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ON THE ENUMERABLE SET OF DIFFERENT CHARACTERISTIC SETS OF SOLUTIONS OF A PFAFFIAN LINEAR SYSTEM

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Consider the Pfaffian linear system

$$\partial x/\partial t_i = A_i(t)t, \quad x \in R^n, \quad t = (t_1, t_2) \in R^2_+, \tag{1}$$

with bounded continuously differentiable matrices $A_1(t)$ and $A_2(t)$ satisfying the following condition of complete integrability:

$$\partial A_1(t)/\partial t_2 + A_1(t)A_2(t) = \partial A_2(t)/\partial t_1 + A_2(t)A_1(t), \quad t \in R^2_+.$$

It is well known [1, p. 34] that the ordinary linear system $dx/dt = A(t)x$, $x \in R^n$, $t \in R^1_+$, with bounded piecewise continuous coefficients has no more than n different characteristic exponents. Let $\lambda[x] = \lambda \in R^2$ be a characteristic vector [2 – 4] of a nontrivial solution x: $R^2_+ \to R^n \backslash \{0\}$ of (1) defined by

$$L_x(\lambda) \equiv \overline{\lim_{t\to\infty}}\, [ln\|x(t)\| - (\lambda, t)]/\|t\| = 0, \; L_x(\lambda - \varepsilon e_i) > 0, \; \forall \varepsilon > 0, \; i = 1, 2.$$

For the characteristic set $\Lambda_x = \bigcup \lambda[x]$ of this solution which is the most natural analog of Lyapunov's characteristic exponent of a one variable vector-function, the essential initial problem about possible number of different * characteristic sets Λ_x of all nontrivial solutions x of (1) remained open. Note also that the set $\{P_x\}$ of different lower characteristic sets $P_x = \bigcup p[x]$ of all nontrivial solutions x of (1) composed of lower characteristic vectors [5, 6] $p[x] = p \in R^2$ defined by

$$l_x(p) \equiv \underline{\lim_{t\to\infty}}\, [ln\|x(t)\| - (p, t)]/\|t\| = 0, \quad l_x(p + \varepsilon e_i) < 0, \quad \forall \varepsilon > 0, \quad i = 1, 2,$$

is nonenumerable and, moreover, the set of the lower characteristic vectors $\bigcup\limits_{x\neq 0} P_x$ of (1) has a positive planar Lebesgue measure [5, 6].

It holds the following

Theorem. For any sequence $C = \{c_m\}$ of pairwise noncollinear vectors there is a complete integrable two-dimensional system (1) with bounded infinitely differentiable coefficients such that all of its solutions $x(t, c_m)$, $m \in N$, have pairwise different characteristic sets $\Lambda(m)$ with a positive linear Lebesgue measure, If $x(t)$ is a solution of (1) linearly independent with any of $x(t, c_m)$, $c_m \in C$, then its characteristic set $\Lambda_x = \underset{m\to\infty}{Lim}\, \Lambda(m)$ also has a positive measure.

1. **Construction of the required system. The preliminary notes.** To an enumerable set $C \subset R^2 \backslash \{0\}$ of the vectors $c_m = (c^1_m, c^1_m) \in R^2$ assign the enumerable set $\alpha = \{\alpha_m\} \subset R$ of different numbers $\alpha_m \equiv -c^2_m/c^1_m \in \in (-\infty, \infty)$, the ratios of the

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*The characteristic sets Λ_x and Λ_y of the solutions $x \neq 0$ and $y \neq 0$ of (1) are different if $\Lambda_x \bigcap \Lambda_y \neq \Lambda_x \bigcup \Lambda_y$.

components of the vector c_m. Without loss of generality it can be assumed that first components c_m^1 of c_m are nonzero.

In the closed first quarter R_+^2 of the plane R^2 we will build the required Pfaffian system by constructing its fundamental (lower-triangular and infinitely differentiable) system of solutions $X(t) = ((x_{ij}(t))_1^2$ with $x_{12}(t) \equiv 0$ for $t \in R_+^2$.

On the interval $(-\infty, \infty)$ define two infinitely differentiable functions [7, p. 54]

$$e_{01}(\eta;\eta_1,\eta_2) = \begin{cases} 0, & \text{if } \eta \in (-\infty, \eta_1], \\ \exp\{-(\eta-\eta_1)^{-2}\exp[-(\eta-\eta_2)^{-2}]\}, & \text{if } \eta \in (\eta_1,\eta_2), \\ 1, & \text{if } \eta \in [\eta_2, \infty), \end{cases}$$

$$e_{11}(\eta;\eta_1,\eta_2) = \begin{cases} 1, & \text{if } \eta \notin (\eta_1,\eta_2), \\ 1-\exp[-(\eta-\eta_1)^{-2}(\eta-\eta_2)^{-2}], & \text{if } \eta \in (\eta_1,\eta_2), \end{cases}$$

where $-\infty < \eta_1 < \eta_2 < +\infty$ are used for constructing of elements of the matrix $X(t)$.

With the help of the numbers $p_0 = 0$, $q_0 = \varepsilon \in (0, 1/8)$, and $q_k = 1 - 2^{-k}$,$p_k = q_k - 2^{-1-k}$, $k \in N$, define the sectors: the closed ones $S_k = \{t \in R_+^2 : p_k \le t_2/t_1 \le g_k\}$ with $k \ge 0$, the open ones $s_k = \{t \in R_+^2 : q_{k-1} < < t_2/t_1 < p_k\}$ with natural $k \ge 1$, and the also sector $s_0 = \{t \in R_+^2 : 0 \le t_1/t_2 \le \varepsilon\}$.

2. The construction of the diagonal elements of the fundamental system. In R_+^2 define the positive function $x_2(t)$ by

$$\ln x_2(t) = \begin{cases} \sqrt{\varepsilon}t_1 + t_2/\sqrt{\varepsilon} - (\sqrt[4]{\varepsilon t_1^2} - \sqrt[4]{t_2^2/\varepsilon})^2 e_{01}(t_2/t_1;0,\varepsilon), & t \in S_0, \\ \sqrt{\varepsilon}t_2 + t_1/\sqrt{\varepsilon} - (\sqrt[4]{\varepsilon t_2^2} - \sqrt[4]{t_1^2/\varepsilon})^2 e_{01}(t_1/t_2;0,\varepsilon), & t \in s_0, \\ 2\sqrt{t_1t_2}, & t \in R_+^2 \setminus (s_0 \bigcup S_0) \equiv S. \end{cases}$$

Put the function $x_1 : R_+^2 \to [1, +\infty)$ be equal to x_2: 1) on a closed sector $\tilde{S} \subset R_+^2$, which is bounded by the bisectrix $t_2 = t_1$ and the positive coordinate semiaxis $t_1 = 0$; 2) on all sectors S_k, $k \ge 0$. In order to define this function on the remaining sectors s_k, $k \in N$, we consider the numbers $r_k \ge er_{k-1}$, $r_0 = 1$, $k \in N$, satisfying

$$r_k > (1 + |\alpha_k| + |\alpha_{k+1}|)\exp 3(q_k - p_k)^{-2}, \quad k \in N; \quad r_1 > (1 + |\alpha_1|)\exp 3\varepsilon^{-2}.$$

In the sector s_k we will define $x_1(t)$ by

$$\ln x_1(t) = 2\sqrt{t_1t_2}\{1 + e_{01}(\|t\|/r_k;1,3/2)[e_{11}(t_2/t_1;q_{k-1},p_k) - 1]\}, t \in s_k, k \in N.$$

Note that by definition of the function $e_{01}(\eta;\eta_1,\eta_2)$ on the whole axis $(-\infty, +\infty)$ we have

$$\ln x_1(t) = 2\sqrt{t_1t_2}e_{11}(t_2/t_1;q_{k-1},p_k), \quad t \in s_k, \quad \text{and} \quad \|t\| \ge 3r_k/2.$$

3. The construction of the off-diagonal elements of the fundamental system. Due to [5, 6], define the off-diagonal element $x_{21}(t)$ of a constructed two-dimesional linear Pfaffian system with bounded infinitely differentiable coefficients and two-dimensional time by the equality $x_{21}(t) = x_2(t)F(t)$, $t \in R_+^2$, where the infinitely differentiable function $F(t)$ is defined by

$$F(t) = \begin{cases} 0, & \text{if } t \in \tilde{S}, \\ \alpha_k e_{01}(\|t\|/r_k;1/2,1), & \text{if } t \in s_k,\ k \in N, \\ \alpha_k e_{01}(\|t\|/r_k;1/2,1) + e_{01}(t_2/t_1;p_k,q_k)[\alpha_{k+1}e_{01}(\|t\|\times & \\ \times r_{k+1}^{-1};1/2,1) - \alpha_k e_{01}(\|t\|/r_k;1/2,1),\ \alpha_0 = 0, & \text{if } t \in S_k, k \ge 0. \end{cases}$$

The infinite differentiability of the functions $x_1(t) \ge 1$, $x_2(t) \ge 1$, and $F(t)$ on R_+^2 follows from the same property of the functions $e_{01}(t_2/t_1;p_k,q_k)$ for $k \ge 0$, $e_{01}(\|t\|/r_k;1/2,1)$ for $k \ge 1$, and $e_{11}(t_2/t_1;q_{k-1},p_k)$ for $k \in N$.

4. The boundedness of coefficient matrices

$$A_i(t) = \frac{\partial X(t)}{\partial t_i} X^{-1}(t) = \begin{pmatrix} x_1^{-1}(t)\frac{\partial x_1(t)}{\partial t_i} & 0 \\ \frac{x_2(t)}{x_1(t)}\frac{\partial F(t)}{\partial t_i} & x_2^{-1}(t)\frac{\partial x_2(t)}{\partial t_i} \end{pmatrix}, \quad i = 1,2$$

of the constructed two-dimensional system (1) is proved by the following statement:

Lemma. *For all $m \in N$ and any (η_1, η_2) with the lengths $\leq 1/2$ there are the estimates*

$$(\eta - \eta_1)^{-m} e_{01}(\eta; \eta_1, \eta_2) \leq [\sqrt{m/2e}\exp(\eta_2 - \eta_1)^{-2}]^m, \quad \eta \in (\eta_1, \eta_2),$$

$$(\eta_2 - \eta)^{-m}\exp[-(\eta_2 - \eta)^{-2}] \leq (\sqrt{m/2e})^m, \quad \eta \in (\eta_1, \eta_2).$$

It is evident, that the infinite differentiability of the matrices $A_i(t)$ in R^2_+ follows from the same property of the nonsingular lower-triangular matrix $X(t)$. Similarly, the infinite differentiability of the fundamental solutions system $X(t)$ ensures the feasibility of the complete integrability conditions (2) for the constructed two-dimensional system (1).

5. The construction of the characteristic set of solutions. First for the characteristic set Λ_{x_2} of the solution $x(t, l_2) = (0, x_2(t))$ of system (1) we obtain the representation $\Lambda_{x_2} = \Lambda = \{(\lambda_1, 1/\lambda_1) \in R^2_+ : \lambda_1 \in [\sqrt{\varepsilon}, 1/\sqrt{\varepsilon}]\}$. Then for the solution $x(t, c_m)$ we establish the relations

$$\|x(t, c_m)\| = x_1(t) = |x_2(t)|^{e_{11}(t_2/t_1; q_{m-1}, p_m)} \equiv \rho_m(t), \quad t \in s_m, \quad \|t\| \geq 3r_m/2;$$
$$\max\{x_1(t), |\alpha_k - \alpha_m| x_2(t)\} \leq \|x(t, c_m)\| \leq$$
$$\leq (1 + |\alpha_k - \alpha_m|) x_2(t), \quad t \in s_k, \quad \|t\| \geq 3r_k/2, k \neq m;$$
$$1 \leq \|x(t, c_m)\|/x_2(t) \leq 1 + |\alpha_k - \alpha_m| + |\alpha_{k+1} - \alpha_k|, \quad t \in S_k, \quad \|t\| \geq r_{k+1}, \quad k \geq 0;$$
$$\|x(t, c_m)\| = \sqrt{1 + \alpha_m^2}\, x_2(t), \quad t \in \tilde{S}.$$

Hence in view of the equality $\lim\limits_{k\to\infty} r_k^{-1}\ln(1 + |\alpha_k| + |\alpha_{k+1}|) = 0$, true by the choice of the numbers r_k, and the uniform in $t \in s_k$ tending of $e_{11}(t_2/t_1; q_{k-1}, p_k)$ as $k \to \infty$, it follows that the characteristic set $\Lambda(m)$ of $x(t, c_m)$ coincides with the characteristic set of the function $\rho_m(t)$, which is equal to $x_2(t)$ outside the sector S_m, $m \in N$. By nontrivial reasonings it established then, that the vector $\lambda_2(\eta) \in R^2$ with the components $\lambda_2(\eta) = = \varphi'_m(\eta)$, $\lambda_1(\eta) = \varphi_m(\eta) - \eta\varphi'_m(\eta)$ for any $\eta \in [\varepsilon, 1/\varepsilon]$ is a characteristic vector of the function $\rho_m(t)$, where the function $\varphi_m(\eta) = 2\sqrt{\eta}e_{11}(\eta; q_{m-1}, p_m)$ is infinitely differentiable and convex up.

Thus we have the representation $\Lambda(m) = \{\lambda(\eta) \in R^2 : \eta \in [\varepsilon, 1/\varepsilon]\}$. The curve $\Lambda(m)$ coincides with the hyperbola Λ at $\lambda_1 \in [\sqrt{\varepsilon}, \sqrt{q_{m-1}}] \bigcup [\sqrt{p_m}, 1/\sqrt{\varepsilon}]$ and is located below this hiperbola at $\lambda_1 \in (\sqrt{q_{m-1}}, \sqrt{p_m})$. In particular, for $\eta = \eta_m \equiv (q_{m-1} + p_m)/2$ we obtain the point $\lambda(\eta_m) \in \Lambda(m)$ with the coordinates $\lambda_1(\eta_m) = \sqrt{\eta_m}(1 - e^{-\gamma_m})$, $\lambda_2(\eta_m) = (1 - e^{-\gamma_m})/\sqrt{\eta_m}$, where $\gamma_m \equiv 16(p_m - q_{m-1})^{-4}$ and their product $\lambda_1(\eta_m)\lambda_2(\eta_m) < 1$. Obviously, $\Lambda(l) \neq \Lambda(m) \neq \Lambda$ for any $l, m \in N$, $l \neq m$, and $\mathop{Lim}\limits_{m\to\infty} \Lambda(m) = \Lambda$. It is not dificult to prove also the equality $\Lambda_x = \Lambda$ for a solution $x(t)$ linearly independent with any of $x(t, c_m)$, $m \in N$, of the system (1).

The construction of the characteristic sets of all solutions of (1) is completed.

Remark. Obviously, from the constructed two-dimensional system (1) it may be possible to obtain an n-dimensional completely integrable system (1) with bounded infinitely differentiable coefficients in R^2_+, which have enumerable number of dofferent characteristic sets of the solutions.

Problem. It ought be to clarified, whether the set $\{\Lambda_x\}$ of different characteristic sets Λ_x of solutions $x : R^2_+ \to R^n$ of a Pfaffian system (1) is finite or enumerable.

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