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COMPARISON THEOREMS FOR DEVIATED DIFFERENTIAL EQUATIONS WITH PROPERTY A

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1. Short Survey of Known Results

Consider the equations

$$u^{(n)}(t) + p(t)u(\tau(t)) = 0, \tag{1.1}$$

$$v^{(n)}(t) + q(t)v(\sigma(t)) = 0, \tag{1.2}$$

where $p, q \in L_{loc}(R_+; R_+)$ and $\tau, \sigma \in C(R_+; R+)$, $\lim_{t\to+\infty} \tau(t) = \lim_{t\to+\infty} \sigma(t) = +\infty$ and σ, τ are nondecreasing.

Comparison theorems play an important role in studying oscillatory properties of differential equations. Sturm [1] was the first who proved a comparison theorem on distribution of zeros for second order linear ordinary differential equations. On the basis of this theorem Kneser [2] obtained effective and somehow optimal criteria for oscillation of all solutions.

For higher order equations, comparison theorems are formulated in terms of the so called Properties A and B, analogues of oscillation of all solutions. In the present note we deal with the case of Property A. Analogous results for Property B will be reported in the nearest future.

Definition 1.1. We say that the equation (1.1) has Property A if any of its proper solutions is oscillatory when n is even and either is oscillatory or satisfies $|u^{(i)}(t)| \downarrow 0$ as $t \uparrow +\infty$ $(i = 1, \dots, n-1)$, when n is odd.

In [3] V. Kondrat'ev first proved a comparison theorem for Property A.

Theorem 1.1 (V. Kondrat'ev). *Let $\tau(t) \equiv \sigma(t) \equiv t$ for $t \in R_+$, $p(t) \geq q(t)$ for $t \in R_+$ and (1.2) have Property A. Then (1.1) also has Property A.*

Basing on this theorem, he obtained an effective condition for (1.1) to have Property A.

Theorem 1.2 (V. Kondrat'ev). *Let $\tau(t) \equiv t$ for $t \in R_+$, $p(t) \geq \frac{M_n+\varepsilon}{t^n}$ for $t \geq 1$, where $\varepsilon > 0$ and $M_n = \max\Big\{ -x(x-1)\cdots(x-n+1) : x \in [n-2, n-1]\Big\}$. Then (1.1) has Property A.*

In this theorem one can not take $\varepsilon = 0$, so it is optimal in a sense.

Generalizing Kondrat'ev's results, T. Chanturia [4] obtained integral conditions.

Theorem 1.3 (T. Chanturia). *Let $\tau(t) \equiv \sigma(t) \equiv t$,*

$$\int_t^{+\infty} s^{n-2} p(s)\, ds \geq \int_t^{+\infty} s^{n-2} q(s)\, ds \quad \textit{for} \quad t \in R_+,$$

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and (1.2) *have Property A. Then* (1.1) *also has Property A.*

Theorem 1.4 (T. Chanturia). *Let* $\tau(t) \equiv t$ *and*

$$\liminf_{t \to +\infty} t \int_t^{+\infty} s^{n-2} p(s)\, ds > M_n,$$

with M_n *defined in Theorem 1.2. Then* (1.1) *has Property A.*

2. Comparison Theorems

Below we give integral comparison theorems of two types for (1.1) and (1.2). Theorems of the first type (Theorems 2.1 and 2.2) generalize T. Chanturia's Theorem 1.3 to deviated differential equations. As to theorems of the second type (Theorems 2.3 and 2.4), as far as we know, they have not been considered earlier even in the case of ordinary differential equations. All the four theorems of this section generalize Kondrat'ev's Theorem 1.1.

Theorem 2.1. *Let* $\sigma(t) \le \tau(t) \le t$, *for sufficiently large* t_0

$$\int_t^{+\infty} \tau^{n-2}(s) p(s)\, ds \ge \int_t^{+\infty} \sigma^{n-2}(s) q(s)\, ds \quad \text{for} \quad t \ge t_0, \tag{2.1}$$

and (1.2) *have Property A. Then* (1.1) *also has Property A.*

Theorem 2.2. *Let* $\tau(t) \ge \max\{t, \sigma(t)\}$ *for* $t \in R_+$. *Let for sufficiently large* t_0

$$\int_t^{+\infty} s^{n-2} p(s)\, ds \ge \int_t^{+\infty} s^{n-2} q(s)\, ds \quad \text{for} \quad t \ge t_0,$$

be satisfied if n *is even, and* (2.1),

$$\int_t^{+\infty} s^{n-3} \tau(s) p(s)\, ds \ge \int_t^{+\infty} s^{n-3} \sigma(s) q(s)\, ds \quad \text{for} \quad t \ge t_0\ .$$

and

$$\int_0^{+\infty} s^{n-1} p(s)\, ds = +\infty. \tag{2.2}$$

be satisfied if n *is odd. Let, moreover,* (1.2) *have Property A. Then* (1.1) *also has Property A.*

Theorem 2.3. *Let* $\tau(t) \le \max\{t, \sigma(t)\}$ *for* $t \in R_+$ *and for sufficiently large* t_0 *there exist* $t_1 = t_1(t_0) \ge t_0$ *such that*

$$\int_{t_0}^{t} s\tau^{n-1}(s) p(s)\, ds \ge \int_{t_0}^{t} s\sigma^{n-1}(s) q(s)\, ds \quad \text{for} \quad t \ge t_1\ . \tag{2.3}$$

Let, moreover, (1.2) *have Property A. Then* (1.1) *also has Property A.*

Theorem 2.4. *Let* $t \le \tau(t) \le \sigma(t)$ *for* $t \in R_+$ *and for sufficiently large* t_0 *there exists* $t_1 = t_1(t_0)$ *such that*

$$\int_{t_0}^{t} s^{n-1} \tau(s) p(s)\, ds \ge \int_{t_0}^{t} s^{n-1} \sigma(s) q(s)\, ds \quad \text{for} \quad t \ge t_1,$$

if n *is even, and* (2.2), (2.3) *and*

$$\int_{t_0}^{t} s^{n-2} \tau^2(s) p(s)\, ds \ge \int_{t_0}^{t} s^{n-2} \sigma^2(s) q(s)\, ds \quad \text{for} \quad t \ge t_1,$$

be fulfilled if n is odd. Let, moreover, (1.2) *have Property A. Then* (1.1) *also has Property A.*

3. Effective Criteria for Property A

Using the above comparison theorems and some results of [5] (see section 7 of Chapter 2), we can obtain new effective criteria for (1.1) to have Property A.

Let $l \in \{1, \dots, n-1\}$, $\alpha \in]0, +\infty[$. Denote

$$M_l(\alpha) = \max\Big\{ -\alpha^{-x} x(x-1)\cdots(x-n+1) : l-1 \le x \le l \Big\}.$$

Theorem 3.1. *Let $\alpha \in]0,1]$, $\alpha t \le \tau(t) \le t$ for $t \in R_+$ and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} \tau^{n-2}(s) p(s)\, ds > \alpha^{n-2} M_{n-1}(\alpha).$$

Then (1.1) *has Property A.*

Theorem 3.2. *Let $\alpha \in [1, +\infty[$, $\tau(t) \ge \alpha t$ for $t \in R_+$. Let*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-2} p(s)\, ds > M_1(\alpha)$$

be fulfilled if n is even, and (2.2) *and*

$$\liminf_{t\to+\infty} t \int_t^{+\infty} s^{n-3} \tau(s) p(s)\, ds > \alpha M_2(\alpha),$$

$$\liminf_{t\to+\infty} t \int_t^{+\infty} \tau^{n-2}(s) p(s)\, ds > \alpha^{n-2} M_{n-1}(\alpha)$$

be fulfilled if n is odd. Then (1.1) *has Property A.*

Theorem 3.3. *let $\alpha \in]0,1]$, $\tau(t) \le \alpha t$ for $t \in R_+$ and*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s\tau^{n-2}(s) p(s)\, ds > \alpha^{n-1} M_{n-1}(\alpha).$$

Then (1.1) *has Property A.*

Theorem 3.4. *Let $\alpha \in [1+\infty[$, $t \le \tau(t) \le \alpha t$. Let*

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^{n-1} \tau(s) p(s)\, ds > \alpha M_1(\alpha)$$

be fulfilled if n is even, and

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s\tau^{n-1}(s) p(s)\, ds > \alpha^{n-1} M_{n-1}(\alpha),$$

$$\liminf_{t\to+\infty} t^{-1} \int_0^t s^{n-2} \tau^2(s) p(s)\, ds > \alpha^2 M_2(\alpha)$$

be fulfilled. Then (1.1) *has Property A.*

Remark. In Theorems 3.1.–3.4 strict inequalities cannot be replaced by nonstrict ones. Moreover, none of the hypotheses of Theorems 3.2–3.4 can be omited without affecting their validity.

Now we consider the case where the condition $\tau(t) \ge \alpha t$ ($\tau(t) \le \alpha t$) is violated.

Theorem 3.5. *Let* $\sigma(t)\le\tau(t)\le t$ *for* $t\in R_+$, $\frac{\sigma(t)}{t}\downarrow 0$ *as* $t\uparrow+\infty$, *and the conditions* (2.2),

$$\int^{+\infty}\frac{dt}{t\sigma(t)}<+\infty,\quad \liminf_{t\to+\infty}\varphi(t)\int_t^{+\infty}\tau^{n-2}(s)p(s)\,ds>0$$

be fulfilled, where $\varphi(t)=\Big(\int_t^{+\infty}\frac{ds}{s\sigma(s)}\Big)^{-1}$. *Then* (1.1) *has Property A.*

Corollary 3.1. *Let* $\alpha\in]0,1[$, $t^\alpha\le\tau(t)\le t$ *for* $t\in R_+$ *and*

$$\liminf_{t\to+\infty}t^\alpha\int_t^{+\infty}\tau^{n-2}(s)p(s)\,ds>0.$$

Then (1.1) *has Property A.*

Theorem 3.6. *Let* $t\le\tau(t)\le\sigma(t)$ *for* $t\in R_+$, $\frac{\sigma(t)}{t}\uparrow+\infty$ *as* $t\uparrow+\infty$, *and* (2.2) *and*

$$\liminf_{t\to+\infty}\varphi(t)\int_0^t s^{n-1}\tau(s)p(s)\,ds>0$$

be fulfilled, where $\varphi(t)=\Big(\int_1^t\frac{\sigma(s)ds}{s}\Big)^{-1}$. *Then* (1.1) *has Property A.*

Corollary 3.2. *Let* $\alpha\in]1,+\infty[$, $t\le\tau(t)\le t^\alpha$ *for* $t\in R_+$ *and*

$$\liminf_{t\to+\infty}t^{-\alpha}\int_0^t s^{n-1}\tau(s)p(s)\,ds>0.$$

Then (1.1) *has Property A.*

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