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HAMILTONIAN REDUCTION AND QUANTIZATION ON SYMPLECTIC MANIFOLDS

Abstract. Hamiltonian reduction scheme based on the analysis of restricted 1-forms in gauge-invariant variables is constructed. The method is applied for several physically interesting gauge invariant models. For the models of Yang-Mills theory a possible mechanism of the confinement is obtained. A quantization method (E-quantization) based on the extension of phase space with further application of the constrained quantization technique is constructed. A problem of scalar product for the constrained systems is investigated. A possible solution to this problem is found. Generalization of the Gupta-Bleuler conditions is done by minimization of quadratic fluctuations of quantum constraints. Connection of E-quantization to the geometric quantization and Berezin quantization is found. The quantum distribution function is introduced. For a pure state distribution the special elliptic type equation is obtained. A possible experimental measuring of the quantum distribution is discussed.

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Key words and Phrases. Phase space, Hamiltonian reduction, gauge theory, symmetry, quantization, coherent state, quantum distribution.

რეზიუმე. აგებულია ჰამილტონური რედუქციის სქემა, რომელიც დაფუძნებულია 1-ფორმების ანალიზზე ყალიბურად ინვარიანტულ ცვლადებში. ეს მეთოდი გამოყენებულია სხვადასხვა, ფიზიკური თვალსაზრისით საინტერესო, ყალიბურად ინვარიანტული მოდელისათვის. იანგ-მილსის თეორიის მოდელთათვის ნაპოვნია კონფაინმენტის შესაძლო მექანიზმი. აგებულია დაქვანტვის მეთოდი, რომელიც დაფუძნებულია ფაზური სივრცის გაფართოვებაზე და რომელიც იყენებს ბმებიანი სისტემების დაქვანტვის ტექნიკას. ბმებიანი სისტემებისათვის შესწავლილია სკალარული ნამრავლის პრობლემა. მოძებნილია ამ პრობლემის შესაძლო გადაწყვეტა. ქვანტური ბმების კვადრატული ფლუქტუაციების მინიმიზაციით მოცემულია გუფთა-ბლეულერის პირობების განზოგადოება. ნაპოვნია დაქვანტვის ახალი მეთოდის კავშირი გეომეტრიულ და ბერეზინის დაქვანტვასთან. შემოყვანილია ქვანტური განაწილების ფუნქცია. წმინდა მდგომარეობის განაწილებისათვის მიღებულია სპეციალური ელიფსური ტიპის განტოლება. განხილულია ქვანტური განაწილების ფუნქციის შესაძლო ექსპერიმენტული გაზომვა.

Introduction

The fundamental physical theories and many corresponding interesting models are described by the gauge-invariant Lagrangians. The well known examples are: electrodynamics, general relativity, Yang-Mills theory, standard model, string theory, *etc.* The gauge invariant Lagrangians are singular and in the Hamiltonian formulation lead to the constrained dynamical systems [1-7].

For the constrained Hamiltonian systems there are, in principle, two ways of quantization. The first is the Dirac method, in which one first quantizes the system ignoring the constraints and then selects the admissible physical states by demanding that they are annihilated by the constraint operators. The second is the reduced phase space method where one first eliminates all unphysical degrees of freedom and then quantizes the resulting unconstrained system.

Unfortunately neither of these methods is universal and several problems arise in practical applications. It should also be noted that the quantum systems constructed by the indicated methods generally are not unitary equivalent to each other [8-11].

In the Hamiltonian formulation of gauge theories one has the first class constraints, which are the generators of the gauge transformations. The constraint functions Φ_a $(a = 1, \ldots, M)$ have the commutation relations of the corresponding Lie algebra, and the key step of the Dirac quantization method is a suitable representation of this algebra. The admissible quantum states we call the physical states and they satisfy the Dirac conditions

$$\hat{\Phi}_a|\Psi_{ph}\rangle = 0 \quad (a = 1, \ldots, M),$$

where $\hat{\Phi}_a$ are the constraint operators. Sometimes these conditions are nontrivial equations and description of the physical states is becoming problematic [12]. Besides that, the Dirac conditions generally have no solutions in the same Hilbert space where the first stage of quantization was performed. Respectively, on the next stage of quantization the problem of scalar product for the physical states arises [13]. Another principal problem of the Dirac method is a problem of unitarity. The latter is connected with the Hermiticity of the observable operators on the physical Hilbert space [14].

The classical reduction procedure of the reduced phase space method implies restriction to the constraint surface $\widetilde{\mathcal{M}} : \Phi_a = 0$ $(a = 1, \ldots, M)$, with subsequent reduction to the physical phase space $\mathcal{M} = \widetilde{\mathcal{M}}/G$ where G is the gauge group and $\widetilde{\mathcal{M}}/G$ is the space of gauge orbits.

Theoretically the set of orbits $(\mathcal{M} = \widetilde{\mathcal{M}}/G\ )$ is a well defined manifold, it possesses a symplectic structure and the coordinates on $\mathcal{M}$ are gauge-invariant true physical degrees of freedom. But in practical applications, this theoretical scheme also encounters the technical problems related to the explicit construction of the physical phase space $\mathcal{M}$ supplied with a symplectic structure. Here, apart from the mathematical difficulties, the physical content of the true degrees of freedom may be quite unpredictable.

A commonly used classical reduction scheme is a gauge fixing procedure when one introduces the additional constraints $\chi_a = 0$ $(a = 1, \dots, M)$ [1], [3]. In simple cases, the explicit form of the true physical variables is obvious and this reduction scheme works perfectly well. But in general, as it was shown in [15], the space of gauge orbits $(\widetilde{\mathcal{M}}/G)$ cannot be obtained by some 'simple' gauge fixing. This problem usually arises for non-abelian gauge groups, and obviously, it is related to a possible non-trivial structure of the physical phase space [16-17].

Another reduction scheme can be based on the introduction of gauge-invariant variables [3], [7]. A complete set of gauge-invariant variables allows us to describe the physical phase space $\mathcal{M}$ and obtain the corresponding symplectic structure. But, from the structure of gauge transformations, one can usually obtain only a part of gauge-invariant variables, and the construction of the complete set, as a rule, is problematic. At the same time, the obtained gauge-invariant variables are not canonically conjugated to each other. For the non-canonical coordinates the Poisson brackets are 'non-linear' and there is no general method for their representation.

Interesting general reduction procedure was proposed in [18-19]. This scheme is based on the analysis of restricted 1-forms in Darboux coordinates, but unfortunately, these coordinates are not so effective in practical applications.

After the classical reduction we arrive at the physical phase space $\mathcal{M}$ with some definite symplectic structure. But, in general, the physical phase space $\mathcal{M}$ could be a symplectic manifold without a global cotangent bundle structure. In such cases, the canonical coordinates exist only locally and the method of canonical quantization is not applicable.

Another drawback of canonical quantization is a crucial dependence of the corresponding quantum theory on the initial choice of coordinates, since a quantum system obtained by canonical quantization is not invariant under the general canonical transformations.

For the generalization of canonical quantization, different methods were developed and geometric quantization [20-26] is accepted as the most general one. This approach is indeed a powerful method for the quantization on arbitrary symplectic manifolds, but its formulation looks like a set of mathematical rules which are not directly derived from the basic physical principles. Besides, the method of geometric quantization has significant difficulties for the field theory generalization. Therefore, in spite of 20 years history, this method is not popular among physicists.

Similar problems arise for other interesting approaches [27-30] which use various mathematical structures for the generalization of canonical quantization.

Rather different method based on the extension of phase space was proposed in [31] (see also [32]). The introduced extended system is equivalent to the initial one on the certain constraint surface. Further, for the quantization of constrained extended system the BFV (BRST) [33-34] quantization was used. The phase space extension method with rather different physical interpretations was considered in other recent papers as well (see [35-38]). In [31] one can find a wide variety of references to different quantization methods, their analyses and a short review for the BFV (BRST) method.

The global Lie group symmetries play an important role for both classical and quantum systems, and representations of the classical symmetries on the quantum level is a key step for any quantization scheme. Symmetries of the classical system provide conservation laws and integrability of some evolution equations. The generators of symmetry group transformations of the quantum system commute with the Hamilton operator, and investigation of the eigen-value problem for these operators helps to obtain the Hamiltonian spectrum and integrate the Schrödinger equation.

If a symmetry group is sufficiently large, the classical system is completely integrable and one can introduce the action-angle variables. These variables are convenient in the classical description, since the dynamics in these variables is trivial. But in the quantum case the completely integrability of the classical system is not so effective for the solution of the quantum problem. The reason is again geometrical: the action-angle variables take values in some restricted domain, and therefore, this variables cannot be used for the canonical quantization, though formally they are canonical.

Obviously, the above mentioned quantization problems are related to each other. The author worked on these problems during several years. The main results obtained in collaboration with the colleagues are published in the papers [39-57]. The present work is a review of those activities.

The work contains introduction, five chapters, conclusion and four appendices.

The first section of Chapter 1 is used for the introduction of notation and definitions of symplectic geometry. Then we consider groups of canonical transformations. In the first two sections we introduce some simple examples which are used in next chapters as a test for the described reduction and quantization methods. In third section we give a probability description of the classical statistical systems using a distribution function on a phase space.

The second chapter is devoted to the Hamiltonian reduction procedure for gauge invariant theories. The general reduction scheme is considered in the first section. This scheme is based on the analysis of restricted 1-forms in gauge-invariant variables. We also consider situations when only a part of the gauge-invariant variables is known. The indicated analysis helps us to obtain the remaining part of the gauge-invariant variables. Here, as an illustration, some simple examples are considered. The additional examples are given in the Appendices A and B.

In Appendix A we consider the reduction scheme for a 6-dimensional model with the rotation symmetry. By reduction procedure we obtain the two dimensional sphere.

In Appendix B we investigate the reduction procedure and a boundary behaviour for $(2+1)$-dimensional massive photodynamics.

In subsequent sections the described method is applied to other more complicated gauge invariant models. In particular, in Section 2.2 we consider the relativistic particle on $SL(2,R)$ group manifold. Here we first consider general manifolds and summarize how the Lagrangian and Hamiltonian formalism is implemented for the reparametrization invariant theory. Then we specialize to manifolds corresponding to semi-simple Lie groups, where there is a left-right Noether symmetry with conserved currents L and R. Here we find that the constraint corresponding to the Minkowski mass-shell condition has a simple form through these conserved quantities. This stipulates our Hamiltonian reduction, and also provides the general solution of the reduced classical equations. We then consider the special group $SL(2,R)$ which is a three dimensional Lorentzian manifold. This group has the property that the above Hamiltonian reduction leads to a split reduced system consisting of two chiral ('left' and 'right') sectors, which are both coadjoint orbits of the group.

In Section 2.3 the reduction scheme is applied to a finite-dimensional system with $SU(2)$ as the gauge group of symmetry. This system can be considered as a toy model for the Yang-Mills theory with fermions. We show the essential difference between the $SU(2)$ and the corresponding $U(1)$ model. The structure of gauge-invariant variables in the case of $SU(2)$ can be interpreted as the confinement phenomenon.

In the last section of the second chapter we study the field theory model with a semi-simple gauge group. The fields take values in the corresponding semi-simple Lie group, and the model is essentially non-linear. For this model one can easily construct the gauge-invariant variables. Respectively, the reduction procedure is rather trivial. We show that the model is equivalent to the Yang-Mills theory with some boundary conditions.

In Chapter 3 we start with the formulation of Dirac quantization principles. Then we give the examples for the partial realization of these principles and describe general problems of quantization. We discuss a possible generalization of canonical quantization based on Mackey-Isham approach.

In the next section we give the corresponding examples. In particular, we consider the quantization on a cylinder and a torus. A quantum system obtained for a cylinder is unitary equivalent to the canonically quantized model. We find that a consistent quantization on a torus requires quantization of the phase space volume. Note that the obtained Hilbert space is finite dimensional. The next example is a reduced system for the particle on $SL(2,R)$ manifold, where quantization is associated with a symmetry group representation.

In Section 3.3 we consider another example of non canonical quantization. Here we study the roots of the phase operators. Correct definition of the corresponding operators leads to the extension of the Hilbert space. In the case of square root the fermionic extension arises. For the k-th root we introduce the new 'particles' whose statistics depends on k. It is shown that the described quantization can be associated with a quantization on k-sheet manifolds.

In Chapter 4 we introduce the scheme of E-quantization. This scheme is based on the extension of the phase space and it uses the technique of the constrained quantization.

The phase space of the extended system, is a cotangent bundle over the initial symplectic manifold. For the quantization of the extended system the canonical method is used and the pre-quantization operators arise as a result of some natural operator ordering. This extension scheme is described in Section 4.1.

In the next section we introduce a constrained surface of the extended system. The constraint functions are characterized by some complete set of observables of the initial system, and they form the set of the second class constraints. Further, the corresponding constraint operators are introduced, and for the restriction of the extended quantum system the Dirac, or the Gupta-Bleuler like, conditions are used. Certainly, the Dirac conditions are used only for a half of commuting constraints. The same number of complex conditions is used in Gupta-Bleuler case as well.

After reduction to the physical Hilbert space, the standard problems of constrained quantization arise. The problem of the correct definition of the observable operators is connected with the deformation procedure. We describe this procedure for the constant symplectic matrix in Appendix C.

In Section 4.3, illustrating the quantization scheme described above, we consider two examples. The first one is a quantization on a plane and the second one on a cylinder.

When we use the Dirac conditions, the problem of scalar product arises. A possible solution to this problem is considered in Section 4.4, where the limiting procedure with normalized physical states is used. Here, the solutions of Dirac conditions are interpreted as the vectors of the space of unbounded functionals on the Hilbert space, and then, the limiting procedure is accomplished in this dual space.

Using the E-quantization method, in Section 4.5 we describe a possible realization of the classical symmetries on the quantum level. For illustration we consider the rotation group on a sphere and its quantum realization.

The last chapter starts with a generalization of the Gupta-Bleuler like conditions, where we use the minimization principle for quadratic fluctuations of quantum constraints. The corresponding technique is described in Appendix D. The obtained generalized condition contains constraint operators of second order, and for a physical wave function it is an elliptic type equation on the phase space.

In Section 5.2 we introduce the coherent states, which are related to some complete set of observables. Coherent states are constructed as the functions on the phase space and, at the same time, they are parametrized by the points of the phase space. Such coherent states minimize uncertainties of those observables to which they are related to. At the end of the section we construct the special coherent states on a cylinder and study their behavior in the limit when the squeezing parameter tends to zero. In this limit we get the eigenstates of the angular-momentum operator.

In the last section we introduce the quantum distribution functions, which are smooth non-negative distributions on the phase space. The corresponding functions satisfy some elliptic type equations. This equation specifies the distribution functions for the pure states. Generalization to the mixed states is done as a convex combination of the pure ones. There are different classes of quantum distribution functions and each class is related to a certain complete set of observables of the system in consideration. We discuss the physical interpretation of these distribution functions. Namely, we interpret them as the distributions obtained in the experiment where we simultaneously measure the indicated complete set of observables. At the end of the paper we discuss a possibility of the formulation of quantum mechanics in terms of quantum distribution functions without referring to the Hilbert space formalism.

In conclusion, we present the main results of the paper.

1. Hamiltonian Dynamics

In this chapter we give a brief review of the Hamiltonian dynamics. We use notations of [20]. For more details see also [58-59] and [23].

1.1. Symplectic manifolds. The Hamiltonian system is described by the phase space, a set of observables, the Poisson structure and the Hamilton function.

The phase space is an even $(2N)$ dimensional manifold $\mathcal{M}$ with smooth, second order tensor field $\omega^{ij}(\xi)$, and this tensor field should satisfy the following three conditions:

1. The matrix $\omega^{ij}(\xi)$ is antisymmetric: $\omega^{ij} = -\omega^{ji}$
2. It is invertible in each point $\xi \in \mathcal{M}$: $\det \omega^{ij} \neq 0$.
3. The identities

$$\omega^{il}\partial_l\omega^{jk} + \omega^{jl}\partial_l\omega^{ki} + \omega^{kl}\partial_l\omega^{ij} = 0 \tag{1.1}$$

are satisfied for any free three indices i, j, k. Here ∂_l denotes the derivatives in some local coordinates $(\xi^1, \ldots, \xi^{2N})$.

The observables $f(\xi), g(\xi), h(\xi), \ldots$ are smooth functions on $\mathcal{M}$ and the space of observables is denoted by $\mathcal{O}(\mathcal{M})$.

The tensor field ω^{ij} defines the Poisson bracket of two observables f and g

$$\{f, g\} \equiv -\partial_i f \omega^{ij} \partial_j g. \tag{1.2}$$

It is easy to check that the Jacobi identity

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0 \tag{1.3}$$

follows from the condition (1.1).

Since the Poisson brackets (1.2) are linear, antisymmetric and satisfy the Jacobi identity, the set of all observables has a structure of Lie algebra, which is called the Poisson-Lie structure.

The Hamiltonian vector field constructed for an observable $f(\xi)$ has the components

$$\mathcal{V}_f^i \equiv \omega^{ij}\partial_j f \tag{1.4}$$

and we associate with this field the first order differential operator V_h

$$V_h := \mathcal{V}_h^i \partial_i \tag{1.5}$$

acting on the space of observables $\mathcal{O}(\mathcal{M})$.

The Hamilton function $H = H(\xi)$ is a generator of dynamics. It defines the phase trajectories (phase flow) as the solutions of the Hamilton equations

$$\dot{\xi}^i = \mathcal{V}_H^i(\xi). \tag{1.6}$$

Respectively, the evolution of any observable f takes the form

$$\dot{f} = \{H, f\} \tag{1.7}$$

The inverse of the matrix ω^{ij} is the co-tensor field ω_{ij} ($\omega^{ij}\omega_{jk} = \delta^i_k$), and from (1.1) we obtain

$$\partial_i\omega_{jk} + \partial_j\omega_{ki} + \partial_k\omega_{ij} = 0. \tag{1.8}$$

Using the co-tensor ω_{ij}, one can construct the 2-form $\omega = 1/2\omega_{ij}d\xi^i \wedge d\xi^j$ which is non-degenerated, antisymmetric and, due to (1.8), closed ($d\omega = 0$). Such a 2-form is called the symplectic form and the corresponding manifold $\mathcal{M}$ is a symplectic manifold.

The co-tensor field ω_{ij} (locally) has the form

$$\omega_{ij} = \partial_i\theta_j - \partial_j\theta_i \tag{1.9}$$

with some co-vector field θ_i (see (1.8)). Then, the Hamilton equations (1.6) can be obtained from the variation of the action

$$S = \int [\theta_i(\xi)\dot{\xi}^i - H(\xi)]dt. \tag{1.10}$$

If the covector field θ_i is global, it defines the 1-form $\theta = \theta_i d\xi^i$ and from (1.9) we have

$$\omega = d\theta. \tag{1.11}$$

Note, that the Poisson structure and the evolution equations (1.1)-(1.6) can be considered for a degenerated matrix w^{ij} as well. But in that case the dynamical equation (1.6) cannot be obtained from the Hamilton's principle of the action (1.10).

Definition (1.2) gives

$$\omega^{ij}(\xi) = -\{\xi^i, \xi^j\}, \tag{1.12}$$

and a choice of the tensor field $\omega^{ij}(\xi)$ in some coordinates ξ^i is equivalent to the postulation of Poisson brackets of those coordinates. Note that, in general, the coordinates ξ^i are not global (sometimes there is no global coordinate system on $\mathcal{M}$) and therefore, in such cases, the relation (1.12) has only a formal meaning.

It is clear that the relations (1.1) are automatically satisfied if at least two indices from (i, j, k) coincide. Therefore, for 2-dimensional manifolds any antisymmetric tensor field with nonvanishing ω^{12} is suitable.

Let us consider some simple examples.

A. $\mathcal{M} = \mathcal{R}^2$. The phase space is a plane with the 'flat' coordinates $(\xi^1 \equiv p, \xi^2 \equiv q)$. For $\omega^{21} = -\omega^{12} = 1$, we get the standard form of Poisson brackets

$$\{f, g\} = \frac{\partial f}{\partial p}\frac{\partial g}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial g}{\partial p}, \tag{1.13}$$

and p and q can be interpreted as the momentum and the coordinate of a particle moving on the axis q.

B. $\mathcal{M} = \mathcal{R}^1 \otimes \mathcal{S}^1$. The phase space is a cylindre with the coordinates $\xi^1 \equiv S \in \mathcal{R}^1, \xi^2 \equiv \varphi \in S^1$ and the natural Poisson structure is given by

$$\{f, g\} = \frac{\partial f}{\partial S}\frac{\partial g}{\partial \varphi} - \frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial S}. \tag{1.14}$$

This system describes a rotator where the variable S has a meaning of the angular momentum.

It should be noted that the coordinate S is a global one, and therefore it can be considered as an observable as well, while the second coordinate φ is only local, and it is not an observable. Observables are periodic functions of φ.

C. $\mathcal{M} = \mathcal{S}^1 \otimes \mathcal{S}^1$. The phase space is a torus with the coordinates $\xi^1 \in \mathcal{S}^1, \xi^2 \in \mathcal{S}^1$. If a_1 and a_2 are the periods of the corresponding coordinates ξ^1 and ξ^2, then any observable f should be periodic: $f(\xi^1, \xi^2) = f(\xi^1 + a_1, \xi^2 + a_2)$. For the Poisson brackets we choose again $\omega^{21} = 1$.

D. $\mathcal{M} = \mathcal{S}^2$. The phase space is a sphere. If r is a radius of the sphere and ϑ, φ are the corresponding spherical angles, then the natural Poisson structure (see the example below) is given by

$$\{f, g\} = \frac{1}{r\,\sin\vartheta}\left(\frac{\partial f}{\partial \varphi}\frac{\partial g}{\partial \vartheta} - \frac{\partial f}{\partial \vartheta}\frac{\partial g}{\partial \varphi}\right). \tag{1.15}$$

The last two examples have no direct analog in classical mechanics since they can not be obtained by the Hamiltonization of regular Lagrangian systems. But such systems can be considered for the description of the internal degrees of freedom [43-49], [60-63]. The corresponding quantum versions are especially interesting, where, due to the compactness of the phase spaces, the 'Hilbert space' of quantum states is finite dimensional (see Chapters 3 and 4).

Examples **A** and **B** have a natural generalization for a symplectic manifold which is the cotangent bundle $\mathcal{M} = T^*Q$ of an N-dimensional manifold Q. In this case $\mathcal{M}$ is a set of pairs (p, q), where $q \in Q$ and p is a covector at q. On T^*Q there is the canonical 1-form $p_a dq^a (a = 1, \ldots, N)$ and the corresponding canonical 2-form

$$\omega = dp_a \wedge dq^a. \tag{1.16}$$

Such systems with the global canonical symplectic structure are most well investigated both on classical and quantum level. In particular, due to the global canonical structure, one can use the standard canonical quantization.

At the same time, the systems with the cotangent bundle structure have some fundamental character since all symplectic manifolds locally have the canonical form. More precisely, according to Darboux's theorem, for an arbitrary point $m \in \mathcal{M}$ there is a neighbourhood U of m and a coordinate system $p_a,\ q^b\ (a, b = 1, \ldots, N)$ on U such that $\omega = dp_a \wedge dq^a$ in U. The proof of this theorem can be found in [19] or [20].

Investigation of integrability is the main problem of the classical mechanics. The Hamiltonian system is called completely integrable if one can find the canonical coordinates $I_a, \varphi^b (a, b = 1, \ldots, N)$, $\theta = I_a d\varphi^a$, such that $H = H(I_1, \ldots, I_N)$. We call the coordinates I_a and φ^a the action and the angle variables respectively. The dynamics of a completely integrable system in the action-angle variables is trivial

$$I_a(t) = I_a \quad \varphi^a(t) = \omega^a t + \varphi_0 \quad \text{where} \quad \omega^a = \frac{\partial H}{\partial I_a}.$$

Note that the integrability is always related to the symmetry groups, and, in general, a Hamiltonian system is not a completely integrable one.

At a formal level, much of the symplectic geometry that underlies the Hamiltonian formulation of the classical mechanics extends to field theories and can be used as the starting point for quantization. The Poisson brackets in a field theory are defined through the variational derivatives of the observables. Note that the symplectic structure can be introduced for the singular fields as well. For example, the phase space of N-singular solutions of the Liouville field theory [39-41]

$$\phi_{tt}(t, x) - \phi_{xx}(t, x) + \exp \phi(t, x) = 0 \tag{1.17}$$

is described by the class of two functions $\phi(x), \pi(x) \equiv \dot{\phi}(x)$ with certain singularities. One can show (see [40-41]) that the corresponding singular solutions of (1.17) can be constructed by the consequent actions of Backlund transformations and the corresponding class is uniquely parametrized by two regular functions $\phi_0(x), \pi_0(x) \equiv \dot{\phi}_0(x)$ and a set of $2N$ variables $p_a, q^b(a, b =$

$1, \ldots, N$). The symplectic form, calculated in regular variables, has the standard canonical form

$$\omega = \int dx\ d\phi_0(x) \wedge \pi_0(x) + \sum_{a=1}^{N} dp_a \wedge dq^a.$$

1.2. Groups of canonical transformations. The first order differential operator (1.5) is invariant under any choice of coordinates ξ^i and its action on the observable $f(\xi)$ is given by the Poisson bracket

$$V_h : f(\xi) \mapsto \{h, f\}. \tag{1.18}$$

Using the Jacobi identity (1.3), we get

$$[V_{h_1}, V_{h_2}] = V_{\{h_1, h_2\}}, \tag{1.19}$$

where [,] denotes the standard commutator of two operators.

The operator V_h generates the group of one parameter transformations T_h^ϵ acting on the space of observables $\mathcal{O}(\mathcal{M})$

$$\begin{aligned} T_h^\epsilon : f(\xi) \mapsto \exp(\epsilon V_h)\ f(\xi) \equiv \\ f + \epsilon\{h, f\} + \frac{\epsilon^2}{2!}\{h, \{h, f\} + \cdots + . \end{aligned} \tag{1.20}$$

In particular, T_H^t is the operator for the time evolution of observables.

We call a transformation T canonical if it preserves the Poisson brackets. From the Jacobi identity (1.3) we obtain that the transformations (1.20) are canonical

$$\{T_h^\epsilon f, T_h^\epsilon g\} = T_h^\epsilon\{f, g\}. \tag{1.21}$$

Let $h_\alpha(\xi)$ ($\alpha = 1, \ldots, M$) be a set of functions which satisfy the relations

$$\{h_\alpha, h_\beta\} = C_{\alpha\beta}^\gamma h_\gamma, \tag{1.22}$$

where $C_{\alpha\beta}^\gamma$ are the structure constants of some Lie group G.

The operators V_{h_α} generate the M-parameter transformations of $\mathcal{O}(\mathcal{M})$

$$U_\epsilon := \exp(\epsilon^\alpha V_{h_\alpha}) : f \mapsto f + \epsilon^\alpha\{h_\alpha, f\} + \frac{\epsilon^\alpha \epsilon^\beta}{2!}\{h_\alpha\{h_\beta, f\}\} + \cdots +, \tag{1.23}$$

and from (1.19) and the Backer-Campel-Hausdorff formula [66] we see that (1.23) is a representation of the group G.

If, in addition to (1.22), all h_α are integrals of motion (see (1.7)),

$$\{H, h_\alpha\} = 0, \tag{1.24}$$

then G is a symmetry group of the classical system since in that case the operators U_ϵ and T_H^t commute with each other.

Generalization of (1.24) is given by

$$\{H, h_\alpha\} = d_\alpha^\beta h_\alpha, \tag{1.25}$$

where d_α^β are constants. In that case we have an extension of the Lie algebra (1.22), and the evolution equations for the observables h_α (1.7) become linear.

Let us consider some examples:

a. The translation group on a plane (see Example **A**).

The functions

$$h_0 = 1, \quad h_1 = p, \quad h_2 = q \tag{1.26}$$

have the commutation relations

$$\{h_0, h_1\} = 0, \quad \{h_0, h_2\} = 0, \quad \{h_1, h_2\} = h_0 \tag{1.27}$$

and according to (1.20), h_1 and h_2 generate the translations on a plane:

$$h_1 : (p, q) \mapsto (p, q + \epsilon) \quad h_2 : (p, q) \mapsto (p - \epsilon, q). \tag{1.28}$$

Note that the constant function h_0 does not generate any transformation.

b. The symplectic group on a plane $Sp(1, R)$ (see Example **A**).

The functions

$$l_0 = \frac{1}{4}(p^2 + q^2), \quad l_1 = \frac{1}{4}(p^2 - q^2), \quad l_2 = \frac{1}{2}pq \tag{1.29}$$

satisfy the commutation relations

$$\{l_0, l_1\} = -l_2, \quad \{l_0, l_2\} = l_1, \quad \{l_1, l_2\} = l_0 \tag{1.30}$$

and they generate the following global linear transformations

$$\begin{aligned} 2l_0 &: (p, q) \mapsto (p \cos \epsilon - q \sin \epsilon, q \cos \epsilon + p \sin \epsilon) \\ 2l_1 &: (p, q) \mapsto (p \cosh \epsilon + q \sinh \epsilon, q \cosh \epsilon + p \sinh \epsilon) \\ 2l_2 &: (p, q) \mapsto (e^{-\epsilon} p, e^{\epsilon} q). \end{aligned} \tag{1.31}$$

The group of linear transformations which preserve the canonical Poisson brackets (1.13) is the definition of the group $Sp(1, R)$.

The commutation relations (1.30) have the form

$$\{l_\mu, l_\nu\} = \varepsilon_{\mu\nu\sigma} g^{\sigma\rho} l_\rho, \tag{1.32}$$

where $g^{\mu\nu} = \mathrm{diag}(+, -, -)$ is the metric tensor of 3-dimensional Minkowski space and $\varepsilon_{\mu\nu\sigma}$ is the corresponding antisymmetric 3-tensor (with $\varepsilon_{012} = 1$). These are the commutation relations of the Lie algebra $so(2, 1)$ (or $sl(2, R)$) and we see the isomorphism of these Lie algebras to $sp(1, R)$ (see [67]).

Note that the functions (1.29) and (1.27) together also form the extended Lie algebra with 6 generators.

c. The translation group on a cylindre (see Example **B**).

Let us consider two different translations on a cylindre:

$$\begin{aligned} T_1^\epsilon &: f(S, \varphi) \mapsto f(S, \varphi + \epsilon), \\ T_2^\epsilon &: f(S, \varphi) \mapsto f(S + \epsilon, \varphi). \end{aligned} \tag{1.33}$$

These transformations are canonical and they are generated by the operators

$$V_1 = \partial_\varphi, \quad V_2 = \partial_S, \tag{1.34}$$

respectively. Note that the first operator V_1 is constructed from the Hamiltonian vector field of the observable S: $V_1 = V_S$, while there is no corresponding global observable for the second one V_2.

Since any observable $f(S,\varphi)$ is periodic in φ: $f(S,\varphi) = f(S,\varphi+2\pi)$, the corresponding transformations T_1^ϵ should be periodic (in ϵ) as well

$$T_1^\epsilon = T_1^{\epsilon+2\pi}. \tag{1.35}$$

It is necessary to stress, that any consistent quantization should take into account geometric peculiarities of the phase space and corresponding transformation groups.

d. The translation group on a torus (see Example **C**).

In this case the translation of ξ^1 and ξ^2 coordinates are again the canonical ones. They are periodic with periods a_1 and a_2 respectively, but these transformations are not generated by any observables.

e. The rotation group on a sphere (see Example **D**).

There is no any global coordinate system on a sphere, but the following three functions

$$J_1 = r\sin\vartheta\cos\varphi, \quad J_2 = r\sin\vartheta\sin\varphi, \quad J_3 = r\cos\vartheta \tag{1.36}$$

are well defined observables. They are, respectively, the X, Y and Z coordinates of a point on the sphere. Note that J_i $(i = 1,2,3)$ are not independent and they are related by

$$J_1^2 + J_2^2 + J_3^2 = r^2. \tag{1.37}$$

From (1.15) we get the commutation relations of the rotation group

$$\{J_k, J_l\} = \varepsilon_{klm} J_m, \quad (\varepsilon_{123} = 1), \tag{1.38}$$

and the observables (1.36) generate the action of the rotation group $SO(3)$ on the sphere. It is easy to see that these are the ordinary rotations on the sphere.

f. The Poincaré group for spinless particle with the mass m.

The phase space is $\mathcal{R}^6$ with the canonical coordinates $(\vec{p},\vec{q})$, where $\vec{p} := (p_k;\ k = 1,2,3)$ and $\vec{q} := (q_k;\ k = 1,2,3)$ are 3-dimensional vectors.

Ten generators of the Poincare group have the form:

$P_0 = \sqrt{\vec{p}^{\,2} + m^2}$ and $P_k = p_k$ are the translation generators,

$J_k = \varepsilon_{klm} q_l p_m$ are the rotation generators and

$$L_k = \sqrt{\vec{p}^{\,2} + m^2}\ q_k \tag{1.39}$$

generate the boosts.

One can easily check that these generators satisfy the commutation relations of the Poincare group Lie algebra. Note that the global transformations generated by (1.39) are nonlinear.

g. A nonlinear realization of the group $SL(2,R)$ (see Example **A**).

Let us consider the following three functions on a plane

$$I_0 = \frac{p^2+q^2}{2} + r,$$
$$I_1 = \sqrt{r + \frac{1}{4}(p^2+q^2)}\; p, \quad I_2 = \sqrt{r + \frac{1}{4}(p^2+q^2)}\; q, \tag{1.40}$$

where r is a positive parameter ($r > 0$), and p, q are the canonical coordinates. The functions I_μ ($\mu = 0, 1, 2$) have the commutation relations (1.32) of $sl(2,R)$ Lie algebra (see Example **b**), but it is clear that the corresponding transformations now are nonlinear.

Investigation of Hamiltonian's symmetries is the main problem of both classical and quantum mechanics. But this problem can be inverted, and one can try to obtain the Hamilton function with a given group of symmetry. When the symmetry group G is generated by the set of functions h_α with the commutation relations (1.22), then the suitable Hamiltonian should satisfy the equations (1.24) (or (1.25)). If the action of the group G on the phase space $\mathcal{M}$ is transitive (as in all above mentioned examples), then the equations (1.24) have only the trivial (constant) solution for the Hamilton function H. Of course, a possibility for (1.25) always remains if the Hamiltonian is equal to one of the generators h_α, as, for example,

$$H = \sqrt{\vec{p}^{\,2} + m^2} \tag{1.41}$$

for the relativistic particle.

1.3. Pure and mixed states in classical mechanics. We usually consider the 'regular' Hamiltonian systems when the Hamilton equations (1.6) have unique global solutions $\xi(t)$ for arbitrary initial conditions $\xi(0) = \xi_0$ and the phase trajectories define the global one-parameter transformations of the phase space $\mathcal{M}$

$$T_H^t : \xi \mapsto \widetilde{\xi}(\xi, t). \tag{1.42}$$

Since the initial data are given by the points of the phase space $\mathcal{M}$, these points are associated with the classical states: each point ξ describes certain state of the classical system. In such description, there is a unique prediction for the evolution of the classical system.

But sometimes, even in the classical mechanics, we consider the probability description as well. In general, we assume that the points of the phase space $\mathcal{M}$ describe the pure states of the system and an arbitrary (mixed) state is given by the distribution function $\rho(\xi)$ on $\mathcal{M}$. The physical meaning of $\rho(\xi)$ is a probability density that the system is in the pure state ξ. This means that if Ω is any domain of the phase space $\mathcal{M}$ and

$$d\mu(\xi) \equiv \sqrt{\omega(\xi)}\; d^{2N}\xi \quad \text{with} \quad \omega(\xi) \equiv \det \omega_{ij}(\xi) \tag{1.43}$$

is the invariant measure on $\mathcal{M}$, then

$$P_\Omega = \int\limits_\Omega \rho(\xi)\; d\mu(\xi) \tag{1.44}$$

is a probability that the system is in some pure state $\xi \in \Omega$. Respectively $P_{\mathcal{M}} = 1$.

We use such probability description for the statistical ensemble of the systems. The distribution functions $\rho(\xi)$ can also be used in the case of possible experimental errors of a measurement procedure.

By definition, the distribution function $\rho(\xi)$ is nonnegative, but, in general, it is a generalized function (rather than the ordinary one). For example, the pure state ξ_0 is described by the δ-function distribution

$$\rho_{\xi_0}(\xi) = \sqrt{\det(\omega^{ij})}\; \delta(\xi - \xi_0). \tag{1.45}$$

Using the definition (1.44), we can calculate the mean value of the observable $f(\xi)$ in the state $\rho(\xi)$

$$\bar{f} = \int\limits_{\mathcal{M}} f(\xi)\rho(\xi)\; d\mu(\xi). \tag{1.46}$$

In the dynamical description of the mean values (1.46), the observable $f(\xi)$ is assumed to be time independent, and obviously, the time evolution of $\rho(\xi)$ takes the form

$$\rho(\xi, t) = \rho_0(T_H^{-t}\xi), \tag{1.47}$$

where T_H^{-t} is the transformation (1.42). Then from (1.6) we obtain the Liouville equation for the distribution function $\rho(\xi, t)$

$$\dot{\rho} = \{\rho, H\}. \tag{1.48}$$

The equations (1.46)-(1.48) define the Liouville picture of statistical systems.

Another description of the statistical systems can be done by a time independent distribution function $\rho(\xi)$ considering the dynamics of the observables f according to (1.7). We call such description the Hamilton picture. It is clear that the Liouville and the Hamilton pictures are, respectively, analogous to the Schrödinger and the Heisenberg pictures of quantum mechanics. One can check that both of these pictures give the same dynamics for the mean values of the observables $f(\xi)$ (see [23]).

2. Hamiltonian Reduction of Gauge Theories

In this chapter we investigate the Hamiltonian reduction procedure of gauge invariant theories. We construct a general reduction scheme and considere several non-trivial examples. The content of this chapter is based on the papers [53-54] and [56].

2.1. Reduction scheme with gauge invariant variables. Any non-singular Lagrangian leads to the standard Hamiltonian description considered in the previous chapter. But if we start from the gauge-invariant Lagrangian $L = L(q_k, \dot{q_k})$ $(k = 1, \ldots, N)$ and use the Dirac procedure [1], or the first order formalism [18]-[19], we arrive at the extended phase space Γ with coordinates (p_k, q_k) and the action

$$S = \int p_k dq_k - [H(p,q) + \lambda_a \phi_a(p,q)]dt, \tag{2.1}$$
$$k = 1, \ldots, N; \quad a = 1, \ldots, M; \ (N > M),$$

where $\phi_a(p,q)$ are the constraints, $H(p,q)$ is the canonical Hamiltonian, and λ_a are the Lagrange multipliers. The constraint surface $\mathcal{M}_c$ is defined by

$$\phi_a(p,q) = 0 \tag{2.2}$$

and the following relations are fulfilled

$$\{H, \phi_a\}_\Gamma = d_a^b \phi_b, \quad \{\phi_a, \phi_b\}_\Gamma = f_{ab}^c, \phi_c \tag{2.3}$$

where f_{ab}^c are the structure constants of the corresponding gauge group. The index Γ on the left hand side indicates that the Poisson brackets are calculated on the extended phase space Γ.

A function $\xi = \xi(p,q)$ is called a gauge-invariant variable if $\xi|_{\mathcal{M}_c} \neq$ const and

$$\{\xi, \phi_a\}_\Gamma = \widetilde{d}_a^b \phi_b, \tag{2.4}$$

where $|_{\mathcal{M}_c}$ denotes the restriction to $\mathcal{M}_c$. The functions $\widetilde{d}_a^b$, as well as d_a^b in (2.3), are assumed to be regular in the neighborhood of $\mathcal{M}_c$. Note that the below described reduction scheme can be generalized for arbitrary first class constraints ϕ_a with non-constant coefficients f_{ab}^c in (2.3).

Each gauge-invariant variable ξ possesses the class $\{\xi\}$ of equivalent gauge-invariant variables on Γ. A gauge-invariant function $\widetilde{\xi}$ is equivalent to ξ if $\widetilde{\xi}|_{\mathcal{M}_c} = \xi|_{\mathcal{M}_c}$. On the other hand, the function $\xi|_{\mathcal{M}_c}$ is constant along the gauge orbit (on $\mathcal{M}_c$) and it defines the function $\widetilde{\widetilde{\xi}}$ on the physical space $\mathcal{M} = \mathcal{M}_c/G$. Thus, $\{\xi\}$, $\xi|_{\mathcal{M}_c}$ and $\widetilde{\widetilde{\xi}}$ denote the gauge-invariant variable ξ in different context. If there is no ambiguity, we will use the notation ξ for all of them.

Maximal number of gauge-invariant variables (2.4) which are functionally independent on the constraint surface $\mathcal{M}_c$ is $2(N-M)$ [3]. Assuming that

$\{\xi^\alpha : \alpha = 1, \ldots, 2(N-M)\}$ is the corresponding complete set of gauge-invariant variables, one can prove (see [18-19]) that

$$
\begin{aligned}
&1. \quad p_k dq_k|_{\mathcal{M}_c} = \theta_1 + \theta_2 \quad \text{with} \\
&\qquad \text{a) } d\theta_1 = 0 \\
&\qquad \text{b) } \theta_2 = \theta_\alpha(\xi) d\xi^\alpha \\
&\qquad \text{c) } \det \omega_{\alpha\beta} \neq 0 \quad \text{where} \quad \omega_{\alpha\beta}(\xi) = \partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha ; \\
&2. \quad H(p,q)|_{\mathcal{M}_c} = h(\xi).
\end{aligned}
\tag{2.5}
$$

The main statement of (2.5) is that after restriction to the constraint surface $\mathcal{M}_c$, the dependence on extra (nonphysical) variables is presented only in the term θ_1, which is a 'total derivative'.

Since $d\theta_1 = 0$, it gives no contribution to the variation of the restricted action and we can neglect it. Then for the reduced system we obtain

$$
S|_{\mathcal{M}_c} \equiv \widetilde{S} = \int \theta_\alpha(\xi) d\xi^\alpha - h(\xi) dt, \tag{2.6}
$$

and the dynamics for gauge-invariant variables are described by the Hamilton equations (see (1.6) and (1.10))

$$
\dot{\xi^\alpha} = \omega^{\alpha\beta}(\xi) \partial_\beta h(\xi), \tag{2.7}
$$

where $\omega^{\alpha\beta}(\xi)$ is the inverse of the symplectic matrix $\omega_{\alpha\beta} = \partial_\alpha \theta_\beta - \partial_\beta \theta_\alpha$.

Thus, the reduced system (2.6)-(2.7) is an ordinary (nonconstrained) Hamiltonian system.

It should be noticed that in the general case any $2(N-M)$ number of gauge-invariant variables play the role of local coordinates on the physical phase space $\mathcal{M}$ and, respectively, (2.6)–(2.7) are defined only locally. A global description can be achieved by a set of gauge-invariant variables which defines the global structure of the physical phase space $\mathcal{M}$. The number of such gauge-invariant variables is greater than $2(N-M)$, but on the constraint surface there are relations among them, and these relations define the geometry of $\mathcal{M}$ (on the phase space geometry of constrained systems, see [16]).

For illustration, let us consider the following example of (2.1)-(2.3) (see [53])

$$
S = \int \vec{p} \cdot d\vec{q} - [\lambda_1 \phi_1 + \lambda_2 \phi_2] dt. \tag{2.8}
$$

Here $\vec{p}$ and $\vec{q}$ are vectors in R^3, the canonical Hamiltonian is zero,

$$
\phi_1 = \vec{p} \cdot \vec{q}, \quad \phi_2 = \vec{p}^{\,2} \vec{q}^{\,2} - (\vec{p} \cdot \vec{q})^2 - r^2 \tag{2.9}
$$

and r is a positive parameter. These constraints are Abelian ($\{\phi_1, \phi_2\} = 0$) and the second constraint ϕ_2 can be written in the form

$$
\phi_2 = \vec{J}^{\,2} - r^2
$$

where $\vec{J} = \vec{q} \times \vec{p}$ is the angular momentum.

It is clear that the physical phase space is two-dimensional and the components of angular momentum $\vec{J}$ are gauge-invariant variables (they commute with the constraints (2.9) since the latter's are $O(3)$ scalars). On the constraint surface these three components are related by $\vec{J} \cdot \vec{J} = r^2$. Therefore, the physical phase space $\mathcal{M}$ is a two-dimensional sphere (see Appendix A and Examples **D** and **e** of the previous chapter), and any two coordinates, as well as the 1-forms θ_1 and θ_2, are defined only locally.

The reduction scheme described in (2.6)-(2.7) can be used if all $2(N - M)$ gauge-invariant variables are known. For practical application of this scheme, one can introduce arbitrary variables $\eta^1, \ldots, \eta^M$, which are complementary to gauge-invariant variables, in order to complete the coordinate system

$$(\xi^1, \ldots, \xi^{2(N-M)}, \eta^1, \ldots, \eta^M)$$

on $\mathcal{M}_c$. Calculating the restricted 1-form $p_k dq_k|_{\mathcal{M}_c}$ in these coordinates and taking its differential, we can find the symplectic form $\omega = \omega_{\alpha\beta}(\xi) d\xi^\alpha \wedge d\xi^\beta$. Note that in actual calculations it is possible to select the 1-form $\theta_2 = \theta_\alpha(\xi) d\xi^\alpha$ and get (2.6).

Application of this procedure to model (2.8)-(2.9) gives $\theta_2 = (s - r) d\varphi$, where s and φ are the cylindrical coordinates on a sphere (see Appendix A)

$$J_1 = \sqrt{r^2 - s^2} \cos\varphi \quad J_2 = \sqrt{r^2 - s^2} \sin\varphi \quad J_3 = s \tag{2.10}$$

It is clear that although $(s - r) d\varphi$ is not a global 1-form, its differential can be continued to the well-defined symplectic form on the sphere [27] (see (A.6))

$$\omega = -\frac{J_1(dJ_2 \wedge dJ_3) + J_2(dJ_3 \wedge dJ_1) + J_3(dJ_1 \wedge dJ_2)}{r^2} = ds \wedge d\varphi.$$

It is easy to check that this symplectic form leads to the Poisson brackets (1.15). Note that a consistent quantum theory of this system exists only for the discrete values of the parameter r (see [20] and Section 4.5).

Generalization of the scheme to the infinite-dimensional case is straightforward (in the Appendix B we present the example of massive photodynamics in $(2+1)$ dimensions). If we use the Dirac observables [68]

$$\psi_{in} = e^{i\Delta^{-1}(\vec{\nabla}\vec{A})}\psi \tag{2.11}$$

in ordinary ED, we will easily obtain the photons in the Coulomb gauge and the 'four-fermion interaction' for the 'dressed fermions' (compare with the example in Section 2.3 and see [18], [69]).

Commutation relations of the complete set of gauge-invariant variables can also be derived by calculation of the Poisson brackets on the extended phase space [3]. This, more standard procedure is based on the fact that the Poisson brackets of any two gauge-invariant variables is again a gauge-invariant one. This procedure and the scheme described in (2.6)-(2.7) are almost equivalent. Sometimes, however, when the complete set of gauge-invariant variables is not a canonical one and the corresponding canonical quantization is problematic, the calculation of differential forms become reliable to apply more general quantization techniques (see: [18], [31], [52], [57]).

In general, from the structure of gauge transformations, one can easily find only a part of gauge-invariant variables, and a construction of a complete set of gauge-invariant variables (2.6) is troublesome. Our approach with differential forms can be effectively used for the solution of this problem as well.

Let us consider the situation when we know a set of gauge-invariant variables $\{\xi^\mu : \mu = 1, \ldots, K\}$, where $N - M \leq k \leq 2(N - M)$. We can add arbitrary variables $\eta^1, \ldots, \eta^{2N-M-K}$ in order to complete the coordinate system on $\mathcal{M}_c$ and calculate the restricted 1-form $p_k dq_k|_{\mathcal{M}_c}$. Assuming that we can select the 'total derivatives' and the differentials $d\xi^\mu$ in the form

$$p_k dq_k|_{\mathcal{M}_c} = dF(\xi, \eta) + \theta_\mu(\xi, \eta) d\xi^\mu \tag{2.12}$$

and using (2.6), we can easily conclude that $\theta_\mu(\xi, \eta)$ will be gauge-invariant variables. Note that passing to gauge-invariant variables ξ^μ is helpful for obtaining the form (2.12). To illustrate this method, we apply it to a relativistic particle [5], where we have the 1-form $\theta = \vec{p} d\vec{q} - p_0 dq_0$, and the constraint surface $\mathcal{M}_c$ with $\phi \equiv p^2 - m^2 = 0, (p_0 > 0)$. The momenta $\vec{p}$ are gauge invariant, and after restriction on $\mathcal{M}_c$, we have

$$\theta|_{\mathcal{M}_c} = \vec{p} d\vec{q} - \sqrt{\vec{p}^{\,2} + m^2}\, dq_0 .$$

One can easily rewrite it in the following form

$$\theta|_{\mathcal{M}_c} = d(\vec{p} \cdot \vec{q} - \sqrt{\vec{p}^{\,2} + m^2}\, q_0) - (\vec{q} - \frac{\vec{p}}{\sqrt{\vec{p}^{\,2} + m^2}} q_0) d\vec{p}.$$

Evidently, the coefficients of the differentials $-d\vec{p}$ are gauge-invariant variables. They are canonically conjugate to $\vec{p}$

$$\vec{Q} = \vec{q} - \frac{\vec{p}}{\sqrt{\vec{p}^{\,2} + m^2}} q_0 .$$

The gauge invariance of $\vec{Q}$ can also be established from the relation

$$\vec{L} = \sqrt{\vec{p}^{\,2} + m^2}\, \vec{Q}, \tag{2.13}$$

where $\vec{L}$ are the generators of Lorentz transformations. Since all generators of the Poincare group $(P_\mu, M_{\mu\nu})$ are gauge-invariant variables, the same property holds for the coordinates $\vec{Q}$. On the constraint surface $\vec{p}^{\,2} - m^2 = 0, (p_0 > 0)$, all of these coordinates are the functions only of the reduced variables $\vec{p}$ and $\vec{Q}$ (see Example **f** in Section 1.2).

The reduced system can be easily quantized in the momentum representation: $\hat{\vec{p}} = \vec{p}$ and $\hat{\vec{Q}} = i\hbar\vec{\nabla}$. The operator ordering problem arises only for the generators (2.13). Note that the standard Lorentz covariant measure in the scalar product

$$\langle\Psi_2|\Psi_1\rangle = \int \frac{d^3\vec{p}}{\sqrt{\vec{p}^{\,2}+m^2}}\bar{\Psi}_2(\vec{p})\Psi_1(\vec{p})$$

corresponds to the ordering $\hat{\vec{L}} = i\hbar\sqrt{\vec{p}^{\,2}+m^2}\vec{\nabla}$.

2.2. Relativistic particle on $SL(2,R)$ group manifold. Let M be a (pseudo) Riemannian manifold with the metric $g_{\mu\nu}(x)$, where x^μ is a local coordinate system on M. Take the familiar action describing a relativistic point particle of the mass $m > 0$ moving freely on the manifold M

$$I_0 = -m\int dt\sqrt{g_{\mu\nu}(x)\,\dot{x}^\mu\dot{x}^\nu}, \tag{2.14}$$

where t is a parameter along the trajectory $x^\mu(t)$ and $\dot{x}^\mu := dx^\mu/dt$. We assume that t increases monotonically, say, from $t=0$ to $t=T$, and that paths under consideration satisfy $g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu > 0$. It is known that, at the classical level, one can replace (2.14) by the quadratic action [70]

$$I = -\frac{1}{2}\int dt\Big[\frac{1}{\lambda}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \lambda m^2\Big] \tag{2.15}$$

with $\lambda = \lambda(t) > 0$ being a Lagrange multiplier. Indeed, if we substitute λ by using its equation of motion, the action I reduces to I_0. Like I_0, the action I is invariant under reparametrizations $t \longrightarrow f(t)$ with

$$\lambda(t) \longrightarrow \lambda'(f(t)) = \Big(\frac{df}{dt}\Big)^{-1}\lambda(t), \quad x^\mu(t) \longrightarrow x^{\mu\prime}(f(t)) = x^\mu(t),$$

where we assume $\dot{f}(t) > 0$ to preserve the monotonic property.

The Hamiltonian that corresponds to the action I is found to be

$$H = -\frac{\lambda}{2}(g^{\mu\nu}p_\mu p_\nu - m^2), \tag{2.16}$$

where p_μ is the momentum conjugate to x^μ. Since the momentum π conjugate to λ vanishes, following Dirac's approach [1] to constrained systems we must have the consistency condition $\dot{\pi} = \{\pi,\, H\} \approx 0$. This leads to

$$\phi := g^{\mu\nu}p_\mu p_\nu - m^2 \approx 0, \tag{2.17}$$

i.e., the Hamiltonian (2.16) be zero. Being first class, the constraint (2.17) generates a local gauge symmetry, which is none other than the reparametrization of the system. Accordingly, the reduced phase space is given by factorizing the constrained surface with respect to the gauge symmetry.

Now we shall consider the case where M is the manifold of a semi-simple Lie group G, which possesses the nondegenerate metric

$$g_{\mu\nu}(x) := \mathrm{Tr}\Big(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\Big) \tag{2.18}$$

where $g = g(x) \in G$ is a group element. The 'Tr' in (2.18) is defined by the matrix trace 'tr' in some irreducible representation multiplied by a constant c, so as to provide an inner product $\langle X\,, Y\rangle := \mathrm{Tr}(XY) = c\mathrm{tr}(XY)$ with a proper sign in the Lie algebra $\mathcal{G}$ of the group. (The constant c possesses a typical scale factor, which we set to unity for brevity.) Choosing a basis $\{T_m\}$ in $\mathcal{G}$, we have the 'flat' metric in the Lie algebra, $\eta_{mn} := \langle T_m\,, T_n\rangle$, which is the metric in the tangent space on the group manifold. As usual, $X_m := \langle T_m\,, X\rangle$ for $X \in \mathcal{G}$ and the indices are raised/lowered as $X^m = \eta^{mn}X_n$ using the inverse η^{mn} of the metric η_{mn}, whence $\langle X\,, Y\rangle = \eta_{mn}X^mY^n$. In terms of the vielbein $e^m_{\ \mu} := \langle T^m\,, g^{-1}\partial_\mu g\rangle$ one has $g_{\mu\nu} = e^m_{\ \mu}e^n_{\ \nu}\eta_{mn}$.

With (2.18) the action (2.14) can be written as

$$I_0 = -m\int dt\sqrt{\mathrm{Tr}(g^{-1}\dot g)^2} \tag{2.19}$$

which is coordinate free and hence globally well-defined over the group manifold. The equations of motion derived from (2.19) are

$$\frac{d}{dt}\Big(\frac{g^{-1}\dot g}{\rho}\Big) = 0, \qquad \text{where} \qquad \rho := \sqrt{\mathrm{Tr}(g^{-1}\dot g)^2}. \tag{2.20}$$

Similarly, the action (2.15) admits the global form

$$I = -\frac{1}{2}\int dt\Big[\frac{1}{\lambda}\mathrm{Tr}(g^{-1}\dot g)^2 + \lambda m^2\Big]. \tag{2.21}$$

A salient feature of M being a group manifold is that, in addition to the reparametrization invariance, the system acquires a chiral invariance. In fact, both of the actions, (2.19) and (2.21), are manifestly invariant under the rigid left-right transformations,

$$g(x) \longrightarrow hg(x) \qquad g(x) \longrightarrow g(x)\widetilde{h} \tag{2.22}$$

for arbitrary elements h, $\widetilde{h} \in G$.

To provide a globally defined Hamiltonian description, let us recall the free Hamiltonian system that can be defined to a semi-simple Lie group G, that is, the system whose phase space $\mathcal{M}$ is given by the cotangent bundle [59], [71]

$$\mathcal{M} = T^*G \simeq G \times \mathcal{G} = \{(g, R)|\; g \in G,\; R \in \mathcal{G}\} \tag{2.23}$$

on which the symplectic 2-form is given by

$$\omega = d\theta \qquad \text{with,} \quad \theta = -\mathrm{Tr} R(g^{-1}dg). \tag{2.24}$$

The non-vanishing Poisson brackets derived from (2.24) are

$$\{R_m\,, R_n\} = f_{mn}{}^l R_l, \qquad \{R_m\,, g_{ij}\} = (gT_m)_{ij}, \tag{2.25}$$

where $f_{mn}{}^l$ are the structure constants appearing in the basis: $[T_m\,, T_n] = f_{mn}{}^l T_l$. The (total) Hamiltonian can be written as $H = -1/2\lambda\phi$ with λ a Lagrange multiplier, and

$$\phi = \mathrm{Tr} R^2 - m^2 \approx 0. \tag{2.26}$$

Then, we get the equations of motion

$$\dot{g} = \{g\,, H\} = \lambda g R, \qquad \dot{R} = \{R, H\} = 0 \tag{2.27}$$

and the constraint (2.26). Since this constraint together with the first equation of (2.27) imply $\lambda^2 = (\rho/m)^2$, the motion equations (2.27) reproduce (2.20).

The conserved 'right' current R appearing in (2.27) is in fact the Noether current associated with the global right symmetry in (2.22) for the action (2.19). Analogously, the 'left' current

$$L := -g\, R\, g^{-1} \tag{2.28}$$

is the conserved Noether current associated with the left symmetry in (2.22), which forms the Poisson brackets

$$\{L_m\,, L_n\} = f_{mn}{}^l L_l, \qquad \{L_m\,, g_{ij}\} = -(T_m g)_{ij} \tag{2.29}$$

and commutes with the right current, $\{L_m\,, R_n\} = 0$. Both of the two currents commute with the constraint (2.26) and are hence gauge invariant.

Although unnecessary so far in the present group manifold case, a local coordinate system may be useful when we wish to find a physical interpretation for the currents. Consider, for example, the normal coordinates [1]

$$g(x) = e^{x^m T_m}, \tag{2.30}$$

where x^m are the 'flat' coordinates specifying the position of the particle. Then, the momentum p_m conjugate to x^m reads

$$p_m = -(g^{-1}\partial_m g)_n\, R^n = (\partial_m g\, g^{-1})_n\, L^n. \tag{2.31}$$

If we now define the 'vector current' V_m by subtracting the two chiral currents, we get

$$V_m = \frac{1}{2}(L_m - R_m) = p_m + \mathcal{O}(x^2), \tag{2.32}$$

[1]This parametrization is available only for a neighbourhood of the identity $g = 1$, but this is not important for our purpose here.

where $\mathcal{O}(x^2)$ denotes a polynomial which is at least quadratic in x^m. This shows that in the vicinity of the origin $g = 1$ the vector current V_m reduces to p_m, but since V_m are conserved (while p_m are not), and since V_m are gauge invariant and survive the reduction, we may regard V_m as the 'momentum' (hence V_0 is the 'energy') of the particle in the chronological gauge $x^0(t) = t$. On the other hand, the 'axial vector current' A_m defined by adding the two currents becomes

$$A_m = \frac{1}{2}(L_m + R_m) = \frac{1}{2} f_{mn}{}^l x^n p_l + \mathcal{O}(x^2). \tag{2.33}$$

As we shall see shortly, for $G = SL(2, R)$ the current A_m will be interpreted as the generator of three dimensional Lorentz transformations (hence A_0 is the 'angular momentum'). The orthogonality $\langle V\,, A\rangle = 0$, which follows from (2.28), is consistent with this interpretation.

We wish to remark at this point on the general solution for the equations of motion (2.20). Thanks to the reparametrization invariance, the general solution can readily be found by choosing the invariant length for the parameter t so that $\rho = 1$. Indeed, the equations of motion (2.20) then reduce to $\frac{d}{dt}(g^{-1}\dot{g}) = 0$, which can be integrated at once to be $g(t) = g(0)e^{-tR/m}$, where $R \in \mathcal{G}$ is a constant satisfying (2.26). The general solution for (2.20) can be obtained simply by returning to the generic parameter by a reparametrization transformation $t \to f(t)$

$$g(t) = g(0)e^{-f(t)R/m}. \tag{2.34}$$

The constant R is in fact the conserved right current determined from the initial condition, $g(0)$ and $\dot{g}(0)$. (The solution can also be given in terms of the left current as $g(t) = e^{f(t)L/m}g(0)$.) Thus, in the normal coordinates (2.30) particle's trajectory is just a straight line for the initial condition $g(0) = 1$.

We now specialize to the case $G = SL(2, R)$ which is a three dimensional Lorentzian manifold isomorphic to $S^1 \times R^2$. We shall work with the following basis $\{T_m\}$ in the algebra $\mathcal{G} = sl(2, R)$

$$T_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.35}$$

Choosing $c = -1/2$, we find that the flat metric becomes

$$\eta_{mn} = \langle T_m\,, T_n\rangle = -\frac{1}{2}\mathrm{tr}(T_m T_n) = \mathrm{diag}\,(+1, -1, -1). \tag{2.36}$$

Since the basis elements satisfy the relation

$$T_m T_n = -\eta_{mn} \cdot 1 + \epsilon_{mn}{}^l T_l \tag{2.37}$$

with $\epsilon_{012} = +1$, we have for X, $Y \in sl(2, R)$ the useful formula

$$XY = -\langle X\,, Y\rangle \cdot 1 + \frac{1}{2}[X\,, Y]\,, \tag{2.38}$$

and, in particular, $XX = -|X|^2 \cdot 1$, where $|X|^2 := \langle X\,, X\rangle$. It is then easy to show that if we write $X = \alpha\hat{X}$ with a 'normalized' vector (*i.e.*, $|\hat{X}|^2 = \pm 1$ or 0), we have

$$e^X = \begin{cases} \cos\alpha \cdot 1 + \sin\alpha \cdot \hat{X}, & \text{if} \quad |\hat{X}|^2 = +1; \\ \cosh\alpha \cdot 1 + \sinh\alpha \cdot \hat{X}, & \text{if} \quad |\hat{X}|^2 = -1; \\ 1 + \alpha\hat{X}, & \text{if} \quad |\hat{X}|^2 = 0. \end{cases} \tag{2.39}$$

We note that the orthochronous Lorentz group $SO^{\uparrow}(2,1)$ in three dimensions is realized by the adjoint action of $SL(2,R)$

$$X \longrightarrow g\,X\,g^{-1} \quad \text{with} \quad g \in SL(2,R). \tag{2.40}$$

More explicitly, the transformations in components induced by the adjoint action (2.40) read

$$X_m \longrightarrow \Lambda_m^{\ n} X_n \quad \text{with} \quad \Lambda_m^{\ n} = \mathrm{Tr}(T_m g\,T^n g^{-1}), \tag{2.41}$$

where the matrices $\Lambda_m^{\ n}$ belong to the group $SO(2,1)$, whereas the property $\Lambda_0^{\ 0} \geq 1$ can be seen by a direct computation. Clearly, the axial vector current (2.33), which now takes the form $A_m = \epsilon_{mn}{}^{l} x^n p_l$, is the generator of the Lorentz transformation (2.41), and in particular A_0 is the angular momentum.

We now carry out the reduction of the Hamiltonian system explicitly by means of the constraint (2.26) in the $SL(2,R)$ case. The first point to be noted is that the reduced phase space $\mathcal{M}_{\mathrm{red}}$ splits up into two coadjoint orbits of the group. To see this, let us first write the variable $R \in \mathcal{G}$ in (2.23) used for the phase space $\mathcal{M}$ as

$$R = h^{-1} K\,h \qquad \text{with} \quad h \in G \quad K \in \mathcal{G}, \tag{2.42}$$

where K is some fixed vector. The parametrization (2.42) is based on the observation that any element in $\mathcal{G} = sl(2,R)$ can be reached from K by an $SO^{\uparrow}(2,1)$ transformation (2.40) with h, if we provide three types of K, that is, time-like $|K|^2 > 0$, space-like $|K|^2 < 0$ and null $|K|^2 = 0$. Since one can write $K = r\hat{K}$ with $r > 0$ and a normalized vector $\hat{K}$, one sees that the phase space $\mathcal{M}$ can be parametrized by the (redundant) set $\{g, h, r; s\}$, where $s := |\hat{K}|^2 = \pm 1$, 0 indicates the type of K. Substituting (2.42) back into (2.24) and renaming gh^{-1} as g, we obtain

$$\theta = \theta_K(g) + \theta_{-K}(h^{-1}), \quad \text{where} \quad \theta_K(g) := -\mathrm{Tr}K(g^{-1}dg). \tag{2.43}$$

If K is constant but not null, then θ_K is just the standard canonical 1-form associated with the coadjoint orbit $\mathcal{O}_K$ of the group G passing through K. But since the constraint (2.26) does indeed render K time-like constant with $r = m$, we see that the reduced phase space is given by the direct product of the two coadjoint orbits

$$\mathcal{M}_{\mathrm{red}} \simeq \mathcal{O}_K \times \mathcal{O}_{-K}, \tag{2.44}$$

where the symplectic structure is carried over to those on the orbits. Accordingly, natural variables parametrizing the reduced phase space $\mathcal{M}_{\rm red}$ are the currents on the coadjoint orbits

$$L = -g\, K\, g^{-1} \qquad \text{and} \qquad R = h^{-1} K\, h \tag{2.45}$$

which form independently an $sl(2,R)$ algebra under the Poisson brackets derived from (2.43).

Further, we point out that for $SL(2,R)$ one can express the symplectic 2-form $\omega_K = d\theta_K$ (or $\omega_{-K} = d\theta_{-K}$) for the coadjoint orbit solely in terms of the chiral current L (or R) in (2.43). For example, in terms of the left current the corresponding symplectic 2-form can be written as

$$\omega_K(g) = -\frac{1}{4m^2}\epsilon^{mnl} L_m dL_n \wedge dL_l. \tag{2.46}$$

To see this, we introduce a parameter $\delta \geq 0$ by $\langle \hat{K}\,,\hat{L}\rangle := -\cosh\delta$ with the normalized left current $\hat{L} := L/\sqrt{|L|^2}$, and construct the three vectors

$$\mathcal{T}_0 := \frac{\hat{K}-\hat{L}}{2\cosh(\delta/2)},\ \ \mathcal{T}_1 := -\frac{\hat{K}+\hat{L}}{2\sinh(\delta/2)},\ \ \mathcal{T}_2 := \frac{[\hat{K}\,,\hat{L}]}{2\sinh\delta}. \tag{2.47}$$

For fixed $\hat{K}$ and $\hat{L}$, these vectors form a new orthonormal basis of the $sl(2,R)$ algebra,

$$\langle \mathcal{T}_m\,,\mathcal{T}_n\rangle = \eta_{mn} \quad \text{and} \quad [\mathcal{T}_m\,,\mathcal{T}_n] = 2\,\epsilon_{mn}{}^{l}\mathcal{T}_l. \tag{2.48}$$

With this basis we consider the Euler angle representation of $SL(2,R)$ elements

$$g = g(\alpha,\beta,\gamma) = e^{\alpha\mathcal{T}_2}\, e^{\beta\mathcal{T}_0}\, e^{\gamma\mathcal{T}_2}. \tag{2.49}$$

Note that among the three parameters is a bounded one $0 \leq \beta < 2\pi$, which is the parameter in the cyclic direction S^1 of the group manifold $SL(2,R)$ (see (2.39)). Observe also that the Lorentz transformation on the vector $\hat{K}$ by the adjoint action of $g(\alpha) = e^{\alpha\mathcal{T}_2}$ is a 'rotation' in the plane spanned by $\hat{K}$ and $\hat{L}$

$$\hat{K} \longrightarrow g(\alpha)\,\hat{K}\,g^{-1}(\alpha) = \frac{\sinh(2\alpha+\delta)}{\sinh\delta}\hat{K} + \frac{\sinh 2\alpha}{\sinh\delta}\hat{L}. \tag{2.50}$$

One then finds that for $\alpha = -\delta/2$ the vector $\hat{K}$ is rotated to $-\hat{L}$, and for $\alpha = -\delta/4$ it is rotated halfway to $-L$, *i.e.*, it directs to $\mathcal{T}_0$. But since the parametrization (2.49) consists of two rotations of the type (2.50) with $g(\alpha)$ and $g(\gamma)$ interrupted by the rotation with $e^{\beta\mathcal{T}_0}$, the parameters fulfilling the relation $L = -gKg^{-1}$ in (2.45) are found to be

$$\alpha = \gamma = -\frac{\delta}{4} \quad \beta = \text{arbitrary}. \tag{2.51}$$

(The appearance of the free parameter β is expected from the counting of degrees of freedom: $SL(2,R)$ is three dimensional while its coadjoint orbit is two-dimensional for $m \neq 0$.) If we now express the canonical 1-form θ_K in (2.43) using (2.49) and (2.51), we get

$$\theta_K(g) = -md\beta + \frac{\langle [\hat{K}\,, L]\,, dL\rangle}{4(m - \langle \hat{K}\,, L\rangle)}. \tag{2.52}$$

Choosing, *e.g.*, $\hat{K} = -T_0$, we find that the corresponding symplectic 2-form ω_K is just the one given in (2.46). Note that from (2.45) this choice implies

$$L_0 > 0 \quad \text{and} \quad R_0 < 0, \tag{2.53}$$

that is, the left current lies in the coadjoint orbit given by an upper hyperboloid in the algebra $sl(2, R)$ whereas the right current lies in the coadjoint orbit given by a lower one.

2.3. Finite dimensional models of ED and YM theories. In this section we consider the finite-dimensional model with the $SU(2)$ gauge group of symmetry. It is difficult to obtain all gauge-invariant variables and therefore, we use the method described at the end of Section 2.1. The obtained structure of gauge-invariant variables is quite unexpected. For comparison, we also present the corresponding $U(1)$ model. These $U(1)$ and $SU(2)$ models can be considered as the toy models of electrodynamics and Yang-Mills theory (with matter), respectively. In the classical description, all 'fields' are assumed to be c-numbers.

A. The model with $U(1)$ symmetry

Let us consider the action

$$S = \int dt[\frac{i}{2}(\psi^*\dot{\psi} - \dot{\psi}^*\psi) - m\psi^*\psi + A_0(\psi^*\psi - kE) + E\dot{A} - \frac{1}{2}E^2], \tag{2.54}$$

where all 'fields' $(\psi^*, \psi, A_0, A, E)$ are functions only of time t; m and k $(k \neq 0)$ are parameters. The similarity to electrodynamics is apparent from the notation. At the same time, (2.54) has the form (2.1), where $A_0 \equiv \lambda(t)$ is a Lagrange multiplier, and $\phi \equiv \psi^*\psi - kE$ is a constraint (we use the time derivatives instead of 1-forms where it is convenient).

The nonzero Poisson brackets are

$$\{\psi, \psi^*\} = i, \quad \{E, A\} = 1$$

and we have the gauge transformations

$$\psi(t) \longrightarrow e^{+i\alpha(t)}\psi(t), \quad \psi^*(t) \longrightarrow e^{-i\alpha(t)}\psi^*(t)$$

and

$$A(t) \longrightarrow A(t) + k\alpha(t), \quad E(t) \longrightarrow E(t).$$

Then,

$$A_0(t) \longrightarrow A_0(t) + \dot{\alpha}(t)$$

provides the invariance of action (2.54).

The reduced system is two-dimensional, and two gauge-invariant variables can be chosen as follows

$$\Psi_{\mathrm{inv}} = e^{\frac{-i}{k}A}\psi \quad \Psi^*_{\mathrm{inv}} = e^{\frac{i}{k}A}\psi^* \tag{2.55}$$

(compare with (2.11)). Here the reduction procedure (2.6) is trivial and we find

$$\widetilde{S} = \int dt \left[\frac{i}{2}(\Psi^*_{\mathrm{inv}}\dot{\Psi}_{\mathrm{inv}} - \dot{\Psi}^*{}_{\mathrm{inv}}\Psi_{\mathrm{inv}}) - m\Psi^*_{\mathrm{inv}}\Psi_{\mathrm{inv}} - \frac{1}{k^2}(\Psi^*_{\mathrm{inv}}\Psi_{\mathrm{inv}})^2\right].$$

Thus, the 'gauge field' A vanishes and the physical 'excitations' are only the 'dressed fields' Ψ_{inv} (with 'four-fermion interactions').

This model has a simple generalization in the case of a multi-component gauge field $\vec{A}$ with the gauge transformations

$$\vec{A} \longrightarrow \vec{A} + \vec{k}\alpha,$$

where $\vec{k}$ are parameters ($\vec{k}^{\,2} \neq 0$). The gauge-invariant variable Ψ_{inv} is constructed similar to (2.55) (or, to (2.11)). Then, after reduction, the 'longitudinal' (to the $\vec{k}$) component of the gauge field $\vec{A}$ vanishes and the physical variables are the 'transverse' ones, together with the constructed 'dressed field' Ψ_{inv}. So, for these Abelian models, the structure of gauge-invariant variables is very similar to the physical observables in electrodynamics [18], [69].

B. The model with $SU(2)$ symmetry

For the model with $SU(2)$ gauge group of symmetry, we consider the action

$$S = \int dt \left[\frac{i}{2}(\psi^*_\alpha\dot{\psi}_\alpha - \dot{\psi}^*_\alpha\psi_\alpha) - m\psi^*_\alpha\psi_\alpha + \vec{A}_0(\vec{j} + \vec{J}) + \vec{E}\dot{\vec{A}} - \frac{1}{2}\vec{E}^2\right]. \tag{2.56}$$

Here ψ_α ($\alpha = 1, 2$) is the 2-component spinor, m is a parameter, $\vec{A}$ and $\vec{E}$ are 3-dimensional vectors, $\vec{A}_0$ are Lagrange multipliers, and the angular momenta $\vec{j}$ and $\vec{J}$ are given by

$$\vec{j} = \psi^*\frac{\vec{\sigma}}{2}\psi \quad \vec{J} = \vec{A} \times \vec{E}, \tag{2.57}$$

where $\vec{\sigma}$ are the standard Pauli matrices. The similarity with Yang–Mills theory is obvious.

The nonzero Poisson brackets are

$$\{\psi_\alpha, \psi^*_\beta\} = i\delta_{\alpha\beta} \quad \{E_m, A_n\} = \delta_{mn} \quad (m, n) = 1, 2, 3 \tag{2.58}$$

and the constraints $\vec{\phi} = \vec{j} - \vec{J}$ generate the gauge transformations

$$\psi \to \omega\psi, \quad \psi^* \to \psi^*\omega^{-1}, \quad A \to \omega A\omega^{-1}, \quad E \to \omega E\omega^{-1},$$

where $\omega(t) \in SU(2)$ and

$$A \equiv \frac{1}{2}\vec{A}\vec{\sigma}, \quad E \equiv \frac{1}{2}\vec{E}\vec{\sigma}. \tag{2.59}$$

Then, for $A_0 \equiv \frac{1}{2}\vec{A}_0\vec{\sigma}$, we get $A_0 \to \omega A_0\omega^{-1} - i\dot{\omega}\omega^{-1}$.

Any scalar product of the vectors $\vec{A}, \vec{E}, \vec{J}, \vec{j}$ will be gauge-invariant variable. But on the constraint surface ($\vec{j} + \vec{J} = 0$), only three of them are functionally independent. If we choose these independent gauge-invariant variables as

$$l_0 = \frac{1}{4}(\vec{A}^{\,2} + \vec{E}^{\,2}), \quad l_1 = \frac{1}{2}(\vec{E}\vec{A}), \quad l_2 = \frac{1}{4}(\vec{A}^{\,2} - \vec{E}^{\,2}), \tag{2.60}$$

then from (2.58) we obtain the $sl(2,R)$ algebra (see (1.32)):

$$\{l_\mu, l_\nu\} = \epsilon_{\mu\nu\sigma}g^{\sigma\rho}l_\rho, \quad \text{where} \quad g^{\mu\nu} = \text{diag}(+,-,-,). \tag{2.61}$$

Since there are three constraints, the physical phase space is four-dimensional. To construct the fourth gauge-invariant variable and find the complete symplectic structure, we use the method of Section 2.1 (see (2.12)).

For parametrization of the constraint surface, we introduce the new variables (j, Φ) and (h, φ)

$$\begin{aligned} j &= \frac{1}{2}(h_1 + h_2) \quad h = \frac{1}{2}(h_1 - h_2) \\ \Phi &= \varphi_1 + \varphi_2 \quad \varphi = \varphi_1 - \varphi_2, \end{aligned} \tag{2.62}$$

where

$$\psi_\alpha = \sqrt{h_\alpha}e^{-i\varphi_\alpha}, \quad \psi^*_\alpha = \sqrt{h_\alpha}e^{i\varphi_\alpha}, \quad (\alpha = 1, 2)$$

Then, for the 1-form, we have

$$\frac{i}{2}(\psi^*_\alpha d\psi_\alpha - \psi_\alpha d\psi^*_\alpha) = jd\Phi + hd\varphi. \tag{2.63}$$

The vector $\vec{j}$ (see (2.57)), in these new coordinates, takes the form

$$\vec{j} = \begin{pmatrix} \sqrt{j^2 - h^2}\cos\varphi \\ \sqrt{j^2 - h^2}\sin\varphi \\ h \end{pmatrix}$$

and $\vec{j}^{\,2} = j^2$. Note that on the constraint surface we have (see (2.60)) $l^\mu l_\mu = j^2/4$, and for fixed j, the commutation relations (2.61) define the well-known symplectic structure on this hyperboloid (see, e.g., [54]).

If we introduce the orthonormal basis ($\vec{e}_i \cdot \vec{e}_k = \delta_{ik}, \quad \vec{e}_i \times \vec{e}_j = \epsilon_{ijk}\vec{e}_k$):

$$\vec{e}_1 = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \quad \vec{e}_2 = -\frac{h}{j}\begin{pmatrix} \cos\varphi \\ \sin\varphi \\ -\frac{\sqrt{j^2-h^2}}{h} \end{pmatrix} \quad \vec{e}_3 = \frac{\vec{j}}{j},$$

then $\vec{A}$ and $\vec{E}$ can be parametrized as follows:

$$\vec{A} = \vec{e}_1 q_1 + \vec{e}_2 q_2, \quad \vec{E} = \vec{e}_1 p_1 + \vec{e}_2 p_2,$$

where

$$p_1 q_2 - p_2 q_1 = j. \tag{2.64}$$

Calculating the restricted 1-form $\vec{E}d\vec{A}|_{\mathcal{M}_c}$ in these new coordinates and using (2.64), we obtain

$$\vec{E}d\vec{A}|_{\mathcal{M}_c} = p_1 dq_1 + p_2 dq_2 - hd\varphi. \tag{2.65}$$

Comparing (2.63) and (2.65), we see that there is a cancellation of the 1-form $hd\varphi$. This means that the corresponding degree of freedom vanishes.

Now it is convenient to introduce the polar coordinates for the two-vectors (q_1, q_2) and (p_1, p_2):

$$\begin{aligned} q_1 &= r\cos\beta \quad p_1 = \rho\cos\gamma, \\ q_2 &= r\sin\beta \quad p_2 = \rho\sin\gamma. \end{aligned}$$

Three of them (r, ρ and $(\beta-\gamma)$) are connected with gauge-invariant variables (2.60)

$$r^2 = 2(l_0 + l_2) \equiv l_+, \quad \rho^2 = 2(l_0 - l_2) \equiv l_-, \quad r\rho\cos(\beta - \gamma) = 2l_1.$$

Using these relations, we finally get the reduced 1-form

$$\theta|_{\mathcal{M}_c} = jd\vartheta + l_1\frac{dl_+}{l_+} \quad \text{where} \quad \vartheta = \Phi - \beta. \tag{2.66}$$

So the coordinate $\vartheta = \Phi - \beta$ is the fourth gauge-invariant variable. Respectively, the reduced Hamiltonian takes the form

$$H|_{\mathcal{M}_c} = 2mj + \frac{j^2 + 4l_1^2}{l_+} \tag{2.67}$$

and this is the complete reduction.

Note that the second part of the reduced 1-form $l_1 d(\ln l_+)$ defines the above mentioned symplectic structure on the hyperboloid $l^\mu l_\mu = \frac{1}{4}j^2$.

We see that the physical picture of this reduced system differs from the corresponding Abelian case. Here, after the reduction procedure, we obtain the vanishing of the part of 'matter field' (ψ) degrees of freedom as well.

Geometric quantization [20] is a natural way of constructing the quantum theory of the reduced system (2.66)–(2.67), but, in principle, one can use the canonical quantization as well. For this purpose, it is convenient to introduce (global) 'creation' and 'annihilation' variables

$$a^+ = \sqrt{j}e^{i\vartheta}, \quad a = \sqrt{j}e^{-i\vartheta}, \tag{2.68}$$

and in quantum theory, we get the discrete eigenvalues for $j = a^+a$. Then, quantization of the system with the canonical 1-form $l_1 d(\ln l_+)$ and the Hamiltonian (2.67) (for the obtained discrete eigenvalues of j), gives the irreducible representations of $SL(2,R)$ group (see, e.g., [54]).

Next, from (2.62), we have the relation $N \equiv \psi^*_\alpha \psi_\alpha = 2j$. It is natural to interpret the corresponding operator ($\hat{N} \equiv 2\hat{j}$) as the ψ 'particle number' operator. In quantum theory, we have

$$[\hat{N}, \hat{a^+}] = 2\hat{a^+},$$

where $\hat{a^+}$ is a physical creation operator (2.68). So among the physical excitations (created by the operator $\hat{a^+}$) there are states with only even numbers of 'fermions'. This fact can also be seen from the structure of the variable a^+ (see (2.68) and (2.62)), since it has the phase factor $e^{i(\varphi_1+\varphi_2)}$. In the quantum case the corresponding operator creates (see [72]) the pairs of 'dressed' ψ-particles. Thus, the vanishing of 'matter field' degrees of freedom in the classical case can be interpreted as the confinement phenomenon of the corresponding quantum theory.

Note that for the similar finite-dimensional constrained systems, such 'confinement' like phenomenon has been derived by the 'first quantize and then reduce' method (see [73]). In that approach, the reduction of the extended states space by the conditions $\hat{\phi}_a|\Psi_{phys}\rangle = 0$ forbids states with certain quantum numbers.

2.4. Field theory model with non-Abelian gauge group. For the finite-dimensional models of the previous section, the gauge group G acts on the configuration space of 'gauge field' A and on the phase space of 'matter field' ψ. This is the standard situation for the Yang-Mills theory.

Using the notations of (2.59), we have

$$\vec{E}d\vec{A} = \langle E, dA\rangle, \tag{2.69}$$

where $\langle\ ,\ \rangle$ is a scalar product in the corresponding Lie algebra $\mathcal{G}$ (see the comments after (2.18)). Thus, the Lie algebra $\mathcal{G}$ can be interpreted as the configuration space of a 'gauge field' $\vec{A}$ and the corresponding cotangent bundle as the phase space.

If one takes a manifold of a semi-simple Lie group G as the configuration space, then there are the natural actions (left and right) of G on this manifold. One can similarly construct the gauge theory where the phase space is a cotangent bundle over G. Generators of the left and right transformation ($g \to \omega g,\ g \to g\omega$) are, respectively, the left and right currents (see (2.28)). Choosing the gauge transformation as the right action, we find that the constraints are $\phi \equiv R = 0$. Thus, the 'gauge field' part in the action takes the form (see (2.24)

$$\int \langle R, g^{-1}dg\rangle - (\langle \Lambda, R\rangle + H(R,g))dt, \tag{2.70}$$

where $\Lambda \in \mathcal{G}$ gives the Lagrange multipliers, H is the gauge-invariant Hamiltonian, and for convenience we change the sign of R.

The field theory generalization of (2.56) is the standard Yang-Mills theory. In this section, we consider the corresponding generalization of (2.70) with the action

$$S = \int dt \left[\int d^{D-1}\vec{x} \left(\sum_{k=1}^{D-1} \langle R_k, g_k^{-1}\dot{g}_k\rangle + e\langle A_0, \phi\rangle \right) - H \right], \tag{2.71}$$

where $g_k(\vec{x},t) \in G$; $R_k(\vec{x},t), A_0(\vec{x},t) \in \mathcal{G}$; H is a gauge-invariant Hamiltonian, A_0 are Lagrange multipliers, $\phi(\vec{x},t) \equiv e\sum_{k=1}^{D-1} R_k(\vec{x},t)$ are constraints, and e is the coupling constant (see below).

The '1-form' $\sum_{k=1}^{D-1}\langle R_k, g_k^{-1}dg_k\rangle$ defines the equal-time Poisson brackets (2.25)

$$\begin{aligned}
\{R_{k,a}(\vec{x}), R_{l,b}(\vec{y})\} &= \delta_{kl}\delta(\vec{x}-\vec{y})f_{ab}^c R_{k,c}(\vec{x}),\\
\{g_k(\vec{x}), R_{l,a}(\vec{y})\} &= \delta_{kl}\delta(\vec{x}-\vec{y})(g_k T_a(\vec{x})),\\
\{g_k(\vec{x}), g_l(\vec{y})\} &= 0,
\end{aligned} \tag{2.72}$$

where the set $\{T_a |\; T_a \in \mathcal{G}\}$ is some basis in the Lie algebra $\mathcal{G}$, and $R_a \equiv \langle T_a, R\rangle$ Thus, for the constraints $\phi_a \equiv \langle T_a, \phi\rangle$, we have

$$\{\phi_a(\vec{x}), \phi_b(\vec{y})\} = \delta(\vec{x}-\vec{y})f_{ab}^c\phi_c(\vec{x}) \tag{2.73}$$

The corresponding gauge transformations are

$$g_k \longrightarrow g_k\omega \quad R_k \longrightarrow \omega^{-1}R_k\omega \tag{2.74}$$

and one can easily construct the following gauge-invariant variables

$$g_{kl} = g_k g_l^{-1} \quad \text{and} \quad L_k = g_k R_k g_k^{-1}. \tag{2.75}$$

The Hamiltonian H in (2.71) is an arbitrary functional of such gauge-invariant variables.

Since Eq. (2.75) gives the sufficient number of gauge-invariant variables, we can use the scheme described in Section 2.1. The first nontrivial case is 3-dimensional space-time. If we introduce $g = g_1 g_2^{-1}$ as the ξ^μ variables, and R_1, R_2 and g_2 as the η variables of the scheme (see (2.12)), then for the '1-form' $\theta = \langle R_1, g_1^{-1}dg_1\rangle + \langle R_2, g_2^{-1}dg_2\rangle$ (where integration over R^2 is assumed), we immediately get

$$\theta = \langle R_1 + R_2, g_2^{-1}dg_2\rangle + \langle g_2 R_1 g_2^{-1}, g^{-1}dg\rangle$$

and after reduction we have

$$\theta|_{\mathcal{M}_c} = \langle r, g^{-1}dg\rangle, \tag{2.76}$$

where $r := g_2 R_1 g_2^{-1}$ is also gauge-invariant variable.

Thus, the structure of the 1-form is the same, and only the number of variables decreasing. One can verify this fact for other dimensions as well.

It is clear that the phase spaces of the systems with 1-forms (2.69) and (2.24) are essentially different, and there is no regular map between them. But in the field theory (where infinite number of such spaces exist), there is a nonlocal transformation (see [74-75])

$$A_k = \frac{1}{e} g_k^{-1}\partial_k g_k \quad E_k = -e g_k^{-1}\partial_k^{-1}(L_k) g_k \tag{2.77}$$

which transforms system (2.71) into the Yang-Mills theory with the same gauge group G. Indeed, from (2.71) and (2.77), one can see that

$$\phi = \sum_{k=1}^{D-1} \partial_k E_k + e[A_k, E_k] \quad \text{(Gauss law)}$$

and

$$\langle E_k, \dot{A}_k\rangle = \langle R_k, g_k^{-1}\dot{g}_k\rangle + \text{ (total derivatives)}. \tag{2.78}$$

To get the corresponding Hamiltonian of the Yang-Mills theory [76]

$$H = \frac{1}{2}\int d^{D-1}\vec{x}\left(\sum_{k=1}^{D-1}\langle E_k, E_k\rangle + \frac{1}{2}\sum_{k,l=1}^{D-1}\langle F_{kl}, F_{kl}\rangle\right)$$

with $F_{kl} = \partial_k A_l - \partial_l A_k + e[A_k, A_l]$, one must choose the following Hamiltonian

$$H = \frac{1}{2}\int d^{D-1}\left[e^2\langle\partial_k^{-1}L_k, \partial_k^{-1}L_k\rangle + \frac{1}{e^2}\langle\partial_k(g_{kl}\partial_l g_{lk}), \partial_k(g_{kl}\partial_l g_{lk})\rangle\right] \tag{2.79}$$

in (2.71). Thus, we can assume that the system (2.71), (2.79) is equivalent to the ordinary Yang-Mills theory with some boundary conditions (which allows us to invert (2.77) and to neglect the total derivatives in (2.78)).

The boundary behavior is a subtle problem even for simple models of the field theory (see, e.g., Appendix B). It is too complicated for the Yang-Mills theory and we do not consider it here.

Unfortunately, the complicated form of the Hamiltonian (2.79) is not simplified after the reduction procedure. For example, in the considered 3-dimensional case, the reduced Hamiltonian acquires the form

$$H = \frac{1}{2}\int d^2x\left[e^2\langle\partial_1^{-1}r, \partial_1^{-1}r\rangle + e^2\langle\partial_2^{-1}l, \partial_2^{-1}l\rangle + \right.$$
$$\left. + \frac{1}{e^2}\langle\partial_1(g\partial_2 g^{-1}), \partial_1(g\partial_2 g^{-1}\rangle\right], \tag{2.80}$$

where $l = grg^{-1}$.

Gribov's ambiguity problem [15] has stimulated many papers on the gauge-invariant description of the Yang-Mills theory. For the literature and new results on this problem see in [7]. The reduced system (2.76), (2.80) gives one of the possible versions of the Hamiltonian reduction for this theory. The main problem in such approaches is the complicated form of the Poincaré generators in terms of gauge-invariant variables [74-77]. For example, Hamiltonian (2.80) is nonlocal in fields and nonanalytical in the coupling constant e. So the standard perturbative quantization is not applicable here.

Note that the Hamiltonian (2.80) and the corresponding symplectic form was obtained in [74-75] by the Dirac bracket formalism.

3. Quantization

In this chapter, we consider general quantization problems. We analize the difficulties of canonical quantization and give the examples of non canonical quantization. The content of this chapter is based mainly on the papers [50], [54-55], but it uses the results of previous papers as well.

3.1. Quantization principles. The classical theory usually plays a role of some approximation to quantum theory, however, it is necessary to stress, that a consistent quantum theory exists not for arbitrary classical one. On the other hand, there is no smooth 'classical' limit ($\hbar \to 0$) for all predictions of quantum theory and the primary role of classical theory is to provide a framework for the interpretation of quantum theory, rather than its approximation. Such a point of view first was indicated by Dirac and he formulated correspondence principles of classical and quantum theories as a similarity between their mathematical structures.

According to Dirac's formulation [78], the quantization of a Hamiltonian system can be considered as a map $\mathcal{Q}$ of the classical observables $f(\xi), g(\xi)$, $h(\xi), \ldots$, into the Hermitian operators $\hat{f}, \hat{g}, \hat{h}, \ldots$, which act on some appropriate Hilbert space $\mathcal{H}$. We call corresponding operators the quantum observables. The map $\mathcal{Q}$ should satisfy the following principles:

P1. Linearity: if a and b are real numbers ($a, b \in \mathcal{R}$), then

$$\mathcal{Q}:\ (af(\xi) + bg(\xi)) \ \mapsto\ a\hat{f} + b\hat{g}.$$

P2. Mapping of the Poisson-Lie structure:

$$\mathcal{Q} : \{f, g\} \ \mapsto\ \frac{i}{\hbar}[\hat{f}, \hat{g}]. \tag{3.1}$$

P3. Mapping of the centre: constant functions are mapped into constant operators

$$\mathcal{Q} : f_0 \ \mapsto\ \hat{I},$$

where $f_0(\xi) \equiv 1$ and $\hat{I}$ is the unit operator.

P4. Irreducibility: any operator $\hat{C}$, which commutes with all operators $\mathcal{Q}\xi^i$ $(i = 1, \ldots, 2N)$ should be proportional to the unit operator

$$\hat{C} = c\hat{I}$$

with some number c.

Here we assume that all coordinates are global and therefore they are observables as well.

In the case of canonical variables p, q, the correspondence (3.1) gives the Heisenberg commutation relation

$$[\hat{q}, \hat{p}] = i\hbar\hat{I} \tag{3.2}$$

and the operators $\hat{q}, \hat{p}$ and $\hat{I}$ form the Heisenberg algebra. From the text books of quantum mechanics it is well known that the commutation relation (3.2) provides the Heisenberg uncertainty principle for the coordinate and momentum $(\delta p)(\delta q) \geq \hbar/2$.

We see that the Dirac's correspondence principles (P1-P4) preserves the Poisson-Lie structure of the classical system and this is a mathematical correspondence of the two theories. Of course, there is some physical motivation for these principles.

It is clear that the linearity principle P1 is related to the superposition principle of quantum mechanics, while the second and third principles are responsible for the realization of quantum uncertainty principle and symmetry group transformations of the quantum system. The principle P4 takes into account a number of physical degrees of freedom, and at the same time, it provides irreducibility of a representation of some fundamental transformation groups.

In quantum description, we use the Dirac's 'bra-ket' notations for vectors of Hilbert space, scalar product and mean values. Thus, $\langle\psi|\hat{f}|\psi\rangle$ denotes the mean value of the quantum observable $\hat{f}$ in the state $|\psi\rangle$.

Unfortunately, in the consistent quantization schemes one can satisfy all four correspondence principles (P1-P4) only for some restricted class of observables, and there is no possibility to extend these principles for all observables [79-80]. Nevertheless, it is interesting to note, that some of these principles can be satisfied for all observables if we refuse other ones. Then, restriction on the class of observables arises, when we try to satisfy all principles together.

For illustration, we consider the Hilbert space $\widetilde{\mathcal{H}} = \mathcal{L}_2(\mathcal{M})$ with the scalar product

$$\langle\Psi_2|\Psi_1\rangle = \int d\mu(\xi)\ \Psi_2^*(\xi)\Psi_1(\xi), \tag{3.3}$$

where $d\mu(\xi)$ is the invariant measure (1.43).

From the equation (1.19) we see that the operators

$$\mathcal{Q}f = -i\hbar V_f \tag{3.4}$$

satisfy the principles P1 and P2 automatically. However, by (3.4), the constant functions are mapped into the zero operators, and there is no irreducibility as well.

In the case of Example A (see (1.13)), the map (3.4) gives

$$\mathcal{Q}p = -i\hbar\partial_q \quad \mathcal{Q}q = i\hbar\partial_p$$

and instead of the Heisenberg algebra we get two commuting operators.

In 1960 Segal [81] proposed the following generalization of (3.4)

$$\mathcal{Q} : f \;\mapsto\; f(\xi) - \theta(V_f) - i\hbar V_f \equiv \hat{f}, \tag{3.5}$$

where $\theta(V_f)$ is the value of the 1-form $\theta = \theta_i(\xi)d\xi^i$ on the Hamiltonian vector field V_f (see (1.11) and (1.5)):

$$\theta(V_f) \equiv \theta_j(\xi)\omega^{jk}(\xi)\partial_k f(\xi).$$

Thus, the operator $\hat{f}$ is constructed from the invariant terms, and it does not depend on the choice of coordinates on $\mathcal{M}$.

We call the operators (3.5) the pre-quantization operators. One can check that the pre-quantization operators are Hermitian with respect to the scalar product (3.3), and they satisfy the principles P1-P3. Then, the problem is only with the irreducibility P4. Of course, this problem is related to the Hilbert space $\mathcal{L}_2(\mathcal{M})$ which is too large for the corresponding quantum system. It is clear that some reduction of $\mathcal{L}_2(\mathcal{M})$ is a natural way to get the irreducibility. The construction of pre-quantization operators and further reduction of the Hilbert space are the basic ideas of geometric quantization [20]. This method is quite general, and it can be used for the quantization of Hamiltonian systems with almost arbitrary symplectic manifolds. However, as it was mentioned in Introduction, despite a 20-years history, geometric quantization is still not popular among physicists due to a comparatively complicated mathematical formulation.

Recall that the standard canonical quantization is applicable only for the manifolds with the global cotangent bundle structure ($\mathcal{M} = T^*Q$) and the global canonical form (1.16). In that case we have a separation of all coordinates ξ^k ($k = 1, \ldots, 2N$) into two canonically conjugated parts. The first part is formed by the 'coordinates' (q^α) (of the configuration space Q), and the second by the corresponding 'momenta' (p_α) ($\alpha = 1, \ldots, N$). Since the latter are unbounded: $-\infty < p_\alpha < +\infty$, one can use the standard quntization rule

$$q^\alpha \longrightarrow \hat{q}^\alpha = q^\alpha, \quad p_\alpha \longrightarrow \hat{p}_\alpha = -i\hbar\frac{\partial}{\partial q^\alpha}. \tag{3.6}$$

According to Darboux's theorem, the canonical coordinates exist on an arbitrary symplectic manifold; but in general, such coordinates exist only locally [20], and there is no global cotangent bundle structure with unbounded momenta. Consistent quantization assumes some equivalence between the set of quantum mean values $Q_f \equiv \{\langle\psi|\hat{f}|\psi\rangle : |\psi\rangle \in \mathcal{H}, \langle\psi|\psi\rangle = 1\}$ and the set of corresponding classical values $C_f \equiv \{f(\xi) : \xi \in \mathcal{M}\}$ [2]. Therefore, in general, the rule (3.6) is not acceptable, since the spectrum of the first order differential operator is unbounded.

After the realization of canonical commutation relations (3.2) by the rule (3.6), the next step of the canonical quantization is a construction of other quantum observables. The corresponding map $\mathcal{Q}$ is given by

$$\mathcal{Q}: \; f(p,q) \; \mapsto \; f(\hat{p},\hat{q}) \tag{3.7}$$

and in general, due to the non-commutativity of the operators $\hat{p}$ and $\hat{q}$, we have to indicate some ordering rule in the classical function $f(p,q)$.

As it was mentioned above, all classical commutation relations cannot be mapped identically for the corresponding quantum observables. In particular, according the Groenwald-van Hove theorem [80]-[79] it is impossible to extend the classical Poisson-Lie algebra to polynomials in the momenta of degree more that two. But one can prove the following important

Lemma. *Let $\mathcal{O}_1$ be the class of observables $f(p,q)$ of the following form*

$$f(p,q) = A^k(q)p_k + B(q), \tag{3.8}$$

where $A^k(q)$ and $B(q)$ are arbitrary smooth functions. If we choose the ordering

$$\hat{f} = \frac{1}{2}\left[A^k(q)\hat{p}_k + \hat{p}_k A^k(q)\right] + B(q), \tag{3.9}$$

where $\hat{p}$ and $\hat{q}$ are the operators of canonical quantization (3.6)*, then:*
I. *$\mathcal{O}_1$ is the subalgebra of $\mathcal{O}(\mathcal{M})$.*
II. *The corresponding operators* (3.9) *are Hermitian.*
III. *There are no anomalies in the commutation relations of the operators* (3.9).

The statements I and II are almost trivial, and III can be checked by the direct calculation using the differential operators $\hat{p} = -i\hbar\partial_q$.

A certain choice of the ordering in (3.7) provides the realization of the classical commutation relations only for the restricted class of observables, and in general, there is no some selected ordering rule. The ordering freedom actually is the ambiguity for any quantum theory. It is clear that this ambiguity vanishes in the classical limit $\hbar \to 0$.

In the case of 'flat' phase space $\mathcal{M} = \mathcal{R}^{2N}$, some natural ordering is the Weyl's one, which implies a complete symmetrization of operators $\hat{p}$ and $\hat{q}$ in the corresponding polynomials [82-83].

Representation of the canonical commutation relations (3.2) is possible only in the infinite-dimensional linear space with unbounded operators.

[2] Q_f and C_f should be the same at least in the classical limit ($\hbar \to 0$).

Such operators have some domain of definition, and they cannot be extended to the whole Hilbert space $\mathcal{H}$. For example, in the 'flat' case, both momentum and coordinate operators in (3.2) are unbounded, and respectively this commutation relation have no sense on the whole Hilbert space $\mathcal{H}$.

With some natural assumptions on the domain of definition of the coordinate and momentum operators one can prove Stone - von-Neumann theorem ([84]-[85]), that all irreducible representations of the canonical commutation relations are unitary equivalent.

If the classical commutation relations (1.22) are preserved for the corresponding operators $\hat{h}_\alpha = \mathcal{Q}h_\alpha$, then these operators generate the group of unitary transformation given by

$$U_\hbar^\epsilon = \exp(\frac{i}{\hbar}\epsilon^\alpha \hat{h}_\alpha). \tag{3.10}$$

In the case of Example **a** (see (1.26)-(1.28)), we get the Weyl group [82] with the elements

$$U(\epsilon, P, Q) = \exp(\frac{i}{\hbar}\epsilon)\exp\left(\frac{i}{\hbar}(P\hat{q} - Q\hat{p})\right), \tag{3.11}$$

where we use the parametrization $\epsilon^0 := \epsilon$, $\epsilon^1 := -Q$, $\epsilon^0 := P$. To obtain the multiplication rule for the Weyl group, one can use Weyl's formula [3]

$$e^{\hat{f}}\ e^{\hat{g}} = e^{\hat{f}+\hat{g}}\ e^{\frac{1}{2}[\hat{f},\hat{g}]} \tag{3.12}$$

which is true when

$$[\hat{f}, [\hat{f}, \hat{g}]] = 0, \quad [\hat{g}, [\hat{f}, \hat{g}]] = 0$$

If we introduce the one parameter transformations $U_1(Q)$ and $U_2(P)$

$$U_1(Q) := \exp\,(-\frac{i}{\hbar}Q\hat{p}) \quad U_2(P) := \exp\,(\frac{i}{\hbar}P\hat{q}), \tag{3.13}$$

then from (3.12) we get the relation of non-commutativity

$$U_2(P)\ U_1(Q) = \exp\,(\frac{i}{\hbar}PQ)U_1(Q)U_2(P) \tag{3.14}$$

which actually defines the multiplication rule for the whole Weyl group

$$\begin{aligned} U(\epsilon_1, P_1, Q_1)U(\epsilon_2, P_2, Q_2) = \\ = U(\epsilon_1 + \epsilon_2 + \frac{1}{2}(P_1Q_2 - P_2Q_1), P_1 + P_2, Q_1 + Q_2). \end{aligned} \tag{3.15}$$

This group can be considered as the 'quantum deformation' of the commutative translation group on the plane (see (1.28)).

[3] Generalization of (3.12) for arbitrary case is the Backer-Cambel-Hausdorf formula [66].

In the classical case, the translation group is generated by the commuting operators (see (1.5))

$$V_p = \partial_q, \quad \text{and} \quad V_q = -\partial_p.$$

Non-commutativity for the Weyl group comes from the Heisenberg algebra (3.2) which is related to the Heisenberg uncertainty principle. Thus, we can summarize that:

a) The basic physical principles for quantum theory are the superposition and uncertainty principles;

b) Dirac's postulates P1-P4 are the mathematical realization of these principles;

c) as a result, we have the Heisenberg algebra and then, for the transformations of quantum system we get the Weyl group.

But one can postulate in another way too: we can take a definition of the Weyl group (3.15) (or of the non-commutativity (3.14)) as the primary principle, and consider the Heisenberg commutation relations (3.2) as the consequence.

This idea can be generalized, and one can consider quantization of a classical system as a suitable unitary representation of the corresponding symmetry group [28-29]. This representation should take into account peculiarities of the phase space and a possible quantum deformation, which is responsible for the uncertainty principle. Necessity for such generalization arises for the systems without the global canonical structure when one cannot apply the ordinary canonical method.

Note that the elements of the Weyl group are the unitary operators and they are well defined for arbitrary vector of the Hilbert space $\mathcal{H}$. Respectively, the non-commutativity relations (3.14) are valid globally on the whole Hilbert space, while the canonical commutation relation (3.2) has no meaning for some vectors of $\mathcal{H}$.

Unfortunately, mathematical principles of quantum mechanics cannot be tested directly, and, by experiments, we usually verify only some consequence of these principles. At the end of this section we consider a possible test for the non-commutativity relation (3.14).

Note that the Weyl group is connected with the transformations of the inertial frames. Indeed, the transformation $U_1(Q) = U(0,0,Q)$ corresponds to the translation in the coordinate space, while the operator $U_2(P))$ can be interpreted as the Galiley transformation (at $t = 0$) to the inertial frame with the velocity $v = P/m$, where m is a mass of a particle. In the classical theory these two transformations commute. This means that two observers, the first moving in the translated system and the second translated in the moving system, see the classical particle in the same state. Note that for a relativistic particle indicated transformations are non-commuting and, in principle, this effect is measurable.

In quantum mechanics we also have the non-commutativity given by (3.14). Then, the wave functions in the indicated classically equivalent frames differ by the phase factor $\exp(\frac{i}{\hbar}PQ)$. Of course, a possible measurement of this effect could be a good test for the non-commutativity relation (3.14) and respectively, for the Weyl group too.

3.2. Examples of non-canonical quantization.

Quantization on a cylindre

The translation group on a cylindre is given by (1.33). We denote the corresponding quantum transformations by $U_1(\phi)$ and $U_2(S)$ where, for convenience, we denote the group parameters by ϕ and S. Formally, these operators are analogous to (3.13)

$$U_1(\phi) \sim \exp(-\frac{i}{\hbar}\phi\hat{S}) \quad U_2(S) \sim \exp(\frac{i}{\hbar}S\hat{\varphi}). \tag{3.16}$$

But, now, we are not introducing the operators $\hat{S}$, $\hat{\varphi}$, and the operators $U_1(\phi)$, $U_2(S)$ are the primary ones.

The quantum non-commutativity rule is postulated by

$$U_2(S)\ U_1(\phi) = \exp(\frac{i}{\hbar}S\phi)U_1(\phi)U_2(S). \tag{3.17}$$

Then, consecutive actions of the operators U_1 and U_2 define the group of transformations. Any element of this group can be written in the form

$$U(\epsilon, S, \phi) = \exp(\frac{i}{\hbar}\epsilon)U_1(\phi)\ U_2(S)$$

and the equation (3.17) gives a multiplication rule for this group (similarly to (3.14)-(3.15)).

The transformations (3.16) are assumed to be unitary and we have the natural conditions

$$U_1(0) = \hat{I} = U_2(0). \tag{3.18}$$

Taking into account the periodicity in ϕ (see (1.35)), we impose the condition

$$U_1(\phi + 2\pi) = U_1(\phi). \tag{3.19}$$

Then, from the equations (3.17)-(3.19) we obtain

$$\exp(\frac{i}{\hbar}2\pi S) = 1 \quad \Longleftrightarrow \quad S = n\hbar,$$

Thus, we conclude that the quantum non-commutativity rule (3.17) restricts the possible values of the parameter S for the quantum transformations $U_2(S)$, and the equations (3.17)-(3.19) can be satisfied simultaneously only if $S/\hbar$ is integer.

One can construct the following irreducible representations of (3.17) with natural physical interpretations:

i) The Hilbert space L_2 is given by the ortho-normal basis $|n\rangle$ $(n \in Z)$, and the transformations $U_1(\phi)$ and $U_2(S)$ act on this basis in the following way:

$$U_1(\phi)|n\rangle = e^{-in\phi}|n\rangle, \quad U_2(m\hbar)|n\rangle = |n+m\rangle. \tag{3.20}$$

ii) The Hilbert space is the space of square-integrable functions on the circle $\mathcal{L}_2(\mathcal{S}^1)$, and the action of the operators $U_1(\phi)$ and $U_2(m\hbar)$ on the function $\psi(\varphi) \in \mathcal{L}_2(\mathcal{S}^1)$ is given by

$$U_1(\phi)\psi(\varphi) = \psi(\varphi - \phi), \quad U_2(m\hbar)\psi(\varphi) = e^{im\varphi}\psi(\varphi). \tag{3.21}$$

One can check that both (3.20) and (3.21) give the realization of non-commutativity (3.17), and the conditions (3.18)-(3.19) are satisfied as well.

The unitary equivalence of the representations i) and ii) is given by the maps $(\psi(\varphi) \leftrightarrow \psi_n)$

$$\psi(\varphi) = \sum_n \frac{e^{in\varphi}}{\sqrt{2\pi}}\psi_n, \quad \psi_n = \int d\varphi\, \frac{e^{-in\varphi}}{\sqrt{2\pi}}\psi(\varphi),$$

where ψ_n are the coefficients of the expansion for the state $|\psi\rangle \in L_2$ in the basis $|n\rangle$

$$|\psi\rangle = \sum_n \psi_n |n\rangle.$$

Quantization on a torus

In the case of a torus (see Examples **C** and **d** in Chapter 1), there are two one-parameter unitary transformations $U_1(\eta_2)$ and $U_2(\eta_1)$, which are both periodic with periods a_2 and a_1, respectively

$$U_1(\eta_2 + a_2) = U_1(\eta_2), \quad U_2(\eta_1 + a_1) = U_2(\eta_1). \tag{3.22}$$

We postulate the non-commutativity rule

$$U_2(\eta_1)\; U_1(\eta_2) = \exp\,(\frac{i}{\hbar}\eta_1\eta_2)U_1(\eta_2)U_2(\eta_1) \tag{3.23}$$

and from (3.22)-(3.23) obtain that both parameters η_1 and η_2 are quantized

$$\eta_1 = \frac{2\pi\hbar n}{a_2} \quad \eta_2 = \frac{2\pi\hbar m}{a_1}, \tag{3.24}$$

where m and n are arbitrary integers. Using again the periodicity conditions (3.22), we find

$$a_1 a_2 = 2\pi\hbar N \tag{3.25}$$

with some fixed positive integer N. Therefore, the group of quantum transformations contains a finite number of elements. We also conclude that the consistent quantum theory on the torus requires quantization of the phase space volume by the condition (3.25). Since the parameters η_1 and η_2 are discrete and periodic, we can denote the quantum translations (3.22) by $U_1(m)$ and $U_2(n)$ $(m, n = 0, \ldots, N-1)$. Note that the condition (3.25) is equivalent to the Bohr-Sommerfeld quantization rule.

Now, similarly to the previous case, one can construct two N dimensional 'ξ^1' and 'ξ^2' representations:

i$'$). The ortho-normal basis of the first representation is formed by the vectors $|n\rangle_1$ $(n = 0, \ldots, N-1)$, and we define

$$U_1(m)|n\rangle_1 = e^{-\frac{2\pi i n m}{N}}|n\rangle_1, \quad U_2(m)|n\rangle_1 = |[n+m]_N\rangle_1, \tag{3.26}$$

where $[n+m]_N$ is the sum of numbers n and m with the module N.

ii$''$). The vectors $|n\rangle_2$ with $(n = 0, \ldots, N-1)$ form the basis of the second representation and

$$U_1(m)|n\rangle_2 = |[n+m]_N\rangle_2, \quad U_2(m)|n\rangle_2 = e^{\frac{2\pi i n m}{N}}|n\rangle_2. \tag{3.27}$$

These representations are unitary equivalent and the corresponding 'transition matrix' has the form

$${}_2\langle n|m\rangle_1 = \frac{e^{\frac{2\pi i m n}{N}}}{\sqrt{N}}.$$

Quantization of the reduced $SL(2,R)$ system

We are now going to discuss the quantization of the system considered in Section 2.2. It was a relativistic particle moving on $SL(2,R)$ group manifold. However, having seen that the reduced phase space consists of the two coadjoint orbits (2.44), the problem actually reduces to the quantization of the system of coadjoint orbits. In other words, the quantization amounts to finding unitary, irreducible representations of the algebra $sl(2,R)$ formed by the chiral currents on the coadjoint orbits, $\mathcal{O}_K$ and $\mathcal{O}_{-K}$. On account of the constraint (2.26) which requires the Casimir $Q = 1/4\mathrm{Tr}L^2$ to be the positive constant $m^2/4$, the irreducible representations [86] (see also, [87]) relevant for our purpose are the discrete series $D_j^{\pm}$ with $2j = 3, 4, \ldots$, for which $q = j(j-1) > 0$. Further, the conditions (2.53) require that the representations for the left sector should be given by D_j^+ while those for the right sector are D_j^-. A simple realization for these representations can be provided by the Holstein-Primakoff method, in which one uses creation/annihilation operators $[a\,, a^\dagger] = 1$ as a basic building block. For instance, for the left sector we have [53]

$$\begin{aligned} L_- &:= L_1 + iL_2 = 2\sqrt{a^\dagger a + 2j}\cdot a, \\ L_+ &:= L_1 - iL_2 = 2a^\dagger \cdot \sqrt{a^\dagger a + 2j}, \\ L_0 &:= 2(a^\dagger a + j). \end{aligned} \tag{3.28}$$

It is straightforward to check that the left current given in (3.29) satisfies the constraint (2.26) as well as the (quantum) commutation relations

$$[L_m\,, L_n] = 2i\,\epsilon_{mn}{}^l L_l. \tag{3.29}$$

In the familiar Fock space consisting of the states $|n_L\rangle$ for $n_L = 0, 1, 2, \ldots$ with

$$a|n_L\rangle = \sqrt{n}_L\,|n_L - 1\rangle, \quad a^\dagger|n_L\rangle = \sqrt{n_L + 1}\,|n_L + 1\rangle, \tag{3.30}$$

we find

$$\begin{aligned} L_-\,|n_L\rangle &= 2\sqrt{(n_L - 1 + 2j)n_L}\,|n_L - 1\rangle, \\ L_+\,|n_L\rangle &= 2\sqrt{(n_L + 2j)(n_L + 1)}\,|n_L + 1\rangle, \\ L_0\,|n_L\rangle &= 2(n_L + j)\,|n_L\rangle. \end{aligned} \tag{3.31}$$

Analogously, one can construct representations for the right sector using another pair of creation/annihilation operators for the right current. Actually, this is equivalent to the formal replacement $\{L_+, L_-, L_0\} \to \{-R_-, -R_+, -R_0\}$ in the above construction, which leads to the Fock space consisting of $|n_R\rangle$ for $n_R = 0, 1, 2, \ldots$, for which

$$\begin{aligned} R_-\,|n_R\rangle &= -2\sqrt{(n_R + 2j)(n_R + 1)}\,|n_R + 1\rangle, \\ R_+\,|n_R\rangle &= -2\sqrt{(n_R - 1 + 2j)n_R}\,|n_R - 1\rangle, \\ R_0\,|n_R\rangle &= -2(n_R + j)\,|n_R\rangle. \end{aligned} \tag{3.32}$$

The full Hilbert space is spanned by the states given by the direct product of the two representations, D_j^+ and D_j^-, sharing the same value for the Casimir. The states are thus labeled by the two integers, $|n_L\,, n_R\rangle = |n_L\rangle \otimes |n_R\rangle$, on which the energy V_0 in (2.32) and the angular momentum A_0 in (2.33) act as

$$\begin{aligned} V_0\,|n_L\,, n_R\rangle &= (n_L + n_R + 2j)\,|n_L\,, n_R\rangle, \\ A_0\,|n_L\,, n_R\rangle &= (n_L - n_R)\,|n_L\,, n_R\rangle. \end{aligned} \tag{3.33}$$

The above result shows that the energy levels are positive definite and spaced integrally, which is in fact expected because of our identification of $x^0 \in S^1$ being 'time', while the angular momentum takes integer values only. The allowed mass of the particle at the quantum level is

$$m = 2\sqrt{j(j-1)} \quad \text{with} \quad 2j = 3, 4, \ldots\,. \tag{3.34}$$

As we have seen in Section 2.2, the basic ingredient underlying the simplicity of the quantization is the chiral split of the reduced system, that is, the split into two coadjoint orbits. In this respect, it is worth mentioning that essentially the same split was discussed (for compact groups) in [88] for the system of the cotangent bundle. This suggests that the Hamiltonian reduction and the subsequent quantization considered for $SL(2, R)$ may be generalized to any higher rank group G with the simplicity intact, by specifying all the Casimir elements of the group in the form of constraints. Whether this yields a physically interesting model or not is however unclear except for $G = SL(2, R)$.

3.3. Roots of the phase operators. The problem of polar-decomposition of the creation and annihilation operators $(\hat{a}^+, \hat{a})$ of a harmonic oscillator goes back to Dirac [89]. First we consider some aspects of this problem (for more details see 72).

Let H be the oscillator hamiltonian

$$H = \frac{1}{2}(p^2 + q^2), \tag{3.35}$$

where p and q are the canonical coordinates ($\{p, q\} = 1$) on $\mathcal{R}^2$.

The polar angle φ is introduced by

$$p = -\sqrt{2H} \sin \varphi, \quad q = \sqrt{2H} \cos \varphi \tag{3.36}$$

and the complex variables a and a^*

$$a = \frac{q + ip}{\sqrt{2}}, \quad a^* = \frac{q - ip}{\sqrt{2}} \tag{3.37}$$

take the form

$$a = \sqrt{H} e^{-i\varphi}, \quad a^* = \sqrt{H} e^{i\varphi}. \tag{3.38}$$

Note that the complex variables a^*, a are the classical functions for the creation and annihilation operators $\hat{a}^+$, $\hat{a}$.

The variable φ is not an observable (it is only a local coordinate), but formally, the Hamiltonian H and the polar angle φ are canonical coordinates since from (3.36) we have

$$\omega = dp \wedge dq = dH \wedge d\varphi. \tag{3.39}$$

The functions $e^{\pm i\varphi}$ are correctly defined on the whole phase space $\mathcal{R}^2$, except the origin, and for the Poisson brackets with the Hamiltonian (3.35) we have

$$\{H, e^{\pm i\varphi}\} = \pm i e^{\pm i\varphi}. \tag{3.40}$$

In the quantum case we choose the normal ordering of the Hamilton operator and define it by

$$\hat{H}_0 = \hat{a}^+ \hat{a} \equiv \frac{\hat{p}^2 + \hat{q}^2}{2} - \frac{\hat{I}}{2}, \tag{3.41}$$

where $\hat{I}$ is the unit operator and, for convenience, we choose the units with $\hbar = 1$.

It is well known that the eigenvalues of $\hat{H}_0$ are nonnegative integers. The corresponding eigenvectors $|n\rangle$ $(n = 0, 1, 2, \dots)$

$$\hat{H}|n\rangle = n|n\rangle \tag{3.42}$$

form the basis of the Hilbert space and satisfy (3.30). For further convenience, this Hilbert space we denote by $\mathcal{H}_B$ (the bosonic space), and we consider the operator $\hat{H}_0$ as the particles (bosonic) number operator:

Let us introduce the operators

$$\hat{E}_-|n\rangle = \begin{cases} |n-1\rangle, & n > 0 \\ 0, & n = 0 \end{cases} \quad \hat{E}_+|n\rangle = |n+1\rangle. \tag{3.43}$$

Then, from (3.30) and (3.42) we get

$$\hat{E}_- = \frac{1}{\sqrt{\hat{H}_0 + \hat{I}}}\hat{a} \quad \hat{E}_+ = \hat{a}^* \frac{1}{\sqrt{\hat{H}_0 + \hat{I}}}, \tag{3.44}$$

where the operator function $\sqrt{\hat{H}_0 + \hat{I}}$ acts on the basis vectors (3.42) as a diagonal operator

$$\frac{1}{\sqrt{\hat{H}_0 + \hat{I}}}|n\rangle = \frac{1}{\sqrt{n+1}}|n\rangle.$$

Comparing (3.43) and (3.38), we can assert that $e^{\pm i\varphi}$ are the classical function of the operators $\hat{E}_\pm$. We call these operators the phase operators.

Note also that the definition (3.43) gives

$$[\hat{H}_0, \hat{E}_\pm] = \pm \hat{E}_\pm$$

and these commutation relations are equivalent to the corresponding classical ones (see (3.40)).

From the definition (3.43), it is clear that the phase operators are mutually conjugate, and they satisfy only the one-side unitary relations

$$\hat{E}_+\hat{E}_- = \hat{I} - |0\rangle\langle 0| \quad \hat{E}_-\hat{E}_+ = \hat{I}, \tag{3.45}$$

where $|0\rangle\langle 0|$ is the projection operator on the vacuum state.

Now we consider the square root of the phase operators $\hat{E}_\pm^{1/2}$. The corresponding classical functions $\exp(\pm i\varphi/2)$ are singular and multi-valued, but formally, from (3.39) we have

$$\{H, e^{\pm i\varphi/2}\} = \pm(i/2)e^{\pm i\varphi/2},$$

which in the quantum case should take the form

$$[\hat{H}_0, \hat{E}_\pm^{1/2}] = \pm(1/2)\hat{E}_\pm^{1/2}. \tag{3.46}$$

Then, the operators $\hat{E}_\pm^{1/2}$ must decrease (or increase) by 1/2 the levels of the oscillator Hamiltonian. However, since such states are absent in the spectrum, relations (3.46) cannot be realized on the Hilbert space $\mathcal{H}_B$.

It is well-known that Dirac's equation describes fermions and Dirac's operator $i\gamma^\mu\partial_\mu$ is connected with the square root of ∂ 'Alamber's operator ∂^2 [3]. Similarly we can try to connect the square root of the phase operators $\hat{E}_\pm^{1/2}$ with the fermionic operators.

For this purpose we introduce the fermionic operators $\hat{f}$ and $\hat{f}^+$ with the anticommutation relations

$$\hat{f}^2 = \hat{f}^{+\,2} = 0 \quad \hat{f}\,\hat{f}^+ + \hat{f}^+\hat{f} = \hat{I}. \tag{3.47}$$

These relations can be irreducibly represented in the two-dimensional space H_F with the basis vectors $|n\rangle_F$ $(n = 0, 1)$

$$\hat{f}|0\rangle_F = 0 \quad \hat{f}|1\rangle_F = |0\rangle_F \quad \hat{f}^+|0\rangle_F = |1\rangle_F \quad \hat{f}^+|1\rangle_F = 0.$$

For the fermionic number operator $\hat{N}_F = \hat{f}^+\hat{f}$ satisfies

$$[\hat{N}_F, \hat{f}] = -\hat{f}, \quad [\hat{N}_F, \hat{f}^+] = \hat{f}^+, \quad \hat{N}_F|n\rangle_F = n|n\rangle_F.$$

Now we consider the exterior product of the spaces $\mathcal{H}_B$ and H_F: $\mathcal{H}_{BF} := \mathcal{H}_B \otimes \mathrm{H}_F$ and introduce the new Hamiltonian $\hat{H}$

$$\hat{H} = \hat{H}_0 \otimes \hat{I}_F + (1/2)\hat{I}_B \otimes \hat{N}_F, \tag{3.48}$$

where $\hat{I}_B$ and $\hat{I}_F$ are the identity operators in the corresponding Hilbert spaces $\mathcal{H}_B$ and H_F.

To simplify the notation we replace the operators $\hat{H}_0 \otimes \hat{I}_F$ and $\hat{I}_B \otimes \hat{N}_F$ respectively by $\hat{H}_0$ and $\hat{N}_F$, i.e.,

$$\hat{H} = \hat{H}_0 + (1/2)\hat{N}_F. \tag{3.49}$$

The eigenstates of the new Hamiltonian $\hat{H}$ are characterized by two numbers $|n_B, n_F\rangle$,

$$\hat{H}|n_B, n_F\rangle = (n_B + \frac{1}{2}n_F)|n_B, n_F\rangle,$$

with $n_B = 0, 1, 2, \ldots;\ n_F = 0, 1$.

Let us introduce the operators $\hat{A}_\pm$

$$\hat{A}_+ = \hat{f}^+ + \hat{E}_+\hat{f} \quad \hat{A}_- = \hat{f} + \hat{E}_-\hat{f}^+. \tag{3.50}$$

Obviously the operator $\hat{A}_+$ increases by $1/2$ and the operator $\hat{A}_-$ decreases by $1/2$ every eigenvalue level of the operator $\hat{H}$ (except the vacuum state which is canceled by $\hat{A}_-$). Thus, the following relations are satisfied

$$[\hat{H}, \hat{A}_\pm] = \pm(1/2)\hat{A}_\pm. \tag{3.51}$$

Considering quadratic combinations of the operators $\hat{A}_\pm$, we obtain

$$\begin{gathered} (\hat{A}_+)^2 = \hat{E}_-, \quad (\hat{A}_-)^2 = \hat{E}_+, \\ \hat{A}_+\hat{A}_- = \hat{I} - |0,0\rangle\langle 0,0|, \quad \hat{A}_-\hat{A}_+ = \hat{I}. \end{gathered} \tag{3.52}$$

Then from (3.50)-(3.52) we conclude that the operators $\hat{A}^\pm$ can be considered as a definition of the operators $\hat{E}_\pm^{1/2}$

$$\hat{E}_\pm^{1/2} := \hat{A}_\pm.$$

The described scheme can be generalized for arbitrary integer k ($k > 2$) considering the k-th root of the phase operators

$$\hat{E}_{\pm}^{1/k}.$$

We introduce the k-dimensional unitary space H_k with the orthonormal basis

$$|0\rangle_k,\ |1\rangle_k,\ |2\rangle_k, \ldots, |k-1\rangle_k \tag{3.53}$$

and define the operators $\hat{f}_k$, $\hat{f}_k^{+}$ by

$$\hat{f}_k|n\rangle_k = \begin{cases} |n-1\rangle_k, & n \geq 1 \\ 0, & n = 0 \end{cases},$$

$$\hat{f}_k^{+}|n\rangle_k = \begin{cases} |n+1\rangle_k, & n \leq k-2 \\ 0, & n = k-1. \end{cases}$$

In the basis (3.53), the operators $\hat{f}_k$ and $\hat{f}_k^{+}$ have the representation

$$f_k = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad f_k^{+} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ . & . & . & \ldots & . & . \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}.$$

It is obvious that

$$(\hat{f}_k)^k = (\hat{f}_k^{+})^k = 0, \tag{3.54}$$

and one can verify that the k^2 'normally ordered' monomes $(\hat{f}_k^{+})^j(\hat{f}_k)^m$, where $0 \leq j, m \leq k-1$, give k^2 linearly independent matrices. Then, the corresponding operators form the basis in the space of linear operators on H_k, and any monome can be written as a linear combination of normally ordered ones. For example, $\hat{f}_k\hat{f}_k^{+}$ takes the form

$$\hat{f}_k\hat{f}_k^{+} = \hat{I} - (\hat{f}_k^{+})^{k-1}(\hat{f}_k)^{k-1} \tag{3.55}$$

which, together with (3.54) is a generalization of the commutation relations (3.47). One can obtain some other useful relations as well:

$$\sum_{j=0}^{k-1}(\hat{f})^j(\hat{f}^{+})^{k-1}(\hat{f})^{k-j-1} = \hat{I}; \quad (\hat{f})^j(\hat{f}^{+})^m = 0, \quad \text{if} \quad j-m \geq k. \tag{3.56}$$

We introduce the particle number operator $\hat{N}_k$ by

$$\hat{N}_k|n\rangle_k = n|n\rangle_k.$$

It satisfies the relations

$$[\hat{N}_k, \hat{f}_k] = -\hat{f}_k \quad [\hat{N}_k, \hat{f}_k^{+}] = \hat{f}_k^{+}$$

and can be written in the normal form

$$\hat{N}_k = \hat{f}_k^+ \hat{f}_k + (\hat{f}_k^+)^2 (\hat{f}_k)^2 + \cdots + (\hat{f}_k^+)^{k-1} (\hat{f}_k)^{k-1}.$$

Now let us consider the exterior product of the spaces $\mathcal{H}_B$ and H_k

$$\mathcal{H}_k = \mathcal{H}_B \otimes \mathrm{H}_k,$$

and define the Hamiltonian on $\mathcal{H}_k$

$$\hat{H} = \hat{H}_0 + \frac{1}{k} \hat{N}_k$$

where we use the above mentioned abbreviation (see (3.48)-(3.49).

The eigenvectors of the new Hamiltonian $\hat{H}$ can be characterized by two quantum numbers $|n_B, n_k\rangle$, where $n_B = 0, 1, 2, \ldots$; $\eta_k = 0, 1, 2, \ldots, k-1$ and we have

$$\hat{H} |n_B, n_k\rangle = (n_B + \frac{1}{k} n_k) |n_B, n_k\rangle. \tag{3.57}$$

The energy levels are non-degenerated and equidistant with the interval $1/k$.

Here, similarly to (3.50), we introduce the operators $\hat{A}_{\pm,k}$:

$$\begin{aligned} \hat{A}_{+,k} &= \hat{f}_k^+ + \hat{E}_+ (\hat{f}_k)^{k-1}, \\ \hat{A}_{-,k} &= \hat{f}_k + \hat{E}_- (\hat{f}_k^+)^{k-1}, \end{aligned} \tag{3.58}$$

This operators shift the energy levels by $1/k$ ($\hat{A}_{+,k}$ increases and $\hat{A}_{-,k}$ decreases) except the vacuum state, which is canceled by the operator $\hat{A}_{-,k}$. The operators $\hat{A}_{\pm,k}$ satisfy the commutation relations

$$[\hat{H}, \hat{A}_{\pm,k}] = \pm \frac{1}{k} \hat{A}_{\pm,k}.$$

Now, using (3.54)-(3.56), we obtain that

$$(\hat{A}_{+,k})^k = \hat{E}_+ \quad (\hat{A}_{-,k})^k = \hat{E}_-$$

and also

$$\hat{A}_{-,k} \hat{A}_{+,k} = \hat{I} \quad \hat{A}_{+,k} \hat{A}_{-,k} = \hat{I} - |0,0\rangle\langle 0,0|.$$

Summarizing our results, we can formulate.

Theorem. *k-th roots of the phase operators $\hat{E}_\pm^{1/k}$ are defined on the Hilbert space $\mathcal{H}_B \otimes H_k$ and have the form*

$$\hat{E}_\pm^{1/k} := \hat{A}_{\pm,k}.$$

The described scheme has some interesting physical interpretation connected with the geometry of the k-sheet surface. If we consider the map of the complex plane (z) to a k-sheet surface

$$a = \frac{z^k}{\sqrt{k(z^* z)^{k-1}}} \quad a^* = \frac{z^{*k}}{\sqrt{k(z^* z)^{k-1}}}$$

then the functions $e^{\pm i\varphi/k}$ (where φ is the polar angle for the variables a, a^*) are 'correctly' defined (see remark after the equation (3.39)).

We can consider the standard symplectic form on the plane z, z^*

$$\omega = dp \wedge dq = -idz^* \wedge dz,$$

and the Hamiltonian

$$h = \frac{1}{k} z^* z. \tag{3.59}$$

A spectrum for the corresponding quantum operator is non-degenerated and equidistant with the interval $1/k$.

The Hamiltonian (3.59) in (a, a^*) variables takes the form (3.35)

$$h(z^*, z) = H(a^*, a) = a^* a.$$

Then, comparing the spectra of the Hamiltonians (3.57) and (3.59), we can conclude that the above described quantization with 'k-statistics particles' [4] can be interpreted as the quantization on the k-sheet surface.

The case $k = 2$ is considered in [48] and [49], where the corresponding quantum system is used for the description of the internal degrees of freedom for a relativistic particle. The case of arbitrary k can be related to the anyon physics and fractional statistics.

4. E-Quantization Scheme

In this chapter we consider the general quantization scheme based on a extension of the phase space with further application of the constrained quantization technique. In our approach the problem of scalar product arises and we investigate this problem in Section 4.4. In the last section we use our quantization method for the realization of classical symmetries on the quantum level. This chapter is based mainly on the papers [51-53] and [57].

[4] In the case $k = 2$ we have the Fermi statistics.

4.1. Quantization on a cotangent bundle. As it was pointed out, the standard canonical quantization can be used only for the systems with the phase space having the cotangent bundle structure. We start with an arbitrary symplectic manifold $\mathcal{M}$. Only, for simplicity, we assume that the symplectic form ω is exact: $\omega = d\theta$ (1.11).

To generalize the canonical method, we introduce some auxiliary Hamiltonian system with the phase space $T^*\mathcal{M}$, where $T^*\mathcal{M}$ is the cotangent bundle over the symplectic manifold $\mathcal{M}$. The new system has $4N$ dimensions, and we choose the 1-form $\Theta = P_k d\xi^k$, where (P_k, ξ^k) are the standard coordinates on the cotangent bundle $T^*\mathcal{M}$: $P_k = P(\partial_{\xi^k})$. So, the coordinates P_k play the role of 'momenta', while the ξ^k are 'coordinates'. The corresponding symplectic form is canonical: $d\Theta = dP_k \wedge d\xi^k$, and for the Poisson brackets of the new system we have (compare with (1.12))

$$\{\xi^k, \xi^l\}_* = 0 = \{P_k, P_l\}_* \quad \{P_k, \xi^l\}_* = \delta_k^l. \tag{4.1}$$

The index $*$ is used to make difference between the Poisson brackets (1.2) and (4.1). Below we denote the initial system by M, and the extended new system by T^*M.

Let us introduce the vector field Φ ($\Phi \in V(\mathcal{M})$) as the solution of the equation

$$\omega(\Phi, \ \cdot\ ) = (\theta - P)\ (\cdot), \tag{4.2}$$

where $\omega(\Phi, \ \cdot\ )$ denotes the contraction of ω with Φ: $\omega(\Phi, \ \cdot\ )_l = \Phi^k \omega_{kl}$. Since the symplectic form ω is non-degenerated, the equation (4.2) defines the vector field Φ uniquely, and the components of this field are given by

$$\Phi^k = \omega^{kl}(P_l - \theta_l). \tag{4.3}$$

Respectively, we get the map $(T^*\mathcal{M} \mapsto V(\mathcal{M}))$ of the cotangent bundle $T^*\mathcal{M}$ to the space of vector fields on $\mathcal{M}$. Using this vector field Φ and some observable $f(\xi) \in \mathcal{O}(\mathcal{M})$, we can construct the function Φ_f on $T^*\mathcal{M}$

$$\Phi_f \equiv \Phi(f) = \Phi^k \partial_k f, \tag{4.4}$$

and from (4.3) we have

$$\Phi_f = \theta(V_f) - P(V_f), \tag{4.5}$$

where V_f is the Hamiltonian vector field (1.5).

The definition of the functions Φ_f by (4.4), at the same time, gives the map

$$\mathcal{O}(\mathcal{M}) \mapsto \mathcal{O}(T^*\mathcal{M})$$

of observables of the system M to a certain class of functions on $T^*\mathcal{M}$. Then, from (4.1)-(4.5) we obtain

$$\{\Phi_f, \Phi_g\}_* = -\{f, g\} - \Phi_{\{f,g\}} \quad \{\Phi_f, g\}_* = -\{f, g\}. \tag{4.6}$$

Note, that these commutation relations are written for the system T^*M, and here for the functions $\{f,g\}$ and g we use the same notation as for the corresponding observables on $\mathcal{M}$. Strictly, of course, we should distinguish between these functions. However, it is generally simpler not to do this except in case of possible confusion.

Now, let us introduce a new map from $\mathcal{O}(\mathcal{M})$ to $\mathcal{O}(T^*\mathcal{M})$

$$f \mapsto R_f \equiv f - \Phi_f \tag{4.7}$$

which in local coordinates (P_k, ξ^k) takes the form

$$R_f = f(\xi) + \partial_k f(\xi)\omega^{kl}(\xi)(P_l - \theta_l(\xi)). \tag{4.8}$$

For further construction, the key step is the following

Lemma. *The map* (4.7) *preserves the Poisson brackets*

$$\{R_f, R_g\}_* = R_{\{f,g\}}. \tag{4.9}$$

This lemma can be checked by the direct calculation, using the Poisson brackets (4.6).

Note that the change of the 1-form θ by an exact form dF: $\theta_k(\xi) \longrightarrow \theta_k(\xi) + \partial_k F(\xi)$ corresponds to the trivial canonical quantization of the system T^*M

$$P_k \longrightarrow P_k - \partial_k F(\xi) \tag{4.10}$$

generated by the function $F(\xi)$.

We choose the Hamiltonian of the extended system T^*M to be equal to R_H, where $H = H(\xi)$ is the initial Hamiltonian. Respectively, for the system T^*M the action (1.10) takes the form

$$S_{T^*\mathcal{M}} = \int [P_k(\xi)\dot{\xi}^k - R_H(P,\xi)]dt. \tag{4.11}$$

The system T^*M can be quantized by the scheme of canonical quantization. This means that the Hilbert space $\widetilde{\mathcal{H}}$ is the space of square integrable functions $\Psi(\xi)$ on $\mathcal{M}$: $\widetilde{\mathcal{H}} = \mathcal{L}_2(\mathcal{M})$. It is convenient to introduce the invariant measure (1.43), and we define the scalar product by (3.3).

For any function $f(\xi)$, according to the scheme of canonical quantization, we have the corresponding operator $\hat{f}$ which acts on a wave function $\Psi(\xi)$ as the multiplication by $f(\xi)$. Taking into account the remarks mentioned above (see after (4.6)), we use the same notation $f(\xi)$ for this operator $\hat{f}$ as well: $\hat{f} \equiv f(\xi)$.

Further, the rule (3.6) defines the Hermitian operators $\hat{P}_k$

$$\hat{P}_k = -i\hbar\partial_k - i\hbar\frac{\partial_k\omega(\xi)}{4\omega(\xi)}, \tag{4.12}$$

where the additional term, proportional to $\partial_k\omega$, arises from the measure (1.43) in (3.3).

Construction of Hermitian operators, in general, has an ambiguity connected with the ordering of coordinate and momentum operators in the functions of corresponding observables. In our case, the ordering problem is only for the term $\partial_k f\omega^{kl}P_l$ (see (4.8)). When the momentum operator is in the first degree, we can use the lemma mentioned in Section 3.1 (see (3.8)-(3.9)), and choose the following symmetric ordering

$$\partial_k f\omega^{kl}P_l \longrightarrow \frac{1}{2}(\partial_k f\omega^{kl}\hat{P}_l + \hat{P}_l\partial_k f\omega^{kl}). \tag{4.13}$$

Using the Jacobi identity (1.1), one can prove that

$$\partial_k\left(\sqrt{\omega(\xi)}\;\omega^{kl}(\xi)\right) = 0. \tag{4.14}$$

Then, from (4.12)-(4.14), we obtain

$$\hat{R}_f = f(\xi) - \theta_k\omega^{kl}\partial_l f - i\hbar\omega^{lk}\,\partial_l f\partial_k. \tag{4.15}$$

Since the operator ordering (4.13) avoids anomalies in the quantum commutation relations, from (4.9) we get

$$[\hat{R}_f, \hat{R}_g] = -i\hbar\hat{R}_{\{f,g\}} \tag{4.16}$$

and this is the most interesting point of the described quantization scheme on the cotangent bundle of a symplectic manifold.

Comparing (4.15) with (3.5), we find that the constructed operators (4.15) (which naturally arise in our scheme) are pre-quantization operators, and the relation (4.16) is a well known fact from the method of geometric quantization [20].

Note that any change of the 1-form θ by an exact form dF corresponds to the unitary transformation of the operators $\hat{R}_f$ (see (4.10))

$$\hat{R}_f \longrightarrow e^{-\frac{i}{\hbar}F(\xi)}\hat{R}_f e^{\frac{i}{\hbar}F(\xi)}.$$

After canonical quantization on the cotangent bundle $T^*\mathcal{M}$, our goal is to use this quantum theory for the quantization of the initial system M, and in the next section we consider the connection between these two systems.

4.2. Constraints on a cotangent bundle. Geometrically there is a standard projection ($\pi : T^*\mathcal{M} \to \mathcal{M}$) of the cotangent bundle $T^*\mathcal{M}$ to the initial phase space $\mathcal{M}$. To find the dynamical relation between these two systems, we introduce a constraint surface on the cotangent bundle $T^*\mathcal{M}$, and define it as the kernel of the mapping $T^*\mathcal{M} \to V(\mathcal{M})$ given by (4.2) and (4.3). It means that on the constraint surface the vector field Φ vanishes ($\Phi = 0$), and if we use the functions $\Phi_f(P,\xi)$ (see (4.4)), this surface can be written as

$$\Phi_f = 0, \quad \forall\; f(\xi) \in \mathcal{O}(\mathcal{M}). \tag{4.17}$$

From (4.6) and (4.7) we have

$$\{R_f, \Phi_g\}_* = \Phi_{\{f,g\}} \tag{4.18}$$

and we see, that (4.17), i.e., the constraint surface, is invariant under the canonical transformations generated by the functions R_f. In particular, it is invariant in dynamics generated by the Hamiltonian R_H.

In local coordinates the surface (4.17) can be written as

$$P_k - \theta_k(\xi) = 0. \tag{4.19}$$

(see (4.3)-(4.5)), and respectively, the momenta P_k are defined uniquely. Hence, the coordinates $\xi^k (k = 1, \dots, 2N)$ can be used for the parametrization of the constraint surface, and this surface is diffeomorphic to the manifold $\mathcal{M}$. Then, the reduction procedure gives (see (4.19) and (4.8))

$$P_k d\xi^k|_{\Phi=0} = \theta_k(\xi) d\xi^k \quad R_H|_{\Phi=0} = H(\xi),$$

and the action (4.11) of the system T^*M is reduced to (1.10). Thus, we conclude that the classical system T^*M restricted on the constraint surface $\Phi_f = 0$ is equivalent to the initial one.

To find the connection on the quantum level too, we have to quantize the system T^*M taking into account the constraints (4.17).

Before beginning the quantum part of the reduction scheme, let us note that the constraints (4.17) are written for an arbitrary observable $f(\xi)$, and since the constraint surface $\Phi = 0$ is $2N$-dimensional, only the finite number of those constraints are independent.

To select the independent constraints, we introduce the complete set of observables on $\mathcal{M}$. The set of observables $\{f_n(\xi) \in \mathcal{O}(\mathcal{M}); (n = 1, \dots, K)\}$ is called complete, if any observable $f(\xi) \in \mathcal{O}(\mathcal{M})$. can be expressed as a function of this set

$$f = \mathcal{F}(f_1, \dots, f_K) \tag{4.20}$$

It is clear that $K \geq 2N$, and we can choose the set with $K = 2N$ only for the manifolds with global coordinates. For $K > 2N$ there are some functional relations for the set $f_1, \dots, f_K$, and locally only 2N of these functions are independent. Then, from (4.4) and (4.20) we have

$$\Phi_f = \frac{\partial \mathcal{F}}{\partial f_n} \Phi_{f_n}, \tag{4.21}$$

and the constraints (4.17) for arbitrary f are equivalent to the K constraints

$$\Phi_{f_n} = 0 \quad (n = 1, \dots, K). \tag{4.22}$$

In particular, in the case of global coordinates we can introduce only the $2N$ constraints Φ_{f_n}, $(n = 1, \dots, 2N)$. If it is not specified, below we are assuming that the manifold $\mathcal{M}$ has global coordinates and a set of functions $f_1, \dots, f_{2N}$ is complete. Note that the constraint surface and the reduced classical system are independent on the choice of such complete set. Using (4.6), we see that on the constraint surface (4.17), the rank of the matrix $\{\Phi_{f_n}, \Phi_{f_m}\}_*$ is equal to $2N$. Therefore these constraints, in Dirac's classification, are the second class constraints.

For constrained dynamical systems there are, actually, two schemes of quantization (see introduction):

A. "First reduce and then quantize".

B. "First quantize and then reduce".

By the scheme A we are returning to the initial problem of quantization on the manifold $\mathcal{M}$. Therefore, it is natural to use the scheme B, especially as the first step of this scheme we have already accomplished.

To justify our strategy, it is necessary to show that the schemes A and B give equivalent quantum theories, when the system M is quantizable by the canonical method, and also it is worthwhile to have a certain general receipt for accounting for the constraints (4.22) on the quantum level.

According to the scheme B, the next step is the construction of Hermitian constraint operators. From (4.5) and (4.13), the operators

$$\hat{\Phi}_f = i\hbar V_f + \theta(V_f) \tag{4.23}$$

are Hermitian, and by direct calculation we obtain

$$[\hat{\Phi}_f, \hat{\Phi}_g] = i\hbar(\{f,g\} + \hat{\Phi}_{\{f,g\}}), \tag{4.24}$$

$$[\hat{R}_f, \hat{\Phi}_g] = -i\hbar\hat{\Phi}_{\{f,g\}}. \tag{4.25}$$

These commutators are quantum versions of the relations (4.6) and (4.18). As it was expected, there are no anomalies for them (see (4.16)).

Now we should make the reduction of the Hilbert space $\mathcal{L}_2(\mathcal{M})$ using the constraint operators (4.23) for some complete set of functions $f_1, \ldots, f_{2N}$. The reduced Hilbert space for the constrained systems is called the physical Hilbert space as well, and we denote it by $\mathcal{H}_{ph}$.

For systems with $2N$ second class constraints there is the following reduction procedure: one has to select a subset of N constraints $\hat{\Phi}_1, \ldots, \hat{\Phi}_N$, which can be treated as the first class constraints (independently from others), and then construct the physical Hilbert space $\mathcal{H}_{ph}$ from the states which satisfy the Dirac conditions $\hat{\Phi}_a|\Psi_{ph}\rangle = 0,\ a = 1, \ldots, N$. Note that we cannot put all constraints equal to zero in strong sense ($\hat{\Phi}_k|\Psi\rangle = 0,\ k = 1, \ldots, 2N$), since it is in contradiction with commutation relations of the second class constraints.

It is easy to see that, in our case, the first class constraints will be commuting (see (4.6)). Then, the described procedure implies the selection of N commuting observables $f_a,\ a = 1, \ldots, N; \{f_a, f_b\} = 0$, and further, solution of the differential equations

$$\hat{\Phi}_{f_a}\ \Psi_{ph}(\xi) = 0 \quad (a = 1, \ldots, N). \tag{4.26}$$

Construction of physical states by selection of N commuting observables is quite natural from the point of view of standard quantum mechanics, and we shall return to this point later.

The equations (4.26) are first order linear differential equations and, in principle, they can be explicitly integrated to describe the corresponding wave functions. But, usually, two significant problems arise at this stage of quantization: the first is connected with a scalar product for the physical states [14], and the second, with definition of the observable operators on these states.

For the first problem, the point is that the Dirac's conditions $\hat{\Phi}_a|\Psi_{ph}\rangle = 0$, in general, have no solutions in the Hilbert space where the first stage of quantization was accomplished.

In our case, the solutions of (4.26), as a rule, are not square integrable on $\mathcal{M}$ (usually they are generalized functions), and the scalar product (3.3) needs modification. On the other hand, a certain measure in scalar product defines the class of functions square integrable by this measure. Thus, a measure for the new scalar product and the class of solutions of (4.26) should be adjusted.

A possible solution of this problem is based on the introduction of complex constraints [91]. Note that classical observables $f(\xi)$ are assumed to be real functions on a phase space, but it is clear that the whole considered construction (except for the self-adjointness) can be naturally extended for complex-valued functions $(f(\xi) = f_1(\xi) + if_2(\xi))$ as well.

Using the remaining part of the constraints $\Phi_{f_{N+1}}, \ldots, \Phi_{f_{2N}}$, one can introduce constraints for the complex functions $Z_a = f_a + i\epsilon f_{N+a}$ and consider the equations

$$\hat{\Phi}_{Z_a}|\Psi_\epsilon\rangle = (\hat{\Phi}_{f_a} + i\epsilon\hat{\Phi}_{f_{N+a}})|\Psi_\epsilon\rangle = 0 \quad (a = 1, \ldots, N). \qquad (4.27)$$

Here, $1 \leq a \leq N$, $\{f_a, f_{N+a}\} \neq 0$ and ϵ is some real parameter. Sometimes we omit the index 'ph' for the physical vectors and the physical Hilbert space, and use the index ϵ only.

The condition (4.27) looks like Gupta-Bleuler quantization [92]-[93], and for normalizable solutions $|\Psi_\epsilon\rangle$ the mean values of corresponding constraints vanish

$$\langle\Psi_\epsilon|\hat{\Phi}_{f_a}|\Psi_\epsilon\rangle = 0 \quad \langle\Psi_\epsilon|\hat{\Phi}_{f_{N+a}}|\Psi_\epsilon\rangle = 0.$$

It turns out that solutions of (4.27) could be square integrable indeed, and then, they form some subspace of the Hilbert space $\mathcal{L}_2(\mathcal{M})$ (see the examples below). We denote the corresponding reduced physical Hilbert space by $\mathcal{H}_\epsilon$. We have $\Psi_\epsilon(\xi) \in \mathcal{H}_\epsilon \subset \mathcal{L}_2(\mathcal{M}) \subset \mathcal{L}_2^*(\mathcal{M})$, where $\mathcal{L}_2^*(\mathcal{M})$ is the space dual to the Hilbert space $\mathcal{L}_2(\mathcal{M})$. If we consider the physical states $|\Psi_\epsilon\rangle$ as the vectors of the dual space $\mathcal{L}_2^*(\mathcal{M})$, then the suitable choice of the norms $||\Psi_\epsilon||$, and some smooth dependence on the parameter ϵ can provide the existence of the limit

$$\lim_{\epsilon\to 0} |\Psi_\epsilon\rangle = |\Psi_{ph}\rangle,$$

where $|\Psi_{ph}\rangle \in \mathcal{H}_{ph} \subset \mathcal{L}_2^*(\mathcal{M})$. The obtained physical states $|\Psi_{ph}\rangle$ specify the class of solutions of (4.26), and the scalar product for them is defined by

$$\langle\Psi_{2ph}|\Psi_{1ph}\rangle = \lim_{\epsilon\to 0}\frac{\langle\Psi_{2\epsilon}|\Psi_{1\epsilon}\rangle}{||\Psi_{2\epsilon}||\ ||\Psi_{1\epsilon}||}, \tag{4.28}$$

where $|\Psi_{1ph}\rangle$ and $|\Psi_{2ph}\rangle$ are the limits of $|\Psi_{1\epsilon}\rangle$ and $|\Psi_{2\epsilon}\rangle$, respectively. More detailed consideration of this problem will be done in Section 4.4.

It should be noted that the choice of physical states by the conditions (4.26) is equivalent to the choice of real polarizations of geometric quantization, while the choice (4.27) gives the complex polarization [20].

The second above mentioned problem arises when the reduced Hilbert space is non-invariant under the action of some pre-quantization operator $\hat{R}_g$. From (4.26), the invariance condition has the form

$$[\hat{R}_g, \hat{\Phi}_{f_a}] = \sum_{b=1}^{N} d_a^b\hat{\Phi}_{f_b} \quad (1 \le a \le N) \tag{4.29}$$

and, using (4.25) we see that this condition is not fulfilled for arbitrary $g(\xi)$. Moreover, even if a pre-quantization operator acts invariantly on $\mathcal{H}_{ph}$, this operator can be non-Hermitian on $\mathcal{H}_{ph}$, when the latter is not a subspace of $\mathcal{L}_2(\mathcal{M})$ and the Hilbert structure is introduced additionally.

For the definition of the corresponding observable operator on the physical Hilbert space, one can deform the pre-quantization operator adding quadratic and higher powers of constraint operators (see Appendix C). Then, using the commutation relations (4.25) and (4.25), one can construct a new Hermitian operator which is invariant on the reduced Hilbert space. Of course, there are different possible deformations, and in general, they define different operators on the physical Hilbert space. In terms of usual canonical quantization, different deformations correspond to different operator ordering. This is the standard ambiguity of quantum theories which vanishes in the classical limit $\hbar \to 0$. Note that the corresponding deformed classical functions are indistinguishable on the constraint surface $\Phi_f = 0$.

We call the described quantization scheme E-quantization scheme. In the next section we consider the application of this scheme to two simple examples considered in Chapter 1. We use these examples also as a test for our approach as well.

4.3. Examples of E-quantization scheme. Example 1. Let $\mathcal{M}$ be a plane $\mathcal{M} \equiv \mathcal{R}^2$ with the standard coordinates $\xi^1 \equiv p$, $\xi^2 \equiv q$ and the symplectic form $\omega = dp \wedge dq$. The coordinates p and q are global and from (4.5) we get

$$\Phi_p = \frac{1}{2}p - P_q, \quad \Phi_q = \frac{1}{2}q + P_p,$$

where, for convenience, we choose the 1-form $\theta = \frac{1}{2}pdq - \frac{1}{2}qdp$. The corresponding constraint operators are

$$\hat{\Phi}_p = \frac{1}{2}p + i\hbar\partial_q, \quad \hat{\Phi}_q = \frac{1}{2}q - i\hbar\partial_p \tag{4.30}$$

and, according to (4.27), for the physical vectors $|\Psi_\epsilon\rangle$ we can write the equation

$$\left(\hat{\Phi}_q - i\epsilon\hat{\Phi}_p\right)|\Psi_\epsilon\rangle = 0. \tag{4.31}$$

The solution of (4.31) is

$$\Psi_\epsilon(p,q) = \exp{(-\frac{\epsilon p^2}{2\hbar})}\exp{(-\frac{ipq}{2\hbar})}\psi(q - i\epsilon p), \tag{4.32}$$

with an arbitrary function ψ. For the square integrability of these solutions the parameter ϵ should be positive ($\epsilon > 0$) and we can specify the class of functions ψ, for example, by

$$\psi(\xi) = \exp{(-\frac{\gamma\xi^2}{2})}P(\xi), \quad (\xi \equiv q - i\epsilon p), \tag{4.33}$$

where γ is some fixed positive parameter ($\gamma > 0$), and $P(\xi)$ is any polynomial. Then, for sufficiently small ϵ the functions (4.32) will be square integrable on the plane and they form the physical subspace $\mathcal{H}_\epsilon$, ($\mathcal{H}_\epsilon \in \mathcal{L}_2(\mathcal{R}^2)$).

To investigate the case $\epsilon = 0$, we consider the limit as $\epsilon \to 0$ in (4.32), and get

$$\Psi_{ph}(p,q) = \exp{(-\frac{ipq}{2\hbar})}\ \psi(q). \tag{4.34}$$

It is clear that these functions are not square integrable on the plane, but they are well-defined elements of the dual space $\Psi_{ph}(p,q) \in \mathcal{L}_2^*(\mathcal{R}^2)$. The functions (4.34) form the physical Hilbert space $\mathcal{H}_{ph}$, and they are solutions of (4.31) with $\epsilon = 0$ (see in the next section as well). Using the rule (4.28), we obtain

$$\langle\Psi_{2ph}|\Psi_{1ph}\rangle = \frac{1}{N_1 N_2}\int \psi_2^*(q)\psi_1(q)\ dq, \tag{4.35}$$

where

$$N_i^2 = \int |\psi_i(q)|^2\ dq, \quad (i = 1, 2;\ N_i > 0).$$

The action of the pre-quantization operators

$$\hat{R}_p = \frac{1}{2}p - i\hbar\partial_q, \quad \hat{R}_q = \frac{1}{2}q + i\hbar\partial_p \tag{4.36}$$

on the physical states (4.34) gives

$$\hat{R}_p\Psi_\epsilon(p,q) = \exp{(-\frac{ipq}{2\hbar})}\ (-i\hbar)\psi'(q), \quad \hat{R}_q\Psi_\epsilon(p,q) = \exp{(-\frac{ipq}{2\hbar})}\ q\psi(q). \tag{4.37}$$

Thus, from (4.35)-(4.37) we have the standard coordinate representation of quantum mechanics. Similarly, one can obtain the momentum representation in the limit as $\epsilon \to \infty$ with corresponding choice of a class of the solutions (4.33).

Let us consider the problem of observable operators on the physical Hilbert space $\mathcal{H}_{ph}$ with the vectors (4.34). It is easy to check that this space is invariant under the action of the pre-quantization operators $\hat{R}_f$, where $f(p,q) = pA(q) + U(q)$, with arbitrary $A(q)$ and $U(q)$. But it turns out that these operators $\hat{R}_f$ are Hermitian (with respect to the scalar product (4.35)) only for the constant function $A(q)$ ($A(q) = c$). Similarly, there is a problem of definition of kinetic energy operator, since the chosen $\mathcal{H}_{ph}$ is not invariant under the action of the corresponding pre-quantization operator [5]. These are the problems mentioned at the end of the previous section, and for the definition of the corresponding observable operators we can make appropriate deformations (see Appendix C). For example, the deformation of the pre-quantization operator of kinetic energy $E = p^2/2m$ by the quadratic term

$$\hat{R}_{p^2/2m} \to \hat{R}_{p^2/2m} + \frac{1}{2m}\hat{\Phi}_p^2 \equiv \hat{E}$$

gives that the corresponding operator $\hat{E}$ is well defined on $\mathcal{H}_{ph}$, and effectively it acts as the standard kinetic energy operator

$$\hat{E} : \psi(q) \mapsto -\frac{\hbar^2}{2m}\psi''(q).$$

Now we return to the physical subspace $\mathcal{H}_\epsilon$ with some fixed positive ϵ. In the complex coordinates

$$z = \frac{q + i\epsilon p}{\sqrt{2\epsilon\hbar}}, \quad z^* = \frac{q - i\epsilon p}{\sqrt{2\epsilon\hbar}} \tag{4.38}$$

the equation (4.31) takes the form

$$\left(\partial_z + \frac{z^*}{2}\right)\Psi_\epsilon(z, z^*) = 0, \tag{4.39}$$

and we obtain the solutions

$$\Psi_\epsilon = \exp\left(-\frac{1}{2}|z|^2\right)F(z^*), \tag{4.40}$$

where $F(z^*)$ is any holomorphic function of z^*. Comparing (4.40) and (4.32), we have $F(z^*) = \exp(1/2\ z^{*2})\ \psi(\sqrt{2\epsilon\hbar}\ z^*)$. The complex coordinates z and z^* (see (4.38)) are the classical functions of annihilation

[5] For this $\mathcal{H}_{ph}$ such problem have functions $f(p,q)$ containing the momentum p in second and higher degrees

and creation operators, respectively (see (3.37). The corresponding prequantization operators

$$\hat{R}_z = \frac{z}{2} + \partial_z^* \quad \hat{R}_{z^*} = \frac{z^*}{2} - \partial_z$$

act invariantly on the physical Hilbert space $\mathcal{H}_\epsilon$, and we have

$$\hat{R}_z \Psi_\epsilon = \exp\,(-\frac{1}{2}|z|^2)F'(z^*) \quad \hat{R}_{z*}\Psi_\epsilon = \exp\,(-\frac{1}{2}|z|^2)z^* F(z^*).$$

Thus, the reduction on $\mathcal{H}_\epsilon$ gives the holomorphic representation of quantum mechanics [94], and we see again that the quantum theory of E-quantization scheme is equivalent to the ordinary canonical one. The physical Hilbert spaces $\mathcal{H}_\epsilon$ for different ϵ are different subspaces of $\mathcal{L}_2(\mathcal{R}^2)$, but, due to the Stone - von-Neumann theorem [84]-[85], the corresponding representations of canonical commutation relations are unitary equivalent to each other.

Further, we have the standard quantum mechanical relation

$$\langle p, q; \epsilon|\Psi\rangle = \int dx\ \langle p, q; \epsilon|x\rangle\psi(x), \tag{4.41}$$

where $\psi(x) \equiv \langle x|\Psi\rangle$ is a wave function of the coordinate representation and $|p, q; \epsilon\rangle$ is a coherent state [95]

$$\hat{a}|p, q; \epsilon\rangle = \frac{q + i\epsilon p}{\sqrt{2\epsilon\hbar}}\ |p, q; \epsilon\rangle. \tag{4.42}$$

The 'matrix element' $\langle p, q; \epsilon|x\rangle$ is given by

$$\langle p, q; \epsilon|x\rangle = \left(\frac{1}{\pi\epsilon\hbar}\right)^{1/4} \exp\,(\frac{i}{2\hbar}pq)\exp\,(-\frac{i}{\hbar}px)\exp\,(-\frac{(x-q)^2}{2\epsilon\hbar}), \tag{4.43}$$

and from (4.41) and (4.43), we obtain

$$\lim_{\epsilon\to 0}\ \langle p, q; \epsilon|\Psi\rangle \left(\frac{1}{4\pi\epsilon\hbar}\right)^{1/4} = \exp\,(-\frac{ipq}{2\hbar})\ \psi(q). \tag{4.44}$$

It is well known that the matrix element $\langle p, q; \epsilon|\Psi\rangle$ is connected with the wave function of holomorphic representation (see [94-95])

$$\langle p, q; \epsilon|\Psi\rangle = \exp\,(-\frac{1}{2}|z|^2)F(z^*), \tag{4.45}$$

where the variables p, q and z, z^* are related by (4.38). On the other hand, from the equivalence of the holomorphic representation and E-quantization scheme, the wave function $\widetilde{\Psi}_\epsilon(p, q) \equiv \langle p, q; \epsilon|\Psi\rangle$ in (4.45) can be considered as a vector of the physical Hilbert space $\mathcal{H}_\epsilon$ (compare (4.40) and (4.45)). Then, (4.41) and (4.44) will be similar to (4.32) and (4.34), respectively. Only it should be noted that the two physical states $\Psi_\epsilon(p, q)$ and $\widetilde{\Psi}_\epsilon(p, q)$, constructed by the same function $\psi(q) \in \mathcal{L}_2(\mathcal{R}^1)$, are different ($\Psi_\epsilon(p, q) \neq \widetilde{\Psi}_\epsilon(p, q)$) (see (4.32) and (4.41)), and they coincide only in the limit $\epsilon \to$

0. This short remark indicates different possibilities of described limiting procedure (for more details see the next section).

Example 2. Let $\mathcal{M}$ be a cylinder $\mathcal{M} \equiv \mathcal{R}^1 \times \mathcal{S}^1$ with the coordinates $\xi^1 \equiv S \in \mathcal{R}^1$, $\xi^2 \equiv \varphi \in S^1$ and the symplectic form $\omega = dS \wedge d\varphi$ (see Examples **B** and **c** in Chapter 1).

Since the cylinder is a cotangent bundle over the circle, the canonical quantization for this model is realized on the space of square integrable functions $\psi(\varphi)$ on the circle ($\psi(\varphi) \in \mathcal{L}_2(\mathcal{S}^1)$). The quantization rule (3.6) gives

$$\hat{S}\ \psi(\varphi) = -i\hbar\partial_\varphi\psi(\varphi),$$
$$\hat{\cos}\,\varphi\ \psi(\varphi) = \cos\varphi\ \psi(\varphi), \quad \hat{\sin}\,\varphi\ \psi(\varphi) = \sin\varphi\ \psi(\varphi), \tag{4.46}$$

and the operator $\hat{S}$ has the discrete spectrum $S_n = n\hbar$, $(n \in Z)$, with the eigenfunctions $\psi_n(\varphi) = 1/\sqrt{2\pi}\ \exp{(in\varphi)}$ (see also (3.21)).

The coordinate φ is not global, and for the 1-form we choose $\theta = Sd\varphi$. The set of functions

$$f_1 = S, \quad f_2 = \cos\varphi, \quad f_3 = \sin\varphi$$

is complete (with the relation $f_2^2 + f_3^2 = 1$), and for the corresponding constraint operators we get

$$\hat{\Phi}_S = S + i\hbar\partial_\varphi, \quad \hat{\Phi}_{\cos\varphi} = i\hbar\sin\varphi\partial_S, \quad \hat{\Phi}_{\sin\varphi} = -i\hbar\cos\varphi\partial_S. \tag{4.47}$$

Note that there is a possibility to have a complete set of observables with only two functions. For example,

$$\widetilde{f}_1 = e^{S/\lambda}\cos\varphi, \quad \widetilde{f}_2 = e^{S/\lambda}\sin\varphi, \tag{4.48}$$

where λ is some constant parameter (with dimension of angular momentum). These functions are global coordinates on the cylinder and they map the cylinder to the plane without the origin: $(\widetilde{f}_1, \widetilde{f}_2) \in \mathcal{R}^2 - \{0\}$.

Using (4.47), we see that the wave functions $\psi(\varphi)$ of 'φ- representation' can be obtained in E-quantization scheme by

$$\hat{\Phi}_{\cos\varphi}\ \Psi_{ph}(S,\varphi) = 0 \quad \text{and (or)} \quad \hat{\Phi}_{\sin\varphi}\ \Psi_{ph}(S,\varphi) = 0. \tag{4.49}$$

But it is clear that these functions are not normalizable on the cylinder. The case of the condition

$$\hat{\Phi}_S\ \Psi_{ph}(S,\varphi) = 0, \tag{4.50}$$

is more complicated, since this equation has no global regular solutions. In the class of generalized functions, one can find solutions of the type

$$\Psi_{ph,n} = \delta(S - n\hbar)\exp{(in\varphi)}, \quad (n \in Z) \tag{4.51}$$

which obviously are not square integrable. To investigate these classes, we need a limiting procedure similar to Example **1**. We consider such a procedure in the next section and also in Chapter 5, where some motivation and generalization of the condition (4.27) will be done. Here, in the remaining part of this section, we construct the example of the physical Hilbert spaces as subspaces of $\mathcal{L}_2(\mathcal{R}^1 \times \mathcal{S}^1)$.

We introduce the complex coordinates related to (4.48)

$$z = \widetilde{f}_1 - i\widetilde{f}_2 = \exp(S/\lambda - i\varphi), \quad z^* = \widetilde{f}_1 + i\widetilde{f}_2 = \exp(S/\lambda + i\varphi) \quad (4.52)$$

and impose the condition (4.27) for $\epsilon = 1$: $\hat{\Phi}_{z^*}\,|\Psi_{ph}\rangle = 0$. The corresponding equation

$$\left(\partial_z + \frac{\lambda}{2\hbar}\frac{\log|z|}{z}\right)\Psi_{ph}(z, z^*) = 0, \quad (4.53)$$

has the solutions

$$\Psi_{ph}(z, z^*) = \exp\left(-\frac{\lambda}{2\hbar}(\log|z|)^2\right)\psi(z^*). \quad (4.54)$$

where $\psi(z^*)$ is any holomorphic function ($\partial_z\psi = 0$) on the plane without origin, and it has the expansion

$$\psi(z^*) = \sum_{n=-\infty}^{\infty} d_n z^{*n}$$

Respectively, in (S, φ) coordinates, the solution (4.54) takes the form

$$\Psi_{ph}(S, \varphi) = \sum_{n=-\infty}^{\infty} c_n \exp\left(-\frac{(S - n\hbar)^2}{2\lambda\hbar}\right)\exp(in\varphi) \quad (4.55)$$

with $c_n = d_n \exp(\hbar n^2/2\lambda)$, and the square integrability gives

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty. \quad (4.56)$$

We have the pre-quantization operator of angular momentum $\hat{R}_S = -i\hbar\partial_\varphi$. It is a well-defined operator on the physical subspace (4.55), and has the same non-degenerated spectrum as the operator $\hat{S}$ of the canonical quantization. Thus, we see the unitary equivalence of these two quantizations.

4.4. The problem of scalar product for the constrained systems. Let f, g be two non-commuting observables and $\hat{\Phi}_f$, $\hat{\Phi}_g$ the corresponding constraint operators (4.23). As it was mentioned, these operators are Hermitian on the Hilbert space $\mathcal{H} \equiv \mathcal{L}_2(\mathcal{M})$. Suppose that the equation (see (4.27))

$$(\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi_\epsilon\rangle = 0 \tag{4.57}$$

has normalizable solutions for any $\epsilon \in (0, \delta)$, where δ is a positive number. The solutions with fixed ϵ form a subspace $\mathcal{H}_\epsilon$ of the Hilbert space $\mathcal{H}$. We assume that each subspace can be represented as $\mathcal{H}_\epsilon = F_\epsilon \mathcal{H}_0$, where $\mathcal{H}_0$ is a linear space, and F_ϵ is a linear invertible map

$$F_\epsilon : \mathcal{H}_0 \to \mathcal{H}_\epsilon, \quad F_\epsilon^{-1} : \mathcal{H}_\epsilon \to \mathcal{H}_0. \tag{4.58}$$

In practical applications the linear space $\mathcal{H}_0$ automatically arises from the form of the general solution of (4.57); only it should be specified from the condition of square integrability of the corresponding functions $\Psi_\epsilon = F_\epsilon\psi$, where $\psi \in \mathcal{H}_0$. For example, in the case of the equation (4.31), the general solution (4.32) and (4.33) is described by the space of polynomials $P(\xi)$, and it can be interpreted as $\mathcal{H}_0$. The representation (4.41) and (4.45) of the same solutions is different, and in that case, the space $\mathcal{H}_0$ is obviously $\mathcal{L}_2(\mathcal{R}^1)$. As for the general solution (4.54) and (4.55), the space $\mathcal{H}_0$ is a space of Fourier modes c_n, $n \in Z$, with $\sum |c_n|^2 < \infty$ (see (4.56)).

The space of linear functionals on the Hilbert space $\mathcal{H}$ is called the dual (to $\mathcal{H}$) space, and we denote it by $\mathcal{H}^*$. From our definitions we have

$$\Psi_\epsilon = F_\epsilon\psi \in \mathcal{H}_\epsilon \subset \mathcal{H} \subset \mathcal{H}^*.$$

Suppose that the set of the vectors $F_\epsilon\psi$ with any fixed $\psi \in \mathcal{H}_0$ has the limit ($\epsilon \to 0$) in the dual space $\mathcal{H}^*$, and this limit defines the vector $\psi_* \in \mathcal{H}^*$

$$\lim_{\epsilon \to 0} F_\epsilon\psi = \psi_*. \tag{4.59}$$

Such linear functional ψ_* usually is unbounded, and the limit in (4.59) means that for any $\Psi \in \mathcal{H}$ we have

$$\lim_{\epsilon \to 0} \langle F_\epsilon\psi|\Psi\rangle = \psi_*(\Psi), \tag{4.60}$$

where $\psi_*(\Psi)$ denotes the value of the functional ψ^* on the corresponding vector $\Psi \in \mathcal{H}$. If we change the map F_ϵ by

$$F_\epsilon \to \widetilde{F}_\epsilon = a(\epsilon)F_\epsilon,$$

where $a(\epsilon)$ is some 'scalar' function of the parameter ϵ, then the new map $\widetilde{F}_\epsilon$ provides a representation of the subspace $\mathcal{H}_\epsilon$ in the same form: $\mathcal{H}_\epsilon = \widetilde{F}_\epsilon\mathcal{H}_0$. It is obvious that the existence of the limit in (4.59) essentially depends on the suitable choice of the normalizable function $a(\epsilon)$.

The action of some operator $\hat{O}$ on the functional ψ_* can be defined by

$$\hat{O}\psi_*(\Psi) \equiv \psi_*(\hat{O}^+\Psi), \tag{4.61}$$

where $\hat{O}^+$ is the Hermitian conjugated to $\hat{O}$.

The norm $||\Psi_\epsilon||$ of the vectors $\Psi_\epsilon = F_\epsilon\psi$, with fixed ψ, usually diverges when $\epsilon \to 0$, but if we assume that

$$\epsilon||F_\epsilon\psi|| \to 0, \tag{4.62}$$

then we can prove that ψ_* satisfies the equation $\hat{\Phi}_f\ \psi_* = 0$. Indeed, from (4.59)-(4.62) we have

$$\begin{aligned}\hat{\Phi}_f\,\psi_*(\Psi) = \psi_*(\hat{\Phi}_f\,\Psi) = \lim_{\epsilon\to 0}\langle F_\epsilon\psi|\hat{\Phi}_f\,\Psi\rangle = \\ = \lim_{\epsilon\to 0}\langle\Psi_\epsilon|\hat{\Phi}_f\,\Psi\rangle = \lim_{\epsilon\to 0} i\epsilon\langle\Psi_\epsilon|\hat{\Phi}_g\Psi\rangle = 0,\end{aligned}$$

where we take into account that the function $\Psi_\epsilon = F_\epsilon\psi$ satisfies (4.57). Thus, (4.59) defines the map $F_* : \mathcal{H}_0 \to \mathcal{H}^*$, and the corresponding functionals $\psi_* = F_*\psi$ satisfy the condition (4.26).

Further, let us assume that $F_*\psi \neq 0$, whenever $\psi \neq 0$. Then, the space $\mathcal{H}_{ph} \equiv F_*\mathcal{H}_0$, as a linear space, will be isomorphic to $\mathcal{H}_0$, and, respectively, isomorphic to each $\mathcal{H}_\epsilon$ as well (see (4.58)).

If for $\forall\ \epsilon_1, \epsilon_2 \in (0, \delta)$ the map

$$F_{\epsilon_2}F_{\epsilon_1}^{-1} : \mathcal{H}_{\epsilon_1} \to \mathcal{H}_{\epsilon_2} \tag{4.63}$$

is a unitary transformation, then one can introduce the Hilbert structure on $\mathcal{H}_0$ and $\mathcal{H}_{ph}$ by definition of the scalar product

$$\langle\psi_2|\psi_1\rangle \equiv \langle F_*\psi_2|F_*\psi_1\rangle \equiv \langle F_\epsilon\psi_2|F_\epsilon\psi_1\rangle \tag{4.64}$$

It is obvious that in the case of unitarity of transformations (4.63), the scalar product (4.64) is independent on the choice of the parameter ϵ, and the corresponding Hilbert structure is a natural one. But, in general, the transformation (4.63) is not unitary, and there is no special Hilbert structure on $\mathcal{H}_0$. Respectively, we have the problem for the scalar product on the space $\mathcal{H}_{ph}$, especially that corresponding functionals are unbounded and have the 'infinite norm' in the Hilbert space $\mathcal{H}$.

Note that for the general solutions (4.32) and (4.33), the corresponding transformation (4.63) is not the unitary one, while the general solution (4.40)-(4.41), (4.45) provides unitarity explicitly

$$\widetilde{\Psi}_{\epsilon_2}(p,q) = \int \frac{dpdq}{2\pi\hbar}\ \langle p,q;\epsilon_2|p',q';\epsilon_1\rangle\widetilde{\Psi}_{\epsilon_1}(p',q').$$

Below we describe some procedure for the solution of the scalar product problem in that general case too.

In ordinary quantum mechanics, a physical state is represented by a ray in a Hilbert space and all vectors on the same ray are physically indistinguishable. So, if we suppose that the vector $|\psi_*\rangle$ has some norm $||\psi_*||$, then the normalized vector

$$|\psi_*\rangle\rangle \equiv \frac{|\psi_*\rangle}{||\psi_*||} \tag{4.65}$$

describes the same physical state. It is just the scalar product of such normalized vectors that has the physical meaning. Up to the phase factor, this scalar product describes the 'angle' between the rays, and defines the probability amplitude.

We introduce the scalar product of such normalized vectors by

$$\langle\langle\psi_{2*}|\psi_{1*}\rangle\rangle \equiv \lim_{\epsilon\to 0}\frac{\langle\Psi_{2\epsilon}|\Psi_{1\epsilon}\rangle}{||\Psi_{2\epsilon}||\,||\Psi_{1\epsilon}||}, \tag{4.66}$$

where the limits of $|\Psi_{1\epsilon}\rangle$ and $|\Psi_{2\epsilon}\rangle$, respectively, are the functionals $|\psi_{1*}\rangle$ and $|\psi_{2*}\rangle$ (see (4.59)), and the latter are related to $|\psi_{1*}\rangle\rangle$ and $|\psi_{2*}\rangle\rangle$ by (4.65). When the limit (4.66) exists, it should define the scalar product of the normalized physical states. Then the scalar product for arbitrary vectors can be obtained uniquely up to a rescaling.

It is obvious that in the case of unitarity of transformations (4.63), the definitions of scalar product (4.64) and (4.66) are equivalent.

Note that the described scheme for the definition of the scalar product of physical states (4.26) can be generalized for other constrained systems as well.

4.5. Symmetry groups in E-quantization scheme. The commutation relations (1.22) and (1.24) define a symmetry for the classical system. The Dirac correspondence principle P2 (see (3.1)) is responsible for the representation of this symmetry on the quantum level. But, as it was mentioned in Chapter 3, in general, there are some problems for the realization of the correspondence principle P2. These problems are of both the technical and principal character [79]-[80].

Realization of the classical commutation relations in the scheme of canonical quantization essentially depends on the choice of operator ordering. For some simple examples one can achieve the non-anomalous quantization by a suitable choice of the ordering. For example, the lemma of Section 3.1 (see (3.8)-(3.9), provides such a choice for the wide class of observables $h_\alpha(p,q)$ which are linear in the momenta

$$h_\alpha(p,q) = A_\alpha^a(q)p_a + B_\alpha(q), \quad (a = 1,\ldots,N; \alpha = 1,\ldots,M).$$

Another example is the Weyl ordering for the quadratic generators

$$h_\alpha(p,q) = A_\alpha^{ab}p_ap_b + B_\alpha^{ab}p_aq_b + C_\alpha^{ab}q_aq_b$$

(see Example **b** in Section 1.2). The absence of anomalies in the corresponding commutation relations can be easily checked by the direct calculation. [6]

[6]Note that in the infinite-dimensional case the Weyl ordering of some quadratic observables is not a well defined one, and we usually use the normal ordering. The latter,

However, in more general cases there are no any special ordering rules for the preservation of the classical commutation relations, and sometimes anomalies are even unavoidable [79]-[80].

In this section we describe a practical scheme for the realization of the correspondence principle P2, using E-quantization scheme.

Let $h_\alpha(\xi)$ $(\alpha = 1,\ldots,M)$ be a set of functions with the commutation relations (1.22), where $C^\gamma_{\alpha\beta}$ are the structure constants of some Lie algebra. According to (4.16), the corresponding pre-quantization operators $\hat{R}_\alpha := \hat{R}_{h_\alpha}$ gives the representation of this algebra on the space $\mathcal{L}_2(\mathcal{M})$

$$[\hat{R}_\alpha, \hat{R}_\beta] = -i\hbar C^\gamma_{\alpha\beta}\hat{R}_\gamma .$$

But the Hilbert space $\mathcal{L}_2(\mathcal{M})$ is too large and we need a representation on the physical Hilbert space $\mathcal{H}_{ph}$. The latter is defined by the conditions (4.26) (or (4.27)) where the functions $f_a(\xi)$ should be commuting

$$\{f_a(\xi), f_b(\xi)\} = 0, \quad (a, b = 1,\ldots,N). \tag{4.67}$$

Obviously we get the non-anomalous quantization if the following two conditions are fulfilled:
a) The physical Hilbert space $\mathcal{H}_{ph}$ is invariant under the action of the pre-quantization operators $\hat{R}_\alpha$.
b) The physical Hilbert space $\mathcal{H}_{ph}$ is a subspace of $\mathcal{L}_2(\mathcal{M})$ (see the previous section).

The invariance condition a) is given by (4.29). Then, using (4.25) and (4.21), we obtain

$$\{h_\alpha, f_a\} = F_{\alpha,a}(f_1,\ldots,f_N), \quad (\alpha = 1,\ldots,M; a = 1,\ldots,N)$$

and since the functions $f_a(\xi)$ satisfy (4.67), we get

$$\{\{h_\alpha, f_a\}, f_b\} = 0, \quad (\alpha = 1,\ldots,M; a, b = 1,\ldots,N). \tag{4.68}$$

Thus, a given set of generators $h_\alpha(\xi)$ $(\alpha = 1,\ldots,M)$ defines the system of differential equations (4.67)-(4.68) for the functions $f_a(\xi)$ $(a = 1,\ldots,N)$. If this system has a solution, one has to investigate the corresponding conditions (4.26) to obtain the physical Hilbert space $\mathcal{H}_{ph}$. When $\mathcal{H}_{ph}$ is a subspace of $\mathcal{L}_2(\mathcal{M})$, we automatically arrive at the non-anomalous quantization. But if $\mathcal{H}_{ph}$ is not a subspace of $\mathcal{L}_2(\mathcal{M})$, then the further investigation is needed to check the Hermiticity of the pre-quantization operators $\hat{R}_\alpha$ with respect to the new scalar product on $\mathcal{H}_{ph}$.

For illustration we consider the rotation group on the sphere (see Examples **D** and **e** in Chapter 1). Poisson bracket (1.15) is equivalent to the

in general, possesses anomalies. For example, the conformal group generators give the Virasoro central extension for the corresponding algebra.

symplectic form (A.6), which is closed, but not exact. The corresponding 1-form exists only locally and in the cylindrical coordinates one can choose

$$\theta = (s - r)d\varphi \tag{4.69}$$

(see (A.5)). The functions (2.10) are generators of the rotation group and they satisfy the commutation relations (1.38).

Then, from the system (4.67)-(4.68) we get three equations for the one function $f_1(s, \varphi)$. The corresponding solution has the form

$$f_1(s, \varphi) = \left(\frac{r-s}{r+s}\right)^{1/2} e^{-i\varphi}, \tag{4.70}$$

and for the additional observable f_2, we have

$$f_2(s, \varphi) = \left(\frac{r-s}{r+s}\right)^{1/2} e^{i\varphi}. \tag{4.71}$$

Note that f_1 and f_2 are the complex coordinates on the plane obtained by the stereographic projection from the lower pole.

In standard notations $f_1 := z^*$ and $f_2 := z$, we get the following components of the 1-form

$$\theta_z = i\frac{rz^*}{1+zz^*}, \quad \theta_{z^*} = -i\frac{rz}{1+zz^*} \tag{4.72}$$

and, respectively,

$$\omega_{z^*z} = -\omega_{zz^*} = \frac{2ir}{(1+zz^*)^2}, \quad \sqrt{\omega} = \frac{2r}{(1+zz^*)^2}. \tag{4.73}$$

Then, the equation (4.26) for physical wave functions is

$$(\hbar\partial_z + \frac{rz^*}{1+zz^*})\Psi_{ph}(z, z^*) = 0, \tag{4.74}$$

and we obtain the solutions

$$\Psi_{ph} = (1+zz^*)^{-r/\hbar}\psi(z^*) \tag{4.75}$$

with an arbitrary holomorphic function $\psi(z^*)$. Using (4.73) and (1.43), we get the scalar product of the physical wave functions

$$\langle\Psi_{ph,2}|\Psi_{ph,2}\rangle = \int \frac{dz^*dz}{(1+zz^*)^{2+2r/\hbar}}\ \psi_2^*(z^*)\psi_1(z^*). \tag{4.76}$$

For the finiteness of the norm we get that the maximum degree K of the polynomial for the holomorphic function $\psi(z^*)$

$$\psi(z^*) = \sum_{n=0}^{K} c_n z^{*n} \tag{4.77}$$

is restricted by
$$K < 1 + \frac{2r}{\hbar}.$$

Further, we have the following pre-quantization operators for the rotation generators (2.10)

$$\hat{R}_+ := \hat{R}_1 + i\hat{R}_2 = rz + \hbar(\partial_{z^*} + z^2\partial_z),$$
$$\hat{R}_- := \hat{R}_1 - i\hat{R}_2 = rz^* - \hbar(\partial_z + {z^*}^2\partial_{z^*}), \quad \hat{R}_3 = r + \hbar(z\partial_z - z^*\partial_{z^*}),$$

and effectively their action on the holomorphic function $\psi(z^*)$ is given by

$$\hat{R}_+ : \psi(z^*) \mapsto \hbar z^*\psi'(z^*), \quad \hat{R}_- : \psi(z^*) \mapsto 2rz^*\psi(z^*) - \hbar {z^*}^2\psi'(z^*),$$
$$\hat{R}_3 : \psi(z^*) \mapsto r\psi(z^*) - \hbar z^*\psi'(z^*), \tag{4.78}$$

We see that the action of the operator $\hat{R}_-$ increases the degree of the polynomial in (4.77), and since the maximum degree is restricted by the number K, we should have
$$\hat{R}_- : {z^*}^K \mapsto 0.$$
Moreover, from (4.78) we find

$$r = \frac{K\hbar}{2}. \tag{4.79}$$

Summarizing, we conclude that a consistent quantum theory on the sphere arises only for the discrete values (4.79) of the radius r of the sphere. The corresponding physical Hilbert space is $K+1$ dimensional and in the basis ${z^*}^n$ $(n = 0, \ldots, K)$ we obtain the standard irreducible representation of the rotation group.

Note that the considered system on the sphere can be obtained by the reduction of the constrained system given in Appendix A. The corresponding generalization in the Minkowski space gives (see the end of Appendix A):

a) When $c > 0$ and $I_0 := s > O$,

$$\theta = (s-c)d\varphi \quad I_0 = s, \quad I_\pm = I_1 \pm iI_2 = \sqrt{s^2 - r^2}e^{\pm i\varphi} \tag{4.80}$$

b) When $c > 0$ and $I_0 := s < O$,

$$\theta = (s+c)d\varphi \quad I_0 = s, \quad I_\pm = I_1 \pm iI_2 = \sqrt{s^2 - r^2}e^{\pm i\varphi}.$$

c) When $c < 0$ and $I_0 := s$,

$$\theta = sd\varphi \quad I_0 = s, \quad I_\pm = I_1 \pm iI_2 = \sqrt{s^2 + r^2}e^{\pm i\varphi}.$$

For all these examples, the symplectic form ω have the same form (A.5) in the cylindrical coordinates. Only the domain of the variable s is different. One can repeat the described scheme for the cases a) and b) and respectively obtain the holomorphic representations of $SL(2,R)$ which are unitary equivalent to the discrete series $D_c^\pm$ [54], [87].

Note that for all three cases a)-c) the reduced phase space has the cotangent bundle structure with the canonical 1-form and the canonical quantization is applicable. For example, in the case a) (see (4.80)) we can fined the 'canonical' 1-form on the plane

$$\theta = (s-c)d\varphi = Hd\varphi = 1/2(pdq - qdp),$$

where $H := s-c \geq 0$, and the coordinates (H,φ) and (p,q) are connected by (3.36). Using the new 'flat' coordinates (p,q), one can check that the generators (4.80) take the form (1.40). When the dependence on the canonical variables (p,q) is such non-polynomial, it is difficult to choose the suitable ordering, but, in this case, one can guess the answer, using the form of the representation D_c^+

$$\hat{I}_0 = \hat{a}^+\hat{a} + c, \quad \hat{I}_+ = \hat{a}^+\sqrt{\hat{a}^+\hat{a} + 2c}, \quad \hat{I}_- = \sqrt{\hat{a}^+\hat{a} + 2c}\ \hat{a},$$

where $\hat{a}$ and $\hat{a}^+$ are the creation and annihilation operators (3.30). In this way we can obtain the Holstein-Primakof representation [67] (see (3.28), (3.31)).

5. Quantum Fluctuations and Uncertainties

This chapter is based on the recent paper [57]. Here we generalize a condition for the selection of admissible physical states. Then we find connection of E-quantization with the Berezin quantization and using the general coherent states. In the last section, we introduce the quantum distribution function and discuss its physical interpretation.

5.1. Minimal fluctuations of quantum constraints. In the case of Example **1** (see Section 4.3), the constraint operators $\hat{\Phi}_p$ and $\hat{\Phi}_q$ have the canonical commutation relations (see (4.30))

$$[\hat{\Phi}_p, \hat{\Phi}_q] = i\hbar. \tag{5.1}$$

Recall that due to quantum uncertainties, we cannot put $\hat{\Phi}_p|\Psi\rangle = 0$ and $\hat{\Phi}_q|\Psi\rangle = 0$ simultaneously. The condition (4.31) is equivalent to the choice of the physical states $|\Psi_\epsilon\rangle$ as the 'vacuum' states in Φ_p, Φ_q variables. Then the mean values of constraints are equal to zero:

$$\langle\Psi_\epsilon|\hat{\Phi}_p|\Psi_\epsilon\rangle = 0, \quad \langle\Psi_\epsilon|\hat{\Phi}_q|\Psi_\epsilon\rangle = 0,$$

and the product of quadratic fluctuations is minimal:

$$\langle\Psi_\epsilon|\hat{\Phi}_p^2|\Psi_\epsilon\rangle\langle\Psi_\epsilon|\hat{\Phi}_q^2|\Psi_\epsilon\rangle = \hbar^2/4.$$

Thus, for this simple example, the meaning of the condition (4.27) (or (4.57)) is that the obtained physical states $|\Psi_\epsilon\rangle$ provide the best realization of the classical constraints $\Phi_p = 0$, $\Phi_q = 0$ on the quantum level.

Let us consider the condition (4.57) in the general case. Note that if two functions f and g are canonically conjugated: $\{f, g\} = 1$, then the corresponding constraint operators have canonical commutation relations (see (4.25)). Therefore, for the construction of the physical states by (4.57), it is natural to choose the function g as canonically conjugated to f, and repeat the calculations of Example **1** in f, g variables. Unfortunately this simple procedure, in general, fails. The reason is that the canonically conjugated variable g usually exists only locally and the corresponding constraint Φ_g is not well defined both on classical and quantum levels. For example, canonically conjugated variable to the harmonic oscillator Hamiltonian $H = 1/2(p^2 + q^2)$ is the polar angle φ (see (3.36), (3.39)). Choosing the 1-form $\theta = 1/2(pdq - qdp) = H d\varphi$, we get

$$\hat{\Phi}_H = H + i\hbar\partial_\varphi ,$$

and for the operator $\hat{\Phi}_\varphi$ one can formally write $\hat{\Phi}_\varphi = -i\hbar\partial_H$, but this operator is not self-adjoint . Then, though the equation

$$(\hat{\Phi}_H + i\epsilon\hat{\Phi}_\varphi)|\Psi\rangle = 0, \tag{5.2}$$

has integrable solutions (for example, $\Psi(p,q) = \exp\left(-H^2/2\epsilon\hbar\right)$), nevertheless they are not acceptable for the physical states, since the mean values of the constraint operators $\hat{\Phi}_H$ and $\hat{\Phi}_\varphi$ do not vanish, and the minimization of quadratic fluctuations is not achieved as well.

For $\epsilon = 0$ one can write the formal solution of (5.2) (like (4.51)): $\Psi = \delta(H - \hbar n)\exp\left(in\varphi\right)$, and since $H \geq 0$, such 'solutions' exist only for $n \geq 0$. Then, the pre-quantization operator $\hat{R}_H = -i\hbar\partial_\varphi$ has the spectrum $H_n = \hbar n,\ n \geq 0$. The situation is similar for any completely integrable system [71]. In action-angle variables I_a, φ_a $(a = 1, \ldots, N)$ we have the 1-form $\theta = I_a d\varphi_a$ and the Hamiltonian $H = H(I_1, \ldots, I_N)$. Then, the constraint and pre-quantization operators take the form

$$\hat{\Phi}_{I_a} = I_a + i\hbar\partial_{\varphi_a} ,$$
$$\hat{R}_{I_a} = -i\hbar\partial_{\varphi_a}, \quad \hat{R}_H = H - \frac{\partial H}{\partial I_a}\hat{\Phi}_{I_a}. \tag{5.3}$$

If φ_a are the cyclic variables $(\varphi_a \in \mathcal{S}^1)$, then by described formal operations we obtain the 'physical states'

$$\Psi_{ph}(I,\varphi) = \prod_{a=1}^{N} \delta(I_a - \hbar n_a)\exp\left(in_a\varphi_a\right), \tag{5.4}$$

as the 'solutions' of the equations

$$\hat{\Phi}_{I_a}\Psi_{ph}(I,\varphi) = 0.$$

The spectra of pre-quantization operators (5.3) on these 'physical states' are

$$(I_a)_{n_a} = \hbar n_a \quad \text{and} \quad H_{n_1,\ldots,n_N} = H(\hbar n_1, \ldots, \hbar n_N),$$

where n_a are integer numbers, and the corresponding admissible values are chosen according to the possible classical values of the variables I_a (as, for example, $n \geq 0$ for the harmonic oscillator). It is remarkable that these formal results correspond to the quantization rule

$$I_a \Delta\varphi_a = \oint p_a dq_a = 2\pi\hbar n_a$$

which is almost the semi-classical one. From these formal operations it seems that the quantum problem is solvable for any completely integrable system; but of course, these expressions have only symbolic meaning, and (5.4) needs further specification, taking account of N other constraints and limiting procedure as well.

After these remarks, let us consider the case where the observables f and g in (4.57) are not canonically conjugated to each other.

It turns out that in general, the equation (4.57) has no normalizable solutions at all, and the choice of sign (or value) of ϵ does not help. If we intend to deal with arbitrary observables and symplectic manifolds, we have to generalize condition (4.27). For this purpose, we introduce the minimization principle for quadratic fluctuations of quantum constraints.

Quadratic fluctuations of two Hermitian operators $\hat{\Phi}_f$ and $\hat{\Phi}_g$ can be characterized by the functional $U(\Psi)$

$$U(\Psi) \equiv \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle, \tag{5.5}$$

where $|\Psi\rangle$ is a vector with the unit norm $\langle\Psi|\Psi\rangle = 1$.

Then, one can postulate the principle that the physical states provide minimization of this functional (5.5). For two arbitrary Hermitian operators, the minimization problem of uncertainties was studied in [96] and [72]. In those papers, the minimization problem was considered for another functional $U_1(\Psi)$

$$U_1(\Psi) \equiv \frac{\langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle}{\langle\Psi|\hat{A}|\Psi\rangle^2} \tag{5.6}$$

as well, where the operator $\hat{A}$ is the commutator

$$\hat{A} = -\frac{i}{\hbar}[\hat{\Phi}_f, \hat{\Phi}_g]. \tag{5.7}$$

In this section we consider only the functional $U(\Psi)$.

Using the results of [96] and [72] and taking variation of the functional (5.5), we get the equation for the physical wave functions $|\Psi_{ph}\rangle$ (see Appendix D)

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi_{ph}\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi_{ph}\rangle = |\Psi_{ph}\rangle \tag{5.8}$$

with the subsidiary conditions

$$a^2 = \langle\Psi_{ph}|\hat{\Phi}_f^2|\Psi_{ph}\rangle, \quad b^2 = \langle\Psi_{ph}|\hat{\Phi}_g^2|\Psi_{ph}\rangle, \tag{5.9}$$

where a and b are some fixed parameters. Possible values of these parameters are defined from the following procedure. At first we have to solve the equation (5.8) with free parameters a, b and select the solutions with unit norm which satisfy (5.9). Usually after this we still have a freedom in a and b. Then we must choose one of those pairs with minimal product of ab (we assume both a and b to be nonnegative). The fixed values of the parameters a and b provide that the solutions of (5.8) form the linear space as the subspace of $\mathcal{L}_2(\mathcal{M})$. This subspace should define the physical Hilbert space $\mathcal{H}_{ph} \equiv \mathcal{H}_{(a,b)}$ of the system.

Thus, instead of the first order differential equation (4.57) with one parameter ϵ, we get the second order equation (5.8) with two parameters a,b and subsidiary conditions (5.9). Note that a possible limiting procedure in (5.8) for $a \to 0$ (or $b \to 0$) can specify the physical states $|\Psi_{ph}\rangle$ with $\hat{\Phi}_f|\Psi_{ph}\rangle = 0$ (or $\hat{\Phi}_g|\Psi_{ph}\rangle = 0$).

For the test of the formulated principle, at first we consider again Example **1**. In this case the constraint operators $\hat{\Phi}_f \equiv \hat{\Phi}_p$ and $\hat{\Phi}_g \equiv \hat{\Phi}_q$ have the canonical commutation relations (5.1). Then (5.8) looks like the harmonic oscillator eigenvalue problem with the frequency $\omega = 1/ab$ and the eigenvalue $E = 1$. Respectively, we get $\hbar(n + 1/2) = ab$. One can check that all the oscillator's eigenstates $|n\rangle$ satisfy the conditions (5.9) and therefore the minimal ab ($ab = \hbar/2$) corresponds to the vacuum state ($n = 0$) given by $(a\hat{\Phi}_q - ib\hat{\Phi}_p)|\Psi_{ph}\rangle = 0$. Thus, for the physical states we arrive again at (4.31) with $\epsilon = b/a$, and the limiting procedure as $a \to 0$ (or $b \to 0$) can be accomplished in a similar way.

Now, let us consider Example **2** with the constraint operators (4.47). For convenience we can construct the operator $\hat{O} \equiv \hat{\Phi}^2_{\sin\varphi} + \hat{\Phi}^2_{\cos\varphi}$ and minimize the product $\langle\Psi|\hat{\Phi}_S^2|\Psi\rangle\langle\Psi|\hat{O}|\Psi\rangle$. From (4.47) we have $\hat{O} = -\hbar^2\partial_S^2$, and we see that this operator is the square of the Hermitian operator $\hat{\Phi}_\varphi \equiv -i\hbar\partial_S$ ($\hat{O} = \hat{\Phi}_\varphi^2$). Then, from the variation principle we get the equation (5.8) with $\hat{\Phi}_f = S + i\hbar\partial_\varphi$ and $\hat{\Phi}_g = -i\hbar\partial_S$. Since these two Hermitian operators have the canonical commutation relations, we arrive again at the oscillator problem. Only, now the 'ground' state should be obtained from the equation

$$(S + i\hbar\partial_\varphi + \frac{a}{b}\hbar\partial_S)|\Psi_{ph}\rangle = 0. \tag{5.10}$$

Hence, for this example, using the minimization principle, we arrive at the equation (5.10). It is interesting to note that the equations (5.10) and (4.53) are equivalent, and the functions (4.55) with $\lambda = a/b$ are the solutions of (5.10) as well.

In (5.10) we can accomplish the limiting procedure to the equations (4.50) (or (4.49)), taking the corresponding limits $a/b \equiv \lambda \to 0$ (or $\lambda \to \infty$).

From (4.55) we see that the functions

$$\Psi_{\lambda,n}(S,\varphi) = \left(\frac{\hbar}{\pi\lambda}\right)^{1/4} \exp\left(-\frac{(S-n\hbar)^2}{2\lambda\hbar}\right) \exp(in\varphi) \tag{5.11}$$

form the ortho-normal basis for the physical states (5.10). The behavior of these basis functions is singular when $\lambda \to 0$ (or $\lambda \to \infty$). But, with some suitable renormalization, the corresponding limits exist in the dual space $\mathcal{L}_2^*(\mathcal{R}^1 \times \mathcal{S}^1)$ (see Section 4.5). Indeed, the limit $\lambda \to 0$ of the function

$$\widetilde{\Psi}_{\lambda,n}(S,\varphi) = \frac{1}{\sqrt{2\hbar}} \left(\frac{1}{\pi\lambda}\right)^{1/4} \Psi_{\lambda,n}(S,\varphi)$$

is the generalized function (4.51), while the limit $\lambda \to \infty$ of the functions

$$\widetilde{\widetilde{\Psi}}_{\lambda,n}(S,\varphi) = \frac{1}{\sqrt{2}} \left(\frac{\lambda}{\pi\hbar}\right)^{1/4} \Psi_{\lambda,n}(S,\varphi)$$

gives the standard basis for 'φ representation' (see (4.46) and (4.49)).

According to the rule (4.28), the obtained physical states (4.51) (similarly, the states $\psi_n(\varphi) = 1/\sqrt{2\pi}\, \exp(in\varphi)$) form an ortho-normal basis of the corresponding reduced Hilbert space.

Any physical state (5.10) with the unit norm can expanded in the basis (5.11):

$$\Psi_\lambda(S,\varphi) = \sum_{n=-\infty}^{\infty} a_n \Psi_{\lambda,n}(S,\varphi) \quad \text{with} \quad \sum_{n=-\infty}^{\infty} |a_n|^2 = 1.$$

Here the numbers a_n can be interpreted as the probability amplitudes for the angular momentum, and it is clear that the functions $\Psi_\lambda(S,\varphi)$ describe the same quantum physical state for all λ. Using the form of the basis functions (5.11), we get

$$\lim_{\lambda\to 0} |\Psi_\lambda(S,\varphi)|^2 = 2\pi\hbar \sum_{n=-\infty}^{\infty} |a_n|^2 \delta(S-n\hbar), \tag{5.12}$$

where we take into account that

$$\lim_{\lambda\to 0} \Psi^*_{\lambda,n}(S,\varphi)\Psi_{\lambda,m}(S,\varphi) = 0, \quad \text{when} \quad m \neq n.$$

We see that the right-hand side of (5.12) describes the angular-momentum distribution function for the corresponding physical state.

We use this property in Section 5.3 for the physical interpretation of wave functions $\Psi_{ph}(\xi)$, and now, in the next section, we return to the condition (4.57) for further investigation.

5.2. Minimal uncertainties and coherent states. The minimization principle for quadratic fluctuations applied for the functional (5.6) $U_1(\Psi)$ gives (see Appendix D)

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi_{ph}\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi_{ph}\rangle - \frac{\hat{A}}{A}|\Psi_{ph}\rangle = 0, \tag{5.13}$$

where $\hat{A}$ is commutator (5.7), A is a parameter, and the solutions $|\Psi_{ph}\rangle$ should satisfy both (5.9) and the condition $\langle\Psi_{ph}|\hat{A}|\Psi_{ph}\rangle = A$.

There is some relation between the minimization of the functional $U_1(\Psi)$ and the condition (4.57). The condition (4.57) is the first order differential equation for the wave function $\Psi_\epsilon(\xi)$. Of course, it is much easier to analyze solutions of (4.57) than to investigate (5.8) (or (5.13)), which are the second order equations with two (or three) free parameters and the subsidiary conditions (5.9). But, to be acceptable for the physical states, the corresponding the solutions of (4.57) should belong to the domain of definition of self-adjoint operators $\hat{\Phi}_f$ and $\hat{\Phi}_g$. Except the finiteness of the norm of $|\Psi_\epsilon\rangle$, this means that the operators $\hat{\Phi}_f$ and $\hat{\Phi}_g$ must be Hermitian on these functions. As it was pointed out, in general, these conditions are not fulfilled, and in that case we have to use the minimization principle for quadratic fluctuations of quantum constraints. But, if for some real ϵ, the solutions of (4.57) satisfy the two conditions mentioned above, then one can derive that (see Appendix D)

$$\langle\Psi_\epsilon|\hat{\Phi}_f^2|\Psi_\epsilon\rangle = \frac{\hbar\epsilon A}{2} \quad \langle\Psi_\epsilon|\hat{\Phi}_g^2|\Psi_\epsilon\rangle = \frac{\hbar A}{2\epsilon}, \quad \text{where} \quad \langle\Psi_\epsilon|\hat{A}|\Psi_\epsilon\rangle = A,$$

and the corresponding physical states $|\Psi_\epsilon\rangle$ provide the minimization of the functional $U_1(\Psi)$: $U_1(\Psi_\epsilon) = \hbar^2/4$. Note (and it is natural) that such functions $|\Psi_\epsilon\rangle$ satisfy (5.13) ($|\Psi_{ph}\rangle = |\Psi_\epsilon\rangle$) with $a^2 = \hbar\epsilon A/2$, $b^2 = \hbar A/2\epsilon$, and $A = \langle\Psi_\epsilon|\hat{A}|\Psi_\epsilon\rangle$. To be convinced, it is sufficient to act on (4.57) by the operator $\hat{\Phi}_f - i\epsilon\hat{\Phi}_g$.

When the commutator $\hat{A}$ in (5.13) is a c-number, the equations (5.13) and (5.8) are equivalent and they define the same physical Hilbert spaces as the subspaces of $\mathcal{L}_2(\mathcal{M})$. But, in general, these subspaces are different, and for their physical interpretation further investigation is required.

Let us return again to the choice of physical states by the condition (4.57). For simplicity we consider the two-dimensional case with the coordinates $\xi \equiv (\xi^1, \xi^2)$, and the observables f and g can play the role of these coordinates as well.

Suppose that for some real ϵ, the solutions of (4.57) satisfy the required conditions, and hence, they are acceptable for the physical states. In complex variables

$$z = f(\xi) - i\epsilon g(\xi), \quad z^* = f(\xi) + i\epsilon g(\xi), \tag{5.14}$$

the condition (4.57) can be written as $\hat{\Phi}_{z^*}|\Psi_{ph}\rangle = 0$. The corresponding differential equation has the form (see (4.39), (4.53) and (4.74))

$$\left(\partial_z - \frac{i}{\hbar}\theta_z\right)\Psi_{ph} = 0, \tag{5.15}$$

where θ_z is the component of the 1-form $\theta = \theta_z dz + \theta_{z^*} dz^*$. The solutions of (5.15) are

$$\Psi_{ph}(\xi) = \exp\left(-\frac{1}{2\hbar}S(\xi)\right)\psi(z^*), \tag{5.16}$$

where $\partial_z S = -2i\theta_z$, the coordinates z and ξ are related through (5.14), and $\psi(z^*)$ is almost arbitrary. Only this holomorphic function $\psi(z^*)$ should provide a finite norm of physical states $\Psi_{ph}(\xi)$.

The pre-quantization operator $\hat{R}_{z^*} = z^* - \hat{\Phi}_{z^*}$ acts invariantly on the physical states (5.16), and this action is given as a multiplication of corresponding wave functions $\psi(z^*)$ by z^*

$$\hat{R}_{z^*}\Psi_{ph}(\xi) = z^*(\xi)\Psi_{ph}(\xi). \tag{5.17}$$

From (4.25) and (4.21) we have

$$[\hat{R}_z, \hat{\Phi}_{z^*}] = -i\hbar\hat{\Phi}_{\{z,z^*\}} = 2\epsilon\hbar\partial_z(\{f,g\})\hat{\Phi}_z + 2\epsilon\hbar\partial_{z^*}(\{f,g\})\hat{\Phi}_{z^*},$$

and if the Poisson bracket $\{f,g\}$ is not a constant, then the physical Hilbert space (5.16) is not invariant under the action of pre-quantization operator $\hat{R}_z$. In this case, the deformation procedure is rather problematic, and to define the operator $\hat{z}$, we use the relation between z, z^* variables. Since the operator $\hat{z}^+ \equiv \hat{R}_{z^*}$ is well defined on the physical states (5.16), it is natural to introduce the operator $\hat{z}$ as Hermitian conjugated to $\hat{R}_{z^*}$: $\hat{z} \equiv (\hat{R}_{z^*})^+$. Respectively, the operators $\hat{f}$ and $\hat{g}$ will be

$$\hat{f} = \frac{1}{2}(\hat{z} + \hat{z}^+), \quad \hat{g} = \frac{i}{2\epsilon}(\hat{z} - \hat{z}^+). \tag{5.18}$$

If $\Psi_{ph,n}(\xi)$ is some ortho-normal basis of the physical Hilbert space (5.16), then the action of the operator $\hat{z}$ on any state $\Psi_{ph}(\xi)$ can be written as

$$\hat{z}\Psi_{ph}(\xi) = \sum_n \Psi_{ph,n}(\xi)\langle\Psi_{ph,n}|\hat{z}|\Psi_{ph}\rangle = \sum_n \Psi_{ph,n}(\xi)\langle\hat{z}^+\Psi_{ph,n}|\Psi_{ph}\rangle =$$

$$= \sum_n \Psi_{ph,n}(\xi)\int d\mu(\xi')\ \Psi^*_{ph,n}(\xi')z(\xi')\Psi_{ph}(\xi'), \tag{5.19}$$

where $d\mu(\xi')$ is the standard measure (1.43).

Let us introduce the wave function $\chi_\zeta(\xi)$:

$$\chi_\zeta(\xi) \equiv \sum_n \Psi^*_{ph,n}(\zeta)\Psi_{ph,n}(\xi), \tag{5.20}$$

where the parameters $\zeta \equiv (\zeta^1, \zeta^2)$ take the same values as the coordinates (ξ^1, ξ^2). So, (5.20) is an expansion of the wave function $\chi_\zeta(\xi)$ in the basis $\Psi_{ph,n}(\xi)$ with the coefficients $\Psi^*_{ph,n}(\zeta)$.

With some assumptions about the analytical structure on $\mathcal{M}$, one can prove (see [82] and [27]) that the function $\chi_\zeta(\xi)$ is well-defined on $\mathcal{L}_2(\mathcal{M})$ and the corresponding norm

$$\int d\mu(\xi)\ |\chi_\zeta(\xi)|^2 = \sum_n \Psi^*_{ph,n}(\zeta)\Psi_{ph,n}(\zeta) = \chi_\zeta(\zeta) \tag{5.21}$$

does not depend on the choice of the basis $\Psi_{ph,n}(\xi)$.

Then for an arbitrary physical state $|\Psi_{ph}\rangle$, (5.20) yields

$$\langle\chi_\zeta|\Psi_{ph}\rangle = \int d\mu(\xi)\ \chi^*_\zeta(\xi)\Psi_{ph}(\xi) = \Psi_{ph}(\zeta). \tag{5.22}$$

If we act with the operator $\hat{z}$ on the state $\chi_\zeta(\xi)$, and use (5.17), (5.19), and (5.22), we obtain

$$\begin{aligned}\hat{z}\chi_\zeta(\xi) = \langle\chi_\xi|\hat{z}|\chi_\zeta\rangle = \langle\chi_\zeta|\hat{z}^+|\chi_\xi\rangle^* = (\hat{z}^+\chi_\xi(\zeta))^* = \\ (z^*(\zeta)\chi_\xi(\zeta))^* = z(\zeta)\ \chi_\zeta(\xi),\end{aligned} \tag{5.23}$$

where the complex valued functions $z(\zeta)$ and $z^*(\zeta)$ are given by (5.14), and we take into account that

$$\chi^*_\xi(\zeta) = \chi_\zeta(\xi)$$

which is apparent from the definition (5.20).

Thus, we see that the function $\chi_\xi(\xi')$ is the eigenstate of the operator $\hat{z}$ with the eigenvalue $z = z(\xi)$. This state is uniquely characterized by the complex parameter z, and we denote the corresponding 'bra' and 'ket' vectors by $\langle z|$ and $|z\rangle$, respectively. We also use the notation $|z\rangle \equiv |\bar{f}, \bar{g}; \epsilon\rangle$, where $\bar{f} = f(\xi)$ and $\bar{g} = g(\xi)$ are the real and the imaginary parts of the complex number ξ.

From the definition (5.20) the set of vectors $|z\rangle$ is a complete one on the physical Hilbert space $\mathcal{H}_{ph}$

$$\int d\mu(\xi)\ |z(\xi)\rangle\langle z(\xi)| = \hat{I}.$$

It is remarkable that this condition of completeness allows us to introduce the covariant and the contravariant symbols of Berezin quantization [27].

Further, for the Hermitian operators (5.18) the relation (5.23) takes the form

$$(\hat{f} - i\epsilon\hat{g})|\bar{f}, \bar{g}; \epsilon\rangle = (\bar{f} - i\epsilon\bar{g})|\bar{f}, \bar{g}; \epsilon\rangle, \tag{5.24}$$

which gives

$$\langle\bar{f}, \bar{g}; \epsilon|\hat{f}|\bar{f}, \bar{g}; \epsilon\rangle = \bar{f}, \quad \langle\bar{f}, \bar{g}; \epsilon|\hat{g}|\bar{f}, \bar{g}; \epsilon\rangle = \bar{g}.$$

Then, using the method described in [96]-[72], we obtain

$$\frac{\langle \bar{f}, \bar{g}; \epsilon|(\hat{f} - \bar{f})^2|\bar{f}, \bar{g}; \epsilon\rangle\langle \bar{f}, \bar{g}; \epsilon|(\hat{g} - \bar{g})^2|\bar{f}, \bar{g}; \epsilon\rangle}{\langle \bar{f}, \bar{g}; \epsilon|\hat{C}|\bar{f}, \bar{g}; \epsilon\rangle^2} = \frac{\hbar^2}{4}, \tag{5.25}$$

where $\hat{C}$ is the commutator

$$\hat{C} = i/\hbar\ [\hat{f}, \hat{g}]. \tag{5.26}$$

One can show (see [96]) that the number $\hbar^2/4$ is the minimal value for the corresponding quadratic fluctuations. Therefore, the quantum state $|\bar{f}, \bar{g}; \epsilon\rangle$ minimizes the quadratic fluctuations of the observables f and g around the values $\bar{f}$ and $\bar{g}$. In this respect, they are very similar to the standard coherent states of quantum mechanics $|p, q, \epsilon\rangle$ (see (4.42)), which minimize the coordinate-momentum uncertainty.

Note that the operators $\hat{f}$ and $\hat{g}$ generally are not the pre-quantization ones, and respectively, the operator $\hat{C}$ has not the form (4.16).

For the considered examples (see Section 4.3) many technical calculations with coherent states can be accomplished explicitly. In the case of a plane, the ortho-normal basis for the physical states (4.40) can be chosen as

$$\Psi_{ph,n}(p, q) = \exp\,(-\frac{1}{2}|z|^2)\frac{z^{*n}}{\sqrt{n!}}.$$

Then, from (5.20) and (5.22)we get

$$\chi_{p_1,q_1}(p, q) = \exp\,(-\frac{1}{2}|z|^2)\exp\,(-\frac{1}{2}|z_1|^2)\exp\,(z_1^* z) = \langle z_1|z\rangle,$$

and these states have the unit norm for arbitrary z_1 (see (5.21)). Comparing (5.22) and (5.24) to (4.45) and (4.42), we see that the states $|z\rangle$ are just the usual coherent states $|p, q, \epsilon\rangle$ mentioned above.

In the case of a cylinder, we have

$$f = e^{S/\lambda}\cos\varphi, \quad g = e^{S/\lambda}\sin\varphi, \quad \epsilon = 1$$

and the corresponding complex variables (4.52). The physical Hilbert space is defined by (4.54) or (4.55), and it has the ortho-normal basis (5.11). Here, we omit the index 'ph', the arguments of the functions, and denote the corresponding basis by $|\Psi_n\rangle$. The states $|\Psi_n\rangle$ are the eigenstates of the operator $\hat{S} \equiv \hat{R}_S = -i\hbar\partial_\varphi$ with the eigenvalues $n\hbar$. Then, from (5.20) and (5.11), we get

$$|z\rangle = \left(\frac{\hbar}{\pi\lambda}\right)^{1/4}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{(S-n\hbar)^2}{2\lambda\hbar}\right)\exp\,(in\varphi)\ ||\Psi_n\rangle. \tag{5.27}$$

This state has the finite norm

$$\langle z|z\rangle = \left(\frac{\hbar}{\pi\lambda}\right)^{1/2}\sum_{n=-\infty}^{\infty}\exp\left(-\frac{(S-n\hbar)^2}{\lambda\hbar}\right),$$

and in the limit as $\lambda \to 0$ we obtain

$$\langle z|z\rangle \to \sum_{n=-\infty}^{\infty} \delta(S/\hbar\ - n). \tag{5.28}$$

Since the operator $\hat{z}^+$ acts as the multiplication by z^* (see (4.52)), for the basis vectors (5.11) we get

$$\hat{z}^+|\Psi_n\rangle = \exp\left(\frac{\hbar n}{\lambda} + \frac{\hbar}{2\lambda}\right)|\Psi_{n+1}\rangle.$$

Respectively, the Hermitian conjugated operator $\hat{z}$ is

$$\hat{z}|\Psi_n\rangle = \exp\left(\frac{\hbar n}{\lambda} - \frac{\hbar}{2\lambda}\right)|\Psi_{n-1}\rangle, \tag{5.29}$$

and we obtain the commutator

$$[\hat{z}, \hat{z}^+] = 2\exp(2\hat{S}/\lambda)\sinh(\hbar/\lambda).$$

Note that the operator $\hat{z}$ is not the pre-quantization one, and the corresponding classical commutation relation is

$$\{\hat{z}, \hat{z}^+\} = \frac{2i}{\lambda}\exp(2S/\lambda).$$

Now, from (5.29) and (5.27), we can check that the states $|z\rangle$ are the eigenstates of the operator $\hat{z}$ with the eigenvalues $z = \exp(S/\lambda\ - i\varphi)$.

The states $|z\rangle$ in (5.27) are defined for arbitrary values of the variable S. At the same time, the states with fixed value of the angular momentum ($\Delta S = 0$), exist only for the discrete values of S ($S_n = \hbar n$). Of course, the states $|z\rangle$ are not the eigenstates of the operator $\hat{S}$, but, from (5.25), it is expected that $\Delta S \to 0$ when $\lambda \to 0$. Therefore, it is interesting to investigate the behavior of the states $|z\rangle$ when $\lambda \to 0$.

Note that the expansion (5.27) can be considered as the definition of the states $|z\rangle$ for a quantum theory of a rotator in abstract Hilbert space; only the basis vectors $|\Psi_n\rangle$ should be the eigenstates of the angular momentum operator $\hat{S}$ with the eigenvalues $S_n = \hbar n$. With this remark we can neglect the dependence on the parameter λ in the basis vectors Ψ_n, and consider the behavior (when $\lambda \to 0$) of the corresponding coefficients only. If we introduce the vector $|S, \varphi; \lambda\rangle$ with the unit norm

$$|S, \varphi; \lambda\rangle \equiv \frac{|z\rangle}{\langle z|z\rangle^{1/2}},$$

then from (5.27) we get

$$|S, \varphi; \lambda\rangle = \sum_{n=-\infty}^{\infty} \frac{d_n(S, \lambda)}{d(S, \lambda)} \exp(in\varphi)\ |\Psi_n\rangle,$$

where

$$d_n(S,\lambda) = \exp\left(-\frac{(S-n\hbar)^2}{2\lambda\hbar}\right), \quad d^2(S,\lambda) = \sum_{n=-\infty}^{\infty} d_n^2, \quad (d>0).$$

In the limit as $\lambda \to 0$, $d_n(S,\lambda)/d(S,\lambda) \to c_n(S)$, and for the coefficients $c_n(S)$ we get:
a. $c_n(S) = 0$, if $S < \hbar(n-1/2)$, or $S > \hbar(n+1/2)$;
b. $c_n(S) = 1/\sqrt{2}$, if $S = \hbar(n-1/2)$ or $S = \hbar(n+1/2)$;
c. $c_n(S) = 1$, if $\hbar(n-1/2) < S < \hbar(n+1/2)$.

We see that $|S,\varphi;\lambda\rangle \to \exp(in\varphi)\ |\Psi_n\rangle$, where n is the nearest integer number to $S/\hbar$. But if $S/\hbar$ is exactly in the middle of two integers: $S/\hbar = n+1/2$, then $|S,\varphi;\lambda\rangle \to 1/\sqrt{2}\ \left(\exp(in\varphi)|\Psi_n\rangle + \exp(i(n+1)\varphi)|\Psi_{n+1}\rangle\right)$. So, when $\lambda \to 0$, all states $|S,\varphi;\lambda\rangle$ with $\hbar(n-1/2) < S < \hbar(n+1/2)$ 'collapse' to the state $|\Psi_n\rangle$.

From (5.22) and (5.28) we see that the obtained behavior of the states $|S,\varphi;\lambda\rangle$ for small λ is in accordance with the corresponding behavior of the wave functions of E-quantization scheme given by (5.12).

5.3. Quantum distribution functions. For the physical interpretation of the wave functions $\Psi_{ph}(\xi)$, we refer to the equation (4.57), where the functions $f(\xi)$ and $g(\xi)$ are two non-commuting observables ($\{f,g\} \neq 0$) on the two dimensional phase space $\mathcal{M}$. We assume that the solutions of (4.57) $\Psi_\epsilon \equiv \Psi_{ph}(\xi)$ define the physical Hilbert space as the subspace of $\mathcal{L}_2(\mathcal{M})$. To emphasize the dependence on the observables f, g and on the parameter ϵ, we denote this physical Hilbert space here by $\mathcal{H}_\epsilon(f,g)$.

On $\mathcal{H}_\epsilon(f,g)$ the operators $\hat{f}$ and $\hat{g}$ have the form (5.18), where the operator $\hat{z}^+$ acts on wave functions $\Psi_{ph}(\xi)$ as the multiplication by $z^*(\xi) = f(\xi) + i\epsilon g(\xi)$, and the operator $\hat{z}$ is its Hermitian conjugated. Then, for mean values of these operators we get

$$\langle\Psi_{ph}|\hat{f}|\Psi_{ph}\rangle = \int d\mu(\xi)\ |\Psi_{ph}(\xi)|^2 f(\xi),$$
$$\langle\Psi_{ph}|\hat{g}|\Psi_{ph}\rangle = \int d\mu(\xi)\ |\Psi_{ph}(\xi)|^2 g(\xi). \tag{5.30}$$

We see that $|\Psi_{ph}(\xi)|^2$ can be interpreted as some 'distribution function' on the phase space $\mathcal{M}$.

For further investigation, we introduce the modulus and phase of wave functions $\Psi_{ph}(\xi)$

$$\Psi_{ph}(\xi) = e^{i\alpha(\xi)}\sqrt{\rho(\xi)}. \tag{5.31}$$

From (4.57) and (5.31), we have two real equations

$$V_f\alpha + \frac{\epsilon}{2}V_g(\log\rho) = \frac{1}{\hbar}\theta(V_f), \quad V_g\alpha - \frac{1}{2\epsilon}V_f(\log\rho) = \frac{1}{\hbar}\theta(V_g), \tag{5.32}$$

where V_f and V_g are the corresponding Hamiltonian vector fields (see (1.5)).

One can check the validity of the following relations

$$[V_f, V_g] = V_{\{f,g\}} = \frac{\{\{f,g\},g\}}{\{f,g\}} V_f - \frac{\{\{f,g\},f\}}{\{f,g\}} V_g$$

and

$$V_f \theta(V_g) - V_g \theta(V_f) = \{f,g\} + \theta(V_{\{f,g\}}).$$

Using these relations, we can exclude the function $\alpha(\xi)$ from (5.32), and obtain the equation only for $\rho(\xi)$

$$\left[\frac{\hbar}{2\epsilon}\left(\frac{1}{\{f,g\}} V_f\right)^2 + \frac{\hbar\epsilon}{2}\left(\frac{1}{\{f,g\}} V_g\right)^2\right] \log \rho = -\frac{1}{\{f,g\}}. \tag{5.33}$$

Note that in variables f, g this equation takes the form

$$\frac{\hbar}{2}\left(\frac{1}{\epsilon}\partial_g^2 + \epsilon \partial_f^2\right) \log \rho = -\frac{1}{\{f,g\}}, \tag{5.34}$$

where the Poisson bracket $\{f,g\}$ can be considered as a function of f and g.

Any solution of (5.33) $\rho(\xi)$ defines the corresponding phase $\alpha(\xi)$ up to the integration constant (see (5.32)). This constant phase factor is unessential for physical states (5.31) and, respectively, there is one-to-one correspondence between the 'distribution functions' $\rho(\xi) = |\Psi_{ph}(\xi)|^2$ and the pure states described by a projection operator $\hat{P}_{\Psi_{ph}} \equiv |\Psi_{ph}\rangle\langle\Psi_{ph}|$

$$\rho(\xi) \longleftrightarrow \hat{P}_{\Psi_{ph}}. \tag{5.35}$$

With this remark we can use the index ρ for corresponding pure states as well: $\hat{P}_{\Psi_{ph}} \equiv \hat{P}_\rho$.

From (5.31) and (5.22), we have

$$\rho(\xi) = |\Psi(\xi)|^2 = \langle z(\xi)|\Psi_{ph}\rangle\langle\Psi_{ph}|z(\xi)\rangle = \langle z(\xi)|\hat{P}_\rho|z(\xi)\rangle, \tag{5.36}$$

where $|z\rangle$ is a coherent state related to the observables f and g (see (5.20) and (5.24)). If one introduces the covariant symbol $P_\rho(\xi)$ of the projection operator $\hat{P}_\rho$

$$P_\rho(\xi) \equiv \frac{\langle z(\xi)|\Psi_{ph}\rangle\langle\Psi_{ph}|z(\xi)\rangle}{\langle z(\xi)|z(\xi)\rangle},$$

then from (5.36) we have $\rho(\xi) = P_\rho(\xi)\langle z(\xi)|z(\xi)\rangle$, and the correspondence (5.35) describes the well known connection between operators and their covariant symbols (see [27]).

Let $F(\xi)$ be any observable on $\mathcal{M}$ and $\hat{F}$ be the corresponding operator on the physical Hilbert space $\mathcal{H}_\epsilon(f,g)$. Standard quantum mechanical mean values are calculated by

$$\langle\Psi_{ph}|\hat{F}|\Psi_{ph}\rangle = Tr(\hat{F}\hat{P}_\rho) \equiv \langle\hat{F}\rangle_\rho. \tag{5.37}$$

We introduce the new mean values $\bar{F}_\rho$:

$$\bar{F}_\rho \equiv \int d\mu(\xi)\ F(\xi)\rho(\xi). \tag{5.38}$$

In general, $\langle \hat{F}\rangle_\rho \neq \bar{F}_\rho$, but for $F = f$ and $F = g$ these mean values are the same for an arbitrary state ρ (see (5.30))

$$\bar{f}_\rho = \langle \hat{f}\rangle_\rho, \quad \bar{g}_\rho = \langle \hat{g}\rangle_\rho. \tag{5.39}$$

Using (5.17) and (5.18), for the operators $\hat{f}^2$ and $\hat{g}^2$ we obtain

$$\bar{f^2_\rho} = \langle \hat{f}^2\rangle_\rho + \frac{\epsilon\hbar}{2}\langle \hat{C}\rangle_\rho, \quad \bar{g^2_\rho} = \langle \hat{g}^2\rangle_\rho + \frac{\hbar}{2\epsilon}\langle \hat{C}\rangle_\rho, \tag{5.40}$$

where the operator $\hat{C}$ is the commutator (5.26).

The quadratic fluctuations calculated for the mean values (5.37) and (5.38), respectively, are

$$(\Delta\hat{F})^2_\rho = \langle \hat{F}^2\rangle_\rho - \langle \hat{F}\rangle^2_\rho, \quad (\Delta F)^2_\rho = \bar{F^2_\rho} - (\bar{F}_\rho)^2. \tag{5.41}$$

Then, from (5.39) and (5.40) we have

$$(\Delta f)^2 = (\Delta\hat{f})^2 + \frac{\epsilon\hbar}{2}\langle \hat{C}\rangle, \quad (\Delta g)^2 = (\Delta\hat{g})^2 + \frac{\hbar}{2\epsilon}\langle \hat{C}\rangle. \tag{5.42}$$

The introduced 'distribution functions' can be generalized for mixed states as well. Any mixed state is described by a density matrix operator $\hat{\rho}$ [97], which is Hermitian ($\hat{\rho} = \hat{\rho}^+$), semi-positive ($\langle\psi|\hat{\rho}|\psi\rangle \geq 0$, for any state $|\psi\rangle$), and it has the unit trace ($\mathrm{Tr}\hat{\rho} = 1$). Respectively, any density matrix operator has the spectral expansion

$$\hat{\rho} = \sum_n c_n |\psi_n\rangle\langle\psi_n|, \tag{5.43}$$

where $|\psi_n\rangle$ are the ortho-normal eigenvectors of $\hat{\rho}$, c_n are the corresponding positive ($c_n > 0$) eigenvalues, and $\sum_n c_n = 1$.

Similarly to (5.36), we define the 'distribution function' $\rho(\xi)$ connected with the covariant symbol of $\hat{\rho}$

$$\rho(\xi) \equiv \langle z(\xi)|\hat{\rho}|z(\xi)\rangle. \tag{5.44}$$

Using the spectral expansion (5.43) we get that a 'distribution function' of a mixed state can be expressed as a convex combination of 'distribution functions' of pure ones

$$\rho(\xi) = \sum_n c_n\rho_n(\xi), \quad (0 < c_n < 1). \tag{5.45}$$

One can easily check that the relations (5.39)-(5.42) are valid for the mixed states as well.

From (5.44)-(5.45) we see that, in general, a 'distribution function' $\rho(\xi)$ is a non-negative function on the phase space $\mathcal{M}$, and it satisfies the standard condition of classical distributions

$$\int d\mu(\xi)\ \rho(\xi) = 1. \tag{5.46}$$

Thus, for a given $f(\xi)$, $g(\xi)$ and ϵ we have 'distribution functions' $\rho(\xi)$ which look like classical ones, and at the same time they describe all possible quantum states uniquely. We call these functions the quantum distribution functions. Sometimes it is convenient to indicate the dependence on the observables f, g and the parameter ϵ explicitly: $\rho(\xi) \equiv \rho(\xi|f, g; \epsilon)$.

We can compare the quantum distribution function $\rho(\xi|f, g; \epsilon)$ to the Wigner function $\rho_w(\xi)$, which is the Weyl symbol of a density matrix operator [98]. For any Wigner function $\rho_w(\xi)$ we have the 'classical' formula for quantum mechanical mean values (see (1.46),

$$\langle \hat{F} \rangle_\rho = \int d\mu(\xi)\ F(\xi)\rho_w(\xi). \tag{5.47}$$

Though this formula is valid for an arbitrary observable $F(\xi)$, nevertheless Wigner functions cannot be interpreted as functions of probability density. Due to the uncertainty principle, there is no such function on the phase space of a quantum system. In general, the Wigner function is even negative in some domain of a phase space. It should also be noted that the Wigner function is defined only for a 'flat' phase space ($\mathcal{M} = \mathcal{R}^{2N}$) and the Cartesian coordinates.

A quantum distribution function $\rho(\xi|f, g; \epsilon)$ can be considered for almost arbitrary 'coordinates' f, g. It is always positive, but the 'classical' formula (5.47) (with substitution ρ_w by ρ) is valid only for the functions $F = f$, or $F = g$ (and their linear combination).

The evolution equation for the Wigner function has the Liouville like form (1.48), where one has to change the Poisson bracket on the right-hand side to the Moyal one (see for example in [83]). The Moyal bracket has an expansion in powers of $\hbar$ and the zero term coincides with the Poisson bracket. A similar type equation can be obtained for the normal symbol of the density matrix operator, and such equation can be generalized for other quantum distribution functions as well. It is clear that the corrections (in powers of $\hbar$) to the Poisson bracket essentially depend on the choice of observables f and g and the parameter ϵ. If we take $f = H$, where $H(\xi)$ is the Hamilton function and take the limit as $\epsilon \longrightarrow 0$, one can expect that all corrections vanish, and the corresponding distribution function satisfies the Liouville equation.

For the physical interpretation of quantum distribution functions $\rho(\xi|f, g;$ $\epsilon)$ we consider again Example **1** (see Section 4.3) with $\mathcal{M} = \mathcal{R}^2$, $f \equiv q$,

$g \equiv -p$. In this case (5.42) takes the form

$$(\Delta q)^2 = (\Delta \hat{q})^2 + \frac{\epsilon\hbar}{2} \quad (\Delta p)^2 = (\Delta \hat{p})^2 + \frac{\hbar}{2\epsilon}, \tag{5.48}$$

where $(\Delta \hat{q})^2$ and $(\Delta \hat{p})^2$ are usual quantum mechanical quadratic fluctuations of coordinate and momentum, and they satisfy the Heisenberg uncertainty relation

$$(\Delta \hat{p})^2 (\Delta \hat{q})^2 \geq \frac{\hbar^2}{4}. \tag{5.49}$$

Suppose that a quantum particle is described by a wave function $\psi(q)$. The function $|\psi(q)|^2$ is a probability density of coordinate distribution, and the quadratic fluctuation $(\Delta \hat{q})^2$ is calculated by the classical formula

$$(\Delta \hat{q})^2 = \int dq\; q^2 |\psi(q)|^2 - \left(\int dq\; q |\psi(q)|^2 \right)^2 .$$

The distribution function $|\psi(q)|^2$, in principle, can be measured. We denote the corresponding experiment by E_q. Theoretically it is assumed that in the experiment E_q the coordinate can be measured with an absolute precision, and the quantum system can be prepared in a given state $\psi(q)$ as many times as it is necessary for a good approximation of the function $|\psi(q)|^2$. A statistical distribution of the coordinate, obtained in such experiment, is the intrinsic property of the quantum system in the given state. In general, in a pure state a definite value has some other observable (for example, energy), but not the coordinate.

Similarly, the momentum distribution for the same state is described by the function $|\widetilde{\psi}(p)|^2$, and for a good approximation of the function $|\widetilde{\psi}(p)|^2$ we need the experiment E_p with the precise measurement of the momentum. Note that the function $\widetilde{\psi}(p)$ is the Fourier transformation of $\psi(q)$.

One possible method for measuring of the coordinate and the momentum of a quantum particle is the scattering of a light on this particle (see [97]). It is well-known that in such an experiment the precise measurement of the coordinate can be achieved by photons with a very short wavelength λ (high energy). On the contrary, photons of low energy are needed for the measurement of the momentum. Thus, E_q and E_p are two essentially different experiments. Theoretically, the experiment E_q is the measurement with photons of 'zero wavelength': $\lambda \to 0$, and the experiment E_p requires photons of 'zero energy': $\lambda \to \infty$.

But real experiments, of course, are with photons of finite and non-zero wavelength λ. we denote the experiment with some fixed wavelength λ by E_λ. In this experiment there are the errors in measuring of both coordinate and momentum. The first one Δ_q is proportional to the photon's wavelength λ, while the error of momentum Δ_p is proportional to the photon's momentum (see [97]) $p_\lambda = 2\pi\hbar/\lambda$. Respectively, we can write

$$\Delta_q = \alpha\lambda, \quad \Delta_p = \beta\frac{\hbar}{\lambda}, \tag{5.50}$$

where α and β are dimensionless parameters of order 1.

Thus, in the experiment E_λ we have two different fluctuations: the first one $((\Delta\hat{q}), (\Delta\hat{p}))$ is the intrinsic property of a quantum system, and the second $((\Delta_q), (\Delta_p))$ is related to the measurement procedure. Then, for the total quadratic fluctuations we can write

$$\begin{aligned}(\Delta_t q)^2 &= (\Delta\hat{q})^2 + (\Delta_q)^2 = (\Delta\hat{q})^2 + \alpha^2\lambda^2, \\ (\Delta_t p)^2 &= (\Delta\hat{p})^2 + (\Delta_p)^2 = (\Delta\hat{p})^2 + \frac{\beta^2\hbar^2}{\lambda^2}.\end{aligned} \tag{5.51}$$

As it was mentioned, the fluctuations $(\Delta\hat{q})$ and $(\Delta\hat{p})$ satisfy the uncertainty relation (5.49). Assuming that for the ideal experiment $\alpha\beta = 1/2$, from (5.50) we get another uncertainty relation

$$\Delta_q\Delta_p = \frac{\hbar}{2}. \tag{5.52}$$

With this assumption, from (5.48) and (5.51), we can write

$$(\Delta_t q)^2 = (\Delta q)^2, \quad (\Delta_t p)^2 = (\Delta p)^2.$$

Then the parameter ϵ is related to the wavelength λ and at the same time it fixes the ratio of the experimental errors

$$\epsilon = \frac{\Delta_q}{\Delta_p}. \tag{5.53}$$

Note that for the total fluctuations we get the uncertainty relation

$$(\Delta p)(\Delta q) \geq \hbar.$$

Since the quadratic fluctuations $(\Delta q)^2$ and $(\Delta p)^2$ are calculated by the mean values of the function $\rho_\epsilon(p,q)$, it is natural to suppose that the quantum distribution function $\rho_\epsilon(p,q) = |\Psi_\epsilon(p,q)|^2$ is the distribution obtained in the experiment E_λ with simultaneous measurements of the coordinate and the momentum.

This idea can be easily generalized assuming that the quantum distribution function $\rho(\xi|f,g:\epsilon)$ is the distribution on the phase space obtained in some ideal experiment with simultaneous measuring of the observables f and g. In such an experiment we have the unavoidable errors Δ_f and Δ_g connected with the measurement procedure with micro-objects. For the corresponding fluctuations there is the additional uncertainty principle (see (5.52)), and the parameter ϵ specifies the experiment by fixing the ratio of the errors $\epsilon = \Delta_f/\Delta_g$.

If the function $\rho(\xi) \equiv \rho(\xi|f,g;\epsilon)$ is really measurable, then in the limit $\epsilon \to 0$ this function $\rho(\xi)$ should describe the experimental distribution of the exact measurement of the observable f. It is obvious that for the observable f with discrete spectrum the corresponding function $\rho(\xi)$ should be localized in the points of this spectrum. Thus, by asymptotics of quantum distribution functions one can obtain the spectrum of the physical observables (see (5.12) and the end of Section 5.2).

We see that quantum distribution functions can play some fundamental role for the interpretation of quantum theory. It is natural to try to formulate quantum mechanics in terms of these distribution functions, especially as they describe all possible states of a quantum system uniquely. But for this purpose it is worthwhile to have an independent (without referring to the Hilbert space) description of the set of functions $\rho(\xi) \equiv \rho(\xi|f,g;\epsilon)$. The corresponding functions are positive, satisfying (5.46), and at the same time they essentially depend on the choice of observables f and g and of the parameter ϵ. On the other hand, the set of physical states is a convex one, where the boundary points are the pure states. So for the description of our set we need to specify the distribution functions of pure states, but the latter are given as the solutions of (5.33). Thus, in this approach equation (5.33) plays an important role. Actually it describes the set of all physical states and, respectively, it contains the information about quantum uncertainties both the intrinsic and the experimental ones.

Note that on the left-hand side of the corresponding equation there is the Laplace operator (see (5.33) and (5.34)) and we have some metric structure induced on the phase space $\mathcal{M}$. It is remarkable, that this metric structure is related to the experimental errors. Indeed, in the case of Example **1** the errors Δ_p, Δ_q and the parameter ϵ are related by (5.53), and we see that the corresponding equation (5.34) takes the form

$$\left(\Delta_q^2 \partial_q^2 + \Delta_p^2 \partial_p^2\right) \log \rho = -1.$$

Note, that a similar 'shadow' metric on a phase space was introduced in [36].

If the equation for the quantum distribution functions of pure states has really the fundamental character, then one might expect that it can be derived from some general principle. A suitable principle could be the minimization of certain functional, and we arrive to the problem of construction of the corresponding functional. Since the minimization should be achieved on pure states, it is natural to interpret such functional as the entropy of a quantum system. Respectively, one candidate for such a functional is the standard quantum mechanical entropy $S = -Tr(\hat{\rho} \log \hat{\rho})$ which can be expressed as a functional of $\rho(\xi)$.

Formulation of quantum mechanics based on the entropy minimization principle for the measurable distribution functions seems quite interesting and it needs further investigation.

Conclusions

In conclusions we present the main results of the work:

1. The practical reduction scheme of gauge invariant theories is constructed. This method is based on the analysis of the restricted 1-forms in gauge-invariant variables, and it is effectively used even if only a part of the gauge invariant variables are known.

2. A complete reduction of the models of Yang-Mills theory is accomplished. A possible mechanism of the confinement is obtained for the finite-dimensional model with $SU(2)$ gauge symmetry. In the infinite dimensional case the reduced Hamiltonian system has a non-analytical dependence on the coupling constant and this system is equivalent to Yang-Mills theory with a certain boundary condition.

3. A quantum theory is constructed for the system of a relativistic particle moving freely on the $SL(2,R)$ group manifold. Applied to the cotangent bundle of $SL(2,R)$, the method of Hamiltonian reduction allows us to split the reduced system into two coadjoint orbits of the group. We find that the Hilbert space consists of states given by a discrete series of unitary irreducible representations of $SL(2,R)$, and only some discrete values of the mass parameter are admissible.

4. The k-th root of the bosonic phase operator is constructed. This construction leads to the extension of the Hilbert space. In the case $k=2$ the ordinary fermionic extension arises. Particles whose statistics depends on k are introduced for other values of k. We show that the constructed quantum system can be considered as a quantization on k-sheet manifold as well.

5. A general quantization method based on the extension of phase space is constructed. This method (E-quantization) turned out to be very similar to geometric quantization, though it is based on essentially different ideas. A connection of E-quantization with the Berezin quantization is found.

6. The problem of scalar product is investigated for the constrained systems and a possible solution to this problem is given.

7. Generalization of the Gupta-Bleuler like conditions is done by minimization of quadratic fluctuations of quantum constraints.

8. The general coherent states are introduced. They minimize uncertainties for some complete set of observables. The special coherent states on the cylinder are constructed and their behaviour is investigated.

9. The positive defined quantum distribution function is introduced. The special elliptic type equations are obtained for the pure state distributions. A possible experimental measuring of the quantum distribution functions is investigated and the corresponding physical interpretation of these distribution is found.

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Appendix A

Let $\vec{J} = (J_x, J_y, J_z)$ be any non-zero vector of 3-dimensional Euclidean space. We associate the following orthonormal frame $\vec{e}_i(\vec{J})$ $(i = 1, 2, 3)$ with this vector $\vec{J}$

$$\vec{e}_3 = \frac{J_z}{J}, \quad \vec{e}_i \cdot \vec{e}_k = \delta_{ik}, \quad \vec{e}_i \times \vec{e}_k = \epsilon_{ikl}\vec{e}_l, \tag{A.1}$$

where $J = (J_x^2 + J_y^2 + J_z^2)^{1/2}$.

Of course, there are different possibilities for the choice of the vectors $\vec{e}_1$ and $\vec{e}_2$, but note that the orthonormal frame $\vec{e}_i(\vec{J})$ cannot be chosen continuously on the sphere $J = r$. For example, if we choose

$$\begin{aligned} \vec{e}_1 &= \frac{1}{J}\left(\frac{J_x^2}{J + J_z} - J,\ \frac{J_x J_y}{J + J_z},\ J_x\right), \\ \vec{e}_2 &= \frac{1}{J}\left(\frac{J_x J_y}{J + J_z},\ \frac{J_y^2}{J + J_z} - J,\ J_y\right), \end{aligned} \tag{A.2}$$

then the continuity is violated at the lower pole $J_z = -J$.

Suppose that the vector $\vec{J}$ is the exterior product of two orthogonal vectors $\vec{q}$ and $\vec{p}$: $\vec{J} = \vec{q} \times \vec{p}$ (see (2.8)-(2.9)). Using the basis (A.1), we get the decomposition for the vectors $\vec{q}$ and $\vec{p}$

$$\vec{q} = a_1\vec{e}_1 + a_2\vec{e}_2, \quad \vec{p} = b_1\vec{e}_1 + b_2\vec{e}_2, \tag{A.3}$$

where the coefficients (a_1, a_2) and (b_1, b_2) satisfy the following conditions

$$a_1b_1 + a_2b_2 = 0, \quad a_1b_2 - a_2b_1 = r,$$

and for the two dimensional vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ we can choose the parametrization

$$a = \rho(\cos\chi,\ \sin\chi), \quad b = \frac{r}{\rho}(-\sin\chi,\ \cos\chi), \quad (\rho > 0). \tag{A.4}$$

The initial phase space with the coordinates $(\vec{p}, \vec{q})$is 6-dimensional. The 4-dimensional constraint surface $\phi_1 := \vec{p} \cdot \vec{q} = 0$, $\phi_2 = \vec{J}^2 - r^2 = 0$ can be parametrized by the coordinates (s, φ, ρ, χ) where $(s := J_z, \varphi)$ are the cylindrical coordinates on the sphere (see (2.10)). Using (A.1)-(A.4), we can calculate the reduction of the 1-form $\vec{p}\cdot d\vec{q}$ on the constraint surface and obtain

$$\vec{p} \cdot d\vec{q}|_{\phi_{1,2}=0} = rd\chi + (s - r)d\varphi. \tag{A.5}$$

In this calculation we use that

$$\vec{e}_1 d\vec{e}_2 = r(r - s)d\varphi$$

which can be derived using the parametrization (A.2).

The differential of the 1-form (A.5) is a well defined symplectic form on the sphere

$$\omega = ds \wedge d\varphi. \tag{A.6}$$

The described reduction scheme can be generalized for the Minkowski space with the metric tensor $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2)$. Here the co-vector I_μ is defined by

$$I_\mu = \epsilon_{\mu\nu\sigma} q^\nu p^\sigma$$

and the constraints have the form $\phi_1 = p_\mu q^\mu = 0$, $\phi_2 = I_\mu I^\mu - c = 0$. It should be noted that the form of the frame connected with the vector I_μ essentially depends on the sign of the constant c, and respectively, different reductions are possible.

Appendix B

The $2+1$-dimensional massive photodynamics is described by the Lagrangian (see e.g., [99])

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m}{4} \epsilon^{\mu\nu\sigma} F_{\mu\nu} A_\sigma. \tag{B.1}$$

We choose $g_{\mu\nu} = \mathrm{diag}(+,-,-)$, $\epsilon^{012} = 1$. Using the first order formalism [18], we obtain

$$S = \int dt \int\limits_{R^2} d^2x [(E_i - \frac{m}{2}\epsilon_{ij}A_j)\dot{A}_i - \frac{1}{2}(E_iE_i + B^2) + A_0(\partial_i E_i - mB)], \tag{B.2}$$

where

$$E_i \equiv F_{0i} \equiv \dot{A}_i - \partial_i A_0 \quad B \equiv \frac{1}{2}\epsilon_{ij}F_{ij} \quad (\epsilon_{ij} \equiv \epsilon^{0ij})$$

and we neglect the boundary term $\int_{R^2} d^2x \partial_i [A_0(\frac{m}{2}\epsilon_{ij}A_j - E_i)]$.

If we use '1-forms' instead of time derivatives (see the comment after (2.54)), the action (B.2) takes the form (2.1) with A_0 playing the role of a Lagrange multiplier.

For the reduction, we choose E_1 and E_2 to be the variables ξ^μ, and A_1 to be the additional variable η (see (2.12)). Then

$$\widetilde{S} = \int dt \int\limits_{R^2} d^2x \left[\frac{1}{m}E_2\dot{E}_1 - \frac{1}{2}[E_iE_i + \frac{1}{m^2}(\partial_k E_k)^2] + \frac{d}{dt}\Theta\right], \tag{B.3}$$

where

$$\Theta = \frac{1}{2}[E_1A_1 + E_2\hat{K}(A_1 + \frac{1}{m}E_2)]$$

and the operator $\hat{K} \equiv \partial_1^{-1}\partial_2$ is assumed to be symmetrical due to the corresponding boundary conditions.

Neglecting the Θ term as the total derivative, we get the local Hamiltonian theory with the canonical commutation relations

$$\{E_2(x), E_1(y)\} = m\delta^{(2)}(x-y) \tag{B.4}$$

and the quadratic Hamiltonian

$$\frac{1}{2}\int\limits_{R^2} d^2x[E_iE_i + \frac{1}{m^2}(\partial_k E_k)^2]. \tag{B.5}$$

The energy-momentum tensor can also be expressed in terms of E_1 and E_2 alone:

$$T_{00} = \frac{1}{2}[E_iE_i + \frac{1}{m^2}(\partial_k E_k)^2], \quad T_{0i} = \frac{1}{m}\epsilon_{ij}E_j(\partial_k E_k). \tag{B.6}$$

Let us briefly discuss the boundary conditions. We can assume that the boundary behavior of the physical variables (E_1, E_2) should provide the Poincaré invariance of the reduced system (B.3)-(B.6), while the boundary behavior of the fields of the initial system (B.1) should allow the outlined reduction procedure.

Generators of the Poincaré group (constructed from the energy-momentum tensor (B.6)) generate transformations of E_1 and E_2 according to the Poisson brackets (B.4). The space of functions $E_1(x)$ and $E_2(x)$ should be invariant under these transformations. It is natural to choose the class of smooth functions rapidly vanishing at infinity.

For diagonalization of the Hamiltonian and momentum, let us make the Fourier transformation

$$E_j(x) = i\int d^2p \frac{e^{-i(p\cdot x)}}{2\pi}\widetilde{E}_j(p).$$

and introduce the longitudinal and transverse components

$$\widetilde{E}_j(p) = \frac{p_j}{|p|}e_1(p) - \frac{\epsilon_{jl}p_l}{|p|}e_2(p),$$

where $|p| = \sqrt{p_1^2 + p_2^2}$.

Then diagonalization of the energy and momentum occurs in the variables

$$a(p) = \frac{\frac{\omega_p}{m}e_1(p) + ie_2(p)}{\sqrt{2\omega_p}}e^{-i\varphi(p)}$$
$$a^*(p) = \frac{\frac{\omega_p}{m}e_1(-p) - ie_2(-p)}{\sqrt{2\omega_p}}e^{i\varphi(p)}$$

with

$$\omega_p = \sqrt{|p|^2 + m^2} \quad \text{and} \quad e^{\pm i\varphi(p)} = \frac{p_1 \pm ip_2}{|p|}.$$

Note that for the chosen class of $E_1(x)$ and $E_2(x)$, the longitudinal and transverse components of $\widetilde{E}_j(p)$ have a singularity at the origin $(p = 0)$, and we need to introduce the phase factor $e^{i\varphi(p)}$ to cancel it. On the other hand, one can easily check that the class of smooth functions $a(p)$, $a^*(p)$ is Poincaré invariant. This phase factor was introduced in [99] to avoid anomalies in the commutation relations of the Poincar e algebra of quantum operators. As we have seen, this phase factor is connected with the Poincare invariance of the classical system, as well.

After describing the class of physical variables, one can go back and find the class of gauge potentials A_μ. One can show that these classes for massive and ordinary photodynamics in (2+1) dimensions are different.

Appendix C

Let us consider a symplectic manifold $\mathcal{M}$ with some global coordinates ξ^k, $(k = 1, \ldots, 2N)$ and constant symplectic matrix: $\partial_j \omega^{kl} = 0$ (see (1.12)). The simple example of such $\mathcal{M}$ is $\mathcal{R}^{2N}$ with canonical coordinates.

For the global coordinates ξ^k we can introduce the corresponding constraint functions Φ_{ξ^k}, and from (4.3)-(4.4) we get

$$\Phi_{\xi^k} = \Phi^k = \omega^{kl}(P_l - \theta_l). \tag{C.1}$$

Then, (4.6) takes the form

$$\{\Phi^k, \Phi^l\}_* = \omega^{kl}, \quad \{f, \Phi^k\}_* = -\omega^{kl}\partial_l f, \tag{C.2}$$

where $f(\xi)$ is any observable on $\mathcal{M}$, but in (C.2) it is considered as a function on $T^*\mathcal{M}$ with the natural extension (see remarks after the equation (4.6)).

Let us add to the function $f(\xi)$ the term linear in constraints Φ^k

$$f(\xi) \to f^{(1)} = f(\xi) + A_l^{(1)}(\xi)\Phi^l \tag{C.3}$$

and choose the functions $A_l^{(1)}(\xi)$ to satisfy the condition

$$\{f^{(1)}, \Phi^k\}_* = B_l^{(1)k}(\xi)\Phi^l. \tag{C.4}$$

This means that the right-hand side of (C.4) should contain the constraints Φ^k only in the first degree. From this condition the functions $A_l^{(1)}(\xi)$ and $B_l^{(1)k}(\xi)$ are defined uniquely

$$A_l^{(1)}(\xi) = -\partial_l f(\xi), \quad B_l^{(1)k}(\xi) = \omega^{kj}\partial^2_{jl} f(\xi). \tag{C.5}$$

It is obvious, that $f^{(1)} = R_f$, and (C.4)-(C.5) are equivalent to (4.18) with constant symplectic matrix ω^{kl}. We can continue this 'deformation' procedure

$$f^{(1)} \to f^{(2)} = f^{(1)} + \frac{1}{2}A_{lj}^{(2)}(\xi)\Phi^l\Phi^j \tag{C.6}$$

demanding

$$\{f^{(2)}, \Phi^k\}_* = B^{(2)k}_{lj}(\xi)\Phi^l\Phi^j .$$

Then, for the functions $A^{(2)}_{lj}(\xi)$ and $B^{(2)k}_{lj}(\xi)$ we have

$$A^{(2)}_{lj}(\xi) = \partial^2_{lj} f(\xi), \quad B^{(2)k}_{lj}(\xi) = -\frac{1}{2}\omega^{ki}\partial^3_{ilj} f(\xi).$$

Generalizing for arbitrary n, we get

$$f(\xi) \to f^{(n)} = f(\xi) + \sum_{a=1}^{n} \frac{1}{a!} A^{(a)}_{k_1\ldots k_a}(\xi)\Phi^{k_1}\ldots\Phi^{k_a}, \tag{C.7}$$

where

$$A^{(a)}_{k_1\ldots k_a}(\xi) = (-)^a \partial^{(a)}_{k_1\ldots k_a} f(\xi) \tag{C.8}$$

and

$$\{f^{(n)}, \Phi^k\}_* = \frac{(-)^{n+1}}{n!}\omega^{kl}\left(\partial^{(n+1)}_{lk_1\ldots k_n} f(\xi)\right)\Phi^{k_1}\ldots\Phi^{k_n}. \tag{C.9}$$

Using this procedure for any observable $f(\xi)$, one can construct a new function $\tilde{f} = \lim f^{(n)}$ $(n \to \infty)$, which commutes with all constraints Φ^k $(k = 1, \ldots, 2N)$, and on the constraint surface ($\Phi^k = 0$ $(k = 1, \ldots, 2N)$) it is equal to $f(\xi)$.

A similar procedure can be accomplished on the quantum level as well, taking into account operators ordering and self-adjoint conditions. But, when the symplectic matrix ω^{kl} depends on coordinates ξ^k, the described procedure fails for some observables $f(\xi)$, even on the classical level. For the illustration, let us consider a simple example on a half plane with coordinates (p, q), $p > 0$, and the canonical 1-form $\theta = pdq$. If we take the coordinates $\xi^1 = p^2/2$, $\xi^2 = q$ (which are global here), then the corresponding constraints $\Phi^1 = p^2 - pP_q$, $\Phi^2 = P_p$ have the commutation relations

$$\{\Phi^2, \Phi^1\} = p + \frac{1}{p}\Phi^1. \tag{C.10}$$

The first deformation of the function $f = q$, as usual, gives $f^{(1)} = R_q = q - P_p$ and we get

$$\{f^{(1)}, \Phi^1\}_* = -\frac{1}{p}\Phi^1, \quad \{f^{(1)}, \Phi^2\}_* = 0. \tag{C.11}$$

Considering the second deformation (C.6)

$$f^{(2)} = f^{(1)} + \frac{1}{2}\left(A_{11}(\xi)(\Phi^1)^2 + 2A_{12}(\xi)\Phi^1\Phi^2 + A_{22}(\xi)(\Phi^2)^2\right)$$

and using commutation relations (C.10)-(C.11), we see that it is impossible to cancel the linear (in the constraints Φ^1 and Φ^2) terms in the Poisson brackets $\{f^{(2)}, \Phi^1\}_*$ and $\{f^{(2)}, \Phi^2\}_*$ simultaneously.

Appendix D

At first, we consider minimization of the product of quadratic fluctuations (see (5.5))

$$U(\Psi) \equiv \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle, \tag{D.1}$$

with the vectors $|\Psi\rangle$ of unit norm

$$\langle\Psi|\Psi\rangle = 1. \tag{D.2}$$

For the minimization of the functional $U(\Psi)$ one can use the variation principle, considering the variation of $|\Psi\rangle$ to be independent of $\langle\Psi|$. Since we have the subsidiary condition (D.2), from the variation of (D.1) we obtain

$$b^2\hat{\Phi}_f^2|\Psi\rangle + a^2\hat{\Phi}_g^2|\Psi\rangle = c|\Psi\rangle, \tag{D.3}$$

where

$$a^2 = \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle, b^2 = \langle\Psi|\hat{\Phi}_g^2|\Psi\rangle. \tag{D.4}$$

Multiplying by $\langle\Psi|$, we get $c = 2a^2b^2$, and the equation (D.3) takes the form

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi\rangle = |\Psi\rangle. \tag{D.5}$$

Thus, the solutions of (D.5) which satisfy the conditions (D.4) can provide minimization of the functional $U(\Psi)$. If there are solutions with different values of the parameters a and b, then we have to choose the solutions with minimal value of the product a^2b^2.

Now we consider minimization of the functional $U_1(\Psi)$ (see (5.6))

$$U_1(\Psi) \equiv \frac{\langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle}{\langle\Psi|\hat{A}|\Psi\rangle^2}. \tag{D.6}$$

For an arbitrary vector $|\Psi\rangle$ and any real parameter ϵ we have

$$\langle\Psi|(\hat{\Phi}_f - i\epsilon\hat{\Phi}_g)(\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi\rangle \geq 0. \tag{D.7}$$

The left-hand side of this inequality is a second order polynomial in ϵ

$$\epsilon^2\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle - \hbar\epsilon\langle\Psi|\hat{A}|\Psi\rangle + \langle\Psi|\hat{\Phi}_f^2|\Psi\rangle,$$

and respectively we have

$$\langle\Psi|\hat{\Phi}_f^2|\Psi\rangle\langle\Psi|\hat{\Phi}_g^2|\Psi\rangle \geq \frac{\hbar^2}{4}\langle\Psi|\hat{A}|\Psi\rangle^2. \tag{D.8}$$

Thus, the minimal value of the functional $U_1(\Psi)$ could be $\hbar^2/4$. If for some ϵ the equation

$$(\hat{\Phi}_f + i\epsilon\hat{\Phi}_g)|\Psi\rangle = 0 \tag{D.9}$$

has a normalizable solution $|\Psi\rangle = |\Psi_\epsilon\rangle$, then, for this $|\Psi_\epsilon\rangle$ we have an equality in (D.7) and (D.8). Respectively, these states $|\Psi_\epsilon\rangle$, provide the minimization of the functional $U_1(\Psi)$. But, as it was indicated in Section 5.1, sometimes equation (D.9) has no normalizable solutions for any real ϵ. In that case, one can consider the minimization problem for the functional $U_1(\Psi)$ by the variation principle, as it was done for the functional $U(\Psi)$ above. Repeating the same procedure, we get the equation

$$\frac{1}{2a^2}\hat{\Phi}_f^2|\Psi\rangle + \frac{1}{2b^2}\hat{\Phi}_g^2|\Psi\rangle - \frac{\hat{A}}{A}|\Psi\rangle = 0, \tag{D.10}$$

where a, b, A are parameters, and the solution $|\Psi\rangle$ should satisfy (D.4) and the additional condition.

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