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S. Chelkak, A. Koshelev, and L. Oganesyan

REGULAR SOLUTIONS OF THE NAVIER–STOKES SYSTEM

Abstract. The problem of existence of regular (continuous, Hölder–continuous) solutions for nonstationary Navier–Stokes systems is one of the important topics in modern mathematical physics. This problem is closely connected with two main issues: the uniqueness and the possibility to apply the methods for numerical analysis and practical computations. We consider here mainly the multidimensional nonstationary problem for finite (not necessarily small) time.

რეზიუმე. ნავიე–სტოქსის არასტაციონარული სისტემების რეგულარული (უწყვეტი, ჰელდერის აზრით უწყვეტი) ამონახსნების არსებობის პრობლემა თანამედროვე მათემატიკური ფიზიკის ერთ–ერთ უმნიშვნელოვანეს საკითხს წარმოადგენს. ეს პრობლემა მჭიდროდაა დაკავშირებული ორ მთავარ მომენტთან: ერთადერთობასა და არსებული მეთოდების რიცხვითი ანალიზისა და პრაქტიკული გამოთვლებისათვის გამოყენებასთან. ნაშრომში ძირითადად განხილულია მრავალგანზომილებიანი არასტაციონარული ამოცანა სასრული (არა აუცილებლად მცირე) დროის მონაკვეთისათვის.

1. Existence of Regular Solutions and Some Estimates

Let Ω be a bounded domain in $\mathbb{R}^m$ with $\partial\Omega \in C^{(1,\kappa)}$. Consider in $Q = (0,\top) \times \Omega$ the Navier–Stokes system

$$\dot{u} - \nu\Delta u + u^{(k)} D_k u + \nabla p + f(t,x) = 0\,, \tag{1.1}$$

$$\operatorname{div} u = 0 \tag{1.2}$$

with initial and boundary conditions

$$u\big|_{t=0} = 0\,, \qquad u\big|_{(0,\top)\times\partial\Omega} = 0\,, \tag{1.3}$$

where D_k denotes the differentiation with respect to x_k,

$$u(t,x) = \{u^{(1)}(t,x), \dots, u^{(m)}(t,x)\}$$

is the unknown velocity vector function, $\nu = \text{const} > 0$, $f(t,x)$ denotes the vector of external forces, p is the pressure which is normed by the equality

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$\int\limits_\Omega p\,dx = 0$, and the summation runs as usual over repeated indices 1 and k from 1 to m.

It is known (E. Hopf [3]) that if $f \in \mathcal{L}_2(Q)$, then the weak solution of (1.1)–(1.3) belongs to $\mathcal{L}_2\{(0,\top);\, W_2^{(1)}(\Omega)\}$ and satisfies the equality

$$\int\limits_Q \Big[-u\,\dot{v} + \nu D_k u D_k v - u^{(k)} D_k v u\Big]\,dxdt +$$

$$+ \int\limits_Q uvdx\Big|_{t=\top} + \int\limits_Q fvdxdt = 0\,, \tag{1.4}$$

where v is an arbitrary smooth function meeting the conditions (1.2) and (1.3).

Up to now for the multidimensional case the theorems of uniqueness and existence are proved simultaneously for different functional classes (see, e.g., Ladyzhenskaya [6]).

We will consider in this paper the so–called regular (or strong) solutions of $u(x,t)$ of (1.1)–(1.3) which should be Hölder–continuous both in x and t. It is assumed that the functions $u(x,t)$ which are given on Q possess at least the second derivatives with respect to x and the first derivative with respect to t which belongs to some normed functional (strong) space X. Assume also that $\nabla p \in X$.

We will use the following notations

$$D'uD'v = D_k u D_k v,\quad D'^2 u D'^2 v = D_{ik} u D_{ik} v,\quad |D'^\ell u|^2 = D'^\ell u D'^\ell u,\quad \ell = 1,2,$$
$$|Du|^2 = |D'u|^2 + |u|^2\,,\quad |D^2 u|^2 = |D^2 u|^2 + |Du|^2,\quad \|u\|^2 = \|D^2 u\|_x^2 + \|u\|_x^2.$$

Suppose that X satisfies the following conditions:

1. $X \subset \mathcal{L}_2\{(0,\top); W_2^{(1)}(\Omega)\}$.
2. $\sup\limits_Q |u(t,x)| \le C_0|u| + C\|u\|_{\mathcal{L}_2}$.
3. For any $f \in X$, the solution of the linear Stokes problem

$$\dot{u} - \nu\Delta u + \nabla p = f\,,\ \operatorname{div} u = 0\,,\ u\big|_{t=0} = u\big|_{(0,\top)\times\partial\Omega} = 0 \tag{1.5}$$

satisfies the inequalities

$$\|u\|_X \le C_X\|f\|_X + C(\|Du\|_{\mathcal{L}_2} + \|p\|_{\mathcal{L}_2}), \tag{1.6}$$

$$\|\nabla p\|_X \le C_X^{(1)}\|f\|_X + C(\|Du\|_{\mathcal{L}_2} + \|p\|_{\mathcal{L}_2}), \tag{1.7}$$

where C_X, $C_X^{(1)}$ and C are some positive constants.

Example. $X = \mathcal{L}_p(Q)$ with $p > m+1$. The inequality (1.6) (for any $p > 1$) was proved by V. Solonnikov [8] but the constant C_X has an implicit form.

In order to make the results explicit and sometimes sharp we took (see A. Koshelev [5]) the space $H^{(1,2)}_{2,\alpha}(Q)$ with the norm

$$\|u\|_{1,2;\alpha} = \left\{ \sup_{x_0 \in \Omega} \int\limits_Q \left[|\dot{u}|^2 + |D^2 u|^2 \right] |x - x_0|^\alpha dx dt \right\}^{\frac{1}{2}} \tag{1.8}$$

with $\alpha = 2 - m - 2\gamma$ $(0 < \gamma < 1)$. In this case $X = \mathcal{L}_{2,\alpha}(Q)$ and the norm is determined by the formula

$$\|u\|_\alpha = \left\{ \sup_{x_0 \in \Omega} \int\limits_Q |u|^2 \, |x - x_0|^\alpha dx dt \right\}^{\frac{1}{2}} . \tag{1.9}$$

The inequalities (1.6), (1.7) are an analytic basis for some existence and uniqueness theorems which were proved in [5]. They follow from analogous estimates in weighted spaces. Let $\gamma > 0$ be sufficiently small and the ball $B = \{x : |x - x_0| < \delta\}$ contained in B. In [5] we proved the following estimates for the solutions of the problem (1.5)

$$\begin{gathered}
\int\limits_{Q_0} |D^2 u|^2 \, |x - x_0|^\alpha dx dt \le \frac{m}{2\nu^2} \left\{ \left[1 + \frac{(m-2)^2}{m-1} + O(\gamma) \right] + 1 \right\}^{\frac{1}{2}} \times \\
\times \left(1 + \frac{m-2}{m+1} \right) \int\limits_{Q_0} |f|^2 \, |x - x_0|^\alpha dx dt + C \int\limits_{Q_0} |f|^2 dx dt, \\
\int\limits_{Q_0} |\nabla p|^2 |x - x_0|^\alpha dx dt \le \qquad (1.10) \\
\le \left[1 + \frac{(m-2)^2}{m-1} + O(\gamma) \right] \int\limits_{Q_0} |f|^2 \, |x - x_0|^\alpha dx dt + C \int\limits_{Q_0} |f|^2 dx dt,
\end{gathered}$$

where $Q_0 = Q \cap [(0, \top) \times B]$. Taking sup over all $x_0 \in \Omega$, we come to (1.6) and (1.7) with

$$C_{\mathcal{L}_{2,\alpha}} = \frac{m}{2\nu^2} \left\{ \left[1 + \frac{(m-2)^2}{m-1} + O(\gamma) \right] + 1 \right\}^{\frac{1}{2}} \left(1 + \frac{m-2}{m+1} \right) \tag{1.11}$$

and

$$C^{(1)}_{\mathcal{L}_{2,\alpha}} = 1 + \frac{(m-2)^2}{m-1} + O(\gamma). \tag{1.12}$$

By the way, the constant (1.12) is sharp.

We consider the existence of Hölder–continuous in both x and t solutions of (1.1) – (1.3) using the following iterative process

$$\varepsilon \, \dot{u}_{n+1} - \nu \Delta u_{n+1} + \varepsilon \nabla p_{n+1} = -\nu \Delta u_n - \varepsilon [- \dot{\nu} \, \Delta u_n + \; u_n^{(k)} D_k u_n + f], \tag{1.13}$$

where $\varepsilon = \text{const} \in (0,1]$ and all u_n satisfy the conditions (1.2) and (1.3). Of course, the initial iteration must be smooth enough. The equations (1.13) can also be written in the weak form. It was proved (S. Chelkak, A. Koshelev [2], see also A. Koshelev [5]) the following

Theorem 1. *Let $f \in \mathcal{L}_{2,\alpha}(Q)$, $\alpha = 2-m-2\gamma$, $\gamma \in (0,1)$ and the quantitiy $\nu^{-\sigma}\mathcal{R}$, where*

$$\mathcal{R} = A^2_{\alpha,m}\,\|f\|^2_{\mathcal{L}_{2,\alpha}(Q)} + C\|f\|^2_{\mathcal{L}_2(Q)}\,, \tag{1.14}$$

be sufficiently small ($A^2_{\alpha,m}$, σ and C are some positive constants). Then the problem (1.1)–(1.3) has a unique solution (u^*,p^*), which belongs to $C^{(0,\delta)}(Q)$ with some $\delta > 0$. This solution satisfies the inequality

$$\|u^*\|_{C^{(0,\delta)}(Q)} \le M\,,$$

where M is determied by $\nu^{-\sigma}\mathcal{R}$. For sufficiently small $\nu^{-\sigma}\mathcal{R}$, the constant M can be calculated explicitly. Here $C^{(0,\delta)}(Q)$ is the space of Hölder–continuous both in x and t functions.

Remark that the constant $A^2_{\alpha,m}$ in (1.14) is given explicitly for the given α and m. The corresponding expressions are given in [5] and [2]. The constant C in (1.14) is given implicitly and we do not even have an explicit bound for it. The expression $\nu^{-\sigma}\mathcal{R}$ is called a strong Reynolds number for the problem (1.1) – (1.3). Then the theorem states that for small strong Reynolds numbers the problem has a Hölder–continuous solution. It is of course very interesting to find out how the regular properties depend on $\mathcal{R}$. Clearly, for this reason we need some numerical experiments and consider such functions $f(t,x)$ when the first term the principal one of the right–hand side of (1.14). Remark that we consider only Hölder–continuous solutions, not obligatorily the classical ones.

2. Axially Symmetric Navier–Stokes Problems

There is no example of the problem when a Hölder–continuous solution looses its regular properties or becomes nonunique. Therefore, we need to consider such a problem where some irregularity can be expected. It is known that for the Taylor example (J. Taylor [9], K.Kirchgässner [4], V.Yudovich [10]), the flow can bifurcate. The domain Q consists for this case of two coaxial cylinders which rotate in opposite directions. This was a reason to consider such kind of a domain and later on to investigate the class of suitable right–hand sides in (1.1). We are not familiar with any algorithm which would allow to find the weak solution of the problem for a general domain Q. Therefore it was decided to find an algorithm for the domain of the Taylor example and to apply the method of finite elements. For the basic elements, bicubical Birkhoff elements were taken. The method was offered and described by L. Oganesyan [7]. In this section we will follow [7].

Let (r, z, φ) be the cylindrical coordinates. We consider the system of Navier-Stokes in a ring $\Omega = \{(r,z) : r_* < r < r^*, 0 < z < z^*\}$. Denote $\dot{u} = \frac{\partial u}{\partial t}$, $u' = \frac{\partial u}{\partial r}$, $u\text{'} = \frac{\partial u}{\partial z}$. Then the stationary problem (1.1)–(1.2) will take the form

$$\begin{cases} \dot{u} + uu' + wu\text{'} - \frac{v^2}{r} = -p' + \nu\big(u'' + u\text{''} + \frac{u'}{r} - \frac{u}{r^2}\big) + f_1\,, \\ \dot{w} + uw' + ww\text{'} = -p\text{'} + \nu\big(w'' + w\text{''} + \frac{w'}{r}\big) + f_2\,, \\ \dot{v} + uv' + wv\text{'} + \frac{uv}{r} = \nu\big(v'' + v\text{''} - \frac{v'}{r} - \frac{v}{r^2}\big) + f_3\,, \\ \frac{1}{r}(ru)' + \frac{1}{r}(wr)\text{'} = 0, \end{cases} \tag{2.1}$$

where $\vec{u} = u\vec{e}_r + w\vec{e}_z + v\vec{e}_\varphi$. Along with (2.1), we consider the boundary conditions

$$u\big|_{t=0} = u_0, \quad u, w\big|_{r=r_*} = u, w\big|_{r=r^*} = 0, \quad v\big|_{r=r_*} = \omega_* r_*, \quad v\big|_{r=r^*} = \omega^* r^*. \tag{2.2}$$

For z we take the periodic conditions: u, v, w, p are periodic along the z-axis with the period z^*. Note that we can choose the conditions of the first boundary value problem also. We assume also that the mean flow along the z-axis is equal to zero. If we consider the flow function ψ, which satisfies the conditions $u = \frac{1}{r}\psi\text{'}$, $w = -\frac{1}{r}\psi'$, we come to much simpler relations. It is clear that $\psi = C_1$ for $r = r_*$ and $\psi = C_2$ for $r = r^*$. The zero flow along the z-axis gives you that $C_1 = C_2 = C$. We can suppose that $C = 0$. For the stationary problem, the flow function ψ satisfies the following integral identity

$$\int_\Omega \frac{1}{r}\left\{\frac{1}{r}\,\triangle_-\psi\{\psi,\Phi\} - v^2\Phi\text{'} + \nu\,\triangle_-\psi\,\triangle_-\Phi\right\} dr dz = \int_\Omega (f_1\Phi\text{'} - f_2\Phi') dr dz, \tag{2.3}$$

where $\{\psi,\Phi\} = \psi\text{'}\Phi' - \psi'\Phi\text{'}$ is the Poisson brackets and $\triangle_-\psi = \psi\text{''} + r(\frac{\psi'}{r})'$.

The identity (2.3) leads to a fourth order partial differential equation and this is the main difficulty of our problem. The function v satisfies the following integral identity

$$\int_\Omega \left[\frac{\psi\text{'}}{r} vV + \nu\left(rv'V' + rv\text{'}V\text{'} + \frac{vV}{r}\right)\right] dr dz - \nu \int_{z_*}^{z^*} rv'V\Big|_{r_*}^{r^*} dz = \int_\Omega f_3 V r dr dz$$

which leads to a second order differential equation. It is known that this problem can be solved numerically by simple methods.

It is clear that the problem to find ψ is a nonlinear one. It can be solved by applying some iterative method. We will consider only the Stokes problem, i.e. the problem

$$\int_\Omega \frac{1}{r}\nu\,\triangle_-\psi\,\triangle_-\Phi dr dz = \int_\Omega (f_1\Phi\text{'} - f_2\Phi') dr dz. \tag{2.4}$$

A linear operator with constant coefficients which is spectrally equivalent to the operator under consideration should be applied. It is known that on the functions, which belong to $\overset{\circ}{H}{}_2^{(2)}$ such an operator is $\triangle^2$. This leads to an iterative process

$$\int\limits_{\omega} \triangle\psi_{n+1} \triangle \Phi dxdz = \int\limits_{\omega} \triangle\psi_n \triangle \Phi dxdz -$$
$$-\varepsilon\Big[\frac{\nu}{2}\int\limits_{\omega}\frac{1}{x}\left(\partial_z^2\psi_n + x\partial_x^2\psi_n\right)\left(\partial_z^2\Phi + x\partial_x^2\Phi\right)dxdz -$$
$$-\int\limits_{\omega}\frac{1}{\sqrt{x}}\left(f_1\partial_z\Phi - \sqrt{x}\partial_x\Phi\right)dxdz\Big], \qquad (2.5)$$

where $n = 0, 1, \dots,$ $\psi_0 = 0$, $x = r^2/4$ and $\omega = \{(x,z) : x_* < x < x^*,\ 0 < z < z^*\}$ is the image of Ω in (x,z). Here $\varepsilon > 0$ is the iterative parameter. The function ψ_n needs to satisfy the boundary conditions.

On each step of the iterative process (2.5) we have to solve a variational problem

$$\begin{cases} \int_\omega \triangle u \triangle \Phi dxdz = \langle F, \Phi\rangle, \\ u|_{x=x_*} = u|_{x=x^*} = \frac{\partial u}{\partial x}\Big|_{x=x_*} = \frac{\partial u}{\partial x}\Big|_{x=x^*} = 0, \\ u(x,z) \text{ is periodic along } z \text{ with the period } z^*, \end{cases} \qquad (2.6)$$

where $\langle\cdot,\cdot\rangle$ is a scalar product corresponding to the right-hand side of (2.5).

Now we consider only (2.6), which is the basic issue of our algorithm. To solve (2.6), we apply the method of finite elements. These elements must belong to $\overset{\circ}{H}{}_2^{(2)}(\omega)$. The bicubical Birkhoff elements satisfy this condition. Therefore we seek the approximate solution of the following form

$$u = \sum_{k=1,l=0}^{\substack{k=M-2,\\ l=N-1}} \left[u_{kl}\varphi_k(x)\varphi_l(z) + p_{kl}\theta_k(x)\varphi_l(z) + q_{kl}\varphi_k(x)\theta_l(z) + r_{kl}\theta_k(x)\theta_l(z)\right],$$

$$\varphi_k(x) = \varphi(\overset{k}{X}), \quad \varphi(t) = \begin{cases} 2(|t|-1)^3 + 3(|t|-1)^2, & |t| < 1, \\ 0, & |t| \ge 1, \end{cases}$$

$$\overset{k}{X} = \begin{cases} \frac{x - x_k}{h_k}, & x > x_k, \\ \frac{x - x_k}{h_{k-1}}, & x \le x_k, \end{cases} \qquad \theta_k(x) = \theta(\overset{k}{X}) \begin{cases} h_k, & \overset{k}{X} > 0, \\ h_{k-1}, & \overset{k}{X} \le 0, \end{cases}$$

$$\theta(t) = \mathrm{sign}(t) \begin{cases} (|t|-1)^2 + (|t|-1)^3, & |t| < 1, \\ 0, & |t| \ge 1, \end{cases}$$

where $h_k = x_{k+1} - x_k$, $x_0 = x_* < x_1 < x_2 < \cdots < x_M = x^*$ are the knots for the coordinate x and $z_0 = 0 < z_1 < z_2 < \cdots < z_N = z^*$ are the knots for the coordinate z. For the functions $\varphi_l(z)$ and $\theta_l(z)$, the steps h_k and

h_{k-1} should be replaced by δ_l and δ_{l-1}, where $\delta_l = z_{l+1} - z_l$. At the same time, the net for z outside of $[0, z^*]$ should be expanded periodically from the segment $[0, z^*]$.

Taking in (2.6) a suitable test function Φ, we come, as usual, to a linear system for the coefficients u_{kl} , p_{kl} , q_{kl} , r_{kl} . This system is written in [7]. The structure of this system is very complicated. It is practically impossible to solve this system by direct methods even for rather small numbers M and N. So it leads to the problem of solving this system by means of an iterative method. Write this system in the form $A\vec{U} = \vec{F}$. The matrix A is positively determined. Denote $a_{mn} = (u_x)_{mn} - p_{mn}$, $b_{mn} = (u_z)_{mn} - q_{mn}$, $c_{mn} = r_{mn} - (q_x)_{mn} - (p_z)_{mn} + (u_{xz})_{mn}$, where u_x, u_{xz} (and further $u_{\overline{x}}$, $u_{x\overline{x}}$) means the ordinary finite differences of the first and second order, respectively. In [7] it is shown that the quadratic form corresponding to the matrix A is equivalent (uniformly with respect to h_m and δ_n) to the following quadratic form

$$\sum_{m,n} \Bigg\{ h_m\delta_n \left(u^2_{x\overline{x}} + u^2_{z\overline{z}}\right) + P_1 \left[h_m\delta_n \left(\frac{h_m^2}{\delta_n^2} + \frac{\delta_n^2}{h_m^2} \right) (a_z)^2_{mn} + \frac{\delta_n}{h_m} a^2_{mn} \right] +$$
$$+ P_2 \left[h_m\delta_n \left(\frac{h_m^2}{\delta_n^2} + \frac{\delta_n^2}{h_m^2} \right) (b_x)^2_{mn} + \frac{h_m}{\delta_n} b^2_{mn} \right] +$$
$$+ P_3 h_m\delta_n \left(\frac{h_m^2}{\delta_n^2} + \frac{\delta_n^2}{h_m^2} \right) c^2_{mn} \Bigg\} \tag{2.7}$$

with the parameters $P_1 = P_2 = P_3 = 1$. The form will not loose the equivalence if we take arbitrary positive numbers for P_j. This equivalence gives a sufficiently simple iterative process for the problem (2.6): the structure of the form (2.7) splits the problem in four problems for u_{mn}, a_{mn}, b_{mn} and c_{mn}, respectively. Thus the problem (2.6) can be solved with the help of the following iterative procedure:

$$\Lambda_1(\overset{s+1}{u} - \overset{s}{u}) = \varepsilon F_1(\overset{s}{\vec{U}}) \tag{2.8}$$

— the problem for u_{mn} with the operator $\Lambda_1 u = \frac{1}{h}(hu_{x\overline{x}})_{x\overline{x}} + \frac{1}{\delta}(\delta u_{z\overline{z}})_{z\overline{z}}$;

$$\Lambda_2(\overset{s+1}{a} - \overset{s}{a}) = \varepsilon F_2(\overset{s}{\vec{U}}) \tag{2.9}$$

— the problem for a_{mn} with the operator $\Lambda_2 a = \left(\left(\frac{h^2}{\delta^2} + \frac{\delta^2}{h^2}\right) a_z\right)_{\overline{z}} + \frac{1}{h^2} a$;

$$\Lambda_3(\overset{s+1}{b} - \overset{s}{b}) = \varepsilon F_3(\overset{s}{\vec{U}}) \tag{2.10}$$

— the problem for b_{mn} with the operator $\Lambda_3 b = \left(\left(\frac{h^2}{\delta^2} + \frac{\delta^2}{h^2}\right) b_x\right)_{\overline{x}} + \frac{1}{\delta^2} b$;

the problem for c_{mn}:

$$\left(\frac{h^2}{\delta^2}+\frac{\delta^2}{h^2}\right)\left(\overset{s+1}{c}-\overset{s}{c}\right)=\varepsilon F_4(\overset{s}{\vec{U}}). \tag{2.11}$$

Here F_i are the corresponding right-hand sides which can be calculated with the help of the problem (2.6). It is worth–while to mention that the problem (2.8) corresponds to a fourth order differential equation and the problem (2.9) and (2.10) correspond to the second order differential equations. The problem (2.11) is a system of linear algebraic equations with a diagonal matrix. It is evident that the equations (2.8)–(2.11) should be accompanied by the corresponding boundary conditions.

It should be stated that the method converges like a geometric progression after the parameter ε was appropriately choosen. Note that this method is a quasioptimal with respect to the number of arithmetical operations. This method was realised and a numerical program was tested in the work of S. Chelkak, G. Konovalov and L. Oganesyan [1].

For the nonstationary Stokes system, the function ψ satisfies the following integral identity

$$\int\limits_{\Omega}\left[\nabla\dot{\psi}\nabla\Phi+\nu\Delta_{-}\psi\Delta_{-}\Phi\right]\frac{drdz}{r}=\int\limits_{\Omega}\left[f_1\Phi\text{'}-f_2\Phi'\right]drdz, \tag{2.12}$$

where $\nabla=\left(\dfrac{\partial}{\partial r},\dfrac{\partial}{\partial z}\right)$. Let $t_k=k\tau,\ k=0,1,\dots,\ \psi_k=\psi\big|_{t=t_k}$. To find ψ_k, we can apply the Krank-Nickolson scheme

$$\int\limits_{\Omega}\left[(\nabla\psi_{\bar{t}})_k\nabla\Phi+\nu\Delta_{-}\psi_k^{+}\Delta_{-}\Phi\right]\frac{drdz}{r}=$$

$$=\int\limits_{\Omega}\left[f_1^{\odot}\Big|_{t=t_k-\frac{\tau}{2}}\Phi\text{'}-f_2^{\odot}\Big|_{t=t_k-\frac{\tau}{2}}\Phi'\right]drdz, \tag{2.13}$$

where $\psi_k^{+}=\dfrac{1}{2}(\psi_k+\psi_{k-1})$ and $f^{\odot}$ is the Steklov average of the function $f(t)$.

On each step in time the approximate value of ψ_k can be found with the help of our algorithm in the spline form Ψ_k. If the functions f_1 and f_2 are sufficiently smooth, then the estimate

$$|||R_k|||^2\equiv\max_{t_k<T}\int\limits_{\Omega}R_i^2\frac{drdz}{r}+\nu\tau\sum_{t_k<T}\int\limits_{\Omega}\left|\Delta_{-}R_i^{+}\right|^2\frac{drdz}{r}\le C\tau^4+\int\limits_{\Omega}R_0^2\frac{drdz}{r}$$

holds, where $R_k=\Psi_k-\psi_k$ and Ψ_0 is a bicubical spline for ψ_0. If ψ_0 is also sufficiently smooth, then $\int\limits_{\Omega}R_0^2\dfrac{drdz}{r}\le Ch^4\|\psi_0\|^2$ and therefore the Krank-

Nickolson scheme has the accuracy $O(h^2) + O(\tau^2)$ with respect of the norm $|||R|||$.

Note that in the process of finding the solution of (2.13) it is worth–while to change a little the iteration operation in (2.13). Such a change can make the convergency of the iterative process better.

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Authors' addresses:

S. Chelkak
Electrotechnical University, Math. 1
Popova 5, St. Petersburg 197022
Russia

A. Koshelev
St. Petersburg University
Math.–Mech. Faculty, Dept. of Elasticity
Petergof Bibliotechnaya 2, St. Petersburg 198904
Russia

L. Oganesyan
Electrotechnical University, Math. 1
Popova 5, St. Petersburg 197022
Russia