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ON OSCILLATORY PROPERTIES OF AN N-TH ORDER SYSTEM OF LINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Consider the system

$$\begin{cases} x_i'(t) = p_i(t)x_{i+1}(\tau_{i+1}(t)) \quad (i = 1, \dots, n-1), \\ x_n'(t) = p_n(t)x_1(\tau_1(t)), \end{cases} \tag{1}$$

where $n \geq 2$, $\tau_i \in C^1(R_+; R)$ are nondecreasing functions, $\lim_{t\to+\infty} \tau_i(t) = +\infty(i = 1, \dots, n)$, $p_i \in L_{\text{loc}}(R_+; R_+)$ $(i = 1, \dots, n-1)$, $p_n \in L_{\text{loc}}(R_+; R)$ and

$$\int^{+\infty} p_i(t)dt = +\infty \quad (i = 1, \dots, n-1). \tag{2}$$

In the present paper, sufficient conditions for the oscillation of all proper solutions of (1) are established. Analogous questions for deviating and general functional differential equations were studied in a great deal of papers, for example, in [1,2], and for ordinary differential equations, in [3,4,5].

Let $t_0 \in R_+$, $t_* = \min\{t_0, \tau_1(t_0), \dots, \tau_n(t_0)\}$. A continuous vector function $x = (x_i)_{i=1}^n : [t_*, +\infty[\to R^n$ is said to be *a proper solution* of the system (1), if it is locally absolutely continuous on $[t_0, +\infty[$, a.e. on this interval satifies (1), and

$$\sup\{\|x(s)\| : s \in [t, +\infty[\} > 0 \quad \text{for} \quad t \in [t_0, +\infty[.$$

A proper solution of (1) is said to be *oscillatory*, if each of its components has a sequence of zeroes tending to $+\infty$. Otherwise the solution is called *nonoscillatory*.

We say that the system (1) has *the property A* provided any of its solutions is oscillatory if n is even, and either is oscillatory or satisfies

$$|x_i(t)| \downarrow 0, \quad \text{for} \quad t \uparrow +\infty \quad (i = 1, \dots, n) \tag{3}$$

if n is odd.

We say that the system (1) has *the property B* provided any of its solutions either is oscillatory or satisfies (3) if n is even, and either is oscillatory or satisfies (3) or

$$|x_i(t)| \uparrow +\infty, \quad \text{for} \quad t \uparrow +\infty \quad (i = 1, \dots, n) \tag{4}$$

if n is odd.

Introduce the notation

$$\tau_{ji}^*(t) = \begin{cases} \tau_j(\tau_{j-1}(\cdots(\tau_{i+1}(t))\cdots)) & \text{for} \quad 1 \leq i < j \leq n+1, \\ t & \text{for} \quad i = j \ (i = 1, \dots, n). \end{cases}$$

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Here we mean that $\tau_{n+1}(t)=\tau_1(t)$.

$$\gamma_{ji}(t)=\inf\Big\{s:\ s\in R_+,\ \ s\ge t,\ \ \tau^*_{ki}(s)\ge t\ \ (k=i,\dots,j)\Big\}\ \ (1\le i\le j\le n);$$

$$I^0=1,\quad I^j(s,t;p_{i+j-1}\dots,p_i)=$$

$$=\int_t^s p_{i+j-1}(\tau^*_{i+j-1,i}(\xi))\tau^{*'}_{i+j-1}(\xi)I^{j-1}(\xi,t;p_{i+j-2},\cdots,p_i)d\xi,$$

$$J^0=1,\quad J^j(t,s;p_i\dots,p_{i+j-1})=$$

$$=\int_s^t p_i(\xi)J^{j-1}(\tau_{i+1}(\xi),\tau_{i+1}(s);\ \ p_{i+1},\cdots,p_{i+j-1})d\xi,$$

$$(i=1,\dots,n-1;\ j=1,\dots,n-i).$$

Besides, everywhere below we set

$$t_{*i}=\gamma_{n-1,i}(0)\quad(i=1,\dots,n-1);$$

$$\alpha_e(t)=\frac{I^{n-e}(t,t_{*e};p_{n-1},\dots,p_e)}{I^{n-e-1}(\tau_{e+1}(t),\tau_{e+1}(t_{*e});p_{n-1},\dots,p_{e+1})};$$

$$z_e(t)=\frac{J^e(t,\gamma_{e1}(0);p_1,\dots,p_e)}{J^1(\tau^*_{e1}(t),0;p_e)}.$$

Theorem 1. *Let* (2) *be satisfied,* $p_n\in L_{\rm loc}(R_+;R_-)$ *and for any* $l\in\{1,\dots,n-1\}$ *with* $l+n$ *odd,*

$$\tau^*_{e1}(\tau^*_{n+1,e}(t))\ge t, \tag{5}$$

$$\lim_{t\to+\infty}\sup\alpha_e(t)\int_t^{+\infty}I^{n-e-1}(\tau_{e+1}(s),\tau_{e+1}(t_{*e});p_{n-1},\cdots,p_{e+1})\times$$
$$\times z_e(\tau_{n+1,e}(s))|p_n(\tau^*_{ne}(s))|\tau^{*'}_{ne}(s)ds>1. \tag{6}$$

If, moreover, the condition

$$\int^{+\infty}I^{n-1}(\xi,t_{*1};p_{n-1},\dots,p_1)|p_n(\tau^*_{n1}(\xi))|\tau^{*'}_{n1}(\xi)d\xi=+\infty, \tag{7}$$

is fulfilled for odd n*, then the system* (1) *has the property A.*

Theorem 2. *Let* (2) *be satisfied,* $p_n\in L_{\rm loc}(R_+;R_-)$ *and for any* $l\in\{1,\dots,n-1\}$ *with* $l+n$ *odd*

$$\tau^*_{e1}(\tau^*_{n+1,e}(t))\le t, \tag{8}$$

$$\lim_{t\to+\infty}\sup\alpha_e(\tau^*_{e1}(\tau^*_{n+1,e}(t)))\int_t^{+\infty}I^{n-e-1}(\tau_{e+1}(s),\tau_{e+1}(t_{*e});p_{n-1},\dots,p_{e+1})\times$$
$$\times|p_n(\tau^*_{ne}(s))|z_e(\tau^*_{n+1,e}(s))\tau^{*'}_{ne}(s)ds>1. \tag{9}$$

If, moreover, the condition (7) *is fulfilled for odd* n*, then the system* (1) *has the property A.*

Theorem 3. *Let* (2) *be satisfied,* $p_n\in L_{\rm loc}(R_+;R_-)$ *and for any* $l\in\{1,\dots,n-1\}$ *with* $l+n$ *odd,* (5) *be fulfilled along with*

$$\lim_{t\to+\infty}\sup\frac{1}{I^1(t,t_{*e};p_e)}\int_{t_{*e}}^t I^{n-e}(s,t_{*e};p_{n-1},\dots,p_e)\times$$
$$\times|p_n(\tau^*_{ne}(s))|z_e(\tau^*_{n+1,e}(s))|I^1(s,t_{*e};p_e)\tau^{*'}_{ne}(s)ds>1. \tag{10}$$

If, moreover, the condition (7) *is fulfilled for odd* n*, then the system* (1) *has the property* A*.*

Theorem 4. *Let* (2) *be satisfied,* $p_n \in L_{\mathrm{loc}}(R_+; R_-)$ *and for any* $l \in \{1, \dots, n-1\}$ *with* $l+n$ *odd,* (8) *be fulfilled along with*

$$\lim_{t\to+\infty} \sup \frac{1}{I^1(t, t_{*e}; p_e)} \int_{t_{*e}}^{t} I^{n-e}(s, t_{*e}; p_{n-1}, \dots, p_e) \times$$
$$\times |p_n(\tau^*_{ne}(s))| z_e(\tau^*_{n+1,e}(s)) | I^1(\tau^*_{e1}(\tau_{n+1,e}(s)), t_{*e}; p_e) \tau^{*'}_{ne}(s) ds > 1. \tag{11}$$

If, moreover, the condition (7) *is fulfilled, then the system* (1) *has the property* A*.*

Theorem 5. *Let* (2) *be satisfied,* $p_n \in L_{\mathrm{loc}}(R_+; R_+)$ *and for any* $l \in \{1, \dots, n-2\}$ *with* $l+n$ *even,* (5) *and* (6) ((8) *and* (9)) *be fulfilled. If, moreover,*

$$\int^{+\infty} J^{n-1}(\tau_1(t), t_{*1}; p_1, \dots, p_{n-1}) p_n(t) dt = +\infty, \tag{12}$$

and, for even n*, the condition* (7) *is fulfilled, then the system* (1) *has the property* B*.*

Theorem 6. *Let* (2) *be satisfied,* $p_n \in L_{\mathrm{loc}}(R_+; R_+)$ *and for any* $l \in \{1, \dots, n-2\}$ *with* $l+n$ *even, the conditions* (5) *and* (10) ((8) *and* (11)) *be fulfilled. If, moreover,* (12) *is satisfied and, for even* n*, the condition* (7) *is fulfilled, then the system* (1) *has the property* B*.*

Now consider the system

$$\begin{cases} x_i'(t) = x_{i+1}(\beta_{i+1} t) \quad (i = 1, \dots, n-1), \\ x_n'(t) = p(t) x_1(\beta_1 t), \end{cases} \tag{13}$$

where $p \in L_{\mathrm{loc}}(R_+; R)$, $\beta_i \in]0; +\infty[$ $(i = 1, \dots, n)$, which is a special case of (1). For this system, the above results can be written in a more effective form.

Theorem 7. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, $\prod_{i=1}^n \beta_i \geq 1$ $\left(\prod_{i=1}^n \beta_i \leq 1\right)$ *and*

$$\overline{\lim_{t\to+\infty}}\, t \int_t^{+\infty} s^{n-2} |p(s)| ds > (n-1)! \prod_{i=2}^{n} \beta_i^{i-1} \quad \left((n-1)! \prod_{i=1}^{n-1} \beta_i^{i-n}\right).$$

Then the system (13) *has the property* A*.*

Theorem 8. *Let* $p \in L_{\mathrm{loc}}(R_+; R_-)$, $\prod_{i=1}^n \beta_i \geq 1$ $\left(\prod_{i=1}^n \beta_i \leq 1\right)$ *and*

$$\overline{\lim_{t\to+\infty}}\, \frac{1}{t} \int_0^{+\infty} s^{n} |p(s)| ds > (n-1)! \prod_{i=2}^{n} \beta_i^{i-1} \quad \left((n-1)! \prod_{i=1}^{n-1} \beta_i^{i-n}\right).$$

Then the system (13) *has the property* A*.*

Theorem 9. *Let* $p \in L_{\mathrm{loc}}(R_+; R_+)$ *and* $\prod_{i=1}^n \beta_i \geq 1$ $\left(\prod_{i=1}^n \beta_i \leq 1\right)$*. Moreover, let*

$$\overline{\lim_{t\to+\infty}}\, t \int_t^{+\infty} s^{n-2} p(s) ds > 2(n-2)! \prod_{i=1}^{n} \beta_i^{i-2} \quad \left(2(n-2)! \prod_{i=1}^{n} \beta_i^{i+1-n}\right),$$

if n is even, and

$$\overline{\lim_{t\to+\infty}}\, t\int_t^{+\infty} s^{n-2}p(s)ds > (n-1)!\prod_{i=2}^{n}\beta_i^{i-1} \quad \Big((n-1)!\prod_{i=1}^{n}\beta_i^{i+1-n}\Big).$$

if n is odd. Then the system (13) *has the property B.*

Theorem 10. *Let $p \in L_{\mathrm{loc}}(R_+;R_+)$ and $\prod_{i=1}^{n}\beta_i \geq 1$ $\Big(\prod_{i=1}^{n}\beta_i \leq 1\Big)$. Moreover, let*

$$\overline{\lim_{t\to+\infty}}\, \frac{1}{t}\int_0^{t} s^{n}p(s)ds > 2(n-2)!\prod_{i=1}^{n}\beta_i^{i-2} \quad \Big(2(n-2)!\prod_{i=1}^{n}\beta_i^{i+1-n}\Big)$$

if n is even, and

$$\overline{\lim_{t\to+\infty}}\, \frac{1}{t}\int_0^{t} s^{n}p(s)ds > (n-1)!\prod_{i=2}^{n}\beta_i^{i-1} \quad \Big((n-1)!\prod_{i=1}^{n}\beta_i^{i+1-n}\Big),$$

if n is odd. Then the system (13) *has the property B.*

References

1. R. G. Koplatadze and T. A. Chanturia, On oscillatory properties of differential equations with deviating arguments. (Russian) *Tbilisi University Press, Tbilisi*, 1977.

2. R. G. Koplatadze, On oscillatory properties of sulutions of functional differential equations. *Mem. Differential Equations Math. Phys.* **3**(1994), 1–177.

3. T. A. Chanturia, On oscillatory properties of systems of nonlinear ordinary differential equations. (Russian) *Trudy Inst. Prikl. Mat. I. N. Vekua* **14**(1983), 163–202.

4. R. G. Koplatadze and N. L. Partsvania, Oscillatory properties of solutions of systems of second order differential equations with deviating arguments. (Russian) *Differentsial'nye Uravneniya* **33**(1997), No. 10.

5. G. Giorgadze, In oscillatory properties of the n-th order system of differential equations with deviating arguments. *Mem. Differential Equations Math. Phys.* **6**(1995), 127–129.

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