1}$ and $b(z)$ is a continuous function in G satisfying $|b(z) - b(0)| < M|z|^{\alpha}$ at least in a sufficiently small neighbourhood of $z = 0$; M, α are positive constants.

Continuous solutions of the systems (1) and (2) are connected by the two–dimensional integral equation

$$w(z) = \Phi(z) + P_G\overline{w}, \tag{3}$$

where $P_G\overline{w} = S_G\left(\frac{b(\zeta)-b(0)}{2\overline{\zeta}}\overline{w(\zeta)}\right)$ and

$$S_Gf = -\frac{1}{\pi}\iint\limits_G [\frac{\Omega_1(z,\zeta)}{\zeta}f(\zeta) + \frac{\Omega_2(z,\zeta)}{\overline{\zeta}}\overline{f(\zeta)}]\,d\xi d\eta.$$

Here $\zeta = \xi + i\eta$ and Ω_1, Ω_2 are certain functions presented in [1]. It is necessary to note that P_G is a completely continuous operator mapping the class $C(G)$ of continuous functions in itself. According to Fredholm's alternatives, the equation (3) will be uniquely solvable, and consequently a one-to-one correspondence between continuous infinitesimal bendings of the surfaces S_0 and S will exist if the following assertion takes place.

Theorem 2. *The homogeneous equation*

$$w^+(z) = P_G\overline{w^+}, \quad z \in G, \tag{4}$$

in the class $C(G)$ has only the zero solution.

Scheme of proof of Theorem 2. Suppose that the equation (4) has a non–trivial solution $w^+(z)$, $z \in G$. Let us note some of its properties. First, we can check that any continuous solution $w^+(z)$ of the equation (4) belongs to the class $D_{1,p}(G)$, $p > 2$, and satisfies the equation (2). In this case, as was shown in [2] (see Theorem 1.1, p.74), we have

$$w^+(z) = O(|z|^{|b(0)|}) \quad \text{as} \quad z \to 0, \tag{5}$$

From (4) it follows that $w^+(z)$ is continuously extended to the domain $E \setminus \overline{G}$ (E is the z–plane and $\overline{G}$ is the closure of G) by a continuous function $w^-(z)$, $z \in E \setminus \overline{G}$, satisfying the equation (1). According to the theory of elliptic systems, $w^-(z)$ is an analytic function in $E\setminus\overline{G}$ with respect to z and $\overline{z}$. In addition, it is established that

$$|w^-(z)| < M\|[b(\zeta)-b(0)]/2\overline{\zeta}\|_{L_p}|z|^{-|b(0)|}, \tag{6}$$

where $M = M(|b(0)|, R_0)$ is a constant depending on $|b(0)|$ and R_0 (a maximal distance from $z=0$ to the boundary of G), and $\|\cdot\|_{L_p}$ denotes the norm of a function $f(z)$ in the space $L_p(\overline{G})$.

Thus on the plane z the continuous function $W(z) = \begin{cases} w^+(z), & z\in\overline{G}, \\ w^-(z), & z\in E\setminus\overline{G} \end{cases}$ is defined. This function is subject to the conditions (5),(6) and satisfies the equation

$$2\overline{z}\partial_{\overline{z}}W - B(z)\overline{W} = 0, \tag{7}$$

in which

$$B(z) = \begin{cases} b(z) & \text{for } z\in\overline{G}, \\ b(0) & \text{for } z\in E\setminus\overline{G}. \end{cases}$$

We establish the following.

Lemma 1. *If $W(z)$ is a function satisfying the above properties, then $W(z)\equiv 0$, $z\in E$.*

From this lemma it follows that $w^+(z)=0$, $z\in G$ and therefore Theorems 1 and 2 are proved.

4. Generalization. Now we consider a surface given in Cartesian coordinates $Ox_1y_1z_1$ by the equation $z_1 = \sum\limits_{k=0}^{n} a_{k,n-k}x_1^ky_1^{n-k} + R(x_1,y_1)$, where n ($n\geq 3$) is an integer and $a_{k,n-k}$ are constants. Let $R(x_1,y_1)$ be a sufficiently regular function; moreover, let $R(x_1,y_1) = O[(x_1^2+y_1^2)^{(n+1)/2}]$ as $x,y\mapsto 0$. Passing over to polar coordinates ($x_1 = r_1\cos\varphi$, $y_1 = r_1\sin\varphi$), we write the surface equation in the form

$$S: \quad z_1 = r_1^n f(\varphi) + R(x_1,y_1), \tag{8}$$

where $f(\varphi) = \sum\limits_{k=0}^{n} a_{k,n-k}(\cos\varphi)^k(\sin\varphi)^{n-k}$. The requirement of positiveness of the curvature in a neighbourhood of the point $(0,0)$ imposes on $f(\varphi)$ the restriction

$$-(n-1)\left(\frac{df}{d\varphi}\right)^2 + nf\frac{d^2f}{d\varphi^2} + n^2f^2 > 0. \tag{9}$$

Besides, we assume $f(\varphi)>0$.

The first summand in the right side of (8) defines the structure of the surface in a neighbourhood of the flat point. The model surface

$$S_0: \quad z_1 = r_1^n f(\varphi) \tag{10}$$

corresponds to it.

Further we will consider the surfaces (8) and (10) under wider assumptions on n and $f(\varphi)$. We assume that n ($n > 2$) is a real number, $f(\varphi)$ is a 2π–periodic function from the class $C^3[0, 2\pi]$, satisfying the inequality (9). It is clear that the surfaces under study are objects with sufficiently general and more complicated structure for a neighbourhood of the flat point $(0, 0)$ compared with those which have been discussed previously.

The problem is to establish a one-to-one correspondence between infinitesimal bendings of surfaces (8) and (10).

As was stated in [3], in a conjugate isometric parametrization $z = x + iy$ and in terms of complex–valued functions $\Phi(z)$ and $w(z)$ introduced earlier, infinitesimal bendings of those surfaces are described by the equations

$$2\overline{z}\partial_{\overline{z}}\Phi - b_0(\varphi)\overline{\Phi} = 0, \tag{11}$$

$$2\overline{z}\partial_{\overline{z}}w - [b_0(\varphi) + B(z)]\overline{w} = 0, \tag{12}$$

where the point $z = 0$ is interior for the domain G, $b_0(\varphi)$ is a 2π–periodic continuous function and $B(z)$ is continuous in G, moreover $B(z) = O(|z|^\alpha)$ as $z \mapsto 0$, $\alpha > 0$.

Apparently, the model equation (11), seeming simpler in comparison with (12), is nevertheless fairly complicated for investigating. This fact is maybe a main reason why a progress in this respect looks such moderate, see [4,5], and the problem of correspondence for infinitesimal bendings of the surfaces (8) and (10) remains unsolved.

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