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ON THE OSCILLATION OF SOLUTIONS OF TWO-DIMENSIONAL LINEAR DIFFERENTIAL SYSTEMS WITH DEVIATED ARGUMENTS

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Consider the system of differential equations

$$u_1'(t) = \sum_{i=1}^{m} p_i(t)\, u_2(\sigma_i(t)), \quad u_2'(t) = -\sum_{i=1}^{m} q_i(t)\, u_1(\tau_i(t)), \tag{1}$$

where $p_i, q_i \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, $\tau_i, \sigma_i \in C(\mathbb{R}_+;\mathbb{R}_+)$, $\sigma_i(t) \le t$ for $t \in \mathbb{R}_+$, $\lim_{t\to+\infty} \tau_i(t) = +\infty$, $\lim_{t\to+\infty} \sigma_i(t) = +\infty$ $(i = 1,\dots,m)$.

Let $t_0 \in \mathbb{R}_+$ and $a_0 = \inf_{t\ge t_0}\{\min(\tau_i(t),\sigma_i(t) : i = 1,\dots,m)\}$. A continuous vector-function (u_1, u_2) defined on $[a_0,+\infty[$ is said to be a *proper* solution of the system (1) in $[t_0,+\infty[$ if it is absolutely continuous on each finite segment contained in $[t_0,+\infty[$, satisfies (1) almost everywhere on $[t_0,+\infty[$, and $\sup\Big\{|u_1(s)| + |u_2(s)| :\ s \ge t\Big\} > 0$ for $t \ge t_0$.

A proper solution (u_1, u_2) of the system (1) is said to be *oscillatory* if both u_1 and u_2 have sequences of zeros tending to infinity. If there exists $t_* \in \mathbb{R}_+$ such that $u_1(t)u_2(t) \ne 0$ for $t \ge t_*$, then (u_1, u_2) is said to be *nonoscillatory*.

In this paper, sufficient conditions are given for the oscillation of proper solutions of the system (1) which make the results contained in [1, 2] more complete.

In the sequel, we will use the following notation: $p(t) = \sum_{i=1}^m p_i(t)$, $q(t) = \sum_{i=1}^m q_i(t)$, $h(t) = \int_0^t p(s)ds$, $h_0(t) = \min\Big\{h(t), h(\tau_i(t)) : i = 1,\dots,m\Big\}$.

Theorem 1. *Let*

$$\int^{+\infty} p(t)dt = +\infty, \quad \int^{+\infty} h_0(t)q(t)dt = +\infty, \tag{2}$$

and there exist a nondecreasing function $\sigma \in C(\mathbb{R}_+;\mathbb{R}_+)$ *such that* $\sigma_i(t) \le \sigma(t) \le t$ $(i = 1,\dots,m)$ *and*

$$\limsup_{t\to+\infty} h\Big(\tau(\sigma(t))\Big)/h(t) < +\infty. \tag{3}$$

If, moreover, there exists $\varepsilon > 0$ *such that for any* $\lambda \in]0,1]$

$$\liminf_{t\to+\infty} h^{\varepsilon}(t)h^{1-\lambda}\Big(\tau(\sigma(t))\Big)\int_{\tau(\sigma(t))}^{+\infty} p(s)h^{-2-\varepsilon}(s)g(s,\lambda)ds > 1,$$

where $\tau(t) = \max\Big(\max\{\tau_i(s),\eta(s) : i = 1,\dots,m\} : 0 \le s \le t\Big)$, $\eta(t) = \sup\{s : \sigma(s) < t\}$, $g(t,\lambda) = \int_0^{\sigma(t)} h(\xi)\sum_{i=1}^m q_i(\xi)h^{\lambda}(\tau_i(\xi))d\xi$, *then every proper solution of the system* (1) *is oscillatory.*

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Theorem 2. *Let the conditions* (2), (3) *be fulfilled, where the function* $\sigma \in C(\mathbb{R}_+;\mathbb{R}_+)$ *is nondecreasing,* $\sigma_i(t) \le \sigma(t) \le t$ *for* $t \in \mathbb{R}_+$ $(i = 1,\dots,m)$. *If, moreover, there exists* $\varepsilon > 0$ *such that for any* $\lambda \in]0,1]$

$$\liminf_{t\to+\infty} h^{-\lambda}(t)\int_0^{\sigma(t)} h(\xi)\sum_{i=1}^m q_i(\xi)h^\lambda(\tau_i(\xi))d\xi > 1-\lambda+\varepsilon,$$

then every proper solution of the system (1) *is oscillatory.*

Theorem 3. *Let*

$$\limsup_{t\to+\infty} h(\tau_i(t))/h(t) < +\infty \quad (i=1,\dots,m) \tag{4}$$

and there exist $\varepsilon > 0$ *such that for any* $\lambda \in]0,1]$

$$\liminf_{t\to+\infty} h^{-1}(t)\int_0^t h^2(\xi)\sum_{i=1}^m q_i(\xi)\big[h(\tau_i(\xi))/h(\xi)\big]^\lambda d\xi > \lambda(1-\lambda)+\varepsilon.$$

Then every proper solution of the system (1) *is oscillatory.*

Corollary 1. *Let* (4) *be fulfilled and* $\alpha_i \in]0,+\infty[$ $(i=1,\dots,m)$, *where*

$$\alpha_i = \liminf_{t\to+\infty} h(\tau_i(t))/h(t) \quad (i=1,\dots,m). \tag{5}$$

If, moreover, there exists $\varepsilon > 0$ *such that for any* $\lambda \in]0,1]$

$$\liminf_{t\to+\infty} h^{-1}(t)\int_0^t h^2(s)\sum_{i=1}^m \alpha_i^\lambda q_i(s)ds > \lambda(1-\lambda)+\varepsilon,$$

then every proper solution of the system (1) *is oscillatory.*

Corollary 2. *Let* (4) *be fulfilled,* $\alpha_i \in]0,+\infty[$, $q_i(t) \ge q_0(t)$ *for* $t \in \mathbb{R}_+$ $(i=1,\dots,m)$, *where* $q_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, α_i $(i=1,\dots,m)$ *are defined by* (5). *Then the condition*

$$\liminf_{t\to+\infty} h^{-1}(t)\int_0^t h^2(s)q_0(s)ds > \max\Big\{\lambda(1-\lambda)\Big(\sum_{i=1}^m \alpha_i^\lambda\Big)^{-1} : \lambda\in[0,1]\Big\}$$

is sufficient for the oscillation of every proper solution of the system (1).

Corollary 3. *Let* $q_0 \in L_{loc}(\mathbb{R}_+;\mathbb{R}_+)$, $\alpha \in]0,1]$ *and* $\liminf_{t\to+\infty} t^{-1}\int_0^t s^{1+\alpha}q_0(s)ds > 0$. *Then every proper solution of the equation* $u''(t) + q_0(t)u(t^\alpha) = 0$ *is oscillatory.*

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References

1. N. Partsvania, On oscillation of solutions of second order systems of deviated differential equations. *Georgian Math. J.* **3**(1996), No. 6, 571–582.

2. R. Koplatadze and N. Partsvania, Oscillatory properties of solutions of two-dimensional differential systems with deviated arguments. (Russian) *Differentsial'nye Uravneniya* **33**(1997), No. 10, 1312–1320.

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