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LAPPO–DANILEVSKI SYSTEMS UNDER LYAPUNOV TRANSFORMATIONS

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Let $A(t)$ be an $n \times n$ matrix of real-valued continuous and bounded functions of real variable t on the non-negative half-line. Consider the linear systems

$$Dx = A(t)x, \quad t \geq 0, \quad D = d/dt. \tag{1_A}$$

It is well known that if A is a Lappo-Danulevskiĭ matrix, i.e. there exists $s \geq 0$ such that for all $t \geq 0$

$$A(t)\int_s^t A(u)du = \int_s^t A(u)du A(t), \tag{2}$$

then a fundamental solution matrix $X_s(t)$ of (1_A) ($X_s(s) = E$, E is the identity matrix) can be represented as

$$X_s(t) = \exp \int_s^t A(u)du. \tag{3}$$

This simple representation of the fundamental solution matrix does explain the fact that the class of Lappo-Danulevskiĭ systems is one of the main and interesting class of linear systems. In this paper we consider a problem of reducibility of (1_A) to the Lappo-Danulevskiĭ system and to the system (1_B) with functional commutative matrix B, where for all $s \geq 0$ and $t \geq 0$

$$B(t)B(s) - B(s)B(t) = [B(t), B(s)] = 0. \tag{4}$$

(The symbol $[.,.]$ is used to indicate the Lie brackets throughout this paper). It is obvious that system (1_B) with functional commutative matrix B is a special case of the Lappo-Danulevskiĭ system.

Note that condition (2) is sufficient but not necessary to represent fundamental solution matrix in the form (3) (see [1],[2]). To verify this fact it is sufficient to consider the system (1_A) with the matrix $A(t) = (a_{ij}(t))$, $i, j = 1, 2$, where $a_{11}(t) = a_{33}(t) = -\mu a(t)$, $a_{31}(t) = -a_{13}(t) = \nu a(t)$, $a_{21}(t) = b(t)$, $a_{12}(t) = a_{22}(t) = a_{32}(t) = a_{23}(t) = 0$, $\mu \pm i\nu$ are roots of the equation $\exp z - z - 1 = 0$, a and b are infinitely differentiable non-analytic functions such that

$$\int_0^s a(u)du = 1, \quad a(t) = 0 \ \ for \ \ t \geq s > 0,$$

$$b(t) = \begin{cases} 0, & t \in [0, s[, \\ b_k(t) \not\equiv 0, & t \in [t_{2k}, t_{2k+1}[, \\ 0 & t \in [t_{2k+1}, t_{2k+2}[, \quad k = 0, 1, ..., \end{cases}$$

((t_k) is an arbitrary sequence of positive numbers such that $t_{k+1} > t_k$ and $t_k \to +\infty$ as $k \to +\infty$.) In this case the fundamental solution matrix $X_0(t)$ of (1_A) may be represented

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as (3) with $s = 0$ but $[A(t), \int_0^t A(u)du] \not\equiv 0$, i.e. $A(t)$ is not a Lappo-Danulevskiĭ matrix. However (4) is the necessary and sufficient condition (see [3]) to represent Cauchy's matrix of (1_B) as $K(t,s) = \exp \int_s^t B(u)du$.

It is well known (see, e.g., [4, p. 274]) that any system (1_A) is an almost reducible to some diagonal system. It is trivial that any diagonal matrix is a functional commutative matrix. Unfortunately, quite different is the case of linear systems under Lyapunov transformations.

A linear transformation $x = L(t)y$ is a Lyapunov transformation if $L(t)$ is a Lyapunov matrix, i.e.

$$\max\{\sup_{t\geq 0} \|L(t)\|, \sup_{t\geq 0} \|L^{-1}(t)\|, \sup_{t\geq 0} \|DL(t)\|\} < +\infty.$$

We follow Yu.S.Bogdanov (see [5]) and say that linear system (1_A) is asymptotically equivalent to (1_B) if there exist a Lyapunov transformation reducing (1_A) to (1_B).

Theorem 1. *There exists a linear system which is not asymptotically equivalent to any system with functional commutative matrix of coefficients.*

Theorem 2. *There exists a linear system which is not asymptotically equivalent to any Lappo-Danulevskiĭ system.*

To prove Theorem 1 and Theorem 2 it is sufficient (see [6,7]) to consider the system (1_A) with $A(t) = diag\{A_0(t), A_1(t)\}$, where $A_0(t) = (a_{ij}(t))$, $i,j = 1,2$, $a_{11}(t) = a_{21}(t) \equiv 0$, $a_{12}(t) = 1$, $a_{22}(t) = (t+1)^{-1}$, $A_1(t)$ is the $(n-2)\times(n-2)$ identity matrix, and to use the following lemmas which describe the structure and the zero distribution for the integrals of the Lappo-Danulevskiĭ matrices.

Lemma 1. *Let the scalar functions φ, ψ be continuous on the half-interval $[a,\, c[$, $a < c \leq +\infty$, and $\int_a^t \psi(u)du \neq 0$, $\forall\ t \in]b,\, c[$, for some b, $a \leq b < c$. If*

$$\varphi(t) \int_a^t \psi(u)du = \psi(t) \int_a^t \varphi(u)du,\ \forall\ t \in [a,\, c[,$$

then there exists δ such that $\varphi(t) = \delta\psi(t)$, $\forall\ t \in [b,\, c[$.

Lemma 2. *Let the scalar functions φ, ψ be continuous on $[a,\, +\infty[$ and*

$$\varphi(t) \int_a^t \psi(u)du = \psi(t) \int_a^t \varphi(u)du, \quad \forall\, t \geq a.$$

If $\sup A = \sup B = +\infty$, where $A = \{\alpha \geq a \,|\, \int_a^\alpha \varphi(u)du = 0\}$ and $B = \{\beta \geq a \,|\, \int_a^\beta \psi(u)du = 0\}$, then $\sup\{A \bigcap B\} = +\infty$.

However, the system mentioned above is a regular system and it can be reduced (Basov-Grobman-Bogdanov's criterion, [8, p. 77]) to the system with functional commutative coefficients by generalized Lyapunov transformation $x = L(t)y$ with the matrix L such that

$$\overline{\lim_{t\to+\infty}}\, t^{-1} \ln \|L(t)\| = \overline{\lim_{t\to+\infty}}\, t^{-1} \ln \|L^{-1}(t)\| = 0.$$

But even if we expand the set of transformations up to the set of generalized Lyapunov's transformations there are statements which are similar to Theorem 1 and Theorem 2.

Theorem 3. *There exists a linear system which is not generalized asymptotically equivalent to any system with functional commutative matrix of coefficients.*

Theorem 4. *There exists a linear system which is not generalized asymptotically equivalent to any Lappo-Danulevskiĭ system.*

The proofs of these theorems are based on Lemma 1, Lemma 2 and on the following lemmas.

Lemma 3. *Let the scalar functions φ_i, $i = 1,2,3$, be continuous on $[a,\, +\infty[$ and*

$$\varphi_i(t) \int_a^t \varphi_j(u)du = \varphi_j(t) \int_a^t \varphi_i(u)du, \quad \forall\, t \geq a, \quad \forall\, i,j = 1,2,3, i \neq j.$$

If $\sup A_1 = \sup A_2 = \sup A_3 = +\infty$, *then* $\sup\{A_1 \bigcap A_2 \bigcap A_3\} = +\infty$, *where* $A_i = \{\alpha \geq a \mid \int_a^\alpha \varphi_i(u)du = 0\}$, $i = 1, 2, 3$.

Lemma 4. *If* $L(t)$ *is a generalized Lyapunov matrix, then*

$$\lim_{t\to+\infty} t^{-1}\ln\|\det L(t)\| = \lim_{t\to+\infty} t^{-1}\ln\|\det L^{-1}(t)\| = 0.$$

Lemma 5. *If* $f(t) = 2 + \sin(\mu\ln t) + \mu\cos(\mu\ln t)$, *where* $\mu > 0$, *then*

$$\underline{\lim}_{t\to+\infty} \int_s^t (\exp \int_s^\tau f(\sigma)d\sigma)d\tau \geq 3\exp(-2\pi/\mu).$$

To prove Theorem 3 and Theorem 4 it is sufficient to consider the system (1_A) with $A(t) = (a_{ij}(t))$, $i, j = 1, 2$, $a_{12}(t) = 1$, $a_{21}(t) = 0$, $a_{22}(t) - a_{11}(t) = 2 + \sin(\mu\ln(t+1)) + \mu\cos(\mu\ln(t+1))$, where $\mu > 0$, $3\exp(-2\pi/\mu) > 2$. In this case (1_A) is generalized asymptotically equivalent neither to a system with functional commutative matrix of coefficients nor to a Lappo-Danulevskiĭ system.

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