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ON BOUNDARY VALUE PROBLEMS FOR N-TH ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSES

Abstract. A boundary value problem is considered for an N-th order functional differential equation with impulses. It is reduced to the same boundary value problem for another equation of the same order without impulses. The reduction is based on constructing of an isomorphism between the space of the functions which are piecewise absolutely continuous up to the $(N-1)$-st derivative and satisfy the impulse conditions, at the discontinuity points and the space of the functions which are absolutely continuous up to the $(N-1)$-st derivative. The approach allows to derive conditions on the sign preservation for the Green function of the considered boundary value problem.

რეზიუმე. განხილულია სასაზღვრო ამოცანა N-ური რიგის ფუნქციონალურ-დიფერენციალური იმპულსებიანი განტოლებისთვის. იგი დაიყვანება იგივე სასაზღვრო ამოცანაზე იმავე რიგის უიმპულსო განტოლებისათვის. დაყვანა ემყარება იზომორფიზმის აგების შემდეგ ორ სივრცეს შორის:ერთის მხრივ, იმ ფუნქციათა სივრცე რომელთაც აქვთ $N-1$ რიგამდე ჩათვლით უბან-უბან აბსოლიტურად უწყვეტი კერძო წარმოებულები და წყვეტის წერტილებში აკმაყოფილებენ იმპულსის პირობებს, ხოლო მეორეს მხრივ $N-1$ რიგამდე ჩათვლით აბსოლუტურად უწყვეტ ფუნქციათა სივრცე. ეს მიდგომა საშუალებას იძლევა დადგენილი იქნას განხილული სასაზღვრო ამოცანის გრინის ფუნქციის ნიშანმუდმივობის პირობები.

1. Introduction

Consider the following equation

$$(\mathcal{L}x)(t) \equiv x^{(n)}(t) + \sum_{j=1}^{k} (T_j x)(t) = f(t), \quad t \in [0,b], \tag{1.1}$$

$$x(t_i) = \beta_i x(t_i - 0), \quad i = 1, 2, \dots, m, \tag{1.2}$$

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where

$$\beta_i > 0, \;\; i = 1, \ldots, m; \quad 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b,$$
$$T_j : C(0, t_1, \ldots, t_m, b) \to L(0, b),$$

are linear bounded Volterra operators acting from the space of piecewise continuous functions $x : [0, b] \to \mathbf{R}$ into the space of summable functions.

Particular cases of the operators T_j are

$$(T_j x)(t) = \int_0^t K_j(t, s) x(s)\, ds,$$
$$(T_j x)(t) = p_j(t) x(h_j(t)), \quad x(\xi) = 0, \;\; \xi < 0,$$

where $p_j \in L(0, b)$, K_j satisfies standard smoothness conditions (see, for example, [6]), h_j are measurable and $h_j(t) \le t$, $t \in [0, b]$.

Equations of the type (1.1)–(1.2) are intensively studied. A large number of publications devoted to these equations has been published recently (see, e.g., [2], [3], [4], [5]).

Let $\mathbf{D}(0, t_1, \ldots, t_m, b)$ stand for the Banach space of all piecewise continuous functions $x : [0, b] \to \mathbf{R}$ having absolutely continuous on every interval $[t_i, t_{i+1})$, $i = 0, 1, \ldots, m$, derivative $x^{(n-1)}$, and satisfying (1.2) at t_i, $i = 1, 2, \ldots, m$, (1.2).

Denote by $\mathbf{D}(0, b)$ the Banach space of all functions $y : [0, b] \to bR$ with absolutely continuous on $[0, b]$ derivative $y^{(n-1)}$.

Let $l_i : \mathbf{D}(0, t_1, \ldots, t_m, b) \to \mathbf{R}$, $i = 1, \ldots, n$, be linear bounded functionals.

If the boundary value problem (1.1), (1.2), (1.3), where

$$l_i x = 0, \quad i = 1, \ldots, n, \tag{1.3}$$

has a unique solution for every $f \in L(0, b)$, then the solution to this problem can be represented in the following integral form

$$x(t) = \int_0^b G(t, s) f(s)\, ds,$$

where $G(t, s)$ is called the Green function of the problem (see [1], [6]).

The aim of this paper is to obtain some positivity (nonpositivity) conditions for the Green function $G(t, s)$ for different boundary value problems to impulsive equations (1.1), (1.2). Positivity of the Green function allows to estimate solution of the boundary value problem. For example, from the inequality

$$(\mathcal{L} z)(t) \le (\mathcal{L} x)(t) \le (\mathcal{L} y)(t), \quad t \in [0, b]$$

and the equalities

$$l_i z = l_i x = l_i y, \quad i = 1, \ldots, n,$$

it follows that $z(t) \le x(t) \le y(t)$, $t \in [0, b]$.

The most important application of the positivity of $G(t,s)$ is connected with the famous regularization scheme for investigation of nonlinear boundary value problems. Different variations on this scheme can be found in ([3], [4], [5]).

To study (1.1), (1.2), (1.3), we compare solutions of the above equation with solutions of some auxiliary equation, constructed using an isomorphism between $\mathbf{D}(0,t_1,\ldots,t_m,b)$ and $\mathbf{D}(\mathbf{0},\mathbf{b})$. Indeed, the following isomorphism between the two spaces can be established:

$$\begin{aligned} x(t) = y(t) &+ \Big[\chi_{[t_1,t_2)}(t) + \chi_{[t_2,t_3)}(t)\beta_2 + \\ &+ \chi_{[t_3,t_4)}(t)\beta_3\beta_2 + \cdots + \chi_{[t_m,t_{m+1})}(t)\beta_m\beta_{m-1}\cdots\beta_2\Big](\beta_1-1)y(t_1) + \\ &+ \Big[\chi_{[t_2,t_3)}(t) + \chi_{[t_3,t_4)}(t)\beta_3 + \chi_{[t_4,t_5)}(t)\beta_4\beta_3 + \cdots + \\ &+ \chi_{[t_m,t_{m+1})}(t)\beta_m\beta_{m-1}\cdots\beta_3\Big](\beta_2-1)y(t_2) + \\ &+ \Big[\chi_{[t_3,t_4)}(t) + \chi_{[t_4,t_5)}(t)\beta_4 + \chi_{[t_5,t_6)}(t)\beta_5\beta_4 + \cdots + \\ &+ \chi_{[t_m,t_{m+1})}(t)\beta_m\beta_{m-1}\cdots\beta_4\Big](\beta_3-1)y(t_3) + \cdots + \\ &+ \big[\chi_{[t_{m-1},t_m)}(t) + \chi_{[t_m,t_{m+1})}(t)\beta_m\big](\beta_{m-1}-1)y(t_{m-1}) + \\ &+ \chi_{[t_m,t_{m+1})}(t)(\beta_m-1)y(t_m). \end{aligned} \tag{1.4}$$

Here

$$\chi_{[t_i,t_{i+1})}(t) = \begin{cases} 1 & \text{if} \quad t_i \le t < t_{i+1} \\ 0 & t \notin [t_i,t_{i+1}) \end{cases}. \tag{1.5}$$

A short expression for (1.4) is:

$$x(t) = y(t) + \sum_{i=1}^{m} q_i(t)y(t_i), \tag{1.6}$$

where

$$q_i(t) = \left[\chi_{[t_i,t_{i+1})}(t) + \sum_{j=1}^{m-i}\chi_{[t_{i+j},t_{i+j+1})}(t)\prod_{k=1}^{j}\beta_{i+j-k}\right](\beta_i-1), \quad i=1,\ldots,m.$$

Note that $q_i(t) = \text{const}$ on each semiinterval $[t_i,t_{i+1})$, $i=1,\ldots,m$.
Substituting (1.6) into (1.1), we obtain

$$y^{(n)}(t) + \sum_{j=1}^{k}(T_jy)(t) + \sum_{j=1}^{k}\Big(T_j\sum_{i=1}^{m}y(t_i)q_i\Big)(t) = f(t), \quad t\in[0,b], \tag{1.7}$$

Changing the order of summation, we get

$$(\widetilde{\mathcal{L}}y)(t) = y^{(n)}(t) + \sum_{j=1}^{k} \Big(T_j y\Big)\Big(t\Big) +$$

$$+ \sum_{i=1}^{m} \Big[\sum_{j=1}^{k} (T_j q_i)(t)\Big] y(t_i) = f(t), \quad t \in [0,b], \tag{1.8}$$

i.e., a non-impulsive functional-differential equation in the space $D(0,b)$.

By the construction, the following result is proved:

Lemma 1.1. *A function $x \in \mathbf{D}(0, t_1, \dots, t_m, b)$ is a solution of* (1.1) – –(1.2) *if and only if the function $y \in \mathbf{D}(0,b)$, corresponding to x by* (1.6), *is a solution of* (1.8).

The lemma reduces the investigation of (1.1)–(1.2) to study of (1.8), which is easier in many cases.

2. On a class of boundary value problems for impulsive equations

Let $l_i : \mathbf{D}(0, t_1, \dots, t_m, b) \to \mathbf{R}$, $i = 1, \dots, n$, be linear bounded functionals.

Add to (1.1), (1.2), boundary conditions

$$l_i x = 0, \quad i = 1, \dots, n, \tag{2.1}$$

and consider the boundary value problem (BVP) (1.1), (1.2), (2.1).

Let us discribe the class of problems under investigation by the following

Definition 2.1. BVP (1.1), (1.2), (2.1) belongs to the class $\mathbf{A}$ if for any y, $z \in D(0, t_1, \dots, t_m, b)$ such that $z(0) = y(0)$, $z^{(i)}(t) = y^{(i)}(t)$, $i = 1, \dots, n-1$, $t \in [0,b]$, the following equalities hold: $l_j z = l_j y$, $j = 1, \dots, n$.

Examples of BVP from the class $\mathbf{A}$ are the problems with the following boundary conditions:

$$x^{(i)}(0) = 0, \quad i = 0, \dots, n-2, \quad x'(b) = 0, \tag{2.2}$$

$$x'(0) = x'(t_1) = x'(t_2) = \cdots = x'(t_{n-2}) = 0. \tag{2.3}$$

Theorem 2.1. *Suppose that*

1) *Impulsive BVP* (1.1), (1.2), (2.1) *belongs to the class* $\mathbf{A}$*;*

2) *The Green function of the non-impulsive BVP* (1.8), (2.1) *preserves its sign;*

3) $\beta_i > 1$, $i = 1, \dots, m$.

Then the Green function $G(t,s)$ of the impulsive BVP (1.1), (1.2), (2.1) *preserves the sign.*

To prove Theorem 2.1, it is sufficient to take into account the isomorphism (1.4) establishing the correspondence between solutions of the impulse equation (1.1), (1.2) and the non-impulsive equation (1.8). The inequalities $\beta_i > 1$, $i = 1, \ldots, m$, guarantee, by virtue of (1.4), that the nonpositivity of y implies the nonpositivity of x.

Consider the equation

$$x^{(n)}(t) + \sum_{j=1}^{k} p_j(t)x(h_j(t)) = f(t), \quad t \in [0,b], \tag{2.4}$$

$$h_j(t) \le t, \quad t \in [0,b],$$

$$x(t_i) = \beta_i x(t_i - 0), \;\; i = 1, \ldots, m, \quad x(\xi) = 0, \;\; \xi < 0. \tag{2.5}$$

Introduce the functions p_j^+ and p_j^-: $p_j(t) = p_j^+(t) - p_j^-(t)$, where $p_j^+ \ge 0$, $p_j^- \ge 0$ and define

$$\tau_j(t) = t - h_j(t), \quad \tau^* = \max_{1 \le j \le m} \operatorname*{vraisup}_{t \in [0,b]} \tau_j(t), \quad \beta^* = \max_{1 \le i \le m} \beta_i.$$

Theorem 2.2. *Let n be an even number, $\beta_i > 1$ for $i = 1, \ldots, m$, and the folowing inequalities be fulfilled:*

$$\tau^* \sqrt[n]{m(\beta^* - 1)\beta_2 \cdots \beta_m + 1}\, \sqrt[n]{\sum_{j=1}^{k} p_j^-(t)} \le \frac{n}{e}, \quad t \in [0,b], \tag{2.6}$$

and

$$\big[m(\beta^* - 1)\beta_2 \cdots \beta_m + 1\big] \sum_{j=1}^{k} p_j^+(t) \le \frac{n!(n-1)}{b^n}, \quad t \in [0,b]. \tag{2.7}$$

Then the Green function of the BVP (2.4), (2.5), (2.2) *is nonpositive in the square t, $s \in [0,b]$.*

For the proof it is enough to mention that inequalities (2.6), (2.7) guarantee, by virtue of [9], [10], that the Green function of the nonimpulsive BVP (1.8), (2.2) is nonpositive in square t, $s \in [0,b]$. Reference to Theorem 2.1 completes the proof. ■

3. On the $(n-1,1)$ boundary value problem

In some cases it is possible, using results on BVP from the class **A**, to obtain statements for some problems not depending on **A**. One of the possible schemes of such kind is given by the following Theorem 3.1.

Consider the two-point boundary value problem (BVP) for (1.1)–(1.2):

$$x^{(i)}(0) = 0, \;\; i = 0, 1, \ldots, n-2, \quad x(b) = 0. \tag{3.1}$$

Let us determine conditions under which the Green function $G(t,s)$, t, $s \in [0,b]$ of (1.1)–(1.2), (3.1) preserves its sign.

Theorem 3.1. *Let $\beta_i > 1$, $i = 1, \ldots, m$, and suppose that the Green function $\widetilde{G}_c(t,s)$, $t, s \in [0,c]$ of the BVP*

$$(\widetilde{\mathcal{L}}y)(t) = f(t), \quad t \in [0,c], \tag{3.2}$$
$$y^{(i)}(0) = 0, \ \ i = 0,1,\ldots,n-2, \quad y'(c) = 0,$$

exists and is nonpositive for every $c \in (0,b)$. Then the Green function $G(t,s)$ of the BVP (1.1)–(1.2), (3.1) *is nonpositive for t, $s \in [0,b]$.*

Proof. Assume, on the contrary, that $G(t,s)$ changes its sign. Then there exists a function $f(t) \geq 0$, $t \in [0,b]$, such that the solution x of the considered problem changes its sign at $t^* \in (0,b)$.

By (1.6), $x^{(i)}(t) = y^{(i)}(t)$, $t \in [0,b]$, $i = 1, \ldots, n-1$. It is evident that there exists $c \in (t^*, b)$ such that $x'(c) = 0$. But it means that also $y'(c) = 0$. The solution y of (3.2) has the representation

$$y(t) = \int_0^c \widetilde{G}_c(t,s) f(s)\, ds.$$

From the nonpositivity of $\widetilde{G}_c(t,s)$ it follows that $y(t) \leq 0$ for $t \in [0,b]$. Using (1.6), we obtain the inequality $x(t) \leq y(t)$ for $t \in [0,b]$, which contradicts our assumption that $x(t)$ changes its sign on $(0,b)$. ■

Corollary 3.1. *Let*
1) $\beta_i > 1$, $i = 1, \ldots, m$;
2) $p_j(t) \geq 0$, $j = 1, \ldots, k$, $t \in [0,b]$;
3) $[m(\beta^* - 1)\beta_2 \cdots \beta_m + 1] \sum_{j=1}^k p_j(t) \leq \frac{n!(n-1)}{b^n}$, $t \in [0,b]$.

Then the Green function $G(t,s)$ of the BVP (2.4)–(2.5), (3.1) *is nonpositive for t, $s \in [0,b]$.*

Proof. Every BVP (3.2) is equivalent to the integral equation

$$y(t) = (\Omega_{0c}y)(t) + \int_0^c G_c(t,\tau) f(\tau)\, d\tau, \quad t \in [0,c].$$

Here $G_c(t,\tau)$, $s \leq \tau < t \leq c$, is the Green function of the boundary value problem

$$y^{(n)}(t) = f(t), \ \ t \in [s,c], \quad y^{(i)}(s) = 0, \ \ i = 0,1,\ldots,n-2, \quad y'(c) = 0.$$

The operator $\Omega_{sc} : \mathbf{C(0,b)} \to \mathbf{C(0,b)}$ ($\mathbf{C(0,b)}$ being the Banach space of continuous functions $y : [0,b] \to \mathbf{R}$ with the Chebyshev norm), is defined by

$$(\Omega_{sc}y)(t) = -\int_s^c G_c(t,\tau) \sum_{j=1}^{m+k} p_j(\tau) y(h_j(\tau))\, d\tau.$$

The condition **2)** guarantees that Ω_{0c} is a positive operator and the condition **3)** guarantees that $|\Omega_{0c}| < 1$. Therefore there exists the positive inverse operator $(I - \Omega_{0c})^{-1}$. Each nonnegative f yields a nonpositive solution x to the BVP (3.2), i.e., the Green function $\widetilde{G}_c(t,s)$ is nonpositive. Employing Theorem 3.1 we complete the proof. ■

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