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ON A CERTAIN BOUNDARY VALUE PROBLEM FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Consider the n-th order ordinary differential equation

$$u^{(n)}+\sum_{k=1}^{n-1}p_k(t)u^{(k)}=f(t,u,u',\dots,u^{(n-1)}) \tag{1}$$

on the interval $[a,+\infty[$, where $a>0$, $n\geq 2$, each of the functions $p_k:[a,+\infty[\to\mathbb{R}$ for $k\in\{1,\dots,n-1\}$ is locally absolutely continuous together with its derivatives up to order $k-1$ inclusively (i.e., the functions $p_k^{(i)}$ $(i=0,\dots,k-1)$ are absolutely continuous on any finite segment contained in $[a,+\infty[$), and the function $f:[a,+\infty[\times\mathbb{R}^n\to\mathbb{R}$ satisfies the local Carathéodory conditions.

Let n_0 be an integer part of the number $\frac{n}{2}$, and let c_i $(i=0,\dots,n_0-1)$ be arbitrary real numbers. Consider the problem on existence of a solution $u:[a,+\infty[\to\mathbb{R}$ of the equation (1), satisfying the conditions

$$u^{(i)}(a)=c_i \quad (i=0,\dots,n_0-1), \qquad \int_a^{+\infty}[u^{(j)}(t)]^2\,dt<+\infty \quad (j=0,\dots,n_0). \tag{2}$$

In the case where $p_k(t)\equiv 0$ $(k=1,\dots,n-1)$, the problems of the type (1), (2) have been investigated by I. Kiguradze [1]. The theorems given below complement the results of this work.

Let μ_i^k $(i=0,1,\dots,n-n_0-1;\ k=2i,2i+1,\dots,n-1)$ be real numbers given by the recurrence relation

$$\mu_0^{i+1}=\frac{1}{2},\quad \mu_i^{2i}=1,\quad \mu_{i+1}^k=\mu_{i+1}^{k-1}+\mu_i^{k-2} \quad (i=0,1,\dots,n-n_0-1;\ k=2i+3,\dots,n-1),$$

and $\mathbb{R}_+=[0,+\infty[$.

Theorem 1. *Let the inequalities*

$$\Big|f(t,x_0,x_1,\dots,x_{n-1})\Big|\leq\varphi\Big(t,|x_0|,|x_1|,\dots,|x_{n_0-1}|\Big), \tag{3}$$

$$(-1)^{n-n_0-1}f(t,x_0,x_1,\dots,x_{n-1})\operatorname{sgn}x_0\geq-\sum_{i=0}^{n_0-1}\alpha_i(t)|x_i|+\alpha(t),$$

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hold on $[a,+\infty[\times\mathbb{R}^n$, *where the functions* $\alpha_0 : [a,+\infty[\to\mathbb{R}$, $\alpha_i : [a,+\infty[\to\mathbb{R}_+$ $(i = 1,\dots,n_0-1)$ *are locally summable,* $\alpha : [a,+\infty[\to\mathbb{R}_+$ *is measurable and*

$$\int\limits_a^{+\infty} t^{2n-4n_0}\alpha^2(t)\,dt < +\infty,$$

while the function $\varphi : [a,+\infty[\times\mathbb{R}_+^{n_0}\to\mathbb{R}_+$ *is locally summable with respect to the first argument, non-decreasing with respect to the last* n_0 *arguments and for any* $\rho_0\in]0,+\infty[$ *satisfies the condition*

$$\lim_{\substack{t\to a\\ \rho\to+\infty}} \frac{1}{\rho^2}\int\limits_a^t \varphi(\tau,\rho_0,\rho,\dots,\rho)\,d\tau = 0. \tag{4}$$

Further, suppose that there exist constants $\gamma_i \ge 0$ $(i=1,\dots,n_0-1)$, $\eta>0$ *and* $\delta>0$ *such that*

$$\mu_{n_0}^n - \sum_{i=1}^{n_0-1}\frac{i\gamma_i}{n_0}\,\eta^{i-n_0} \ge \delta$$

and the inequalities

$$\sum_{k=2i}^{n-1}(-1)^{n-n_0+k-i-1}\mu_i^k\left[t^{n-2n_0}p_k(t)\right]^{(k-2i)} + t^{n-2n_0}\alpha_i(t) \le \gamma_i \quad (i=1,\dots,n_0-1),$$

$$\sum_{k=1}^{n-1}(-1)^{n-n_0+k}\mu_0^k\left[t^{n-2n_0}p_k(t)\right]^{(k)} - t^{n-2n_0}\alpha_0(t) \ge$$

$$\ge \sum_{i=1}^{n_0-1} t^{n-2n_0}\alpha_i(t) + \sum_{i=1}^{n_0-1}\frac{(n_0-i)\gamma_i}{n_0}\,\eta^i + \delta$$

hold on $[a,+\infty[$. *Then there exists at least one solution of the problem* (1), (2).

Corollary 1. *Let the inequalities* (4) *and*

$$(-1)^{n-n_0-1}f(t,x_0,\dots,x_{n-1})\,\mathrm{sgn}\,x_0 \ge \gamma(t)|x_0|^\lambda$$

hold on $[a,+\infty[\times\mathbb{R}^n$, *where* $\lambda>1$, *the function* φ *be taken as it were in Theorem 1 and* $\gamma : [a,+\infty[\to]0,+\infty[$ *is a measurable function satisfying the condition*

$$\int\limits_a^{+\infty} t^{n-2n_0}[\gamma(t)]^{-\frac{2}{\lambda-1}}\,dt < +\infty.$$

Further, suppose that there exists a constant $r\in]0,+\infty[$ *such that the inequalities*

$$\sum_{k=2i}^{n-1}(-1)^{n-n_0+k-i-1}\mu_i^k\left[t^{n-2n_0}p_k(t)\right]^{(k-2i)} < r \quad (i=1,\dots,n_0-1),$$

$$\sum_{k=1}^{n-1}(-1)^{n-n_0+k}\mu_0^k\left[t^{n-2n_0}p_k(t)\right]^{(k)} > -r$$

hold on $[a,+\infty[$. *Then the problem* (1), (2) *is solvable.*

Corollary 2. *Let all the conditions of Corollary* 1, *except* (4) *be fulfilled. Then the problem* (1), (2) *has an* n_0*-parametric family of solutions satisfying the conditions*

$$\int_a^{+\infty} [u^{(j)}(t)]^2\,dt < \infty \quad (i = 0, \dots, n_0).$$

Remark 1. In the case where $n = 2n_0$, $p_{n-2}(t) \equiv 1$ and $p_k(t) \equiv 0$ ($k \neq n-2$; $k = 1, \dots, n-1$), from Corollary 1 it follows Theorem 1.2 of the paper [2].

References

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