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ON THE SOLVABILITY OF THE WEIGHTED INITIAL VALUE PROBLEM FOR HIGH ORDER EVOLUTION SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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In the present paper on the basis of the results obtained in [1, 2] optimal, in a certain sense, sufficient conditions for solvability of the weighted initial value problem

$$u^{(n)}(t) = f(u)(t), \tag{1}$$

$$\lim_{t\to a} \frac{u^{(k)}(t)}{h^{(k)}(t)} = 0 \quad (k = 0, \dots, n-1) \tag{2}$$

are established, where $f \in C^{n-1}([a,b];\mathbb{R}^m) \to L_{loc}(]a,b];\mathbb{R}^m)$ is a continuous Volterra operator and $h : [a,b] \to [0,+\infty[$ is an $(n-1)$-times continuously differentiable function such that

$$h^{(k)}(a) = 0 \ \ (k = 0, \dots, n-2), \quad h^{(n-1)}(t) > 0 \ \text{ for } \ a < t \le b. \tag{3}$$

The problem (1), (2) for the case $n = 1$ has been investigated in [1, 2]. Therefore below we will assume that $n \ge 2$.

Throughout the paper the use will be made of the following notation.

$\mathbb{R}^m$ is the space of m-dimensional column vectors $x = (x_i)_{i=1}^m$ with real components x_i $(i = 1, \dots, m)$ and the norm $\|x\| = \sum\limits_{i=1}^m |x_i|$.

$\mathbb{R}^m_\rho = \{x \in \mathbb{R}^n : \ \|x\| \le \rho\}$.

If $x = (x_i)_{i=1}^m \in \mathbb{R}^m$, then $\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^m$.

$x \cdot y$ is the scalar product of the vectors x and $y \in \mathbb{R}^m$.

$C^{n-1}([a,b];\mathbb{R}^m)$ is the space of $(n-1)$-times continuously differentiable vector functions $x : [a,b] \to \mathbb{R}^m$ with the norm

$$\|x\|_{C^{n-1}} = \max\left\{ \sum_{k=1}^{n-1} \|x^{(k-1)}(t)\| : \ \ a \le t \le b \right\}.$$

$C^{n-1}_h([a,b];\mathbb{R}^m)$ is the set of $u \in C^{n-1}([a,b];\mathbb{R}^m)$ such that

$$\sup\left\{ \frac{\|u^{(k)}(t)\|}{h^{(k)}(t)} \ : \ \ a < t \le b \right\} < +\infty \quad (k = 0, \dots, n-1).$$

$C^{n-1}_{h,\rho}([a,b];\mathbb{R}^m)$ is the set of $u \in C^{n-1}([a,b];\mathbb{R}^m)$ satisfying the inequalities

$$|u^{(k)}(t)| \le \rho h^{(k)}(t) \ \text{ for } \ a < t \le b \quad (k = 0, \dots, n-1).$$

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If $x :]a,b] \to \mathbb{R}^m$ is a bounded function and $a \le s < t \le b$, then

$$\nu(x)(s,t) = \sup\Big\{\|x(\xi)\| : \ s < \xi < t\Big\}.$$

$L_{loc}(]a,b];\mathbb{R}^m)$ is the space of vector functions $x :]a,b] \to \mathbb{R}^m$ which are summable on each segment from $]a,b]$ with the topology of convergence in the mean on each segment from $]a,b]$.

Definition 1. $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{loc}(]a,b];\mathbb{R}^m)$ is called *a Volterra operator* if the equality $f(x)(t) = f(y)(t)$ holds almost everywhere on $]a,t_0[$ for any $t_0 \in]a,b]$ and any vector functions x and $y \in C^{n-1}([a,b];\mathbb{R}^m)$ satisfying the condition $x(t) = y(t)$ for $a \le t \le t_0$.

Definition 2. We will say that the operator $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{loc}(]a,b];\mathbb{R}^m)$ satisfies the local Carathéodory conditions if it is continuous and there exists a nondecreasing with respect to the second argument function $\gamma :]a,b] \times [0,+\infty[\to [0,+\infty[$ such that $\gamma(\cdot,\rho) \in L_{loc}(]a,b];\mathbb{R})$ for any $\rho \in]0,+\infty[$, and the inequality

$$\|f(x)(t)\| \le \gamma(t, \|x\|_{C^{n-1}})$$

is fulfilled for any $x \in C^{n-1}([a,b];\mathbb{R}^m)$ almost everywhere on $]a,b[$.

Definition 3. If $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{loc}(]a,b];\mathbb{R}^m)$ is a Volterra operator and $b_0 \in]a,b]$, then:

(i) for any $u \in C^{n-1}([a,b_0];\mathbb{R}^m)$ by $f(u)$ is understood the vector function given by the equality $f(u)(t) = f(\overline{u})(t)$ for $a \le t \le b_0$, where

$$\overline{u}(t) = \begin{cases} u(t) & \text{for} \quad a \le t \le b_0, \\ \displaystyle\sum_{k=1}^{n} \frac{(t-b_0)^{k-1}}{(k-1)!}\, u^{(k-1)}(b_0) & \text{for} \quad b_0 < t \le b; \end{cases}$$

(ii) a function $u \in C^{n-1}([a,b_0];\mathbb{R}^m)$ is called a *solution of the equation* (1) on the segment $[a,b_0]$ if $u^{(n-1)}$ is absolutely continuous on each segment contained in $]a,b_0]$ and $u^{(n)}(t) = f(u)(t)$ almost everywhere on $]a,b_0[$;

(iii) a solution u of the equation (1) on the segment $[a,b_0]$, satisfying the initial conditions (2) is called a *solution of the problem* (1), (2) on the segment $[a,b_0]$.

Definition 4. The problem (1), (2) is said to be *locally solvable* (*globally solvable*) if it has at least one solution on a segment $[a,b_0] \subset [a,b[$ (on the segment $[a,b]$).

In what follows, we will assume that $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{loc}(]a,b];\mathbb{R}^m)$ is a continuous Volterra operator satisfying the local Carathéodory conditions.

Theorem 1. *Let there exist a positive number ρ and summable functions $p_k : [a,b] \to [0,+\infty[$ $(k = 0,\dots,n-1)$ and $q : [a,b] \to [0,+\infty[$ such that*

$$\limsup_{t\to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_a^t p_k(s)\,ds\right) < 1, \quad \lim_{t\to a}\left(\frac{1}{h^{(n-1)}(t)} \int_a^t q(s)\,ds\right) = 0 \tag{4}$$

and for any $u \in C^{n-1}_{h,\rho}([a,b];\mathbb{R}^m)$ the inequality

$$f(u)(t)\cdot \operatorname{sgn}(u^{(n-1)}(t)) \le \sum_{k=0}^{n-1} p_k(t)\nu\Big(\frac{u^{(k)}}{h^{(k)}}\Big)(a,t) + q(t) \tag{5}$$

is fulfilled almost everywhere on $]a,b[$. Then the problem (1), (2) is locally solvable.

Proof. For any $x \in C([a,b];\mathbb{R}^m)$ assume

$$w(x)(t) = \frac{1}{(n-2)!}\int_a^t (t-s)^{n-2} x(s)\,ds, \quad \widetilde{f}(x)(t) = f(w(x))(t). \tag{6}$$

Then by (3)

$$h(t) = w(h^{(n-1)})(t) = \frac{1}{(n-2)!} \int_a^t (t-s)^{n-2} h^{(n-1)}(s)\, ds. \tag{7}$$

Obviously, $\widetilde{f} : C([a,b];\mathbb{R}^m) \to L_{loc}(]a,b];\mathbb{R}^m)$ is a continuous Volterra operator satisfying the local Carathéodory conditions.

Assume now that $y \in C([a,b];\mathbb{R}^m)$, $\|y\|_C \le \rho$ and

$$u(t) = w(h^{(h-1)}y)(t). \tag{8}$$

Then by virtue of (6) and (7)

$$\begin{aligned} \|u^{(k)}(t)\| &= \frac{1}{(n-2-k)!} \left\| \int_a^t (t-s)^{n-2-k} h^{(n-1)}(s) y(s)\, ds \right\| \le \\ &\le \frac{1}{(n-2-k)!} \left(\int_a^t (t-s)^{n-2-k} h^{(n-1)}(s)\, ds \right) \nu(y)(a,t) = \\ &= h^{(k)}(t)\nu(y)(a,t) \ \text{ for } \ a < t \le b \quad (k = 0, \dots, n-2), \end{aligned}$$

and

$$u^{(n-1)}(t) = h^{(n-1)}(t) y(t), \quad \|u^{(n-1)}(t)\| \le h^{(n-1)}(t)\nu(y)(a,t) \ \text{ for } \ a < t \le b.$$

Therefore

$$u \in C^{n-1}_{h,\rho}([a,b]), \quad \operatorname{sgn}(u^{(n-1)}(t)) = \operatorname{sgn}(y(t)), \tag{9}$$

$$\nu\left(\frac{u^{(k)}}{h^{(k)}}\right)(a,t) \le \nu(y)(a,t) \ \text{ for } \ a < t \le b \quad (k = 0, \dots, n-1). \tag{10}$$

On the basis of the conditions (5), (6) and (8)–(10), almost everywhere on $]a,b[$ the inequality

$$\begin{aligned} &\widetilde{f}(h^{(n-1)}y)(t) \cdot \operatorname{sgn}(y(t)) = f(u)(t) \cdot \operatorname{sgn}(u^{(n-1)}(t)) \le \\ &\le \sum_{k=0}^{n-1} p_k(t) \nu\left(\frac{u^{(k)}}{h^{(k)}}\right)(a,t) + q(t) \le \sum_{k=0}^{n-1} p_k(t)\nu(y)(a,t) + q(t), \end{aligned}$$

is fulfilled, that is,

$$\widetilde{f}(h^{(n-1)}y)(t) \operatorname{sgn}(y(t)) \le p(t)\nu(y)(a,t) + q(t), \quad \text{where} \quad p(t) = \sum_{k=0}^{n-1} p_k(t).$$

On the other hand, as it follows from (4),

$$\limsup_{t \to a} \left(\frac{1}{h^{(n-1)}(t)} \int_a^t p(s)\, ds \right) < 1.$$

Hence all the conditions of Theorem 2.1 from [1] are fulfilled for the problem

$$\frac{dx(t)}{dt} = \widetilde{f}(x)(t), \quad \lim_{t \to a} \frac{x(t)}{h^{(n-1)}(t)} = 0. \tag{11}$$

Therefore this problem is locally solvable.

Let x be a solution of the problem (11) on a segment $[a, b_0]$, and $u(t) = w(x)(t)$. Then, owing to (6), the function u is a solution of the problem (1), (2) on $[a, b_0]$. □

Applying Corollary 1 of [2] and repeating the arguments used in proving Theorem 1, we convince ourselves that the following theorem is valid.

Theorem 2. *Let for any* $u \in C_h^{n-1}([a,b];\mathbb{R}^n)$ *the inequality*

$$f(u)(t)\cdot \operatorname{sgn}(u^{(n-1)}(t)) \le \sum_{k=0}^{n-1} p_k(t,\rho_0(u)(t))\nu\Big(\frac{u^{(k)}}{h^{(k)}}\Big)(a,t)+q(t,\rho_0(u)(t))$$

be fulfilled almost everywhere on $]a,b[$, *where*

$$\rho_0(u)(t)=\sum_{j=0}^{n-1}\nu\Big(\frac{u^{(j)}}{h^{(j)}}\Big)(a,\tau(t)),$$

$\tau:[a,b]\to[a,b]$ *is a continuous function,* p_k $(k=0,\dots,n-1)$ *and* $q:[a,b]\times[0+\infty[\to[0,+\infty[$ *are summable with respect to the first argument and continuous and nondecreasing with respect to the second argument. Let furthermore* $\tau(t)<t$ *for* $a<t\le b$ *and*

$$\limsup_{t\to a}\Big(\frac{1}{h^{(n-1)}(t)}\sum_{k=0}^{n-1}\int_a^t p_k(s,\rho)\,ds\Big)<1,\quad \lim_{t\to a}\Big(\frac{1}{h^{(n-1)}(t)}\int_a^t q(s,\rho)\,ds\Big)=0$$

for some positive constant ρ. *Then the problem* (1), (2) *is globally solvable.*

A particular case of the equation (1) is the vector differential equation with delay

$$u^{(n)}(t)=f_0\Big(t,u(\tau_{10}(t)),...,u^{(n-1)}(\tau_{1\,n-1}(t)),...,u(\tau_{l0}(t)),...,u^{(n-1)}(\tau_{l\,n-1}(t))\Big),\ (12)$$

where $f_0\ :]a,b]\times\mathbb{R}^{lmn}\to\mathbb{R}^m$ satisfies the local Carathéodory conditions, and $\tau_{ik}:[a,b]\to[a,b]$ are measurable functions such that $\tau_{ik}(t)\le t$ for $a\le t\le b$ $(i=1,\dots,l;$ $k=0,\dots,n-1)$.

Theorems 1 and 2 result in the following

Corollary 1. *Let* $\tau_{l\,n-1}(t)\equiv t$ *and there exist a positive number* ρ, *summable functions* $p_{ik}:[a,b]\to[0,+\infty[$ $(i=1,\dots,l;\ k=0,\dots,n-1)$ *and* $q:[a,b]\to[0,+\infty[$ *such that*

$$\limsup_{t\to a}\Big(\frac{1}{h^{(n-1)}(t)}\sum_{k=0}^{n-1}\sum_{i=1}^{l}\int_a^t p_{ik}(s)\,ds\Big)<1,\quad \lim_{t\to a}\Big(\frac{1}{h^{(n-1)}(t)}\int_a^t q(s)\,ds\Big)=0.$$

Let furthermore the inequality

$$f_0\Big(t,h(\tau_{10}(t))x_{10},\dots,h^{(n-1)}(\tau_{1n}(t))x_{1\,n-1},\dots,$$
$$h(\tau_{l0}(t))x_{l0},\dots,h^{(n-1)}(\tau_{l\,n-1}(t))x_{l\,n-1}\Big)\cdot\operatorname{sgn}(x_{l\,n-1})\le\sum_{k=0}^{n-1}\sum_{i=1}^{l}p_{ik}(t)\|x_{ik}\|+q(t)$$

be fulfilled on $]a,b]\times\mathbb{R}_\rho^{lmn}$. *Then problem* (12), (2) *is locally solvable.*

Corollary 2. *Let there exist a number* $l_0\in\{1,\dots,l-1\}$ *and a continuous function* $\tau:[a,b]\to[a,b]$ *such that* $\tau_{l_0\,n-1}(t)\equiv t$,

$$\tau_{ik}(t)\le\tau(t)<t\ \text{ for }\ a<t\le b\quad (i=l_0-1,\dots,l;\ \ k=0,\dots,n-1)$$

and let the inequality

$$f_0\Big(t,h(\tau_{10}(t))x_{10},\dots,h^{(n-1)}(\tau_{1n}(t))x_{1\,n-1},\dots,$$
$$h(\tau_{l0}(t))x_{l0},\dots,h^{(n-1)}(\tau_{l\,n-1}(t))x_{l\,n-1}\Big)\cdot\operatorname{sgn}(x_{l_0\,n-1})\le$$
$$\le\sum_{k=0}^{n-1}\sum_{i=1}^{l_0}p_{ik}\Big(t,\sum_{j=0}^{n-1}\sum_{i=l_0+1}^{l}\|x_{ij}\|\Big)|x_{ik}|+q\Big(t,\sum_{j=0}^{n-1}\sum_{i=l_0+1}^{l}\|x_{ij}\|\Big)$$

be fulfilled on $]a,b] \times \mathbb{R}^{lmn}$*, where the functions* $p_{ik} : [a,b] \times [0,+\infty[\to [0,+\infty[$ $(i = 1,\dots,l_0;\ k = 0,\dots,n-1)$*,* $q : [a,b] \times [0,+\infty[\to [0,+\infty[$ *are summable with respect to the first argument and continuous and nondecreasing with respect to the second argument. Let furthermore*

$$\limsup_{t\to a}\left(\frac{1}{h^{(n-1)}(t)}\sum_{k=0}^{n-1}\sum_{i=1}^{l_0}\int_a^t p_{ik}(s,\rho)\,ds\right) < 1, \quad \lim_{t\to a}\left(\frac{1}{h^{(n-1)}(t)}\int_a^t q(s,\rho)\,ds\right) = 0$$

for some positive constant ρ*. Then problem* (12), (2) *is globally solvable.*

Remark 1. Under the conditions of the above-mentioned propositions the right sides of differential equations may have singularities of arbitrary orders. Indeed, as an example let us consider on the interval $[a,b]$ the scalar differential equation

$$u^{(n)}(t) = \sum_{k=0}^{n-1}\left[\frac{\alpha_k}{t^{(\lambda-k)\mu_k+n-\lambda}}\,u^{(k)}(t^{\mu_k}) + \frac{\beta_k}{t^{(\lambda-k)\mu_k\gamma_k+n-k}}|u^{(k)}(t^{\mu_k})|^{\gamma_k}\right] - \\ - \sum_{k=1}^{k_0} g_k\Big(t,u(t),\dots,u^{(n-1)}(t)\Big)u^{(n-1)}(t) + ct^{\lambda_0-n} \tag{13}$$

with the initial conditions

$$\lim_{t\to a}\frac{u^{(k)}(t)}{t^{\lambda-k}} = 0 \quad (k = 0,\dots,n-1) \tag{14}$$

where $b \in]0,1[$, α_k and $\beta_k \in \mathbb{R}$, $\mu_k > 1$, $\gamma_k > 1$, $c \in \mathbb{R}$, $\lambda_0 > \lambda$, $g_k :]0,b] \times \mathbb{R}^n \to [0,+\infty[$ are continuous functions. By Corollary 2, for the global solvability of problem (13), (14) it is sufficient that

$$\sum_{k=0}^{n-1}\frac{|\alpha_k|}{(\lambda-k)\cdots(\lambda-n+1)} < 1.$$

Remark 2. There exists an example which shows that condition (4) in Theorem 1 is optimal and it cannot be replaced by the condition

$$\limsup_{t\to a}\left(\frac{1}{h^{(n-1)}(t)}\sum_{k=0}^{n-1}\int_0^t p_k(s)\,ds\right) \le 1.$$

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