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THE BROTHERS RIESZ THEOREM IN $\mathbb{R}^n$ AND LAPLACE SERIES

Abstract. In the present paper the concept of series conjugate to a Laplace series on the $(n-1)$-dimensional sphere is introduced and a Brothers Riesz Theorem for such series is proved.

რეზიუმე. ნაშრომში შემოტანილია $n-1$ განზომილებიან სფეროზე განსაზღვრული ლაპლასის მწკრივის შეუღლებულის ცნება და ასეთი მწკრივებისათვის დამტკიცებულია ძმები რისების თეორემა.

1. Introduction

It is well known that the classical Brothers Riesz theorem can be stated in terms of Fourier series. Namely, if a trigonometric series and its conjugate series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k\cos k\vartheta+b_k\sin k\vartheta)\ , \qquad \sum_{k=1}^{\infty}(a_k\sin k\vartheta-b_k\cos k\vartheta)$$

are both Fourier-Stieltjes series, then they are ordinary Fourier series (see, e.g., [8], p.285). This means that, if α and β are two real measures such that

$$\int_0^{2\pi}\cos k\vartheta\, d\alpha=\int_0^{2\pi}\sin k\vartheta\, d\beta,\quad \int_0^{2\pi}\sin k\vartheta\, d\alpha=-\int_0^{2\pi}\cos k\vartheta\, d\beta\quad (k=1,2,\dots),\qquad (1.1)$$

then α and β have to be absolutely continuous, i.e., there exist $f,\,g\in L^1(0,2\pi)$ such that

$$\alpha(B)=\int_B f(\vartheta)\,d\vartheta,\qquad \beta(B)=\int_B g(\vartheta)\,d\vartheta$$

for any Borel set $B\subset[0,2\pi]$. As far as generalizations of this theorem in higher real dimensions are concerned, we recall that Bochner [1] proved a result of this kind in the theory of multiple Fourier series. In [7], Muckenhoupt and Stein have deeply studied some series arising from ultraspherical

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expansions of functions having appropriate rotational invariance. This has led to consider some series generalizing the trigonometric ones. In this context, they gave also a concept of conjugacy for such series and they proved a result which is an analogue of the theorem of F. and M. Riesz. Because of the rotational invariance, the series they have considered are one-dimensional.

The purpose of this paper is to prove theorem IV, which provides a Brothers Riesz theorem for Laplace series. We remark that no rotational invariance is supposed. The series conjugate to a Laplace series, which is introduced in Section 3, is not a series of scalar functions, but of differential forms of degree $n-2$.

2. An Overview of Previous Results

Let Ω be a domain in $\mathbb{R}^n$. By $C^1_k(\Omega)$ we denote the space of differential forms of degree k (briefly k-forms) defined in Ω such that their coefficients are of class C^1. The differential and the co-differential of u are denoted by du and δu, respectively, while $*u$ denotes the adjoint of u with respect to the usual metric of $\mathbb{R}^n$. Namely, if $u = \frac{1}{k!} u_{i_1\ldots i_k} dx^{i_1}\ldots dx^{i_k}$, where $u_{i_1\ldots i_k} \in C^1(\Omega)$ are the components of a skew-symmetric covariant tensor, then

$$du = \frac{1}{k!}\frac{\partial u_{i_1\ldots i_k}}{\partial x_j} dx^j dx^{i_1}\ldots dx^{i_k}\ , \quad \delta u = (-1)^{n(k+1)+1} * d * u\ ,$$

$$*u = \frac{1}{k!(n-k)!}\ \delta^{1\ldots\ldots\ldots\ldots n}_{s_1..s_k i_1..i_{n-k}} u_{s_1\ldots s_k}\ dx^{i_1}\ldots dx^{i_{n-k}}\ .$$

We say that non–homogeneous differential form $U \in C^1_0(\Omega)\oplus\ldots\oplus C^1_n(\Omega)$ is *self-conjugate* if $dU = \delta U$ in Ω. This means that, if $U = \sum_{k=0}^n u_k$ (u_k being a k-form), then $\delta u_1 = 0$, $du_k = \delta u_{k+2}$ ($k = 0,\ldots,n-2$), $du_{n-1} = 0$.

It is possible to show that holomorphic functions of one complex variable, solutions of the Moisil-Theodorescu system, quaternionic hyperholomorphic functions, harmonic vectors (i.e. vectors w such that $div\, w = 0$, $curl\, w = 0$) can be identified with particular self-conjugate forms (for the details, see [4]).

In [3], [4] it is showed that several results of the theory of holomorphic functions of one complex variable hold true for self-conjugate forms in $\mathbb{R}^n$.

Hereafter Ω denotes a bounded domain of $\mathbb{R}^n$ such that its boundary Σ is a Lyapunov boundary and $M_k(\Sigma)$ is the space of k-measures on Σ (see [5]).

I. *If a self-conjugate form $U \in C^1_0(\Omega)\oplus\ldots\oplus C^1_n(\Omega)$ is such that U and $*U$ admit traces on Σ in $M_0(\Sigma)\oplus\ldots\oplus M_{n-1}(\Sigma)$, then these traces are absolutely continuous.*

This theorem provides a generalization of the classical Brothers Riesz Theorem in $\mathbb{R}^n$ and in some domains it can be stated also in the following way, where $\omega_h^{i_1\ldots i_k}$ denotes the k-form $\omega_h(x)dx^{i_1}\ldots dx^{i_k}$ and $\{\omega_h\}$ is a complete system of homogeneous harmonic polynomials.

II. *Let Ω be such that $\mathbb{R}^n - \overline{\Omega}$ is connected. If* $\alpha = (\alpha^0, \dots, \alpha^{n-1})$, $\widetilde{\alpha} = (\widetilde{\alpha}^n, \dots, \widetilde{\alpha}^1) \in M_0(\Sigma) \oplus \dots \oplus M_{n-1}(\Sigma)$ *are such that*

$$\int_{+\Sigma} [\alpha^k \wedge *d\omega_h^{i_1 \dots i_k} - \delta\omega_h^{i_1 \dots i_k} \wedge \widetilde{\alpha}^k + d\omega_h^{i_1 \dots i_k} \wedge \widetilde{\alpha}^{k+2} - \alpha^{k-2} \wedge *\delta\omega_h^{i_1 \dots i_k}] = 0$$

(where $\alpha^k = \widetilde{\alpha}^{n-k} = 0$, $k = -2, -1, n$*) for any* $1 \le i_1 < \dots < i_k \le n$, $h = 1, 2, \dots$, $k = 0, 1, \dots, n$*, then* $\alpha, \widetilde{\alpha}$ *are absolutely continuous.*

For the details and the proofs of I and II we refer to [3].

3. The Series Conjugate to a Laplace Series

It is well known that if u is a harmonic function in the unit ball $B = \{x \in \mathbb{R}^n \mid |x| < 1\}$, then it can be expanded by means of harmonic polynomials

$$u(x) = \sum_{h=0}^{\infty} |x|^h \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk} \left(\frac{x}{|x|} \right),$$

where $p_{nh} = (2h + n - 2)\frac{(h+n-3)!}{(n-2)!h!}$ and $\{Y_{hk}\}$ is a complete system of ultraspherical harmonics. We suppose $\{Y_{hk}\}$ orthonormal, i.e.,

$$\int_{\Sigma} Y_{hk} Y_{rs} \, d\sigma \begin{cases} = 1 & \text{if } h = r \text{ and } k = s, \\ = 0 & \text{otherwise.} \end{cases}$$

The "trace" of u on $\Sigma = \{x \in \mathbb{R}^n \mid |x| = 1\}$ is given by the expansion

$$\sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} a_{hk} Y_{hk}(x) \qquad (|x| = 1). \tag{3.1}$$

If the coefficients a_{hk} are

$$a_{hk} = \int_{\Sigma} f \, Y_{hk} \, d\sigma \qquad (a_{hk} = \int_{\Sigma} Y_{hk} \, d\mu),$$

we say that (3.1) is the Laplace series of the function f (of the measure μ).

Let us consider the 2-form

$$v = \sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} \, dY_{hk} \left(\frac{x}{|x|} \right) \wedge d(|x|^{h+2}) \tag{3.2}$$

and its adjoint

$$*v = \sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(h+2)(n+h-2)} * \left(dY_{hk} \left(\frac{x}{|x|} \right) \wedge d(|x|^{h+2}) \right). \tag{3.3}$$

It is possible to show that $dv = 0$, $\delta v = du$ in B, i.e., the non–homogeneous form $u + v$ is self–conjugate.

If $n = 2$, then the series which is obtained by taking $|x| = 1$ in (3.3) is just the series conjugate to (3.1); roughly speaking, it, represents the "trace" of the harmonic conjugate function on Σ. So in general, for any n, we say that (3.3) (with $|x| = 1$) is the series conjugate to (3.1). It represents the "restriction" of $*v$ on Σ, while the "restriction" of v, provided it does exist, is equal to 0, as it follows from (3.2).

In order to write the conjugate series (3.3) more explicitly, let us consider the polar coordinates in $\mathbb{R}^n$: $x_h = x_h(\varrho, \varphi_1, \dots, \varphi_{n-1})$, $h = 1, \dots, n$, and the relevant metric tensor

$$g_{ij} = \frac{\partial x}{\partial \varphi_i} \times \frac{\partial x}{\partial \varphi_j},\ g_{ni} = g_{in} = \frac{\partial x}{\partial \varphi_i} \times \frac{\partial x}{\partial \varrho},\ i, j = 1, \dots, n-1;\ g_{nn} = \frac{\partial x}{\partial \varrho} \times \frac{\partial x}{\partial \varrho}.$$

Let $\{g^{ij}\}$ be the inverse matrix of $\{g_{ij}\}$ (i.e., $g^{ij} g_{js} = \delta^i_s$) and let us set $g = \det(g_{ij})_{i,j=1,\dots,n}$. We remark that we have $g_{nn} = g^{nn} = 1$; $g_{ij} = 0$ if $i \neq j$ and that $g = \det(g_{ij})_{i,j=1,\dots,n-1}$.

If $v = v_j d\varphi^j d\varrho$, we have

$$*v \frac{1}{(n-2)!} \delta^{1\dots\dots n}_{jns_1..s_{n-2}} \sqrt{g}\, g^{jj} g^{nn} v_j d\varphi^{s_1} \dots d\varphi^{s_{n-2}} =$$
$$= \frac{1}{(n-2)!} \delta^{1\dots..n-1}_{s_1\dots s_{n-1}} \sqrt{g}\, g^{s_{n-1}s_{n-1}} v_{s_{n-1}} d\varphi^{s_1} \dots d\varphi^{s_{n-2}},$$

and then we may write

$$*v = \sum_{j=1}^{n-1} (-1)^{n-1-j} \sqrt{g}\, g^{jj} v_j\, d\varphi^1 \dots \widehat{j} \dots d\varphi^{n-1} \tag{3.4}$$

(where $\widehat{j}$ indicates that $d\varphi^j$ is omitted). Therefore the restriction $*v|_\Sigma$ is given by (3.4), where g, g^{jj}, v_j are considered for $\varrho = 1$. In particular, taking (3.2) as v, we find that the series conjugate to (3.1) can be written as

$$\sum_{h=0}^{\infty} \sum_{k=1}^{p_{nh}} \frac{a_{hk}}{(n+h-2)} \sum_{j=1}^{n-1} (-1)^{n-1-j} \sqrt{g}\, g^{jj} \frac{\partial Y_{hk}}{\partial \varphi_j}\, d\varphi^1 \dots \widehat{j} \dots d\varphi^{n-1}\ . \tag{3.5}$$

Let us consider now the space $L^2_{n-2}(\Sigma)$ endowed with the scalar product

$$(\gamma, \psi) = \int_{+\Sigma} \gamma \wedge \underset{\Sigma}{*} \psi,$$

where $\underset{\Sigma}{*}\psi$ denotes the adjoint of ψ on Σ with respect to the usual metric on Σ. Namely, if $\psi = \frac{1}{(n-2)!} \psi_{s_1\dots s_{n-2}}\, d\varphi^{s_1} \dots d\varphi^{s_{n-2}}$ then

$$\underset{\Sigma}{*}\psi = \frac{\sqrt{g}}{(n-2)!} \delta^{1\dots..n-1}_{s_1\dots s_{n-1}} g^{s_1 i_1} \dots g^{s_{n-2} i_{n-2}} \psi_{i_1\dots i_{n-2}} d\varphi^{s_{n-1}}\ . \tag{3.6}$$

Let us introduce the following system of $(n-2)$-forms

$$\psi_{hk} = \frac{1}{(h+2)\sqrt{h(n+h-2)}} * (dY_{hk} \wedge d\varrho^{h+2}) =$$
$$\frac{1}{\sqrt{h(n+h-2)}} \varrho^{h+1} \sum_{j=1}^{n-1} (-1)^{n-1-j} \sqrt{g}\, g^{jj} \frac{\partial Y_{hk}}{\partial \varphi_j}\, d\varphi^1 \ldots \widehat{j} \ldots d\varphi^{n-1} \ .$$

III. *The system $\{\psi_{hk}\}$ is orthonormal in $L^2_{n-2}(\Sigma)$.*

If

$$\psi = \sum_{j=1}^{n-1} \psi_j\, d\varphi^1 \ldots \widehat{j} \ldots d\varphi^{n-1}, \qquad \widetilde{\psi} = \sum_{j=1}^{n-1} \widetilde{\psi}_j\, d\varphi^1 \ldots \widehat{j} \ldots d\varphi^{n-1} \ ,$$

taking into account (3.6), we get

$$\underset{\Sigma}{*}\widetilde{\psi} = \sum_{j=1}^{n-1} (-1)^{n-1-j} \sqrt{g}\, g^{11} .. \widehat{j} .. g^{n-1,n-1} \widetilde{\psi}_j\, d\varphi^j =$$
$$= \sum_{j=1}^{n-1} (-1)^{n-1-j} \frac{1}{\sqrt{g}\, g^{jj}}\, \widetilde{\psi}_j\, d\varphi^j \ , \tag{3.7}$$

from which easily follows

$$\int_{+\Sigma} \psi \wedge \underset{\Sigma}{*}\widetilde{\psi} = \int_{\Sigma} \sum_{j=1}^{n-1} \frac{1}{g\, g^{jj}}\, \psi_j\, \widetilde{\psi}_j\, d\sigma \ .$$

This implies

$$(\psi_{hk}, \psi_{rs}) = \frac{1}{\sqrt{hr(n+h-2)(n+r-2)}} \int_{\Sigma} \sum_{j=1}^{n-1} g^{jj} \frac{\partial Y_{hk}}{\partial \varphi_j} \frac{\partial Y_{rs}}{\partial \varphi_j}\, d\sigma \ .$$

On the other hand,

$$\int_{\Sigma} \sum_{j=1}^{n-1} g^{jj} \frac{\partial Y_{hk}}{\partial \varphi_j} \frac{\partial Y_{rs}}{\partial \varphi_j}\, d\sigma = -\int_{\Sigma} \sum_{j=1}^{n-1} \frac{1}{\sqrt{g}} \frac{\partial}{\partial \varphi_j} \left(\sqrt{g}\, g^{jj} \frac{\partial Y_{hk}}{\partial \varphi_j} \right) Y_{rs}\, d\sigma =$$
$$= -\int_{\Sigma} Y_{rs}\, \Delta_\Sigma Y_{hk}\, d\sigma = h(n+h-2) \int_{\Sigma} Y_{rs} Y_{hk}\, d\sigma$$

(Δ_Σ being the Laplace-Beltrami operator on Σ) shows that $\{\psi_{hk}\}$ is orthonormal in $L^2_{n-2}(\Sigma)$.

We finally remark that the conjugate series (3.5) (or (3.3) with $|x| = 1$) can be written also as

$$\sum_{h=1}^{\infty} \sum_{k=0}^{p_{nh}} \sqrt{\frac{h}{n+h-2}}\, a_{hk} \psi_{hk}. \tag{3.8}$$

4. The Brothers Riesz Theorem for Laplace Series

If (3.1) is a Laplace series of a function $f \in L^2(\Sigma)$, then the conjugate series (3.8) is a "Fourier series" of an $(n-2)$-form $g \in L^2_{n-2}(\Sigma)$. This follows immediately from Fischer-Riesz theorem since

$$\sum_{h=1}^{\infty} \sum_{k=0}^{p_{nh}} \frac{h}{n+h-2}\, a_{hk}^2 \leq \sum_{h=1}^{\infty} \sum_{k=0}^{p_{nh}} a_{hk}^2 < +\infty$$

implies that there exists $g \in L^2_{n-2}(\Sigma)$ such that

$$a_{hk} = \sqrt{\frac{n+h-2}{h}} \int_{+\Sigma} g \wedge \underset{\Sigma}{*}\, \psi_{hk}$$

and therefore the conjugate series is the "Fourier series" of g with respect to the system $\{\psi_{hk}\}$: $\sum\limits_{h=1}^{\infty} \sum\limits_{k=0}^{p_{nh}} (g, \psi_{hk})\, \psi_{hk}$. Moreover, we may suppose g such that $(g, \gamma) = 0\ \forall\ \gamma \in L^2_{n-2}(\Sigma)\ : (\gamma, \psi_{hk}) = 0$ $(h = 1, 2, \ldots ; k = 1, \ldots, p_{nh})$.

Because of a completeness theorem proved in [2] (p.195), these orthogonality conditions are equivalent to the following ones: $(g, \gamma) = 0\ \forall\ \gamma \in C^{\infty}_{n-2}(\mathbb{R}^n)\ : d\gamma = 0$ on Σ. A similar question for measures is more delicate and the theorem II makes possible to prove the following result, which can be considered as the Brothers Riesz theorem for Laplace series:

IV. *Let (3.1) be a Laplace series of a measure $\mu \in M(\Sigma)$. If its conjugate series (3.8) is a "Fourier series" of an $(n-2)$-measure, i.e., if there exists $\beta \in M_{n-2}(\Sigma)$ such that*

$$a_{hk} = \sqrt{\frac{n+h-2}{h}} \int_{+\Sigma} \beta \wedge \underset{\Sigma}{*}\, \psi_{hk} \quad (h = 1, 2, \ldots ; k = 1, \ldots, p_{nh}), \tag{4.1}$$

and if

$$\int_{+\Sigma} \beta \wedge \underset{\Sigma}{*}\, \gamma = 0 \qquad \forall\ \gamma \in C^{\infty}_{n-2}(\mathbb{R}^n)\ :\ d\gamma = 0 \text{ on } \Sigma\ , \tag{4.2}$$

then μ and β are absolutely continuous.

We remark that in the case $n = 2$, (4.1) are nothing but (1.1), while (4.2) is not restrictive. Observe now that we may write

$$Y_{hk} d\sigma = \frac{1}{h} \frac{\partial}{\partial \varrho} (\varrho^h Y_{hk}) \Big|_{\varrho=1} d\sigma = \frac{1}{h} * d(\varrho^h Y_{hk})\ .$$

On the other hand, because of (3.7),

$$\underset{\Sigma}{*}\psi_{hk} = \frac{1}{\sqrt{h(n+h-2)}} \sum_{j=1}^{n-1} \frac{\partial Y_{hk}}{\partial \varphi_j}\, d\varphi_j = \frac{1}{\sqrt{h(n+h-2)}}\, dY_{hk},$$

and the conditions (4.1) can be written as

$$\int_{+\Sigma} \widetilde{\mu} \wedge *d\omega = \int_{+\Sigma} *d\omega \wedge \widetilde{\mu} = \int_{+\Sigma} \beta \wedge d\omega = (-1)^n \int_{+\Sigma} d\omega \wedge \beta \tag{4.3}$$

for any harmonic polynomial ω. Here $\widetilde{\mu} \in M_0(\Sigma)$ is defined by

$$\int_{+\Sigma} \widetilde{\mu} \wedge w\, d\sigma = \int_{\Sigma} w\, d\mu \qquad \forall\ w \in C^0(\Sigma).$$

In order to apply the theorem II, we have also to show that

$$(-1)^n \int_{+\Sigma} \delta(\omega dx^p dx^s) \wedge \beta + \int_{+\Sigma} \widetilde{\mu} \wedge *\delta(\omega dx^p dx^s) = 0 \tag{4.4}$$

for any harmonic polynomial ω and for any $1 \le p < s \le n$. Since

$$\delta(\omega dx^p dx^s) = \omega_{,s} dx^p - \omega_{,p} dx^s\,;\ *\delta(\omega dx^p dx^s) = x_p\omega_{,s} - x_s\omega_{,p}\ (\text{on } \Sigma) \tag{4.5}$$

(where $\omega_{,p} = \frac{\partial \omega}{\partial x_p}$) and taking into account that if ω is a harmonic homogeneous polynomial of degree h, then $x_p\omega_{,s} - x_s\omega_{,p}$ is a harmonic homogeneous polynomial of the same degree, it follows from (4.3)

$$\int_{+\Sigma} \widetilde{\mu} \wedge *\delta(\omega dx^p dx^s) = \int_{\Sigma} (x_p\omega_{,s} - x_s\omega_{,p})\, d\mu =$$
$$= \frac{1}{h} \int_{+\Sigma} \widetilde{\mu} \wedge *d(x_p\omega_{,s} - x_s\omega_{,p}) = \frac{1}{h} \int_{+\Sigma} \beta \wedge d(x_p\omega_{,s} - x_s\omega_{,p}). \tag{4.6}$$

Moreover (s and p are fixed),

$$d(x_p\omega_{,s} - x_s\omega_{,p}) = \omega_{,s} dx^p - \omega_{,p} dx^s + [x_p\omega_{,js} - x_s\omega_{,jp}]dx^j =$$
$$= \omega_{,s} dx^p - \omega_{,p} dx^s + [x_p\omega_{,ps} - x_s\omega_{,pp}]dx^p +$$
$$+ [x_p\omega_{,ss} - x_s\omega_{,sp}]dx^s + \sum_{\substack{j=1 \\ j \ne p,s}}^{n} [x_p\omega_{,js} - x_s\omega_{,jp}]dx^j\ .$$

Since

$$x_p\omega_{,ps} - x_s\omega_{,pp} = x_j\omega_{,js} + \sum_{\substack{k=1 \\ k \ne p,s}}^{n} x_s\omega_{,kk} - \sum_{\substack{k=1 \\ k \ne p,s}}^{n} x_k\omega_{,ks} =$$

$$= (h-1)\omega_{,s} + \sum_{\substack{k=1 \\ k\neq p,s}}^{n} (x_s\omega_{,kk} - x_k\omega_{,sk}),$$

and analogously

$$x_p\omega_{,ss} - x_s\omega_{,sp} = -(h-1)\omega_{,p} - \sum_{\substack{k=1 \\ k\neq p,s}}^{n} (x_p\omega_{,kk} - x_k\omega_{,pk}) \ ,$$

we may write $d(x_p\omega_{,s} - x_s\omega_{,p}) = h(\omega_{,s}dx^p - \omega_{,p}dx^s) + \Lambda$, where

$$\Lambda = \sum_{\substack{j=1 \\ j\neq p,s}}^{n} \left[(x_p\omega_{,sj} - x_s\omega_{,pj})dx^j + (x_s\omega_{,jj} - x_j\omega_{,sj})dx^p + (x_j\omega_{,pj} - x_p\omega_{,jj})dx^s\right] .$$

Because of (4.5) and (4.6), we have

$$\int\limits_{+\Sigma} \widetilde{\mu} \wedge *\delta(\omega dx^p dx^s) = \int\limits_{+\Sigma} \beta \wedge \delta(\omega dx^p dx^s) + \frac{1}{h}\int\limits_{+\Sigma} \beta \wedge \Lambda \ . \qquad (4.7)$$

Let us fix j, s, p and consider the 1-form $\Theta = (x_p w_{,s} - x_s w_{,p})\, dx^j + (x_s w_{,j} - x_j w_{,s})\, dx^p + (x_j w_{,p} - x_p w_{,j})\, dx^s$, where w is a scalar function. A direct computation (we omit for lack of space) shows that, up to a multiplicative constant, $\underset{\Sigma}{*}\,\Theta = \delta^{1\,\ldots\ldots\,n}_{jspi_1..i_{n-3}} dw \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_{n-3}}$. Then $d\underset{\Sigma}{*}\,\Theta = 0$ and therefore $d\underset{\Sigma}{*}\,\Lambda = 0$. By virtue of (4.2), we must have $\int_{+\Sigma} \beta \wedge \Lambda = 0$. Now (4.4) follows from (4.7). By theorem II, $\widetilde{\mu}$ and $(-1)^n\beta$ are absolutely continuous and this completes the proof. Finally notice that, without (4.2), the theorem is false (if $n > 2$).

References

1. B. Bochner, Boundary values of analytic functions in several variables and of almost periodic functions. *Ann. of Math.* 45(1944), 708–722.

2. A. Cialdea, Sul problema della derivata obliqua per le funzioni armoniche e questioni connesse. *Rend. Acc. Naz. delle Scienze detta dei XL* 12(1988), 181–200.

3. A. Cialdea, The Brothers Riesz theorem for conjugate differential forms in $\mathbb{R}^n$. *Appl. Anal.* (*to appear*).

4. A. Cialdea, On the theory of self-conjugate differential forms. (*to appear*).

5. G. Fichera, Spazi lineari di k-misure e di forme differenziali. *In: Proc. of Intern. Symposium on Linear Spaces, Jerusalem* 1960, *Israel Ac. of Sciences and Humanities. Pergamon Press,* 1961, 175–226.

6. G. Fichera and L. De Vito, Funzioni analitiche di una variabile complessa. *Veschi, Roma,* 1971.

7. B. Muckenhoupt and E. M. Stein, Classical expansions and their relation to conjugate harmonic functions. *Trans. Amer. Math. Soc.* 118(1965), 17–92.

8. A. Zygmund, Trigonometric Series. *Cambridge University Press,* 1979.

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