Mem. Differential Equations Math. Phys. 12(1997), 57–61

ROBERT FINN AND JIANAN LU

SOME REMARKABLE PROPERTIES OF H-GRAPHS

ABSTRACT. It is proved that if $H(u)$ is non-decreasing and if $H(-\infty) \neq H(+\infty)$, then if $u(x)$ describes a graph over a disk $B_R(0)$, with (upward oriented) mean curvature $H(u)$, there is a bound on the gradient $|D\,u(0)|$ that depends only on R, on $u(0)$, and on the particular function $H(u)$. As a consequence a form of Harnack's inequality is obtained, in which no positivity hypothesis appears. The results are qualitatively best possible, in the senses that a) they are false if H is constant, and b) the dependences indicated are essential (If $H(-\infty) = -\infty$, $H(+\infty) = \infty$, then the dependence on $u(0)$ can be deleted). The demonstrations are based on an existence theorem for a nonlinear boundary problem with singular data, which is of independent interest.

რეზიუმე. ვთქვათ, $H(u)$ არაკლებადია და $H(-\infty) \neq H(+\infty)$. დამტკიცებულია, რომ თუ $u(x)$ აღწერს $B_R(0)$ წრეზე განსაზღვრული ფუნქციის გრაფიკს (ზევით მიმართული) $H(u)$ საშუალო სიმრუდით, მაშინ არსებობს $|D_u(0)|$ გრადიენტის შეფასება, რომელიც დამოკიდებულია მხოლოდ R-ზე, $u(0)$-ზე და $H(u)$ ფუნქციაზე. შედეგის სახით მიღებულია ჰარნაკის უტოლობის ერთი ნაირსახეობა, რომელშიც არ ფიგურირებს დადებითობის პირობა. შედეგები თვისებრივად გაუუმჯობესებადია იმ აზრით, რომ ა) ისინი არაა სამართლიანი, თუ H მუდმივია, და ბ) ზემოთ მითითებული დამოკიდებულებები არსებითია (თუ $H(-\infty) = -\infty$, $H(+\infty) = \infty$, მაშინ $u(0)$-ზე დამოკიდებულებას ადგილი არა აქვს). დამტკიცებები ეყრდნობა გარკვეული არაწრფივი სინგულარული სასაზღვრო ამოცანისათვის არსებობის ერთ, თავისთავადაც საინტერესო თეორემას.

In this note, we summarize and improve in some detail the material of [1]. We consider the equation

$$\operatorname{div} Tu = 2H(u), \quad Tu = \frac{1}{\sqrt{1+u_x^2+u_y^2}}\langle u_x, u_y\rangle \tag{1}$$

whose solutions are graphs $u(x,y)$ of mean curvature $H(u)$. Since (1) is of elliptic type, the qualitative behavior of its solutions may be expected to emulate what happens for the Laplace equation $\Delta u = 0$, which is usually taken

---

1991 *Mathematics Subject Classification.* 53A10, 35J60, 35B45.

Key words and phrases. Mean curvature, H-graph, gradient estimate, Harnack inequality, moon surface, capillarity.

as prototype for elliptic equations. Notably, one expects an a-priori bound on gradient of solutions, depending only on a bound for the magnitude of the solution and on distance R to the boundary of the domain of definition. Further, one expects for positive solutions a Harnack inequality, bounding the values of a solution in a neighborhood of a point in terms of the value at that point. Such formally analogous estimates have in fact been demonstrated, see for example [2]–[6], but to restrict attention to them would obscure important distinctions in behavior, as the solutions of (1) exhibit much stronger regularity properties than are suggested by the analogies. In fact, it was shown in [6] that if $H \equiv H_0 \neq 0$ and if R exceeds a critical value $R_0 = (0.535...)/|H_0|$, then there is a gradient bound depending only on R; such behavior is qualitatively very different from what happens with uniformly elliptic equations.

We intend to show that if H is not constant then still more striking differences in behavior can occur, exhibiting properties of the solutions that differ basically from what happens when $H \equiv H_0$. We assert:

Theorem 1. *Assume $H'(u) \geq 0$, $H(-\infty) \neq H(+\infty)$. Let $u(x,y)$ be a solution of* (1) *over a disk $B_R(0)$. Then $|\nabla u(0)|$ is bounded, depending only on R and on $u(0)$. If $H(-\infty) = -\infty$, $H(+\infty) = +\infty$, then the bound depends only on R.*

The significant new feature of this result is that the bound requires no information on the magnitude of u, except perhaps at the single point of evaluation. The result cannot be improved, in the sense that the dependences on R and on $u(0)$ both are necessary. Additionally, the theorem is false if $H \equiv$ const. In contrast to that case, there is no requirement that R be sufficiently large.

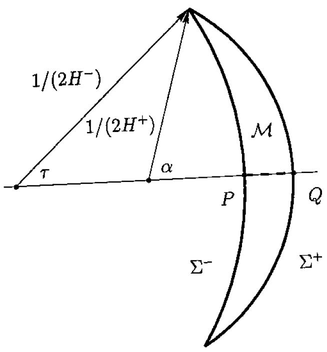

Figure 1: Moon domain; $H^- \neq H^+$.

From Theorem 1 we obtain by a formal integration a version of Harnack's Inequality, in which the one sided bound required in the usual form of that inequality does not appear:

Theorem 2. *Assume $H'(u) \geq 0$, $H(-\infty) \neq H(+\infty)$. Then there exist a positive function $\rho^+(u_0; R) \leq R$ and a continuous function $U^+(u_0; R; \rho)$ with $U^+(u_0; R; 0) = u_0$ such that if $u(x,y)$ satisfies* (1) *in $B_R(0)$ and $u(0) =$*

u_0, then $u \leq U^+$ throughout $B_{\rho^+}(0)$. There exist a positive $\rho^-(u_0; R) \leq R$ and a continious $U^-(u_0; R; \rho)$ with $U^-(u_0; R; 0) = u_0$ such that $u \geq U^-$ throughout $B_{\rho^-}(0)$. If $H(-\infty) = -\infty$, $H(+\infty) = +\infty$, then the functions $U^+ - u_0$ and $U^- - u_0$ do not depend on u_0, and additionally $\rho^+ = \rho^- = R$.

Again the result is qualitatively best possible.

Our proofs of these results are obtained by comparison with the solution of a singular nonlinear boundary problem, that has an independent interest as a singular problem of capillarity theory (see, e.g., [7], Chapters 6, 7). Set $H^- = H(-\infty)$, $H^+ = H(+\infty)$. We assume at first that H^-, H^+ are finite. Figure 1 illustrates the case $H^-, H^+ > 0$. If either is negative, the orientation of the corresponding arc reverses.

Theorem 3. *Assume $H'(u) \geq 0$, $H^- \neq H^+$. Then in any "moon domain" (Figure 1) bounded by the two circular arcs Σ^- of radius $1/2H^-$ and Σ^+ of radius $1/2H^+$, there exists a unique solution $w(x, y)$ of (1), such that $\nu \cdot Tw = -1$ on Σ^-, and $\nu \cdot Tw = +1$ on Σ^+, ν being the exterior unit normal. There holds $w(x, y) = w(x, -y)$. On the symmetry line $\overline{PQ}$, w increases monotonely from $-\infty$ to $+\infty$. As the size of $\mathcal{M}$ tends to zero (that is, $\tau, \alpha \to 0$), $w'(x, 0) \to \infty$ uniformly on the entire segement $\overline{PQ}$.*

Geometrically, Theorem 3 asserts the existence of a solution surface $w(x, y)$ of (1) that is tangent downward to vertical walls over Σ^-, and tangent upward to vertical walls over Σ^+. We note that it is essential for this theorem that $H^- \neq H^+$. The boundary conditions require $|\nabla w| = \infty$ on Σ^- and on Σ^+, and it can be shown that w itself necessarily becomes infinite on the two arcs. However, interior to the segment $\overline{PQ}$, $|\nabla w|$ is bounded depending only on τ and on the point of evaluation. The solution can be regarded as the natural analogue of the vertical cylinder which is a limiting case of solutions in the case of constant H.

The proof of Theorem 3 can be reduced to Theorem 7.10 of [7]. It should be noted that it is essential that the two boundary arcs have exactly the given curvatures; arbitrarily small smooth perturbations of these arcs can lead to configurations for which there is no solution to the prescribed problem.

To prove Theorem 1, we compare the given solution $u(x, y)$ in $B_R(0)$ with the solution $w(x, y)$ in a moon domain, that is chosen to lie interior to $B_R(0)$ in such a way that $w(0) = u(0)$ and $\nabla w(0)$ is directed parallel to $\nabla u(0)$. This is always possible following a suitable choice of τ. See Figure 2.

We assert that then $|\nabla w(0)| > |\nabla u(0)|$. For if not, the size of $\mathcal{M}$ could be decreased by decreasing τ, until equality is obtained. In that event we would have two solutions u and w of (1) that agree to first order derivatives at the point O, and it can be shown that a finite number $N \geq 2$ of dictinct level curves of the difference function $\varphi = w - u$ must then pass through O. It follows that there would be at least four domains interior to $\mathcal{M}$ and sharing the common boundary point O, in which, alternatively, $w < u$ and $w > u$, see Figure 3. Since the boundary of $\mathcal{M}$ decomposes into only two sets in which respectively $\varphi = \pm\infty$, and two singular points E, F, we obtain contradiction with an extended form of the maximum principle, see [1]. This contradiction yields the initial statement of Theorem 1.

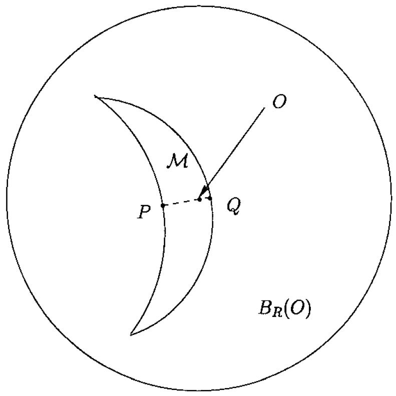

Figure 2: Comparison procedure.

To complete the proof of the theorem, we show that if $H(-\infty) = -\infty$, $H(+\infty) = +\infty$ then there is a uniform bound for all solutions of (1) in $B_R(0)$. In fact, for any $R' < R$, we consider a lower hemisphere $\mathcal{S}_{R'} : v(r)$ of radius R', situated over $B_{R'}(0)$ and lying above the surface $u(x, y)$. We now lower $\mathcal{S}_{R'}$ until a first point P of contact occurs. Any such P must be interior to $\mathcal{S}_{R'}$, as $v'(r) = \infty$ on the boundary $r = R'$. Therefore $H(u) \leq H(v) = 1/R'$ at P. There follows $u \leq \max\{t : H(t) \leq 1/R'\}$ which is finite since $H(+\infty) = +\infty$. Similarly a bound from below is obtained. The desired bound now follows by letting $R' \to R$ and observing that u lies below S_R throughout $B_R(0)$.

Thus, if $H^- = -\infty$, $H^+ = \infty$, we may truncate these functions without affecting the solutions; Theorem 3 then leads to the second assertion of Theorem 1.

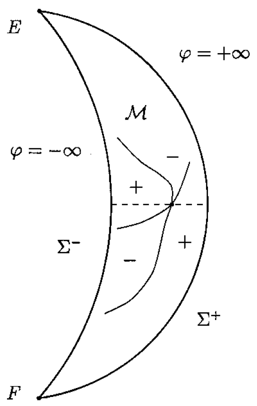

Figure 3: Division into subregions.

For solutions $u(x, y)$ of (1) in a general domain $\mathcal{D}$, the conclusion of Theorem 1 can be expressed as an inequality

$$\left|\nabla u(p)\right| \leq \mathcal{F}(d, u), \tag{2}$$

where d denotes the distance to the boundary. We consider $u(p)$ as being defined in a disk $B_d(p)$, and observe that along a ray from p to the boundary, (2) yields

$$\left|\frac{du}{ds}\right| \leq \mathcal{F}(d - s, u). \tag{3}$$

This equation may be integrated locally, yielding Theorem 2 as result. We emphasize again that the one sided bound essential for the classical Harnack inequality does not appear in Theorem 2. If H is constant, such a result would be false, as can be shown by example. It has to be expected in general that $\rho^+, \rho^- < R$, as occurs when H is constant, see [4] or [5].

Acknowledgement

The work was supported in part by the National Aeronautics and Space Administration, and in part by the National Science Foundation. The former author wishes to thank the Max-Planck Institut für Mathematik in Leipzig for its hospitality during the preparation of the manuscript.

References

1. R. Finn and J. Lu, Regularity Properties of H-graphs. *Stanford University Preprint,* 1997.
2. R. Finn, On equations of minimal surface type. *Ann. of Math.* 60(1954), 397–416.
3. E. Bombieri and E. Giusti, Local estimates for the gradient of non-parametric surfaces of prescribed mean curvature. *Comm. Pure Appl. Math.* 26(1973), 381–394.
4. J. B. Serrin, The Dirichlet problem for surfaces of constant mean curvature. *Proc. London. Math. Soc.* 21(1970), 361–384.
5. R. Finn, The inclination of an H-graph. *Springer-Verlag Lecture Notes* 1340(1988), 40–60.
6. R. Finn and E. Giusti, On nonparametric surfaces of constant mean curvature. *Ann. Scuola Norm. Sup. Pisa* 4(1977), 13–31.
7. R. Finn, Equilibrium capillary surfaces. *Springer-Verlag, New York,* 1986.

(Received 25.06.1997)

Authors' addresses:

Robert Finn
Mathematics Departament
Stanford University
California 9405-2125
USA

Jianan Lu
International Business Machines Corporation
1055 Joaquin Road
Mountain View, CA 94043
USA