Memoirs on Differential Equations and Mathematical Physics
Volume 15, 1998, 29–58

Otar Chkadua and Roland Duduchava

ASYMPTOTICS OF SOLUTIONS TO SOME BOUNDARY VALUE PROBLEMS OF ELASTICITY FOR BODIES WITH CUSPIDAL EDGES

Abstract. The main purpose of the paper is to obtain complete asymptotic expansion of solutions to boundary value problems of elasticity of Dirichlet, Neumann and mixed type for an n-dimensional ($n \geq 2$) composed body in $\mathbb{R}^n$. The body is composed of two anisotropic bodies with smooth boundaries stick together along parts of their boundaries. Therefore the body has a closed smooth cuspidal edge, along which the Dirichlet and Neumann conditions in the mixed problem collide. Asymptotics of solutions are obtained near the cuspidal edge (L_p–theory), with precise description of exponents and of logarithmic terms of the expansion.

1991 Mathematics Subject Classification. 47A68, 35J25, 35J55.

Key words and phrases. Dirichlet, Neumann and mixed problems, anisotropic homogeneous media, pseudodifferential operators, asymptotic of solutions, Wiener-Hopf method.

რეზიუმე. ნაშრომში შესწავლილია n–განზომილებიანი ($n \geq 2$) დრეკადობის ერთგვაროვანი ანიზოტროპული თეორიის დირიხლეს, ნეიმანისა და შერეული სასაზღვრო ამოცანების ამონახსნების სრული ასიმპტოტიკა ჩაკეტილი უკუქცევის წიბოს მიდამოში. უკუქცევის წიბოს შესაბამისი ორწახნაგა კუთხე 2π–ს ტოლია. გამოთვლილია ამოცანის ამონახსნის ასიმპტოტიკის მთავარი წევრის კოეფიციენტი შესაბამისი ფსევდოდიფერენციალური განტოლების ამონახსნის ასიმპტოტიკის პირველი კოეფიციენტის საშუალებით.

Introduction

In the present paper, we study asymptotics of solutions of the classical (Dirichlet, Neumann and mixed) boundary value problems of anisotropic elasticity in an n-dimensional domain $\Omega \subset \mathbb{R}^n$ composed of two subdomains $\Omega = \Omega_1 \cup \Omega_2$ with smooth boundaries $\partial\Omega_1$ and $\partial\Omega_2$. $\overline{S}_0 = \partial\Omega_1 \cap \partial\Omega_2$ is assumed to be a smooth surface. Thus, the domain Ω has a smooth, closed cuspidal edge with the angle 2π viewed from Ω. For a mixed problem, the cuspidal edge is the place where the Dirichlet and Neumann conditions collide. In the case of the plane $n = 2$ the cuspidal edge degenerates into cuspidal points, the so-called interior peaks (any finite number of interior peaks can be treated).

Interior and exterior Dirichlet and Neumann boundary value problems for the Laplace equation as well as for the Lamé equation of isotropic elasticity in plane domains ($n = 2$) were studied by V. Maz'ya and A. Solov'yev [22–25]. They obtained conditions for unique solvability (which is non-trivial for exterior peaks with the angle $\gamma = 0$) and established asymptotics of solutions near these peaks.

In [6] the existence and uniqueness of solutions of the above-mentioned problems were investigated in the Bessel potential and Besov spaces on the basis of the potential method and the Wiener-Hopf method for pseudodifferential equations on open manifolds. The obtained results enable one to establish a priori smoothness of solutions which is restricted due to the presence of a cuspidal edge even for the Dirichlet and Neumann problem, although the solutions are C^α-smooth, where $\alpha < \frac{1}{2}$ for the Dirichlet and Neumann problems, and $\alpha < \frac{1}{4}$ for the mixed problem.

G. Eskin and J. Bennish applied the Wiener-Hopf method and obtained complete asymptotic expansion of solutions for elliptic pseudodifferential equation on a manifold with a smooth boundary (the L_2-theory) (see [1], [15]). In [7] we have developed this techniques and obtained more precise asymptotics (the L_p-theory). Particular results in this direction can be found in [13] and [14].

Having in hand asymptotics of solutions for the boundary pseudodifferential equation on the boundary surface (such as a crack surface or the interface between two anisotropic materials), we still need spatial asymptotics of solutions for the original boundary value problem which is representable, as usual, by layer potentials with densities being solutions of boundary pseudodifferential equations and thus having definite asymptotic expansion near crack fronts or other geometric peculiarities of boundary manifolds. These investigations were carried out in [8] in the most general form: spatial asymptotics was established for functions representable by layer potentials with prescribed asymptotics of density; exact formulae were found relating the coefficients of spatial asymptotics and asymptotics on the corresponding boundary surface. These formulae simplify substantially the calculation of coefficients of spatial asymptotics and allow one to express them by coefficients of asymptotics on the surface. The latter can be found from the boundary pseudodifferential equation with less dimension than that of the corresponding boundary value problem.

The obtained results can be successfully applied in calculating stress intensity factors (SIF) which play an important role in crack propagation criteria.

In the present paper we demonstrate the results obtained in [7] and [8] for the above-mentioned boundary value problems of elasticity and write a complete asymptotic expansion of solutions near the cuspidal edge. Formulae relating to the SIF-coefficients (coefficients of the leading term of asymptotics) of spatial and surface asymptotics are written out.

For different applications of the results dealing with asymptotics from [7] and [8], the reader can be referred to [13], [11] and [4], [5].

A quite different approach to the problem of asymptotics, based on the Mellin transform as well as on the calculus of boundary value problems (a direct approach to spatial asymptotics) has been initiated by a pilot paper of V. Kondrat'yev [18]. This method was developed in many outstanding papers and monographs (see, e.g., [9], [10], [17], [19], [21], [27], [28]) and encompass boundary value problems in domains with sophisticated singularities occuring on the boundary (edges, wedges, conical singularities and their arbitrary combinations). Although the Wiener-Hopf method cannot (so far) be applied to the above-mentioned cases with singularities, in crack and mixed type problems it demonstrates more precise asymptotics and provides us with formulae for the exponents and coefficients.

1. Formulation of the Problems

Let a domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, be either finite or infinite but have a compact boundary $S = \partial\Omega$, and let there exist a surface $\overline{S}_0$ of the class C^∞ of dimension $n-1$ which divides the domain Ω into two subdomains Ω_1 and Ω_2 with C^∞-boundaries $\partial\Omega_1$ and $\partial\Omega_2$ ($\Omega_1 \cap \Omega_2 = \varnothing$, $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \overline{S}_0$). Then ∂S_0, the boundary of the surface S_0 ($\partial S_0 \subset \partial\Omega$), represents an $(n-2)$-dimensional closed cuspidal edge and $\partial\Omega_1 = S_1 \cup \overline{S}_0$, $\partial\Omega_2 = S_2 \cup \overline{S}_0$.

Assume Ω is filled with anisotropic homogeneous elastic material.

The basic static equations of elasticity for anisotropic homogeneous elastic media written in terms of displacement components are of the form

$$\mathbf{A}(D_x)u + F = 0 \quad \text{in} \quad \Omega \tag{1.1}$$

([20], [16], [3]), where $u = (u_1, \dots, u_n)$ is the displacement vector, $F = (F_1, \dots, F_n)$ is the volume force acting on Ω and $\mathbf{A}(D_x)$ is an $n \times n$-matrix differential operator

$$\mathbf{A}(D_x) = \| \sum_{i,l=1}^{n} a_{ijlk} \partial_i \partial_l \|_{n \times n}\,, \quad \partial_l := \frac{\partial}{\partial x_l}\,, \quad D_l := -i\partial_l\,, \tag{1.2}$$

a_{ijlk} being elastic constants satisfying $a_{ijlk} = a_{lkij} = a_{ijkl}$.

The quadratic form

$$\sum_{i,j,l,k=1}^{n} a_{ijlk}\xi_{ij}\xi_{lk}, \qquad \xi_{ij}=\xi_{ji}\,, \tag{1.3}$$

is assumed to be positive–definite with respect to the variables ξ_{ij}.

We introduce the differential stress operator

$$\mathcal{T}(D_z,n(z)) = \|\mathcal{T}_{jk}(D_z,n(z))\|_{n\times n}, \quad \mathcal{T}_{jk}(D_z,n(z)) = \sum_{i,l=1}^{n} a_{ijlk}n_i(z)\partial_l\,,$$

where $n(z)=(n_1(z),\ldots,n_n(z))$ is the unit normal vector to the manifold $S_1\cup S_2$ at the point $z\in S_1\cup S_2$, exterior to the domain Ω. For convenience in the sequel we will use the short notation $\mathcal{T}=\mathcal{T}(D_z,n(z))$.

From the symmetry of the coefficients a_{ijlk} and the positive definiteness of the quadratic form (1.1) it follows (the operator $\mathbf{A}(D_x)$ is a strongly elliptic formally self-adjoint differential operator [16]) that the symbol

$$\mathcal{A}(\xi) = \Big\| \sum_{i,l=1}^{n} a_{ijlk}\xi_i\xi_l \Big\|_{n\times n}, \quad \xi=(\xi_1,\ldots,\xi_n)\in\mathbb{R}^n$$

is positive definite, i.e. the inequality

$$\big(\mathcal{A}(\xi)\eta,\eta\big) = \big(\mathcal{A}(\xi)\eta,\eta\big) \ge P_0|\xi|^2|\eta|^2 \quad \text{for all} \quad \xi\in\mathbb{R}^n \quad \text{and} \quad \eta\in\mathbb{C}^n$$

holds with $P_0 = const > 0$ depending only on the elastic constants.

For the spaces we follow the notation suggested in [29] and in the case of an infinite domain Ω we will invoke the spaces $\mathbb{H}^s_{p,loc}(\Omega)$, $\mathbb{B}^s_{p,t,loc}(\Omega)$, $\mathbb{H}^s_{p,comp}(\Omega)$, $\mathbb{B}^s_{p,t,comp}(\Omega)$.

Let $u\in W^1_p(\Omega)=\mathbb{H}^1_p(\Omega)$ ($W^1_{p,loc}(\Omega)=\mathbb{H}^1_{p,loc}(\Omega)$). Then $r_iu\in W^1_p(\Omega_i)$ ($r_iu\in W^1_{p,loc}(\Omega_i)$), where r_i is the operator of restriction to Ω_i, $i=1,2$. From the theorem on traces (see [29]) there follows that the trace of the function γ_iu on $\partial\Omega_i$ exists and $\gamma_iu\in\mathbb{B}^{1/p'}_{p,p}(\partial\Omega_i)$, $i=1,2$, $p'=p/(p-1)$. Let $u\in W^1_{p,loc}(\Omega)$ be such that $\mathbf{A}(D_x)u\in L_{p,comp}(\Omega)$ (if Ω is compact, we simply ignore the subscripts "loc" and "comp"). Then the trace $\gamma_i\{\mathcal{T}(r_iu)\}^+$ is correctly defined by the following Green formula (see [12], [26])

$$\int_{\Omega_i}\big[\overline{v}^{(i)}\mathbf{A}(D_x)(r_iu)+E(r_iu,v^{(i)})\big]\,dx = \Big\langle\gamma_i\mathcal{T}(r_iu),\gamma_iv^{(i)}\Big\rangle_{\partial\Omega_i}$$
$$\text{for all} \quad v^{(i)}\in W^1_{p'}(\Omega_i)\ \ \big(v^{(i)}\in W^1_{p',comp}(\Omega_i)\big),\ \ i=1,2\,;$$

here

$$E(r_iu,v^{(i)}) = \sum_{m,j,l,k=1}^{n} a_{mjkl}\partial_m(r_iu)_j\partial_l\overline{v}^{(i)}_k\,;$$

the symbol $\langle\cdot,\cdot\rangle$ denotes the duality between the spaces $\mathbb{B}^{-1/p}_{p,p}(\partial\Omega_i)$, $\mathbb{B}^{1/p}_{p',p'}(\partial\Omega_i)$ and

$$\langle\psi,\varphi\rangle_{\partial\Omega_i}=\int_{\partial\Omega_i}\psi\overline{\varphi}\,dS \quad \text{for} \quad \psi,\varphi\in C^1(\partial\Omega_i), \quad i=1,2.$$

If $u\in W^1_{p,loc}(\Omega)$ is a solution (in the sense of distributions) of (1.1) with $F\in L_{q,comp}(\Omega)$, then $\mathbf{A}(D_x)u\in L_{q,comp}(\Omega)$, $q\geq\frac{np}{n+p}$. It is easy to ascertain that the functions

$$\begin{aligned}&\gamma_{S_i}u=\pi_i\{\gamma_i(r_iu)\} \quad \text{on} \quad S_i,\\ &\gamma_{S_i}\mathcal{T}u=\pi_i\{\gamma_i\mathcal{T}(r_iu)\} \quad \text{on} \quad S_i, \quad i=1,2\,,\end{aligned}$$

where π_i denotes the restriction from $\partial\Omega_i$ to S_i, $\quad i=1,2$, are all correctly defined.

In the case of an infinite domain Ω, we require that a solution of (1.1) satisfies the following condition

$$\begin{aligned}&u(x)=o(1) \quad \text{for} \ |x|\to\infty \ \text{if} \ n>2,\\ &u(x)=O(1) \quad \text{for} \ |x|\to\infty \quad \text{if} \ n=2.\end{aligned} \tag{1.4}$$

It is known (see [2]), that for any solution of (1.1) under condition (1.4) has the following asymptotics at infinity

$$\partial^\mu u(x)=\begin{cases} O(|x|^{2-n-|\mu|}) & \text{for} \quad |x|\to\infty \quad \text{if} \ n>2,\\ O(|x|^{-|\mu|-1}) & \text{for} \quad |x|\to\infty \quad \text{if} \ n=2\end{cases}$$

with an arbitrary multi–index $\mu\in\mathbb{N}_0^n$.

We will study the asymptotics of a function $u\in W^1_{p,loc}(\Omega)$, which vanishes at infinity (see condition (1.4)) and solves one of the following boundary value problems:

Dirichlet Problem:

$$\begin{cases}\mathbf{A}(D_x)u=0 & \text{in} \quad \Omega,\\ \quad \gamma_{S_i}u=\varphi_i & \text{on} \quad S_i\,,\end{cases}$$

where $\varphi\in\mathbb{B}^{1/p'}_{p,p}(S_i)$, $\;i=1,2,\;\; 1<p<\infty,\;\; p':=\dfrac{p}{p-1}$.

Neumann Problem:

$$\begin{cases}\mathbf{A}(D_x)u=0 & \text{in} \quad \Omega,\\ \quad \gamma_{S_i}\mathcal{T}u=\psi_i & \text{on} \quad S_i\,,\end{cases}$$

where $\psi_i\in\mathbb{B}^{-1/p}_{p,p}(S_i)$, $\quad i=1,2,\;\; 1<p<\infty$.

Mixed Problem:

$$\begin{cases}\mathbf{A}(D_x)u=0 & \text{in}\ \Omega,\\ \gamma_{S_1}u=\varphi_1 & \text{on}\ S_1,\\ \gamma_{S_2}\mathcal{T}u=\varphi_2 & \text{on}\ S_2,\end{cases}$$

where $\varphi_1\in\mathbb{B}^{1/p'}_{p,p}(S_1)$, $\;\varphi_2\in\mathbb{B}^{-1/p}_{p,p}(S_2)$, $\;1<p<\infty$.

2. Asymptotics of Solutions to the Dirichlet Boundary Value Problem

The simple layer potential

$$\mathbf{V}^{(i)}(g)(x) = \int\limits_{\partial\Omega_i} H(x-y)g(y)d_y S, \quad x \in \Omega_i, \quad i = 1, 2,$$

where $H(x)$ is the fundamental solution of (1.1), and the composition $(\mathcal{T}\mathbf{V}^{(i)})(g)(x)$ have the following traces on the surface

$$\gamma_i \mathbf{V}^{(i)}(g)(z) = \int\limits_{\partial\Omega_i} H(z-y)g(y)d_y S,$$

$$\gamma_i(\mathcal{T}\mathbf{V}^{(i)})(g)(z) = -\frac{1}{2}g(z) + \int\limits_{\partial\Omega_i} \mathcal{T}(\partial_z, n(z))H(z-y)\big)g(y)d_y S,$$

$$z \in \partial\Omega_i, \quad i = 1, 2\,.$$

Let us use the notation

$$\mathbf{V}_{-1}^{(i)}(g)(z) = \int\limits_{\partial\Omega_i} H(z-y)g(y)d_y S,$$

$$\overset{*}{\mathbf{V}}{}_0^{(i)}(g)(z) = \int\limits_{\partial\Omega_i} \mathcal{T}(\partial_z, n(z))H(z-y)g(y)d_y S, \quad z \in \partial\Omega_i, \quad i = 1, 2\,,$$

for the direct values of the corresponding potential operators.

In [6] a solution to the Dirichlet boundary value problem is represented by the simple layer potential

$$r_i u = \mathbf{V}^{(i)} g_i \quad \text{in} \quad \Omega_i, \quad i = 1, 2\,.$$

Let $\Phi_0^{(i)} \in \mathbb{B}_{p,p}^{1/p'}(\partial\Omega_i)$ be some fixed continuation of the function $\varphi_i \in \mathbb{B}_{p,p}^{1/p'}(S_i)$ to $\partial\Omega_i = S_i \cup \overline{S}_0$, $i = 1, 2$. Then any continuation $\Phi_{(i)}$ of the function φ_i to $\partial\Omega_i$ has the form $\Phi^{(i)} = \Phi_0^{(i)} + \varphi_0^{(i)}$, where $\varphi_0^{(i)} \in \widetilde{\mathbb{B}}_{p,p}^{1/p'}(S_0)$, $i = 1, 2$.

The Dirichlet boundary value problem can be reduced to the following system of pseudodifferential equations on the manifold with boundary S_0:

$$\begin{cases} \varphi_0^{(1)} - \varphi_0^{(2)} = g, \\ \pi_0(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)})(\mathbf{V}_{-1}^{(1)})^{-1}\varphi_0^{(1)} + \pi_0(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)})(\mathbf{V}_{-1}^{(1)})^{-1}\varphi_0^{(2)} = f, \end{cases} \tag{2.1}$$

where

$$g = \pi_0\Phi_0^{(2)} - \pi_0\Phi_0^{(1)},$$

$$f = -\pi_0\big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)}\big)\big(\mathbf{V}_{-1}^{(2)}\big)^{-1}\Phi_0^{(2)} - \pi_0\big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)}\big)\big(\mathbf{V}_{-1}^{(1)}\big)^{-1}\Phi_0^{(1)}\,;$$

here π_0 is the operator of restriction to S_0 and $\mathcal{I}$ is the identity.

For the system (2.1) with $\varphi_i \in \mathbb{B}^s_{p,r}(S_i)$, $g \in \mathbb{B}^s_{p,r}(S_0)$, $i = 1, 2$, $1/p - 1/2 < s < 1/p + 1/2$ (in particular, for $s = 1/p'$, $r = p$) to be solvable we require the compatibility condition

$$\exists \Phi_0^{(i)} \in \mathbb{B}^s_{p,r}(\partial\Omega_i) \;\; i = 1, 2, \; : \; g \in \widetilde{\mathbb{B}}^s_{p,r}(S_0)\,. \tag{2.2}$$

Note that when $1/p - 1 < s < 1/p$, this condition is fulfilled automatically (see [29, Theorem 2.10.3(c)]).

In the case $1/p < s < 1/p + 1$ the compatibility condition acquires the form

$$\gamma_{S_0}\varphi_2 = \gamma_{S_0}\varphi_1\,.$$

In the case where $s = 1/p$ the compatibility condition looks rather cumbersome (see [6, Remark 5.7]).

The system (2.1) is reduced to a pseudodifferential equation on an open manifold S_0

$$\pi_0 \mathbf{A} \varphi_0^{(1)} = \mathbf{\Psi},$$

where

$$\mathbf{\Psi} \in \mathbb{H}^{s-1}_p(S_0) \quad (\mathbb{B}^{s-1}_{p,r}(S_0)),$$

$$\mathbf{A} = \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)}\right)(\mathbf{V}_{-1}^{(1)})^{-1} + \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)}\right)(\mathbf{V}_{-1}^{(2)})^{-1}\,.$$

The pseudodifferential operator $\pi_0\mathbf{A}$ is positive definite and the following proposition holds for it.

Theorem 2.1 (see [6, Theorem 4.2]). *Let $1 < p < \infty$, $1 \le r \le \infty$. Then the operator*

$$\pi_0\mathbf{A} : \widetilde{\mathbb{H}}^s_p(S_0) \to \mathbb{H}^{s-1}_p(S_0)$$

is Fredholm if and only if the inequality

$$\frac{1}{p} - \frac{1}{2} < s < \frac{1}{p} + \frac{1}{2} \tag{2.3}$$

holds.

If (2.3) is the case, then the operator

$$\begin{aligned} \pi_0\mathbf{A} \quad &: \quad \widetilde{\mathbb{H}}^s_p(S_0) \to \mathbb{H}^{s-1}_p(S_0) \\ &: \quad \widetilde{\mathbb{B}}^s_{p,r}(S_0) \to \mathbb{B}^{s-1}_{p,r}(S_0) \end{aligned}$$

is invertible in both cases.

It is worth noting that PsDO $\pi_0\mathbf{A}$ is invertible in the anisotropic Bessel potential spaces with weight $\widetilde{\mathbb{H}}^{(\mu,s),k}_p(S_0) \to \mathbb{H}^{(\mu,s-1),k}_p(S_0)$ for all $\mu \in \mathbb{R}$, $k \in \mathbb{N}_0$ (see [7]) provided the conditions (2.3) hold.

Now let us formulate the main theorems about uniqueness, existence and smoothness for solutions to the Dirichlet problem (see [6, Theorems 4.3, 4.4, 4.5 and Remark 5.7]).

Theorem 2.2. *Let $4/3 < p < 4$ and the compatibility conditions* (2.2) *hold for $s = 1 - 1/p$. Then the Dirichlet boundary value problem has a unique solution of the class $W^1_{p,loc}(\Omega)$, (with the condition* (1.4) *at infinity); this solution is given by the formula*

$$r_i u = \mathbf{V}^{(i)}(\mathbf{V}^{(i)}_{-1})^{-1}(\Phi^{(i)}_0 + \varphi^{(i)}_0)), \quad q = 1, 2,$$

where $\Phi^{(i)}_0 \in \mathbb{B}^{1/p'}_{p,p}(\partial\Omega_i)$ is a fixed continuation of the function φ_i to $\partial\Omega_i$, satisfying condition (2.2) *and $\varphi^{(i)}_0 \in \widetilde{\mathbb{B}}^{1/p'}_{p,p}(S_0)$, $i = 1, 2$, is a solution to the system* (2.1).

Theorem 2.3. *Let $4/3 < p < 4$, $1 < t < \infty$, $1 \leq r \leq \infty$, $1/t - 1/2 < s < 1/t + 1/2$, the compatibility condition* (2.2) *with t instead of p be fulfilled. Let $u \in W^1_p(\Omega)$ ($u \in W^1_{p,loc}(\Omega)$ with the condition* (1.4) *at infinity) be a solution of the Dirichlet problem. In that case:*

- *If $\varphi_i \in \mathbb{B}^s_{t,t}(S_i)$, $i = 1, 2$, then $u \in \mathbb{H}^{s+1/t}_t(\Omega)$ $(\mathbb{H}^{s+1/t}_{t,loc}(\Omega))$;*
- *If $\varphi_i \in \mathbb{B}^s_{t,r}(S_i)$, $i = 1, 2$, then $u \in \mathbb{B}^{s+1/t}_{t,r}(\Omega)$, $(\mathbb{B}^{s+1/t}_{t,r,loc}(\Omega))$;*
- *If $\varphi_i \in C^\alpha(\overline{S}_i)$, $i = 1, 2$, $\alpha \in]0, 1/2]$, then $u \in \bigcap\limits_{\alpha'<\alpha} C^{\alpha'}(\overline{\Omega})$.*

Now we will write the asymptotics of the Dirichlet boundary value problem. It will be assumed below that the boundary data of the Dirichlet problem are sufficiently smooth, namely, $\varphi_i \in \mathbb{H}^{(\infty,s+2M+1),\infty}_p(S_i)$, $i = 1, 2$, (see [7]).

The following equalities hold for the symbols of the operators $\mathbf{V}^{(i)}_{-1}$ and $\overset{*}{\mathbf{V}}{}^{(i)}_0$ (see [6]):

$$\begin{aligned} &\sigma_{\mathbf{V}^{(1)}_{-1}}(z, \xi') = \sigma_{\mathbf{V}^{(2)}_{-1}}(z, \xi') \quad &\text{for} \quad z \in \overline{S}_0, \\ &\sigma_{\overset{*}{\mathbf{V}}{}^{(1)}_0}(z, \xi') = -\sigma_{\overset{*}{\mathbf{V}}{}^{(2)}_0}(z, \xi') \quad &\text{for} \quad z \in \overline{S}_0. \end{aligned} \tag{2.4}$$

The symbol $\sigma_{\mathbf{A}}(x', \xi')$ of the pseudodifferential operator $\mathbf{A}$ has the form

$$\sigma_{\mathbf{A}}(x', \xi') = \sigma^{-1}_{-\mathbf{V}^{(1)}_{-1}}(x', \xi') = \sigma^{-1}_{-\mathbf{V}^{(2)}_{-1}}(x', \xi').$$

The symbol $\sigma_{-\mathbf{V}^{(i)}_{-1}}(x', \xi')$ $(i = 1, 2)$ is an even matrix–function with respect to ξ' and therefore all eigenvalues of the matrix $(\sigma_{\mathbf{A}}(x', 0, +1))^{-1}$ $\sigma_{\mathbf{A}}(x', 0, -1) = I$ are trivial $\lambda_j(x') = 1$, $j = 1, \ldots, n$.

Let us consider a local system of coordinates $(x'', x_{n-1}) \in S_0$, where $x'' \in \partial S_0$ is a parameter which ranges along the cuspidal edge, while $x_{n,+} = \mathrm{dist}(x, \partial S_0)$ denotes the distance to the edge along the surface S_0.

Applying a result on strongly elliptic pseudodifferential equation (see [7, Theorem 2.1]) and taking into account the first equation in (2.1), we obtain the following asymptotic expansion for the function $\varphi^{(i)}_0$, $i = 1, 2$:

$$\varphi^{(i)}_0(x'', x_{n-1,+}) = c_0(x'') x^{\frac{1}{2}}_{n-1,+}$$

$$+\sum_{k=1}^{M} x_{n-1,+}^{\frac{1}{2}+k} B_k(x'', \log x_{n-1,+}) + \varphi_{M+1}^{(i)}(x'', x_{n-1,+}), \qquad (2.5)$$

where $c_0 \in C^\infty(\partial S_0)$ and the remainder $\varphi_{M+1}^{(i)} \in \mathbb{H}_p^{(\infty,s+M+1),\infty}(S_\varepsilon^+)$, $i = 1,2$, $M \in \mathbb{N}$, $S_\varepsilon^+ = \partial S_0 \times [0,\varepsilon]$.

$B_k(x'', t)$ in (2.5) is a polynomial of degree k with respect to the variable t and has $C^\infty(\partial S_0)$–smooth vector coefficients on the cuspidal edge $x'' \in \partial S_0$.

Thus, recalling that solutions of the Dirichlet boundary value problem are represented by a potential–type function (see Theorem 2.2) and using asymptotic expansion of such functions from [8, Theorem 2.2 and 2.3]), assuming $\Phi_0^{(i)} \in \mathbb{H}_p^{(\infty,s+2M+1),\infty}(S_i)$, $i = 1, 2$, we obtain the following asymptotic expansion of the solution to the Dirichlet boundary value problem:

$$(r_i u)(x'', x_{n-1}, x_n) = \sum_{s=1}^{l(n)} \operatorname{Re}\Bigg\{ \sum_{j=0}^{n_s-1} [d_{sj}^{(i)}(x'', +1) x_n^j z_{s,+1}^{1/2-j} -$$

$$-d_{sj}^{(i)}(x'', -1) x_n^j z_{s,-1}^{1/2-j}] + \sum_{\vartheta=\pm1} \sum_{\substack{l,k=0}}^{M+2} \sum_{\substack{j+p=1 \\ l+k+j+p\neq 0}}^{M+2-l} x_{n-1}^l x_n^j z_{s,\vartheta}^{\frac{1}{2}+p+k} B_{slkjp}^{(i)}(x'', \log z_{s,\vartheta}) \Bigg\} +$$

$$+u_{M+1}^{(i)}(x'', x_{n-1}, x_n) \quad \text{for} \quad M > \frac{n-1}{p} - \min\{[s-1], 0\}, \ \ i = 1, 2,$$

with the coefficients $d_{sj}^{(i)}(\cdot, \pm 1) \in C^\infty(\partial S_0)$ and $u_{M+1}^{(i)} \in C^{M+1}(\overline{\Omega}_i)$, $i = 1, 2$. Here

$$z_{s,+1} = -x_{n-1} - x_n \tau_{s,+1}, \quad z_{s,-1} = x_{n-1} - x_n \tau_{s,-1},$$
$$-\pi < \operatorname{Arg} z_{s,\pm1} < \pi, \quad \tau_{s,\pm1} \in C^\infty(\partial S_0),$$

$\{\tau_{s,\pm1}\}_{s=1}^{l(n)}$ are all different roots of the polynomial $\det \mathbf{A}(J_\varkappa^\top(x'', 0)(0, \pm1, \tau))$ of multiplicity n_s, $s = 1, \ldots l(n)$, in the complex lower half-plane; $J_\varkappa$ is the Jacoby matrix of the mapping $\varkappa$ (see [8]). Again, $x'' \in \partial S_0$, $x_{n-1} = \operatorname{dist}(x_{S_0}, \partial S_0)$, $x_n = \operatorname{dist}(x, S_0)$, where x_{S_0} is the projection of $x \in \Omega$ onto the hyperplane containing S_0.

$B_{slkjp}^{(i)}(x'', t)$ is a polynomial of order $\nu_{kjp} = k + p + j$ with respect to t, with vector coefficients depending on the variable x''. The coefficients $d_{sj}^{(i)}(x'', \pm1)$ have the following form:

$$d_{sj}^{(1)}(x'', +1) = \mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, +1) \sigma_{V_{-1}^{(1)}}^{-1}(x'', 0, 0, +1) c^{(j)}(x''),$$

$$d_{sj}^{(1)}(x'', -1) = -i\mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, -1) \sigma_{V_{-1}^{(1)}}^{-1}(x'', 0, 0, +1) c^{(j)}(x''),$$

$$d_{sj}^{(2)}(x'', +1) = \mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, +1) \sigma_{V_{-1}^{(2)}}^{-1}(x'', 0, 0, +1) c^{(j)}(x''),$$

$$d_{sj}^{(2)}(x'', -1) = i\mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, -1) \sigma_{V_{-1}^{(2)}}^{-1}(x'', 0, 0, +1) c^{(j)}(x''),$$

$$s=1,\dots,l(n),\quad j=0,\dots n_s-1,$$

where $\mathcal{G}_\varkappa$ is the square root from the Gramm determinant of the diffeomorphisms $\varkappa$ (see [8]);

$$V^{(s)}_{-1,j}(x'',0,0,\pm 1)=$$
$$=-\frac{i^{j+1}}{j!(n_s-1-j)!}\,\frac{d^{n_s-1-j}}{d\tau^{n_s-1-j}}\,(\tau-\tau_{s,\pm 1})^{n_s}A^{-1}(J^\top_\varkappa(x'',0)(0,\pm 1,\tau))\Big|_{\tau=\tau_{s,\pm 1}},$$
$$c^{(j)}(x)=\frac{i^{j+1}}{4\sqrt{\pi}}\Gamma\Big(j-\frac{1}{2}\Big)\,c_0(x'')$$

and $c_0(x'')$ is the first coefficient of the asymptotic expansion in (2.5).

3. Asymptotics of Solutions to the Neumann Boundary Value Problem

Let $\Psi^{(i)}_0\in\mathbb{B}^{-1/p}_{p,p}(\partial\Omega_i)$ be some fixed continuation of a function $\psi_i\in\mathbb{B}^{-1/p}_{p,p}(S_i)$ on $\partial\Omega_i=S_i\cup\overline{S}_0$. Then any continuation $\Phi^{(i)}\in\mathbb{B}^{-1/p}_{p,p}(\partial\Omega_i)$ of ψ_i on $\partial\Omega_i$ has the form

$$\Psi^{(i)}=\Psi^{(i)}_0+\psi^{(i)}_0,$$

where $\psi^{(i)}_0\in\widetilde{\mathbb{B}}^{-1/p}_{p,p}(S_0)$, $i=1,2$.

In [6], a solution to the Neumann boundary value problem is sought in the form of a simple-layer potential

$$r_i v=\mathbf{V}^{(i)}(\mathbf{V}^{(i)}_{-1})^{-1}g_i\quad\text{in }\ \Omega_i,\quad i=1,2.$$

For unknown densities g_1,g_2 and functions $\psi^{(1)}_0,\psi^{(2)}_0$ the following system of boundary pseudodifferential equations was obtained (see [6]):

$$\mathbf{N}\begin{pmatrix} g_1\\ g_2\\ \psi^{(1)}_0\\ \psi^{(2)}_0\end{pmatrix}=\begin{pmatrix}\Psi^{(1)}_0\\ \Psi^{(2)}_0\\ 0\\ -\pi_0\Psi^{(2)}_0-\pi_0\Psi^{(1)}_0\end{pmatrix},\tag{3.1}$$

where

$$\mathbf{N}=\begin{pmatrix}(-\frac{1}{2}\mathcal{I}+\overset{*}{\mathbf{V}}{}^{(1)}_0)(\mathbf{V}^{(1)}_{-1})^{-1} & 0 & -\mathcal{I} & 0\\ 0 & (-\frac{1}{2}\mathcal{I}+\overset{*}{\mathbf{V}}{}^{(2)}_0)(\mathbf{V}^{(2)}_{-1})^{-1} & 0 & -\mathcal{I}\\ \pi_0\mathcal{I} & -\pi_0\mathcal{I} & 0 & 0\\ 0 & 0 & \mathcal{I} & \mathcal{I}\end{pmatrix}.$$

It is almost obvious that the system (3.1) has a solution if and only if the following compatibility conditions

$$\exists\Psi^{(i)}_0\in\mathbb{B}^{s-1}_{p,r}(\partial\Omega_i)\ i=1,2:\ \pi_0\Psi^{(2)}_0+\pi_0\Psi^{(1)}_0\in\widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0)\tag{3.2}$$

hold for $\psi_i \in \mathbb{B}^{s-1}_{p,r}(S_i)$, $\pi_0\Psi^{(2)}_0 + \pi_0\Psi^{(1)}_0 \in \mathbb{B}^{s-1}_{p,r}(S_0)$, $i = 1, 2$, $1 \le r \le \infty$, $1/p - 1/2 < s < 1/p + 1/2$ (cf. [6]).

Note that the compatibility conditions hold automatically provided $1/p - 1/2 < s < 1/p$ or $1/p < s < 1/p + 1/2$ (cf. [29]). Unfortunately, when $s = 1/p$ we can not provide the compatibility condition in explicit form.

Consider the operator

$$\begin{aligned}
\mathbf{N}_M &= \begin{pmatrix} \mathbf{B}^{(1)}_M & 0 & -\mathcal{I} & 0 \\ 0 & \mathbf{B}^{(2)}_M & 0 & -\mathcal{I} \\ \pi_0\mathcal{I} & -\pi_0\mathcal{I} & 0 & 0 \\ 0 & 0 & \mathcal{I} & \mathcal{I} \end{pmatrix}, \\
\mathbf{B}^{(i)}_M : &= (-\mathbf{V}^{(i)}_{-1})^M + (-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(i)}_0)(\mathbf{V}^{(i)}_{-1})^{-1}, \\
& \quad i = 1, 2, \quad M = 0, 1, 2, \dots,
\end{aligned}$$

which differs from $\mathbf{N}_M$ by a compact operator.

The system of equations corresponding to the operator $\mathbf{N}_M$ has the form

$$\left\{ \begin{aligned}
\big[(-\mathbf{V}^{(1)}_{-1})^M + (-\tfrac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(1)}_0)(\mathbf{V}^{(1)}_{-1})^{-1}\big]\widetilde{g}_1 - \widetilde{\psi}^{(1)}_0 &= \widetilde{\Psi}^{(1)}_0, \\
\big[(-\mathbf{V}^{(2)}_{-1})^M + (-\tfrac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(2)}_0)(\mathbf{V}^{(2)}_{-1})^{-1}\big]\widetilde{g}_2 - \widetilde{\psi}^{(2)}_0 &= \widetilde{\Psi}^{(2)}_0, \\
\pi_0\widetilde{g}_1 - \pi_0\widetilde{g}_2 &= G_1, \\
\widetilde{\psi}^{(1)}_0 + \widetilde{\psi}^{(2)}_0 &= G_2,
\end{aligned} \right. \tag{3.3}$$

where

$$\begin{gathered}
\widetilde{\Psi}^{(i)}_0 \in \mathbb{H}^{s-1}_p(\partial\Omega_i) \ \big(\mathbb{B}^{s-1}_{p,r}(\partial\Omega_i)\big), \quad i = 1, 2, \\
G_1 \in \mathbb{H}^s_p(S_0) \ \big(\mathbb{B}^s_{p,r}(S_0)\big), \quad G_2 \in \widetilde{\mathbb{H}}^{s-1}_p(S_0) \ \big(\widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0)\big).
\end{gathered}$$

Note that while studying the Neumann problem in [6], the system (3.1) was reduced to the system of equations corresponding to the operator $\mathbf{N}_0$, i.e., only the case $M = 0$ was considered. Here we have introduced the operator $\mathbf{N}_M$ to obtain a complete asymptotics both for solutions to the system (3.1) and for solutions to the Neumann problem.

We need the following auxiliary proposition which is proved similarly to [6, Lemmata 5.2, 6.2].

Lemma 3.1. *Let $1 < p < \infty$, $1 \le r \le \infty$, $s \in \mathbb{R}$. Then the pseudodifferential operators*

$$\begin{aligned}
(-\mathbf{V}^{(i)}_{-1})^M + \Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(i)}_0\Big)\big(\mathbf{V}^{(i)}_{-1}\big)^{-1} \ &: \ \mathbb{H}^s_p(\partial\Omega_i) \to \mathbb{H}^{s-1}_p(\partial\Omega_i) \\
&: \ \mathbb{B}^s_{p,r}(\partial\Omega_i) \to \mathbb{B}^{s-1}_{p,r}(\partial\Omega_i)
\end{aligned}$$

are invertible for $i = 1, 2$ and for $M = 0, 1, 2, \dots$.

Therefore, after defining $\widetilde{g}_1$, $\widetilde{g}_2$ from the first and the second equation of the system (3.3) and inserting them into the third and the fourth equation in (3.3), we obtain the system of pseudodifferential equations on the open manifold S_0 with unknown functions $\widetilde{\psi}_0^{(1)}$ and $\widetilde{\psi}^{(2)}$:

$$\begin{cases} \pi_0 \left[(-\mathbf{V}_{-1}^{(1)})^M + \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)} \right) (\mathbf{V}_{-1}^{(1)})^{-1} \right]^{-1} \widetilde{\psi}_0^{(1)} & - \\ -\pi_0 \left[(-\mathbf{V}_{-1}^{(2)})^M + \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)} \right) (\mathbf{V}_{-1}^{(2)})^{-1} \right]^{-1} \widetilde{\psi}_0^{(2)} & = \widetilde{G}_1, \\ \widetilde{\psi}_0^{(1)} + \widetilde{\psi}_0^{(2)} & = G_2, \end{cases} \tag{3.4}$$

where

$$\begin{aligned} \widetilde{G}_1 &= G_1 - \pi_0 \left[(-\mathbf{V}_{-1}^{(1)})^M + \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)} \right) (\mathbf{V}_{-1}^{(1)})^{-1} \right]^{-1} \widetilde{\mathbf{\Psi}}_0^{(1)} + \\ &\quad + \pi_0 \left[(-\mathbf{V}_{-1}^{(2)})^M + \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)} \right) (\mathbf{V}_{-1}^{(2)})^{-1} \right]^{-1} \widetilde{\mathbf{\Psi}}_0^{(2)}. \end{aligned}$$

Th system (3.4) yields a pseudodifferential equation with respect to $\widetilde{\psi}_0^{(1)}$:

$$\pi_0 \mathbf{B} \widetilde{\psi}_0^{(1)} = G^*,$$

where

$$\begin{aligned} \mathbf{B} &= \left[(-\mathbf{V}_{-1}^{(1)})^M + \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)} \right) (\mathbf{V}_{-1}^{(1)})^{-1} \right]^{-1} + \\ &+ \left[(-\mathbf{V}_{-1}^{(2)})^M + \left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)} \right) (\mathbf{V}_{-1}^{(2)})^{-1} \right]^{-1}. \end{aligned}$$

The pseudodifferential operator $\pi_0 \mathbf{B}$ is positive definite and the following proposition is proved in [6, Theorem 5.3].

Theorem 3.2. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/2 < s < 1/p + 1/2$. Then the pseudodifferential operator*

$$\begin{aligned} \pi_0 \mathbf{B} \ &: \ \widetilde{\mathbb{H}}_p^{s-1}(S_0) \to \mathbb{H}_p^s(S_0), \\ &: \ \widetilde{\mathbb{B}}_{p,r}^{s-1}(S_0) \to \mathbb{B}_{p,r}^s(S_0) \end{aligned}$$

is invertible in both cases.

It is worth noting that PsDO $\pi_0 \mathbf{B}$ is invertible in the anisotropic Bessel potential spaces with weight $\widetilde{\mathbb{H}}_p^{(\mu,s-1),k}(S_0) \to \mathbb{H}_p^{(\mu,s),k}(S_0)$ (see [7]).

Theorem 3.2 implies the following (see [6, Theorem 5.4])

Theorem 3.3. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/2 < s < 1/p + 1/2$. Then the operator*

$$\mathbf{N}: \begin{array}{c} \mathbb{H}^s_p(\partial\Omega_1) \\ \oplus \\ \mathbb{H}^s_p(\partial\Omega_2) \\ \oplus \\ \widetilde{\mathbb{H}}^{s-1}_p(S_0) \\ \oplus \\ \widetilde{\mathbb{H}}^{s-1}_p(S_0) \end{array} \longrightarrow \begin{array}{c} \mathbb{H}^{s-1}_p(\partial\Omega_1) \\ \oplus \\ \mathbb{H}^{s-1}_p(\partial\Omega_2) \\ \oplus \\ \mathbb{H}^s_p(S_0) \\ \oplus \\ \widetilde{\mathbb{H}}^{s-1}_p(S_0) \end{array} \quad \left(\begin{array}{c} \mathbb{B}^s_{p,r}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}^s_{p,r}(\partial\Omega_2) \\ \oplus \\ \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \\ \oplus \\ \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \end{array} \longrightarrow \begin{array}{c} \mathbb{B}^{s-1}_{p,r}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}^{s-1}_{p,r}(\partial\Omega_2) \\ \oplus \\ \mathbb{B}^s_{p,r}(S_0) \\ \oplus \\ \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \end{array} \right)$$

is Fredholm and has index zero: $\operatorname{Ind}\mathbf{N} = 0$.

Now we will formulate theorems about uniqueness, existence and smoothness of solutions to the Neumann problem (see [6, Theorems 5.5, 5.6 and Remark 5.7]).

Theorem 3.4. *Let $4/3 < p < 4$ and the compatibility condition (3.2) be fulfilled. Then the Neumann boundary value problem has solutions of the class $W^1_p(\Omega)$ in the bounded domain Ω if and only if the condition*

$$\int_{\partial\Omega} \psi \cdot (az + b) ds = 0$$

holds for any constant antisymmetric $n \times n$ matrix a and any constant n-dimensional vector b.

If Ω is an infinite domain and $n > 2$, then the Neumann boundary value problem has a unique solution of the class $W^1_{p,loc}(\Omega)$, provided the solution vanishes at infinity (see the first condition in (1.4)).

If Ω is an infinite domain and $n = 2$, then the Neumann boundary value problem has a unique solution of the class $W^1_{p,loc}(\Omega)$, provided the solution has a finite limit at infinity (see the second condition in (1.4)) and the condition

$$\int_{\partial\Omega} \psi \, ds = 0$$

holds.

Solutions, if they exist, are given by the formulae

$$r_i u = \mathbf{V}^{(i)}(\mathbf{V}^{(i)}_{-1})^{-1} g_i \quad in \quad \Omega_i, \qquad i = 1, 2,$$

where $g_i \in \mathbb{H}^{1/p'}_p(\partial\Omega_i)$, $i = 1, 2$, are found from the system (3.1).

Theorem 3.5. *Let $4/3 < p < 4$, $1 < t < \infty$, $1 \le r \le \infty$, $1/t - 1/2 < s < 1/t + 1/2$, the compatibility condition (3.2) with t instead of p be fulfilled, $u \in W^1_p(\Omega)$ ($W^1_{p,loc}(\Omega)$ and conditions (1.4) hold at infinity). If we solve the Neumann problem, then:*

- $\psi_i \in \mathbb{B}^{s-1}_{t,t}(S_i)$, $i = 1, 2$, *ensures* $u \in \mathbb{H}^{s+1/t}_t(\Omega)$ ($\mathbb{H}^{s+1/t}_{t,loc}(\Omega)$);

- $\psi_i \in \mathbb{B}^{s-1}_{t,r}(S_i)$, $i=1,2$, *ensures* $u \in \mathbb{B}^{s+1/t}_{t,r}(\Omega)$, $(\mathbb{B}^{s+1/t}_{t,r,loc}(\Omega))$;
- $\psi_i \in \mathbb{B}^{\alpha-1}_{\infty,\infty}(\overline{S}_i)$, $i=1,2$, $\alpha \in]0,1/2]$ *ensures* $u \in \bigcap\limits_{\alpha'<\alpha} C^{\alpha'}(\overline{\Omega})$.

Now let us investigate asymptotics of the Neumann boundary value problem. The boundary data of the Neumann problem are sufficiently smooth, i.e., $\psi_i \in \mathbb{H}_p^{(\infty,s+2M),\infty}(S_i)$, $i=1,2$.

In view of the equality (2.5), we can write the symbol $\sigma_{\mathbf{B}}(x',\xi')$ of the pseudodifferential operator $\mathbf{B}$ as follows

$$\begin{aligned}\sigma_{\mathbf{B}}(x',\xi') = \; & \Big[(-\frac{1}{2}\mathcal{I} + \sigma_{\overset{*}{\mathbf{V}}{}^{(1)}_0}(x',\xi'))(\sigma_{\mathbf{V}^{(1)}_{-1}}(x',\xi'))^{-1}\Big]^{-1} + \\ & + \Big[(-\frac{1}{2}\mathcal{I} - \sigma_{\overset{*}{\mathbf{V}}{}^{(1)}_0}(x',\xi'))(\sigma_{\mathbf{V}^{(1)}_{-1}}(x',\xi'))^{-1}\Big]^{-1}.\end{aligned}$$

Since the symbol $\sigma_{\overset{*}{\mathbf{V}}{}^{(1)}_0}(x',\xi')$ is an odd matrix–function with respect to ξ', while the symbol $\sigma_{\mathbf{V}^{(1)}}(x',\xi')$ is an even matrix–function. Therefore one can easily ascertain that the symbol $\sigma_{\mathbf{B}}(x',\xi')$ is even with respect to the variable ξ', i.e.

$$\sigma_{\mathbf{B}}(x',-\xi') = \sigma_{\mathbf{B}}(x',\xi')$$

and all eigenvalues of the matrix $(\sigma_{\mathbf{B}}(x',0,-1))^{-1}\sigma_{\mathbf{B}}(x',0,-1) = \mathcal{I}$ are trivial $\lambda^{(i)}_{\mathbf{B}} = 1$, $j=1,\dots,n$.

Let us consider a local system of coordinates $(x'',x_{n-1,+}) \in S_0$ (see (2.5)). Applying a result on strongly elliptic pseudodifferential equations (see [7, Theorem 2.1]) and taking into account the second equation in (3.4), we obtain the following result on asymptotic expansion of the function $\widetilde{\psi}^{(i)}_0$, $i=1,2$:

$$\begin{aligned}\widetilde{\psi}^{(i)}_0(x'',x_{n-1}) = (-1)^{i+1}c_0(x'')x^{-1/2}_{n-1,+} + \\ + \sum_{k=1}^{M} x^{-1/2+k}_{n-1,+} B^{(i)}_k(x'',\log x_{n-1,+}) + \widetilde{\psi}^{(i)}_{M+1},\end{aligned} \qquad (3.5)$$

where $c_0 \in C^\infty(\partial S_0)$, and the remainder $\widetilde{\psi}^{(i)}_{M+1} \in \mathbb{H}_p^{(\infty,s+M+1),\infty}(S^+_\varepsilon)$, $i=1,2$, $M\in\mathbb{N}$.

$B^{(i)}_k(x'',t)$ in (3.5) is a polynomial of degree k with respect to the variable t and has $C^\infty(\partial S_0)$–smooth vector coefficients on the cuspidal edge $x'' \in \partial S_0$.

Let $(g_1,g_2,\psi^{(1)}_0,\psi^{(2)}_0)$ be a solution of system (3.1), i.e.,

$$\mathbf{N}(g_1,g_2,\psi^{(1)}_0,\psi^{(2)}_0) = \Psi, \qquad (3.6)$$

where $\Psi = (\Psi^{(1)}_0,\Psi^{(2)}_0,0,-(\pi_0\Phi^{(1)}_0+\pi_0\Phi^{(2)}_0))$.

By adding to both parts of the system (3.6) the expression

$$\mathbf{T}_{2M}\begin{pmatrix} g_1 \\ g_2 \\ \psi_0^{(1)} \\ \psi_0^{(2)} \end{pmatrix} = \begin{pmatrix} (\mathbf{V}_{-1}^{(1)})^{2M} & 0 & 0 & 0 \\ 0 & (\mathbf{V}_{-1}^{(2)})^{2M} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ \psi_0^{(1)} \\ \psi_0^{(2)} \end{pmatrix},$$

we obtain the equality

$$\mathbf{N}_{2M}(g_1, g_2, \psi_0^{(1)}, \psi_0^{(2)}) = \Psi^*. \tag{3.7}$$

Here $\overset{*}{\Psi} = (\Psi_0^{(1)} + (\mathbf{V}_{-1}^{(1)})^{2M} g_1, \Psi_0^{(2)} + (\mathbf{V}_{-1}^{(2)})^{2M} g_2, 0, -(\pi_0\Phi_0^{(1)} + \pi_0\Phi_0^{(2)}))$. The system (3.7) takes the form

$$\begin{cases} \mathbf{B}_{2M}^{(i)} g_i - \psi_0^{(i)} = \Psi_0^{(i)} + (\mathbf{V}_{-1}^{i})^{2M} g_i & \text{on } \partial\Omega_i, \quad i = 1, 2, \\ \pi_0 g_1 - \pi_0 g_2 = 0 & \text{on } S_0, \\ \psi_0^{(1)} + \psi_0^{(2)} = 0 & \text{on} S_0, \end{cases} \tag{3.8}$$

where

$$\mathbf{B}_{2M}^{(i)} = (\mathbf{V}_{-1}^{(i)})^{2M} + \Big(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(i)} \Big)(\mathbf{V}_{-1}^{(i)})^{-1}, \quad i = 1, 2.$$

As it is clear from the foregoing arguments, the system can be reduced to a pseudodifferential equation with the positive definite operator.

From the first two equations of the system (3.8) we find that

$$g_i = (\mathbf{B}_{2M}^{(i)})^{-1}\psi_0^{(i)} + F_i, \quad i = 1, 2,$$

where

$$F_i = (\mathbf{B}_{2M}^{(i)})^{-1}\Psi_0^{(i)} + (\mathbf{B}_{2M}^{(i)})^{-1}(\mathbf{V}_{-1}^{(i)})^{2M} g_i,$$
$$F_i \in \mathbb{H}_p^{(\infty, s+2M+1), \infty}(\partial\Omega_i), \quad i = 1, 2.$$

Therefore we can write

$$r_i u = \mathbf{V}^{(i)}(\mathbf{V}_{-1}^{(i)})^{-1}(B_{2M}^{(i)})^{-1}\psi_0^{(i)} + G_i, \quad i = 1, 2;$$

here $G_i = \mathbf{V}^{(i)}(\mathbf{V}_{-1}^{(i)})^{-1}F_i$, $\;G_i \in C^{M+1}(\overline{\Omega}_i)$.

Thus by the asymptotic expansion of the functions $\psi_0^{(i)}$, $i = 1, 2$, (see (3.5)) and the asymptotic expansion of functions represented by potentials (see [8, Theorems 2.2 and 2.3]) and $\Psi_0^{(i)} \in \mathbb{H}_p^{(\infty, s+2M), \infty}(\partial\Omega_i)$, $i = 1, 2$, we obtain the following asymptotics of the solutions of the Neumann boundary value problems in the local coordinates

$$(r_i u)(x'', x_{n-1}, x_n) = \sum_{s=1}^{l(n)} \mathrm{Re}\left\{ \sum_{j=0}^{n_s - 1} [d_{sj}^{(i)}(x'', +1) x_n^j z_{s,+1}^{1/2-j} - d_{sj}^{(i)}(x'', -1) \times \right.$$

$$\times x_n^j z_{s,-1}^{1/2-j}] + \sum_{\vartheta=\pm 1} \sum_{l,k=0}^{M+1} \sum_{\substack{j+p=1 \\ l+k+j+p\neq 1}}^{M+2-l} x_{n-1}^l x_n^j z_{s,\vartheta}^{-\frac{1}{2}+p+k} B_{slkjp}^{(i)}(x'', \log z_{s,\vartheta})\Big\} +$$

$$+ u_{M+1}^{(i)}(x'', x_{n-1}, x_n) \quad \text{for} \quad M > \frac{n-1}{p} - \min\{[s], 0\}, \quad i = 1, 2,$$

with the coefficients $d_{sj}^{(i)}(\cdot, \pm 1) \in C^\infty(\partial S_0)$ and the remainder $u_{M+1}^{(i)} \in C^{M+1}(\overline{\Omega}_i)$, $i = 1, 2$. Here

$$z_{s,+1} = -x_{n-1} - x_n \tau_{s,+1}, \quad z_{s,-1} = x_{n-1} - x_n \tau_{s,-1},$$
$$-\pi < \operatorname{Arg} z_{s,\pm 1} < \pi, \quad \tau_{s,\pm 1} \in C(\partial S_0),$$

$\{\tau_{s,\pm 1}\}_{s=1}^{l(n)}$ are all different roots of the polynomial $\det \mathbf{A}(J_\varkappa^\top(x'', 0)(0, \pm 1, \tau))$ of multiplicity n_s, $s = 1, \dots l(n)$, in the complex lower half-plane;

$B_{slkjp}^{(i)}(x'', t)$ is a polynomial of order $\nu_{kjp} = k + p + j - 1$ with respect to t, with vector coefficients depending on the variable $x'' \in \partial S_0$. The coefficients $d_{sj}^{(i)}(x'', \pm 1)$ have the form (see [8, Theorem 2.3])

$$d_{sj}^{(1)}(x'', +1) = \mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, +1) \sigma^{-1}_{-\frac{1}{2}\mathcal{I} + \overset{*}{V}{}_0^{(1)}}(x'', 0, 0, +1) c^{(j)}(x''),$$

$$d_{sj}^{(1)}(x'', -1) = i\mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, -1) \sigma^{-1}_{-\frac{1}{2}\mathcal{I} + \overset{*}{V}{}_0^{(1)}}(x'', 0, 0, -1) c^{(j)}(x''),$$

$$d_{sj}^{(2)}(x'', +1) = -\mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, +1) \sigma^{-1}_{-\frac{1}{2}\mathcal{I} + \overset{*}{V}{}_0^{(2)}}(x'', 0, 0, -1) c^{(j)}(x''),$$

$$d_{sj}^{(2)}(x'', -1) = i\mathcal{G}_\varkappa(x'', 0) V_{-1,j}^{(s)}(x'', 0, 0, -1) \sigma^{-1}_{-\frac{1}{2}\mathcal{I} + \overset{*}{V}{}_0^{(2)}}(x'', 0, 0, +1) c^{(j)}(x''),$$

$$s = 1, \dots, l(n), \quad j = 0, \dots, n_s - 1,$$

where $\mathcal{G}_\varkappa$ is the square root from the Gramm determinant of the diffeomorphisms $\varkappa$,

$$V_{-1,j}^{(s)}(x'', 0, 0, \pm 1) =$$
$$= -\frac{i^{j+1}}{j!(n_s - 1 - j)!} \frac{d^{n_s-1-j}}{d\tau^{n_s-1-j}} (\tau - \tau_{s,\pm 1})^{n_s} (A^{-1}(J_\varkappa^\top(x'', 0)(0, \pm 1, \tau))\big|_{\tau=\tau_{s,\pm 1}},$$

$$c^{(j)}(x) = \frac{i^j}{2\sqrt{\pi}} \Gamma\Big(j - \frac{1}{2}\Big) c_0(x''),$$

and $c_0(x'')$ is the first coefficient of the asymptotic expansion in (3.5).

4. Asymptotics of Solutions for the Mixed Boundary Value Problems

In [6], a solution of the mixed boundary value problem is sought in the form of a simple layer potential

$$r_i u = \mathbf{V}^{(i)} g_i \quad \text{in} \quad \Omega_i, \quad i = 1, 2.$$

Any continuation $\Phi^{(1)} \in \mathbb{B}^{1/p'}_{p,p}(\partial\Omega_1)$ of the function φ_1 onto the entire boundary $\partial\Omega_1 = S_1 \cup \overline{S}_0$ has the form

$$\Phi^{(1)} = \Phi^{(1)}_0 + \varphi^{(1)}_0,$$

where $\Phi^{(1)}_0$ is a fixed continuation of the function φ_1, and $\varphi^{(1)}_0 \in \widetilde{\mathbb{B}}^{1/p'}_{p,p}(S_0)$.

Similarly, any extension $\Phi^{(2)}_0 \in \mathbb{B}^{-1/p}_{p,p}(\partial\Omega_2)$ of the function φ_2 onto the entire boundary $\partial\Omega_2 = S_2 \cup \overline{S}_0$ has the form

$$\Phi^{(2)} = \Phi^{(2)}_0 + \varphi^{(2)}_0,$$

where $\Phi^{(2)}$ is a fixed continuation of the function φ_2, and $\varphi^{(2)}_0 \in \widetilde{\mathbb{B}}^{-1/p}_{,p}(S_0)$.

The mixed boundary value problem can be reduced to the following system of equations (see [6]):

$$\mathbf{N}\begin{pmatrix} g_1 \\ g_2 \\ \varphi^{(1)}_0 \\ \varphi^{(2)}_0 \end{pmatrix} = \begin{pmatrix} \Phi^{(1)}_0 \\ \Phi^{(2)}_0 \\ -\pi_0\Phi^{(1)}_0 \\ -\pi_0\Phi^{(2)}_0 \end{pmatrix}, \tag{4.1}$$

where

$$\mathbf{N} = \begin{pmatrix} \mathbf{V}^{(1)}_{-1} & 0 & -\mathcal{I} & 0 \\ 0 & -\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(2)}_{0} & 0 & -\mathcal{I} \\ 0 & -\pi_0\mathbf{V}^{(2)}_{-1} & \mathcal{I} & 0 \\ \pi_0(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(1)}_{0}) & 0 & 0 & \mathcal{I} \end{pmatrix}.$$

Consider the combination

$$\mathbf{D} \circ \mathbf{N},$$

where $\mathbf{D}$ is an invertible operator of the form

$$\mathbf{D} = \begin{pmatrix} \mathcal{I} & 0 & 0 & 0 \\ 0 & \mathbf{V}^{(2)}_{-1} & 0 & 0 \\ 0 & 0 & \mathcal{I} & 0 \\ 0 & 0 & 0 & -\mathcal{I} \end{pmatrix}.$$

Now consider the operator

$$\mathbf{N}_M = \begin{pmatrix} \mathbf{V}^{(1)}_{-1} & 0 & -\mathcal{I} & 0 \\ 0 & (-\mathbf{V}^{(2)}_{-1})^M + \mathbf{V}^{(2)}_{-1}(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(2)}_{0}) & 0 & -\mathbf{V}^{(2)}_{-1} \\ 0 & -\pi_0\mathbf{V}^{(2)}_{-1} & \mathcal{I} & 0 \\ \pi_0(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}^{(1)}_{0}) & 0 & 0 & \mathcal{I} \end{pmatrix},$$

$$M = 2, 3, \dots,$$

which differs from the composition $\mathbf{D} \circ \mathbf{N}$ by a compact operator.

A system of equations corresponding to $\mathbf{N}_M$ has the form

$$\begin{cases} \mathbf{V}_{-1}^{(1)} h_1 - \psi_0^{(1)} = \Psi_0^{(1)}, \\ [(-\mathbf{V}_{-1}^{(2)})^M + \mathbf{V}_{-1}^{(2)}(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)})] h_2 - \mathbf{V}_{-1}^{(2)} \psi_0^{(2)} = \Psi_0^{(2)}, \\ -\pi_0 \mathbf{V}_{-1}^{(2)} h_2 + \psi_0^{(1)} = F_1, \\ \pi_0(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)}) h_1 - \psi_0^{(2)} = F_2, \end{cases} \tag{4.2}$$

where

$$\Psi_0^{(i)} \in \mathbb{H}_p^s(\partial\Omega_i) \quad \big(\Psi_0^{(i)} \in \mathbb{B}_{p,r}^s(\partial\Omega_i)\big), \quad i = 1, 2,$$
$$F_1 \in \mathbb{H}_p^s(S_0) \quad \big(F_1 \in \mathbb{B}_{p,r}^s(S_0)\big), \quad F_2 \in \mathbb{H}_p^{s-1}(S_0) \quad \big(F_2 \in \mathbb{B}_{p,r}^{s-1}(S_0)\big).$$

Note that the system (4.1) emerged in [6] while studying the mixed problem in the case $M = 2$.

We have the following auxiliary proposition which is proved similarly to that Lemma 6.2 from [6].

Lemma 4.1. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le r \le \infty$. Then the pseudodifferential operator*

$$(-\mathbf{V}_{-1}^{(2)})^M + \mathbf{V}_{-1}^{(2)}\left(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)}\right) \quad : \quad \mathbb{H}_p^{s-1}(\partial\Omega_2) \to \mathbb{H}_p^s(\partial\Omega_2)$$
$$: \quad \mathbb{B}_{p,r}^{s-1}(\partial\Omega_2) \to \mathbb{B}_{p,r}^s(\partial\Omega_2)$$

is invertible for any $M = 0, 1, 2, \ldots$.

Defining h_1 and h_2 by the first and the second equations of the system (4.2), substituting them into the third and the fourth equations of system (4.2), we obtain a system of pseudodifferential equations on the open manifold S_0

$$\mathbf{Q}\begin{pmatrix} \psi_0^{(1)} \\ \psi_0^{(2)} \end{pmatrix} = \begin{pmatrix} G^* \\ F^* \end{pmatrix}$$

with unknown $\psi_0^{(1)}$ and $\psi_0^{(2)}$, where

$$\mathbf{Q} = \begin{pmatrix} \mathcal{I} & -\pi_0 \mathbf{A}_2 \\ \pi_0 \mathbf{A}_1 & \mathcal{I} \end{pmatrix},$$

$$\mathbf{A}_1 = (-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)})(\mathbf{V}_{-1}^{(1)})^{-1},$$
$$\mathbf{A}_2 = \mathbf{V}_{-1}^{(2)}[(-\mathbf{V}_{-1}^{(2)})^M + \mathbf{V}_{-1}^{(2)}(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)})]^{-1} \mathbf{V}_{-1}^{(2)},$$
$$G^* = F_1 + \pi_0 \mathbf{V}_{-1}^{(2)}[(-\mathbf{V}_{-1}^{(2)})^M + \mathbf{V}_{-1}^{(2)}(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)})]^{-1} \Psi_0^{(2)},$$
$$F^* = F_2 - \pi_0(-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(1)})(\mathbf{V}_{-1}^{(1)})^{-1} \Psi_0^{(1)},$$

The operator $\mathbf{A}_2$ can be written in a more simple form

$$\mathbf{A}_2 = [(-\mathbf{V}_{-1}^{(2)})^{M-2} + (-\frac{1}{2}\mathcal{I} + \overset{*}{\mathbf{V}}{}_0^{(2)})(\mathbf{V}_{-1}^{(2)})^{-1}]^{-1}, \quad M = 2, 3, \dots .$$

Consider the operator

$$\mathbf{P} = \mathbf{Q} \circ \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix} = \begin{pmatrix} \pi_0 \mathbf{A}_2 & \mathcal{I} \\ -\mathcal{I} & \pi_0 \mathbf{A}_1 \end{pmatrix}.$$

Since the operators $\pi_0 \mathbf{A}_i$, $i = 1, 2$ (see [6]) are positive definite, we obtain a strong Gårding inequality for the operator $\mathbf{P}$, i.e., we have

Lemma 4.2 (see [6, Lemma 6.3]). *For the pseudodifferential operator* $\mathbf{P}$ *there exists a constant* $c > 0$ *such that*

$$\mathrm{Re}\langle \mathbf{P}\chi, \chi\rangle_{S_0} \geq c\|\chi\|^2_{\widetilde{\mathbb{H}}_2^{-1/2}(S_0)\oplus\widetilde{\mathbb{H}}_2^{1/2}(S_0)}, \quad \forall \chi \in \widetilde{\mathbb{H}}_2^{-1/2}(S_0) \oplus \widetilde{\mathbb{H}}_2^{1/2}(S_0),$$

where the symbol $\langle\cdot,\cdot\rangle$ *denotes the duality between the spaces* $\mathbb{H}_2^{1/2}(S_0) \oplus \mathbb{H}_2^{-1/2}(S_0)$ *and* $\widetilde{\mathbb{H}}_2^{-1/2}(S_0) \oplus \widetilde{\mathbb{H}}_2^{1/2}(S_0)$.

From now on the investigation of the operator $\mathbf{P}$ is continued by using a local rectification of the manifold and by "freezing" the coefficients. There arises a matrix operator with components of different orders. Therefore it is convenient first to reduce the orders.

Let $\mathbf{P}(x', D')$ be a pseudodifferential operator with the symbol $\sigma_{\mathbf{P}}(x', \xi')$ ($\xi' = (\xi_1, \dots, \xi_{n-1})$), "frozen" at the point and written in some local coordinate system of the manifold S_0.

Let $\mathbf{\Lambda}_\pm$ be pseudodifferential operators (Bessel potentials) whose symbols in the local coordinate system have the form

$$\Lambda_\pm(\xi') = \xi_{n-1} \pm i \pm i|\xi''|, \quad \xi' = (\xi'', \xi_{n-1}).$$

Now we reduce the orders, i.e.,

$$\mathbf{R}(x', D') = \begin{pmatrix} \mathbf{L}_- & 0 \\ 0 & \mathcal{I} \end{pmatrix} \circ \mathbf{P}(x', D) \circ \begin{pmatrix} \mathbf{L}_+ & 0 \\ 0 & \mathcal{I} \end{pmatrix},$$

where $\mathbf{L}_+ = \mathrm{diag}\,\mathbf{\Lambda}_+$, $\mathbf{L}_- = \mathrm{diag}\,\pi_+\mathbf{\Lambda}_-$ are $n \times n$ matrix operators, π_+ is the operator of restriction onto $\mathbb{R}^+_{n-1}$, and ℓ is the continuation operator.

The operators

$$\begin{pmatrix} \mathbf{L}_\pm & 0 \\ 0 & \mathcal{I} \end{pmatrix}$$

are invertible in the respective spaces [15], [29].

Now we will formulate the statements whose proofs are given in [6, Lemma 6.7, Theorems 6.8–6.12].

Lemma 4.3. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/4 < s < 1/p + 1/4$. Then the operator*

$$\begin{aligned}\mathbf{R}(x', D') \quad : \quad & \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+_{n-1}) \oplus \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+_{n-1}) \to \mathbb{H}^{s-1}_p(\mathbb{R}^+_{n-1}) \oplus \mathbb{H}^{s-1}_p(\mathbb{R}^+_{n-1}) \\ & \big(\widetilde{\mathbb{B}}^s_{p,r}(\mathbb{R}^+_{n-1}) \oplus \widetilde{\mathbb{B}}^s_{p,r}(\mathbb{R}^+_{n-1}) \to \mathbb{B}^{s-1}_{p,r}(\mathbb{R}^+_{n-1}) \oplus \mathbb{B}^{s-1}_{p,r}(\mathbb{R}^+_{n-1})\big)\end{aligned}$$

is Fredholm with the zero index.

It is worth noticing that PsDO $\mathbf{R}(x', D')$ is Fredholm in the anisotropic Bessel potential spaces with the weight

$$\widetilde{\mathbb{H}}^{(\mu,s),k}_p(\mathbb{R}^+_{n-1}) \oplus \widetilde{\mathbb{H}}^{(\mu,s),k}_p(\mathbb{R}^+_{n-1}) \to \mathbb{H}^{(\mu,s-1),k}_p(\mathbb{R}^+_{n-1}) \oplus \mathbb{H}^{(\mu,s-1),k}_p(\mathbb{R}^+_{n-1})$$

for all $\mu \in \mathbb{R}$ and $k = 0, 1, \dots$ (see [7]).

Lemma 4.4. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/4 < s < 1/p + 1/4$. Then the operator*

$$\begin{aligned}\mathbf{Q} \quad : \quad & \widetilde{\mathbb{H}}^s_p(S_0) \oplus \widetilde{\mathbb{H}}^{s-1}_p(S_0) \to \mathbb{H}^s_p(S_0) \oplus H^{s-1}_p(S_0) \\ & \big(\widetilde{\mathbb{B}}^s_{p,r}(S_0) \oplus \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \to \mathbb{B}^s_{p,r}(S_0) \oplus \mathbb{B}^{s-1}_{p,r}(S_0)\big)\end{aligned}$$

is invertible.

Theorem 4.5. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/4 < s < 1/p + 1/4$, $M = 2, 3, \dots$. Then the operator*

$$\mathbf{N}_M : \begin{array}{c} \mathbb{H}^{s-1}_p(\partial\Omega_1) \\ \oplus \\ \mathbb{H}^{s-1}_p(\partial\Omega_2) \\ \oplus \\ \widetilde{\mathbb{H}}^s_p(S_0) \\ \oplus \\ \widetilde{\mathbb{H}}^{s-1}_p(S_0) \end{array} \longrightarrow \begin{array}{c} \mathbb{H}^s_p(\partial\Omega_1) \\ \oplus \\ \mathbb{H}^s_p(\partial\Omega_2) \\ \oplus \\ \mathbb{H}^s_p(S_0) \\ \oplus \\ \mathbb{H}^{s-1}_p(S_0) \end{array} \left(\begin{array}{c} \mathbb{B}^{s-1}_{p,r}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}^{s-1}_{p,r}(\partial\Omega_2) \\ \oplus \\ \widetilde{\mathbb{B}}^s_{p,r}(S_0) \\ \oplus \\ \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \end{array} \longrightarrow \begin{array}{c} \mathbb{B}^s_{p,r}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}^s_{p,r}(\partial\Omega_2) \\ \oplus \\ \mathbb{B}^s_{p,r}(S_0) \\ \oplus \\ \mathbb{B}^{s-1}_{p,r}(S_0) \end{array} \right)$$

is invertible.

Theorem 4.6. *Let $1 < p < \infty$, $1 \le r \le \infty$, $1/p - 1/4 < s < 1/p + 1/4$. Then the operator*

$$\mathbf{N} : \begin{array}{c} \mathbb{H}^{s-1}_p(\partial\Omega_1) \\ \oplus \\ \mathbb{H}^{s-1}_p(\partial\Omega_2) \\ \oplus \\ \widetilde{\mathbb{H}}^s_p(S_0) \\ \oplus \\ \widetilde{\mathbb{H}}^{s-1}_p(S_0) \end{array} \longrightarrow \begin{array}{c} \mathbb{H}^s_p(\partial\Omega_1) \\ \oplus \\ \mathbb{H}^{s-1}_p(\partial\Omega_2) \\ \oplus \\ \mathbb{H}^s_p(S_0) \\ \oplus \\ \mathbb{H}^{s-1}_p(S_0) \end{array} \left(\begin{array}{c} \mathbb{B}^{s-1}_{p,r}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}^{s-1}_{p,r}(\partial\Omega_2) \\ \oplus \\ \widetilde{\mathbb{B}}^s_{p,r}(S_0) \\ \oplus \\ \widetilde{\mathbb{B}}^{s-1}_{p,r}(S_0) \end{array} \longrightarrow \begin{array}{c} \mathbb{B}^s_{p,r}(\partial\Omega_1) \\ \oplus \\ \mathbb{B}^{s-1}_{p,r}(\partial\Omega_2) \\ \oplus \\ \mathbb{B}^s_{p,r}(S_0) \\ \oplus \\ \mathbb{B}^{s-1}_{p,r}(S_0) \end{array} \right)$$

is invertible.

Theorem 4.7. *Let $8/5 < p < 8/3$. Then the mixed problem has a unique solution in the class $W_p^1(\Omega)$ (in $W_{p,loc}^1(\Omega)$, provided the condition (1.4) is satisfied at infinity); the solution is given by the formula*

$$r_i u = \mathbf{V}^{(i)} g_i \quad in \quad \Omega_i \quad i = 1, 2,$$

where g_i, $i = 1, 2$, are defined from the system (4.1).

Theorem 4.8. *Let $8/5 < p < 8/3$, $1 < t < \infty$, $1 \leq r \leq \infty$, $1/t - 1/4 < s < 1/t + 1/4$, $u \in W_p^1(\Omega)$ (in $W_{p,loc}^1(\Omega)$, provided the condition (1.4) is satisfied at infinity) be a solution of the mixed problem. Then:*

- *If $\varphi_1 \in \mathbb{B}_{t,t}^s(S_1)$, $\varphi_2 \in \mathbb{B}_{t,t}^{s-1}(S_2)$, we have $u \in \mathbb{H}_t^{s+1/t}(\Omega)$ $(\mathbb{H}_{t,loc}^{s+1/t}(\Omega))$;*
- *If $\varphi_1 \in \mathbb{B}_{t,r}^s(S_1)$, $\varphi_2 \in \mathbb{B}_{t,r}^{s-1}(S_2)$, we have $u \in \mathbb{B}_{t,r}^{s+1/t}(\Omega)$, $(\mathbb{B}_{t,r,loc}^{s+1/t}(\Omega))$;*
- *if $\varphi_1 \in C^\alpha(\overline{S}_1)$, $\varphi_2 \in \mathbb{B}_{\infty,\infty}^{\alpha-1}(\overline{S}_2)$, $\alpha \in]0, 1/2]$, we have $u \in \bigcap\limits_{\alpha' < \alpha} C^{\alpha'}(\overline{\Omega})$.*

Theorem 4.7 implies that a solution of the mixed problem belongs to the class C^α for arbitrary $\alpha < \frac{1}{4}$, provided the problem data are sufficiently smooth.

The principal homogeneous symbol of the pseudodifferential operator $\mathbf{R}(x', D')$ is written as

$$\sigma_{\mathbf{R}}(x', \xi') = \begin{pmatrix} (\xi_{n-1} - i|\xi''|)\sigma_{\mathbf{A}_2}(x', \xi')(\xi_{n-1} + i|\xi''|) & (\xi_{n-1} - i|\xi''|)\mathcal{I} \\ -(\xi_{n-1} + i|\xi''|)\mathcal{I} & \sigma_{\mathbf{A}_1}(x', \xi') \end{pmatrix},$$

where $\sigma_{\mathbf{A}_1}(x', \xi')$ and $\sigma_{\mathbf{A}_2}(x', \xi')$ are the principal homogeneous symbols of the pseudodifferential operators $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively written in the given local coordinate system, and $\mathcal{I}$ is the identity matrix.

Let $\lambda_k(x')$, $k = 1, \dots, 2n$, be the eigenvalues of the matrix

$$\big(\sigma_{\mathbf{R}}(x', 0, +1)\big)^{-1} \sigma_{\mathbf{R}}(x', 0, -1), \tag{4.3}$$

where

$$\sigma_{\mathbf{R}}(x', 0, -1) = \begin{pmatrix} \sigma_{\mathbf{A}_2}(x', 0, -1) & -\mathcal{I} \\ \mathcal{I} & \sigma_{\mathbf{A}_1}(x', 0, -1) \end{pmatrix},$$
$$\sigma_{\mathbf{R}}(x', 0, +1) = \begin{pmatrix} \sigma_{\mathbf{A}_2}(x', 0, +1) & \mathcal{I} \\ -\mathcal{I} & \sigma_{\mathbf{A}_1}(x', 0, +1) \end{pmatrix}.$$

The following propositions are valid.

Lemma 4.9 (see [6, Lemma 6.5]). *Let β_k, $k = 1, \dots, n$, be the eigenvalues of the matrix $\sigma_{\overset{*}{\mathbf{V}}_0} = \sigma_{\overset{*}{\mathbf{V}}_0}(x', 0, +1)$. Then $\beta_k \in]-1/2; 1/2[$, $k = 1, \dots, n$, and for $n = 2l$ we have $\beta_k = b_k$, $\beta_{k+1} = -b_k$, $k = 1, \dots, l$, while for $n = 2l + 1$ we have $\beta_1 = 0$, $\beta_k = b_k$, $b_{k+1} = -b_k$, $k = 1, \dots, l$, where $b_1 > 0, \dots, b_l > 0$.*

Theorem 4.10 (see [6, Theorem 6.6]). *Let $\lambda_k(x')$, $k = 1,\dots,2n$, be the eigenvalues of the matrix (4.3). Then*

$$\lambda_k(x') = \begin{cases} i\sqrt{\dfrac{1-2\beta_k(x')}{1+2\beta_k(x')}}, & \text{if } k = 1,\dots,n, \\ -i\sqrt{\dfrac{1-2\beta_{k-n}(x')}{1+2\beta_{k-n}(x')}}, & \text{if } k = n+1,\dots,2n, \quad x' \in \overline{S}_0, \end{cases}$$

where $\beta_k \in]-\frac{1}{2};\frac{1}{2}[$ are the eigenvalues of the matrix $\sigma_{\overset{*}{\mathbf{V}}_0}$.

Note that Theorem 4.10 plays an important role in proving Lemmata 4.3, 4.4 and Theorems 4.5–4.8. Let $m_1,\dots,m_{2\ell}$ be algebraic multiplicities of the eigenvalues $\lambda_1,\dots,\lambda_{2\ell}$, $\sum\limits_{j=1}^{2\ell} \lambda_j = 2n$.

We introduce the notation

$$b_{\mathbf{R}}(x'') = \big(\sigma_{\mathbf{R}}(x'',0,+1)\big)^{-1}\sigma_{\mathbf{R}}(x'',0,-1,).$$

Let

$$b_{0\mathbf{R}}(x'') = \mathcal{K}^{-1}(x'')b_{\mathbf{R}}(x'')\circ\mathcal{K}(x''), \quad x'' \in \partial S_0,$$

be a canonical Jordan form, where $\mathcal{K}$ is some non-degenerate matrix function, $\det\mathcal{K}(x'') \neq 0$, $x'' \in \partial S_0$ and $\mathcal{K} \in C^\infty(\partial S_0)$.

Asymptotics of the solutions for a strongly elliptic pseudodifferential equation (see [7]) implies that the solution $\chi = (\chi_1,\chi_2)^\top$ of the pseudodifferential equation

$$\mathbf{R}(x',D')\chi = \Psi, \quad \Psi \in \mathbb{H}_p^{(\infty,s+M),\infty}(S_\varepsilon^+)$$

has the following asymptotic expansion:

$$\begin{aligned} \chi(x'',x_{n-1,+}) = {} & \mathcal{K}(x'')x_{n-1,+}^{\frac{1}{4}+\Delta(x'')}\mathbb{B}^0_{a_{pr}}(-\tfrac{1}{2\pi i}\log x_{n-1,+})\mathcal{K}^{-1}(x'')c_0(x'') + \\ & + \sum_{k=1}^{M}\mathcal{K}(x'')x_{n-1,+}^{\frac{1}{4}+\Delta(x'')+k}\mathbb{B}_k(x'',\log x_{n-1,+}) + \chi_{M+1}(x'',x_{n-1,+}) \end{aligned} \tag{4.4}$$

for all sufficiently small $x_{n-1,+} > 0$; here $c_0 \in C^\infty(\partial S_0)$ and $\chi_{M+1} \in \widetilde{\mathbb{H}}_p^{(\infty,s+M+1),\infty}(S_\varepsilon^+)$; exact expansion for $\mathbb{B}^0_{a_{pr}}(t) = \operatorname{diag}\{B^0_{a_{pr}}(t), B^0_{a_{pr}}(t)\}$, where $B^0_{a_{pr}}(t)$ is a triangular block-diagonal matrix function defined in [7]; the vector function $\mathbb{B}_k(x'',t)$ is a polynomial of order $\nu_k = k(2m_0-1)+m_0-1$, $m_0 = \max\{m_1,\dots,m_{2\ell}\}$ with respect to the variable t with $2n$-dimensional vector coefficients which depend on the variable x'', and

$$\Delta(x'') = (\Delta_1(x''),\Delta_2(x''));$$

here

$$\Delta_j(x'') = (\underbrace{\delta_1^{(j)}(x''),\dots,\delta_1^{(j)}(x'')}_{m_1\text{-times}},\dots,\underbrace{\delta_\ell^{(j)}(x''),\dots,\delta_\ell^{(j)}(x'')}_{m_\ell\text{-times}}), \quad j = 1,2,$$

$$\delta_k^{(1)}(x'') = i\alpha_k(x''), \quad \delta_k^{(2)}(x'') = \frac{1}{2} + i\alpha_k(x''),$$
$$\alpha_k(x'') = -\frac{1}{2\pi} \log|\lambda_k(x'')|, \quad k = 1, \dots, \ell.$$

Hence one can write asymptotic expansion for the functions χ_1 and χ_2 separately. In fact, let

$$\mathcal{K}(x'') = \begin{pmatrix} \mathcal{K}_{11}(x'') & \mathcal{K}_{12}(x'') \\ \mathcal{K}_{21}(x'') & \mathcal{K}_{22}(x'') \end{pmatrix}_{2n\times 2n}$$

and

$$\mathcal{K}^{-1}(x'')c_0(x'') = (c_0^{(1)}(x''), c_0^{(2)}(x''))^\top, \tag{4.5}$$

where $\mathcal{K}_{ij}(x'')$, $i, j = 1, 2$, are $n\times n$-matrices; $c_0^{(i)}$, $i = 1, 2$, are n-dimensional vector functions. Then

$$\begin{aligned}\chi_i(x'', x_{n-1,+}) = &\sum_{j=1}^{2} \mathcal{K}_{ij}(x'') x_{n-1,+}^{\frac{1}{4}+\Delta_j(x'')} B^0_{a_{pr}}(-\tfrac{1}{2\pi i}\log x_{n-1,+})c_0^{(i)}(x'') + \\ &+ \sum_{j=1}^{2}\sum_{k=1}^{M} \mathcal{K}_{ij}(x'') x_{n-1,+}^{\frac{1}{4}+\Delta_j(x'')+k} B_{kj}^{(i)}(x'', \log x_{n-1,+}) + \\ &+ \chi_{M+1}^{(i)}(x'', x_{n-1,+}), \quad i = 1, 2,\end{aligned} \tag{4.6}$$

where $B_{kj}^{(i)}(x'', t)$ is a polynomial of order $\nu_k = k(2m_0 - 1) + m_0 - 1$ with respect to the variable t with n-dimensional vector coefficients which depend on the variable x''.

Note that the boundary data of the mixed problem are assumed to be sufficiently smooth, i.e., $\varphi_1 \in \mathbb{H}_p^{(\infty,s+2M+1),\infty}(S_1)$, $\varphi_2 \in \mathbb{H}_p^{(\infty,s+2M),\infty}(S_2)$.

Let $(g_1, g_2, \varphi_0^{(1)}, \varphi_0^{(2)})$ be a solution of the system (4.1), i.e.,

$$\mathbf{N}(g_1, g_2, \varphi_0^{(1)}, \varphi_0^{(2)}) = \Phi,$$

where $\Phi = (\Phi_0^{(1)}, \Phi_0^{(2)}, -\pi_0\Phi_0^{(1)}, -\pi_0\Phi_0^{(2)})$. Then

$$\mathbf{D}\circ\mathbf{N}(g_1, g_2, \varphi_0^{(1)}, \varphi_0^{(2)}) = \Psi; \tag{4.7}$$

here $\Psi = (\Phi_0^{(1)}, \mathbf{V}_{-1}^{(2)}\Phi_0^{(2)}, -\pi_0\Phi_0^{(1)}, -\pi_0\Phi_0^{(2)})$.

Adding the expression

$$\mathbf{T}_{2M+1}\begin{pmatrix} g_1 \\ g_2 \\ \varphi_0^{(1)} \\ \varphi_0^{(2)} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -(\mathbf{V}_{-1}^{(2)})^{2M+1} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} g_1 \\ g_2 \\ \varphi_0^{(1)} \\ \varphi_0^{(2)} \end{pmatrix}$$

to the both parts of the system (4.7), we obtain the equality

$$\mathbf{N}_{2M+1}(g_1, g_2, \varphi_0^{(1)}, \varphi_0^{(2)}) = \widetilde{\Psi}, \tag{4.8}$$

where $\widetilde{\Psi}=(\Phi_0^{(1)},\mathbf{V}_{-1}^{(2)}\Phi_0^{(2)}-(\mathbf{V}_{-1}^{(2)})^{2M+1}g_2,-\pi_0\Phi_0^{(1)},-\pi_0\Phi_0^{(2)})$.

Thus we can obtain $((-\mathbf{L}_+)^{-1}\varphi_0^{(2)},\varphi_0^{(1)})^\top$ which in some local coordinate system would satisfy the pseudodifferential equation

$$\mathbf{R}(x',D')\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}=F,$$

where $F=(\mathbf{L}_-F_1,F_2)^\top$ and

$$\begin{aligned}
F_1&=-\pi_0\Phi_0^{(1)}+\pi_0\mathbf{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbf{V}_{-1}^{(2)}\Phi_0^{(2)}-\\
&\quad-\pi_0\mathbf{V}_{-1}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}(\mathbf{V}_{-1}^{(2)})^{2M+1}g_2,\\
F_2&=\pi_0\Phi_0^{(2)}-\pi_0\Big(-\frac{1}{2}\mathcal{I}+\overset{*}{\mathbf{V}}{}_0^{(1)}\Big)(\mathbf{V}_{-1}^{(1)})^{-1}\Phi_0^{(1)},\\
F_i&\in\mathbb{H}_p^{(\infty,s+2M),\infty}(S_\varepsilon^+),\quad i=1,2,\quad\text{and}\\
\mathbf{B}_{2M+1}^{(2)}&=-(\mathbf{V}_{-1}^{(2)})^{2M+1}+\mathbf{V}_{-1}^{(2)}\Big(-\frac{1}{2}\mathcal{I}+\overset{*}{\mathbf{V}}{}_0^{(2)}\Big).
\end{aligned}$$

Consequently, we can obtain the asymptotic expansions (4.6) for the functions $(\mathbf{L}_+)^{-1}\varphi_0^{(2)}$ and $\varphi_0^{(1)}$.

Using the first two equations of the system (4.8), we can define g_1 and g_2:

$$g_1=(\mathbf{V}_{-1}^{(1)})^{-1}\varphi_0^{(1)}+(\mathbf{V}_{-1}^{(1)})^{-1}\Phi_0^{(1)},\tag{4.9}$$

$$g_2=(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbf{V}_{-1}^{(2)}\varphi_0^{(2)}+(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbf{V}_{-1}^{(2)}\Phi_0^{(2)}+G,\tag{4.10}$$

where $G=-(\mathbf{B}_{2M+1}^{(2)})^{-1}(\mathbf{V}_{-1}^{(2)})^{2M+1}g_2,\quad G\in\mathbb{H}_p^{(\infty,s+2M),\infty}(\partial\Omega_2)$.

Expressions (4.9) and (4.10) result in the following representations: the solutions of the mixed boundary value problems can be expressed by the potential type functions

$$r_1u=\mathbf{V}^{(1)}(\mathbf{V}_{-1}^{(1)})^{-1}\varphi_0^{(1)}+R_1,\tag{4.11}$$

$$r_2u=\mathbf{V}^{(2)}(\mathbf{B}_{2M+1}^{(2)})^{-1}\mathbf{V}_{-1}^{(2)}(-\mathbf{L}_+)[(-\mathbf{L}_+)^{-1}\varphi_0^{(2)}]+R_2,\tag{4.12}$$

where $R_i\in C^{M+1}(\overline{\Omega}_i)$, $i=1,2$.

Thus, taking into account (4.11), (4.12), invoking the asymptotic expansions of the functions $(-\mathbf{L}_+)^{-1}\varphi_0^{(2)}$ and $\varphi_0^{(1)}$ (see (4.6)) and also that of the functions represented by the potentials (see [8, Theorems 2.2 and 2.3]), keeping in mind that $\Phi_0^{(1)}\in\mathbb{H}_p^{(\infty,s+2M+1),\infty}(\partial\Omega_1)$, $\Phi_0^{(2)}\in\mathbb{H}_p^{(\infty,s+2M),\infty}(\partial\Omega_2)$, we derive the following asymptotics of solutions of the mixed boundary value problem under consideration:

$$\begin{aligned}
(r_iu)(x'',x_{n-1},x_n)=\sum_{j=1}^{2}\sum_{s=1}^{l(n)}\operatorname{Re}\bigg\{\sum_{m=0}^{n_s-1}x_n^m\Big[d_{sjm}^{(i)}(x'',+1)z_{s,+1}^{1/4+\Delta_j(x'')-m}\times\\
\times B_{a_{pr}}^0\big(-\tfrac{1}{2\pi i}\log[(-1)^{i+1}z_{s,+1}]\big)-d_{sjm}^{(i)}(x'',-1)z_{s,-1}^{1/4+\Delta_j(x'')-m}\times
\end{aligned}$$

$$\times B^0_{a_{pr}}(-\tfrac{1}{2\pi i}\log[(-1)^{i+1}z_{s,-1}])\Big] c_{ijm}(x'') +$$
$$+\sum_{\vartheta=\pm1}\sum_{l,k=0}^{M+2}\sum_{\substack{p+m=0\\ l+p+m+k\neq 0}}^{M+2-l} x^l_{n-1}x^m_n d^{(i)}_{slmpj}(x'',\vartheta) z_{s,\vartheta}^{\frac14+\Delta_j(x'')+p+k}\times$$
$$\times B^{(i)}_{skmpj}(x'',\log z_{s,\vartheta})\Big\} + u^{(i)}_{M+1}(x'',x_{n-1},x_n)$$
$$\text{for}\quad M > \frac{n-1}{p} - \min\{[s-1],0\},\quad i=1,2, \tag{4.13}$$

with the coefficients $d^{(i)}_{sjm}(\cdot,\pm1)$, c_{ijm}, $d^{(i)}_{slmpj}(\cdot,\pm1) \in C^\infty(\partial S_0)$ and the remainder $u^{(i)}_{M+1} \in C^{M+1}(\overline{\Omega}_i)$; here

$$z_{s,+1} = -x_{n-1} - x_n\tau_{s,+1},\quad z_{s,-1} = x_{n-1} - x_n\tau_{s,-1},$$
$$-\pi < \operatorname{Arg} z_{s,\pm1} < \pi,\quad \tau_{s,\pm1}\in C^\infty(\partial S_0),$$

$\{\tau_{s,\pm1}\}_{s=1}^{l(n)}$ are all different roots of the polynomial $\det \mathbf{A}(J^\top_\varkappa(x'',0)(0,\pm1,\tau))$ of multiplicity n_s, $s=1,\dots,l(n)$, in the complex lower half-plane.

In choosing the corresponding branches, we assume here that the equalities $(-z_{s,\pm1})^{1/4+\Delta_j(x'')-m} = e^{i\pi(1/4+\Delta_j(x'')-m)} z_{s,\pm1}^{1/4+\Delta_j(x'')-m}$ are fulfilled.

$B^{(i)}_{skmpj}(x'',t)$ is a polynomial of order $\nu_{kmp} = \nu_k + p + m$, $\nu_k = k(2m_0 - 1) + m_0$, $m_0 = \max\{m_1,\dots,m_\ell\}$, $\sum\limits_{j=1}^{\ell} m_j = n$, with respect to the variable t with vector coefficients depending on the variable x''.

The following relation between the leading (first) coefficients of the asymptotic expansions (4.13) and (4.6) holds (see [8, Theorem 2.3]):

$$d^{(1)}_{sjm}(x'',+1) = \frac{1}{2\pi}\mathcal{G}_\varkappa(x'',0)V^{(s)}_{-1,m}(x'',0,0,+1)\sigma^{-1}_{V^{(1)}_{-1}}(x'',0,0,+1)\mathcal{K}_{2j}(x''),$$
$$d^{(1)}_{sjm}(x'',-1) = \frac{1}{2\pi}\mathcal{G}_\varkappa(x'',0)V^{(s)}_{-1,m}(x'',0,0,-1)\sigma^{-1}_{V^{(1)}_{-1}}(x'',0,0,+1)\times$$
$$\times\mathcal{K}_{2j}(x'')e^{i\pi(-\frac14-\Delta_j(x''))},$$
$$d^{(2)}_{sjm}(x'',+1) = -\frac{1}{2\pi}\mathcal{G}_\varkappa(x'',0)V^{(s)}_{-1,m}(x'',0,0,+1)\times$$
$$\times\sigma^{-1}_{-\frac12\mathcal{I}+\overset{*}{V}{}^{(2)}_0}(x'',0,0,-1)\mathcal{K}_{1j}(x''), \tag{4.14}$$
$$d^{(2)}_{sjm}(x'',-1) = \frac{1}{2\pi}\mathcal{G}_\varkappa(x'',0)V^{(s)}_{-1,m}(x'',0,0,-1)\times$$
$$\times\sigma^{-1}_{-\frac12\mathcal{I}+\overset{*}{V}{}^{(2)}_0}(x'',0,0,+1)\mathcal{K}_{1j}(x'')e^{i\pi(\frac14+\Delta_j(x''))},$$
$$j=1,2,\quad s=1,\dots,l(n),\quad m=0,\dots,n_s-1;$$

here $\mathcal{G}_\varkappa$ is the square root from the Gramm determinant, and

$$V^{(s)}_{-1,m}(x'',0,0,\pm1) =$$
$$= -\frac{i^{m+1}}{m!(n_s-1-m)!}\,\frac{d^{n_s-1-m}}{d\tau^{n_s-1-m}}\,(\tau-\tau_{s,\pm1})^{n_s}\big(A(J^\top_\varkappa(x'',0)(0,\pm1,\tau))\big|^{-1}_{\tau=\tau_{s,\pm1}}.$$

The coefficients $c_{ijm}(x'')$ in (4.13) are defined as follows:

$$c_{1jm}(x'') = a_{jm}(x'')b_j(x'')c_0^{(2)}(x''),$$
$$c_{2jm}(x'') = a_{jm}(x'')b_j(x'')c_0^{(1)}(x''), \quad j = 1,2,$$

where

$$b_1(x'') = \operatorname{diag}\Big\{b^{m_1}\Big(\frac{1}{4}+i\alpha_1(x'')\Big),\dots,b^{m_\ell}\Big(\frac{1}{4}+i\alpha_l(x'')\Big)\Big\},$$
$$b_2(x'') = \operatorname{diag}\Big\{b^{m_1}\Big(\frac{3}{4}+i\alpha_1(x'')\Big),\dots,b^{m_\ell}\Big(\frac{3}{4}+i\alpha_l(x'')\Big)\Big\},$$
$$b^{m_r}(t) = \|b^{m_r}_{kp}(t)\|_{m_r\times m_r},$$
$$b^{m_r}_{kp}(t) = \begin{cases} \Big(\dfrac{1}{2\pi i}\Big)^{p-k}\dfrac{(-1)^{p+k}}{(p-k)!}\dfrac{d^{p-k}}{dt^{p-k}}(\Gamma(t+1)e^{\frac{i\pi(t+1)}{2}}), & k\le p,\\ 0, & k>p,\end{cases}$$
$$p = 0,\dots,m_r-1, \quad r = 1,\dots,\ell.$$

Further,

$$a_{jm}(x'') = \operatorname{diag}\{a^{m_1}(\lambda_1^{(j)}),\dots,a^{m_\ell}(\lambda_\ell^{(j)})\}, \quad j = 1,2,$$
$$\lambda_r^{(1)}(x'') = -\frac{5}{4} - i\alpha_r(x'') + m, \quad \lambda_r^{(2)}(x'') = -\frac{7}{4} - i\alpha_r(x'') + m,$$
$$m = 0,1,\dots,n_s-1,$$
$$\alpha_r(x'') = -\frac{1}{2\pi}\log|\lambda_k(x'')|, \quad r = 1,\dots,\ell;$$
$$a^{m_r}(\lambda_r^{(j)}) = \|a^{m_r}_{kp}(\lambda_r^{(j)})\|_{m_r\times m_r},$$

where

$$a^{m_r}_{kp}(\lambda_r^{(j)}) = \begin{cases} -i\displaystyle\sum_{l=k}^{p}\frac{(-1)^{p+k}(2\pi i)^{l-p}b^{m_r}_{kl}(\mu_r^{(j)})}{(\lambda_r^{(j)}+1)^{p-l+1}}, & m=0, \ k\le p,\\ (-1)^{p+k}b^{m_r}_{kp}(\lambda_r^{(j)}), & m=1,\dots,n_s-1, \ k\le p,\\ 0, & k>p,\end{cases}$$

j=1,2; here $\lambda_r^{(j)} = -1+m+\mu_r^{(j)}$, $\mu_r^{(1)} = -\frac{1}{4} - i\alpha_r(x'')$, $\mu_r^{(2)} = -\frac{3}{4} - i\alpha_r(x'')$, $r = 1,\dots,\ell$, and $c_0^{(1)}(x'')$, $c_0^{(2)}(x'')$ are defined by using the first coefficients of the asymptotic expansion of the functions $(-\mathbf{L}_+)^{-1}\varphi_0^{(2)}$ and $\varphi_0^{(1)}$, respectively (see (4.5)).

Remark 4.11. Lemma 4.9 and Theorem 4.10 readily imply that if $n = 2$ or $n = 3$, then the eigenvalues λ_k, $k = 1, \dots, 2n$, of the matrix (4.3) are different; therefore there exists a nondegenerate infinitely differentiable matrix $\mathcal{K}$ such that the matrix $b_{0\mathcal{R}}$ is diagonal. Then $B^0_{a_{pr}} = \mathcal{I}$, $\nu_k = k$, and expansion (4.13) of the solutions of the mixed boundary value problem can be written in a simple form:

$$
\begin{aligned}
(r_i u)(x'', x_{n-1}, x_n) = \sum_{j=1}^{2} \sum_{s=1}^{l(n)} \operatorname{Re} \Bigg\{ \sum_{m=0}^{n_s-1} x_n^m \Big[d^{(i)}_{sjm}(x'', +1) z_{s,+1}^{1/4+\Delta_j(x'')-m} - \\
- d^{(i)}_{sjm}(x'', -1) z_{s,-1}^{1/4+\Delta_j(x'')-m} \Big] c_{ijm}(x'') + \\
+ \sum_{\vartheta=\pm 1} \sum_{\substack{l,k=0 \\ l+p+m+k\neq 0}}^{M+2} \sum_{p+m=0}^{M+2-l} x_{n-1}^l x_n^m d^{(i)}_{slmpj}(x'', \vartheta) z_{s,\vartheta}^{\frac{1}{4}+\Delta_j(x'')+p+k} \times \\
\times B^{(i)}_{skmpj}(x'', \log z_{s,\vartheta}) \Bigg\} + u^{(i)}_{M+1}(x'', x_{n-1}, x_n), \\
u^{(i)}_{M+1} \in C^{M+1}(\overline{\Omega}_i), \quad i = 1, 2, \quad \text{for } M > \frac{n-1}{p} - \min\{[s-1], 0\},
\end{aligned}
$$

where $B^{(i)}_{skmpj}(x'', t)$ is a polynomial of order $\nu_{kmp} = k + p + m$. The coefficients $d^{(i)}_{sjm}(x'', \pm 1)$ have the same form as in (4.14), and

$$
\begin{aligned}
c_{1jm}(x'') = \operatorname{diag} \left\{ \frac{\Gamma(\mu_r^{(j)} + 1)\Gamma(-\mu_r^{(j)} + 1)}{\lambda_r^{(j)} + 1} \right\}_{r=1}^{n} i^{m+1} c_0^{(2)}(x''), \\
c_{2jm}(x'') = \operatorname{diag} \left\{ \frac{\Gamma(\mu_r^{(j)} + 1)\Gamma(-\mu_r^{(j)} + 1)}{\lambda_r^{(j)} + 1} \right\}_{r=1}^{n} i^{m+1} c_0^{(1)}(x''), \\
j = 1, 2, \quad m = 0, 1, \dots, n_s - 1.
\end{aligned}
$$

Acknowledgement

This work was supported by:

- Georgian Academy of Sciences within the GRANT No 1.3 (1997).

References

1. J. Bennish, Asymptotics for elliptic boundary value problems for systems of pseudodifferential equations. *J. Math. Anal. Appl.* **179**(1993), 417–445.

2. T. Buchukuri and T. Gegelia, On the uniqueness of solutions of the basic problems of the elasticity for infinite domain. (Russian) *Differentsial'nye Uravneniya* **25**(1989), No. 9, 1556–1565.

3. T. Burchuladze and T. Gegelia, Development of the method of a potential in the theory of elasticity. (Russian) *Tbilisi, Metsniereba,* 1985.

4. O. Chkadua, The Nonclassical boundary-contact problems of elasticity for homogeneous anisotropic media. *Math. Nach.* **172**(1995), 49–64.

5. O. Chkadua, Some boundary-contact problems of the elasticity theory with mixed boundary conditions outside the contact surface. *Math. Nach.* **188**(1997), 23–48.

6.O. Chkadua, The Dirichlet, Neumann and mixed boundary value problems of the theory of elasticity in n–dimensional domains with boundaries containing closed cuspidal edges. *Math. Nach.* **189**(1998).

7. O. Chkadua and R. Duduchava, Pseudodifferential equations on manifolds with boundary: Fredholm property and asymptotics. *Preprint, Univarsität Stuttgart, Sonderforschungsbereich* 404, *Bericht* **98/11**.

8. O. Chkadua and R. Duduchava, Asymptotics of functions represented by potentials. *Preprint, Univarsität Stuttgart, Sonderforschungsbereich* 404, *Bericht* **98/12**.

9. M. Costabel and M. Dauge, General edge asymptotics of solutions of second order elliptic boundary value problems, I–II. *Proc. Royal Soc. Edinburgh* **123 A**, 1993, 109–155, 157–184.

10. M. Dauge, Elliptic boundary value problems in corner domains. Smoothness and asymptotics of solutions. *Lecture Notes in Math.* **1341**, *Springer-Verlag, Berlin,* 1988.

11. R. Duduchava and D. Natroshvili, Mixed crack type problems in anisotropic elasticity. *Math. Nach.* **191**(1998), 83–107.

12. R. Duduchava, D. Natroshvili, and E. Shargorodsky, Boundary value problems of the mathematical theory of cracks. (Russian) *Trudy Inst. Prikl. Mat. I. N. Vekua,* **39**(1990), 63–84.

13. R. Duduchava, A. Sändig, and W. Wendland, Interface cracks in anisotropic composites. *Universität Stuttgart, Sonderforschungsbereich* 404, *Bericht* **97/20**, 1–49, 1997.

14. R. Duduchava and W. Wendland, The Wiener–Hopf method for systems of pseudodifferential equations with an application to crack problem. *Integral Equation Operator Theory* **23**(1995), 294–335.

15. G. Eskin, *Boundary value problems for elliptic pseudo–differential equations. Translations of Mathematical Monographs, vol.* 52, *AMS, Providence, Rhode Island* 1981.

16. G. Fichera, Existence theorems in elasticity. Boundary value problems of elasticity with unilateral constraints. *Handbuch der Physik, Band* 6 a/2, *Springer–Verlag, Berlin,* 1972.

17. P. Grisvard, Elliptic problems in non-smooth domains. *Pitman, London, Boston,* 1985.

18. V. Kondrat'ev, Boundary problems for elliptic equations in domains with conical or angular points. (Russian) *Transactions Moscow Mathematical Society* **16**(1967), 227–313.

19. V. Kozlov and V. Maz'ya, On stress singularities near the boundary of a polygonal crack. *THD–Preprint* 1289, *TH Darmstadt,* 1990.

20. S. G. Lekhnitskiĭ, Theory of elasticity of an anisotropic body. (Russian) *Nauka, Moscow,* 1977.

21. V. Maz'ya and B. Plamenevskiĭ, On elliptic boundary value problems in a domain with piecewise smooth boundary. (Russian) *Trudy Simposiuma po Mekhanike Sploshnoĭ Sredy i Rodstvennim Problemam Analiza* I, 171–181, *Metsniereba, Tbilisi,* 1971.

22. V. Maz'ya and A. Soloviev, On the integral equation of the Dirichlet problem in a plane domain with cusps on the boundary. (Russian) *Mat. Sb.* **180**(1989), No. 6, 1211–1233.

23. V. Maz'ya and A. Soloviev, On the boundary integral equation of the Neumann problem in a plane domain with a peak. LiTH–MAT–R–91–16. *Linköping University,* 1991.

24. V. Maz'ya and A. Soloviev, On solvability of boundary integral equations of the elasticity theory in domains with outward peaks. LiTH–MAT–R–91–21. *Linköping University,* 1991.

25. V. Maz'ya and A. Soloviev, Asymptotics of the solution of the integral equation of the Newmann problem in a plane domain with cusps on the boundary. (Russian) *Soobshch. Akad. Nauk Gruz. SSSR* **130(1)**(1998), 17–20.

26. D. Natroshvili, O. Chkadua, and E. Shargorodsky, Mixed problems for homogeneous anisotropic elastic media. (Russian) *Trudy Inst. Prikl. Mat. I.N. Vekua,* **39**(1990), 133–181.

27. S. Nazarov and B. Plamenevsky, Elliptic problems in domains with piecewise-smooth boundaries. (Russian) *Nauka, Moscow,* 1991.

28. B. W. Schulze, Crack problems in the edge pseudo-differential calculus. *Appl. Anal.* **45**(1992), 333–360.

29. M. Triebel, Interpolation theory, function spaces, differential operators. *North-Holland, Amsterdam,* 1978.

(Received 6.05.1998)

Authors' address:
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 3880093
Georgia