Memoirs on Differential Equations and Mathematical Physics
Volume 15, 1998, 81–108

Avtandil Gagnidze

HEAT DISTRIBUTION IN A MEDIUM WITH CONCENTRATED PERTURBATION OF DENSITY

Abstract. In the paper a parabolic equation with a small parameter is considered, the vanishing of the patameter implying the concentration of perturbations of the coefficients in a small neighbourhood. Full asymptotic expansions with respect to rational powers of the small parameter are constructed.

1991 Mathematics Subject Classification. 35K05.

Key words and phrases. Small parameter, asymptotic expansion.

რეზიუმე. ნაშრომში განხილულია მცირე პარამეტრის შემცველი მეორე რიგის პარაბოლური განტოლება, პარამეტრის ქრობას თან მოსდევს კოეფიციენტების შეშფოთების თავმოყრა მცირე მიდამოში. აგებულია ამონახსნის სრული ასიმპტოტური გაშლები მცირე პარამეტრის რაციონალური ხარისხების მიმართ.

Introduction

In mathematics, equations with a small parameter have been being considered for a long time. The works of M. I. Vishik and L. A. Lyusternik [1]–[2] exerted great influence on the investigation of problems with a small parameter. In their works, the above-mentioned authors systematized different classes of problems with a small parameter and gave general principles and methods for their solution. Then followed many excellent works dealing with the theory of equations with a small parameter. We mention here only the works of E. Sanchez-Palencia [3], N. S. Bakhvalov and G. A. Panasenko [4], A. M. Il'in[5], O. A. Oleĭnik, G. A. Iosifyan and A. S. Shamaev [6], V. G. Maz'ya, S. A. Nazarov and B. A. Plamenevskiĭ [7], S. A. Nazarov [8], A. S. Demidov [9], etc. A range of problems studied in these and in many other works is too wide, but we will restrict ourselves to the consideration of those problems which are closely connected with the problems studied in the present paper.

E. Sanchez-Palencia and H. Tchatat ([10], [11]) were the first who focused their attention on the problems appearing in mechanics, physics and engineering. They treated the eigenvalue problem on for the Laplace operator in a medium with density perturbing in a small neighborhood of the origin.

Subsequently, O. A. Oleĭnik, S. A. Nazarov, Yu. D. Golovatyĭ, T. S. Soboleva, G. A. Iosifyan and A. S. Shamaev [6], [12]–[17] elaborated methods for the solution of such problems. In particular, the eigenvalue problem for the second order elliptic equation of in a medium with apparent additional density is studied in [12]. This problem corresponds to the physical problem dealing with proper oscillations of a fixed string with apparent additional mass.

In the present work we consider the problem on heat distribution in a medium with density perturbing in a small neighborhood of the origin.

1. Asymptotic Expansion of the Solution of Heat Equation with a Weak Concentrated Perturbation of Density

In the domain $\Omega = (-1,1)\times(0,T)$ let us consider the initial boundary value problem for the heat equation of the kind

$$\Big(1+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)u_t = u_{xx}, \tag{1.1}$$

with the boundary conditions

$$u(-1,t) = u(1,t) = 0 \tag{1.2}$$

and the initial condition

$$u(x,0) = u_0(x), \tag{1.3}$$

where $\varepsilon \in (0,1)$, $m < 1$ is some rational number, and χ is a function satisfying the following conditions: $\chi(\xi) = 0$ for $|\xi| > 1$, $\chi(\xi) > 0$, $|\xi| \le 1$ and $\int_{-1}^{1} \chi(\xi) d\xi = M = \mathrm{const} > 0$.

It is assumed that the initial function u_0 is continuous on $[-1,1]$, satisfies the conditions $u_0(\pm 1) = 0$ and in the neighborhood of $x = 0$ can be expanded in Taylor series. We can easily see that in this case

$$\lim_{\varepsilon \to 0} \varepsilon^{-m} \int_{-\varepsilon}^{\varepsilon} \chi\Big(\frac{x}{\varepsilon}\Big),$$

and just because of it we call the perturbation of the coefficient a weak perturbation.

Under a solution of the problem (1.1)–(1.3) will be meant a function u which under the conditions (1.2) and (1.3) satisfies the equation (1.1) in Ω for $x \ne \pm\varepsilon$, and at the points of discontinuity $x = \pm\varepsilon$ of the function χ (there are only two points of discontinuity) satisfies continuous "sewing" conditions

$$\begin{aligned} u(\varepsilon+0,t) = u(\varepsilon-0,t), \quad u_x(\varepsilon+0,t) = u_x(\varepsilon-0,t), \\ u(-\varepsilon+0,t) = u(-\varepsilon-0,t), \quad u_x(-\varepsilon+0,t) = u_x(-\varepsilon-0,t). \end{aligned} \tag{1.4}$$

According to O.A. Oleĭnik's work [18], the problem (1.1)–(1.3) is uniquely solvable in the domain Ω.

Let $m = \frac{l}{p}$, where $p > 0$ and $l < p$ (here l and p are integers). The use will be made of the following notation: $\xi = \frac{x}{\varepsilon}$, $\Omega_+^\varepsilon = (\varepsilon, 1) \times (0,T)$, $\Omega_-^\varepsilon = (-1,-\varepsilon) \times (0,T)$, $\Omega_+ = (0,1) \times (0,T)$, $\Omega_- = (-1,0) \times (0,T)$.

Construct a complete asymptotic expansion of the solution u_ε of the problem (1.1)–(1.3) into power series $\delta = \varepsilon^{\frac{1}{p}}$ as $\varepsilon \to 0$. A solution is sought in the form

$$u_\varepsilon(x,t) \sim \sum_{i=0}^{\infty} \delta^i v_i^{\pm}(x,t), \quad (x,t) \in \Omega_\pm^\varepsilon, \tag{1.5}$$

$$u_\varepsilon(x,t) \sim \sum_{i=0}^{\infty} \delta^i w_i(\frac{x}{\varepsilon},t), \quad (x,t) \in (-\varepsilon,\varepsilon) \times (0,T). \tag{1.6}$$

We have treated this problem in [20], where the detailed construction of the complete asymptotic expansion of the form (1.5)–(1.6) is given.

Denote

$$U_N(x,t) = \begin{cases} \sum\limits_{i=0}^{N} \delta^i v_i^{\pm}(x,t), & |x| > \varepsilon, \\ \sum\limits_{i=0}^{N} \delta^i w_i\Big(\dfrac{x}{\varepsilon},t\Big), & |x| < \varepsilon. \end{cases} \tag{1.7}$$

In [20] the following theorem is proved:

Theorem 1.1. *Let u_ε be a solution of the problem* (1.1)–(1.3)*, and let U_N be a partial sum of the formal asymptotic series* (1.5)–(1.6) *which is defined by the formula* (1.7)*. Then the following inequality is valid:*

$$\|u_\varepsilon - U_N\|_{L_2(\Omega)} \le \widetilde{M}\delta^{N+1},$$

where the constant $\widetilde{M}$ does not depend on δ and N.

2. Asymptotics of the Solution of Heat Equation with Delta-Shaped Perturbation of Density

In the domain $\Omega = (-1,1) \times (0,T)$ consider the initial boundary value problem for the heat equation of the kind

$$\Big(1+\varepsilon^{-1}\chi\big(\frac{x}{\varepsilon}\big)\Big)u_t = u_{xx} \tag{2.1}$$

under the boundary conditions

$$u(-1,t) = u(1,t) = 0, \tag{2.2}$$

and the initial conditions

$$u(x,0) = u_0(x), \tag{2.3}$$

where the functions χ and u_0 are the same as in [20].

Obviously, in this case we have the equality

$$\varepsilon^{-1}\int_{-\varepsilon}^{\varepsilon}\chi\big(\frac{x}{\varepsilon}\big)dx = M,$$

and such a perturbation is called a delta-shaped perturbation.

As to the initial function, we assume that it is continuous on $[-1,1]$ and in the neighborhood of the point $x=0$ can be expanded in Taylor seties.

Just as in [20], under a solution of the problem (2.1)–(2.3) it will be understood a function u which in the neighborhood of the point Ω, with the possible exception of the points $x=\pm\varepsilon$, satisfies the equation (2.1), whereas at the points of discontinuity the "sewing" conditions

$$\begin{aligned} u(\varepsilon+0,t) &= u(\varepsilon-0,t), \quad u_x(\varepsilon+0,t) = u_x(\varepsilon-0,t),\\ u(-\varepsilon+0,t) &= u(-\varepsilon-0,t), \quad u_x(-\varepsilon+0,t) = u_x(-\varepsilon-0,t) \end{aligned} \tag{2.4}$$

are fulfilled.

It follows as in Section 1 of Oleĭnik's work [18] that the problem (2.1)–(2.3) is uniquely solvable.

A formal asymptotic expansion will be sought in the form

$$u_\varepsilon(x,t) \sim \begin{cases} \displaystyle\sum_{i=0}^{\infty}\varepsilon^i v_i^{\pm}(x,t), & |x|>\varepsilon,\\ \displaystyle\sum_{i=0}^{\infty}\varepsilon^i w_i\Big(\frac{x}{\varepsilon},t\Big), & |x|<\varepsilon. \end{cases} \tag{2.5}$$

Similarly to [20] we get that the functions $v_i^\pm$ and w_i satisfy the following initial and boundary conditions:

$$v_0^\pm(x,0) = u_0(x), \quad v_i^\pm(x,0) = 0, \quad i \geq 1, \tag{2.6}$$

$$w_i(\xi,0) = \frac{\xi^i}{i!}\frac{d^i}{dx^i}u_0(0), \quad \xi = \frac{x}{\varepsilon}, \quad i \geq 0, \tag{2.7}$$

$$v_i^\pm(\pm 1,t) = 0 \quad i \geq 0. \tag{2.8}$$

Substituting the formal expansion (2.5) into the equation (2.1), we find as in [20] that the functions $v_i^\pm$ and $w_i^\pm$ satisfy the equations

$$\frac{\partial}{\partial t}v_i^\pm(x,t) - \frac{\partial^2}{\partial x^2}v_i^\pm(x,t) = 0, \quad i \geq 0, \tag{2.9}$$

$$\frac{\partial}{\partial t}w_{i-2}(\xi,t) + \chi(\xi)\frac{\partial}{\partial t}w_{i-1}(\xi,t) - \frac{\partial^2}{\partial t^2}w_i(\xi,t) = 0, \quad i \geq 0, \tag{2.10}$$

where the functions with negative indices are absent.

Assume now that the functions $v_i^\pm$ in the neighborhood of the points $(0,t)$ are represented by the Taylor series:

$$v_i^\pm(x,t) \sim \sum_{S=0}^{\infty}\frac{x^S}{S!}\frac{\partial^S}{\partial x^S}v_i^\pm(0,t), \quad |x| > \varepsilon.$$

Then the formal expansion (2.5) results in

$$u_\varepsilon(x,t) \sim \sum_{i=0}^{\infty}\varepsilon^i\sum_{S=0}^{\infty}\frac{x^S}{S!}\frac{\partial^S}{\partial x^S}v_i^\pm(0,t), \quad |x| > \varepsilon,$$

$$\frac{u_\varepsilon}{\partial x}(x,t) \sim \sum_{i=0}^{\infty}\varepsilon^i\sum_{S=1}^{\infty}\frac{x^{S-1}}{(S-1)!}\frac{\partial^S}{\partial x^S}v_i^\pm(0,t), \quad |x| > \varepsilon,$$

$$u_\varepsilon(x,t) \sim \sum_{i=0}^{\infty}\varepsilon^i w_i(\xi,t), \ |\xi| < 1, \ \frac{u_\varepsilon}{\partial x}(x,t) \sim \sum_{i=0}^{\infty}\varepsilon^{i-1}\frac{\partial w_i}{\partial \xi}(\xi,t), \ |\xi| < 1.$$

Substitute $x = \pm\varepsilon$ and $\xi = \pm 1$ in the latter expansion. Taking into account that the function u_ε must satisfy the "sewing" condition, as in [20] we find that

$$\begin{gathered}\frac{\partial w_0}{\partial \xi}(\pm 1,t) = 0, \ \ w_i(\pm 1,t) = \sum_{S=0}^{i}\frac{(\pm 1)^S}{S!}\frac{\partial^S}{\partial x^S}v_{i-S}^\pm(\pm 0,t), \ i \geq 0, \\ \frac{\partial w_{i+1}}{\partial \xi}(\pm 1,t) = \sum_{S=0}^{i}\frac{(\pm 1)^S}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}}v_{i-S}^\pm(\pm 0,t), \quad i \geq 0.\end{gathered} \tag{2.11}$$

Show how one can construct successively the functions v_i and w_i (in what follows, instead of $v_i^\pm$ we will write v_i).

I. First step. The equations (2.10) and the conditions (2.11) for the function w_0 result in $\frac{\partial^2 w_0}{\partial \xi^2}(\xi,t) = 0$, $\frac{\partial w_0}{\partial \xi}(\pm 1,t) = 0$. This implies that

$w_0(\xi,t) = a_1(t)$. Since form (2.11) we have $w_0(\pm 1,t) = v_0(\pm 0,t)$, it is obvious that $w_0(\xi,t) = v_0(0,t)$ and $v_0(+0,t) - v_0(-0,t) = 0$. For the function w_1, from (2.10) we obtain $\frac{\partial^2 w_1}{\partial \xi^2}(\xi,t) = \chi(\xi)\frac{\partial}{\partial t}w_0(\xi,t)$, whence $\frac{\partial^2 w_1}{\partial \xi^2}(\xi,t) = \chi(\xi)\frac{\partial}{\partial t}v_0(0,t)$. Consequently,

$$\frac{\partial w_1}{\partial \xi}(\xi,t) = \frac{\partial}{\partial t}v_0(0,t)\int_{\xi_0}^{\xi}\chi(S)dS + a_2(t). \tag{2.12}$$

For the function w_1, from (2.11) we get

$$\frac{\partial w_1}{\partial \xi}(\pm 1,t) = \frac{\partial}{\partial x_0}v_0(\pm 0,t). \tag{2.13}$$

Then we find

$$\frac{\partial v_0}{\partial x}(+0,t) - \frac{\partial v_0}{\partial x}(-0,t) = \frac{\partial v_0}{\partial t}(0,t)\int_{-1}^{1}\chi(\xi)d\xi = M\frac{\partial v_0}{\partial t}(0,t).$$

Thus for the function v_0 we obtain the problem

$$\frac{\partial v_0}{\partial t}(x,t) = \frac{\partial^2 v_0}{\partial x^2}(x,t), \quad x \neq 0,$$
$$v_0(\pm,t) = v_0(-1,t) = 0, \;\; v_0(x,0) = u_0(x), \;\; v_0(+0,t) - v_0(-0,t) = 0,$$
$$\frac{\partial v_0}{\partial x}(+0,t) - \frac{\partial v_0}{\partial x}(-0,t) = M\frac{\partial}{\partial t}v_0(0,t)$$

with the discontinuity conditions at $x = 0$. This problem is uniquely solvable according to [18]. Hence the function v_0 is defined uniquely. Then the function w_0 can be defined uniquely by the formula $w_0(\xi,t) = v_0(0,t)$.

Revert now to the function w_1. It is easily seen that

$$w_1(\xi,t) = \frac{\partial v_0}{\partial t}(0,t)\int_{\xi_0}^{\xi}\int_{\eta_0}^{\eta}\chi(S)dSd\eta + a_2(t)\xi + a_3(t).$$

So we have to determine the functions a_2 and a_3. From (2.12) and (2.13) we have

$$\frac{\partial v_0}{\partial t}(0,t)\int_{\xi_0}^{1}\chi(\xi)d\xi + a_2(t) = \frac{\partial v_0}{\partial x}(+0,t),$$
$$\frac{\partial v_0}{\partial t}(0,t)\int_{\xi_0}^{-1}\chi(\xi)d\xi + a_2(t) = \frac{\partial v_0}{\partial x}(-0,t).$$

After addition we obtain

$$a_2(t) = \frac{1}{2}\left[\frac{\partial v_0}{\partial x}(+0,t) + \frac{\partial v_0}{\partial x}(-0,t) + \frac{\partial v_0}{\partial t}(0,t)\left(\int_{\xi_0}^{1}\chi(\xi)d\xi + \int_{\xi_0}^{-1}\chi(\xi)d\xi\right)\right],$$

and then the function w_1 is defined to within the summand $a_3(t)$. Thus at the first step we have defined v_0 and w_0 uniquely, whereas w_1 has been defined to within a summand c_1 depending only on t.

II. Second step. It follows from (2.9) and (2.6)–(2.8) that the function v_1 satisfies the equation

$$\frac{\partial v_1}{\partial t}(x,t) = \frac{\partial^2 v_1}{\partial x^2}(x,t), \quad x \neq 0,$$

and the conditions $v_1(x,0) = 0$, $v_1(-1,t) = v_1(1,t) = 0$. We have to find the conditions of conjugation at the point $x+0$.

The conditions (2.11) for the function w_1 yield

$$w_1(\pm 1,t) - v_1(\pm 0,t) = \pm\frac{\partial}{\partial x} v_0(\pm 0,t),$$

whence

$$v_1(+0,t) - v_1(-0,t) = f_1(1,t) - f_1(-1,t) - \frac{\partial v_0}{\partial x}(+0,t) + \frac{\partial v_0}{\partial x}(-0,t),$$

$$C_1(t) = \frac{1}{2}\left[v_1(+0,t)+v_1(-0,t)-f_1(1,t)-f_1(-1,t)+\frac{\partial v_0}{\partial x}(+0,t)-\frac{\partial v_0}{\partial x}(-0,t)\right],$$

where $f_1(\xi,t) = w_1(\xi,t) - C_1(t)$ is the given function. Thus we have defined the difference $v_1(+0,t) - v_1(-0,t)$.

The equations (2.10) and the conditions (2.11) for the function w_2 result in the following problem:

$$\frac{\partial^2 w_2}{\partial \xi^2}(\xi,t) = \frac{\partial w_0}{\partial t}(\xi,t) + \chi(\xi)\frac{\partial w_1}{\partial t}(\xi,t), \tag{2.14}$$

$$\frac{\partial w_2}{\partial \xi}(\pm 1,t) = \frac{\partial v_1}{\partial x}(\pm 0,t) \pm \frac{\partial^2}{\partial x^2} v_0(\pm 0,t). \tag{2.15}$$

The condition for the solvability of the problem (2.14)–(2.15) consists in that the difference $\frac{\partial w_2}{\partial \xi}(1,t) - \frac{\partial w_2}{\partial \xi}(-1,t)$ be the same for (2.14) and (2.15). Since from (2.14) we have

$$\frac{\partial w_2}{\partial \xi}(1,t) - \frac{\partial w_2}{\partial \xi}(-1,t) = \int_{-1}^{1} \frac{\partial w_0}{\partial t}(\xi,t)d\xi + \int_{-1}^{1} \chi(\xi)\frac{\partial f_1}{\partial t}(\xi,t)d\xi + MC_1'(t)$$

and the condition (2.15) gives

$$\frac{\partial w_2}{\partial \xi}(1,t) - \frac{\partial w_2}{\partial \xi}(-1,t) = \frac{\partial v_1}{\partial x}(+0,t) + frac\partial v_1 \partial x(-0,t) + \\ + \frac{\partial^2}{\partial x^2} v_0(+0,t) + \frac{\partial^2}{\partial x^2} v_0(-0,t),$$

we obtain

$$\frac{\partial v_1}{\partial x}(+0,t) - \frac{\partial v_1}{\partial x}(-0,t) + \frac{\partial^2}{\partial x^2} v_0(+0,t) + \frac{\partial^2}{\partial x^2} v_0(-0,t) = \\ = \int_{-1}^{1} \frac{\partial w_0}{\partial t}(\xi,t)d\xi + \int_{-1}^{1} \chi(\xi)\frac{\partial f_1}{\partial t}(\xi,t)d\xi + MC_1'(t).$$

Taking into account the above-obtained expression for C_1, we find that

$$\frac{\partial v_1}{\partial x}(+0,t)-\frac{\partial v_1}{\partial x}(-0,t)-\frac{M}{2}\Big[\frac{\partial}{\partial t}v_1(+0,t)+\frac{\partial}{\partial t}v_1(-0,t)\Big]=\Psi_1(t),$$

where Ψ_1 is a known function.

Thus for the function v_1 we have the problem with the following conditions of conjugation:

$$\frac{\partial v_1}{\partial t}(x,t)=\frac{\partial^2 v_1}{\partial x^2}(x,t),\quad x\neq 0,$$
$$v_1(x,0)=0,\ \ v_1(-1,t)=v_1(1,t)=0,\ \ v_1(+0,t)-v_1(-0,t)=h_1(t),$$
$$\frac{\partial v_1}{\partial x}(+0,t)-\frac{\partial v_1}{\partial x}(-0,t)-\frac{M}{2}\Big[\frac{\partial v_1}{\partial t}(+0,t)+\frac{\partial v_1}{\partial t}(-0,t)\Big]=\Psi_1(t),$$

where h_1 and Ψ_1 are known functions.

According to [18], this problem is uniquely solvable.

Having defined the function v_1, we can easily determine C_1 and hence the function w_1.

Now, to determine the function w_2, we first obtain the equation

$$\frac{\partial^2 w_2}{\partial \xi^2}(\xi,t)=a_4(\xi,t),$$

where a_4 is a known function, and then find that

$$\frac{\partial w_2}{\partial \xi}(\xi,t)=\int_{\xi_0}^{\xi}a_4(S,t)dS+a_5(t),$$
$$w_2(\xi,t)=\int_{\xi_0}^{\xi}\int_{\eta_0}^{\eta}a_4(S,t)dSd\eta+a_5(t)\xi+a_6(t).$$

But from (2.15) it follows

$$\frac{\partial v_1}{\partial x}(+0,t)+\frac{\partial^2}{\partial x^2}v_0(+0,t)=\int_{\xi_0}^{1}a_4(S,t)dS+a_5(t),$$
$$\frac{\partial v_1}{\partial x}(-0,t)-\frac{\partial^2}{\partial x^2}v_0(-0,t)=\int_{\xi_0}^{1}a_4(S,t)dS+a_5(t).$$

Thus we have defined uniquely the function a_5. The function w_2 is defined to within the summand $C_2(t)=a_6(t)$ which depends only on t.

Consequently, the functions v_0, w_0, v_1, w_1 are defined exactly, whereas the function w_2 is defined to within the summand $C_2=C_2(t)$.

III. n-th step. Suppose that the functions v_i w_j are defined uniquely for all $i\leq n$, and the function w_{n+1} is defined in the form $w_{n+1}(\xi,t)=f_{n+1}(\xi,t)+C_{n+1}(t)$, where f_{n+1} is a known function.

It follows from (2.9) and (2.6)–(2.8) that the function v_{n+1} satisfies the equation

$$\frac{\partial v_{n+1}}{\partial t}(x,t)=\frac{\partial^2 v_{n+1}}{\partial x^2}(x,t),\quad x\neq 0,$$

and also the conditions $v_{n+1}(x,0)$, $v_{n+1}(-1,t) = v_{n+1}(1,t) = 0$. We have to find the conditions of conjugation at the point $x = 0$.

For the function w_{n+1} we find from (2.11) that

$$w_{n+1}(1,t) - v_{n+1}(+0,t) = \sum_{S=1}^{n+1} \frac{1}{S!}\frac{\partial^S}{\partial x^S} v_{n+1-S}(+0,t),$$

$$w_{n+1}(-1,t) - v_{n+1}(-0,t) = \sum_{S=1}^{n+1} \frac{(-1)^S}{S!}\frac{\partial^S}{\partial x^S} v_{n+1-S}(-0,t).$$

After addition and subtraction of these two expressions, we obtain respectively

$$\begin{gathered} 2C_{n+1}(t) = v_{n+1}(+0,t) + v_{n+1}(-0,t) - f_{n+1}(-1,t) - f_{n+1}(1,t) + \\ \sum_{S=1}^{n+1} \frac{1}{S!}\frac{\partial^S}{\partial x^S} v_{n+1-S}(+0,t) + \sum_{S=1}^{n+1} \frac{(-1)}{S!}\frac{\partial^S}{\partial x^S} v_{n+1-S}(-0,t); \qquad (2.16) \\ v_{n+1}(+0,t) - v_{n+1}(-0,t) = f_{n+1}(1,t) - f_{n+1}(-1,t) + \\ \sum_{S=1}^{n+1} \frac{1}{S!}\frac{\partial^S}{\partial x^S} v_{n+1-S}(+0,t) - \sum_{S=1}^{n+1} \frac{(-1)}{S!}\frac{\partial^S}{\partial x^S} v_{n+1-S}(-0,t). \qquad (2.17) \end{gathered}$$

The right-hand side of (2.17) contains known values, and therefore

$$v_{n+1}(+0,t) - v_{n+1}(-0,t) = h_{n+1}(t),$$

where h_{n+1} is a known function.

The equations (2.10) and the conditions of conjugation (2.11) result for the function w_{n+2} in the following problem:

$$\frac{\partial^2 w_{n+2}}{\partial \xi^2}(\xi,t) = \frac{\partial w_n}{\partial t}(\xi,t) + \chi(\xi)\frac{\partial w_{n+1}}{\partial t}(\xi,t), \qquad (2.18)$$

$$\frac{\partial w_{n+2}}{\partial \xi}(\pm 1,t) = \frac{\partial v_{n+1}}{\partial x}(\pm 0,t) + \sum_{S=1}^{n+1} \frac{(\pm 1)^S}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}} v_{n+1-S}(\pm 0,t). \quad (2.19)$$

For the problem (2.18)–(2.19) to be solvable, it is necessary that the difference $\frac{\partial w_{n+2}}{\partial \xi}(1,t) - \frac{\partial w_{n+2}}{\partial \xi}(-1,t)$ be the same for (2.18) and (2.19). Similarly to the second step, we arrive at the condition

$$\frac{\partial v_{n+1}}{\partial x}(+0,t) - \frac{\partial v_{n+1}}{\partial x}(-0,t) - \frac{M}{2}\Big[\frac{\partial v_{n+1}}{\partial t}(+0,t) + \frac{\partial v_{n+1}}{\partial t}(-0,t)\Big] = \Psi_1(t),$$

where Ψ_{n+1} is a known function.

Thus for determination of the function v_{n+1} we obtain the following problem:

$$\frac{\partial v_{n+1}}{\partial t}(x,t) = \frac{\partial^2 v_{n+1}}{\partial x^2}(x,t), \quad x \neq 0, \quad v_{n+1}(x,0) = 0,$$
$$v_{n+1}(-1,t) = v_{n+1}(1,t) = 0, \quad v_{n+1}(+0,t) - v_{n+1}(-0,t) = h_{n+1}(t),$$
$$\frac{\partial v_{n+1}}{\partial x}(+0,t) - \frac{\partial v_{n+1}}{\partial x}(-0,t) - \frac{M}{2}\Big[\frac{\partial v_{n+1}}{\partial t}(+0,t) + \frac{\partial v_{n+1}}{\partial t}(-0,t)\Big] = \Psi_1(t),$$

where h_{n+1} and Ψ_{n+1} are known functions.

According to [18], this problem is uniquely solvable. The function C_{n+1} is defined from (2.16), and thus the function w_{n+1} is defined uniquely. It remains to notice that in the same way as we have defined the functions w_1 and w_2 at the first and the second steps, we can now define the function w_{n+2} in the form $w_{n+2}(\xi,t) = f_{n+2}(\xi,t) + C_{n+2}(t)$, where f_{n+2} is a known function.

Thus the functions v_i and w_i are now defined uniquely for all $i \leq n+1$, and the function w_{n+2} is defined to within the summand C_{n+2} which depends only on t.

Consequently, we have constructed by induction the formal asymptotic series (2.5).

Introduce the notation

$$U_N(x,t) = \begin{cases} \sum\limits_{i=0}^{N} \varepsilon^i v_i^{\pm}(x,t), & |x| > \varepsilon, \\ \sum\limits_{i=0}^{N} \varepsilon^i w_i\Big(\dfrac{x}{\varepsilon}, t\Big), & |x| < \varepsilon. \end{cases}$$

Theorem 2.1. *Let u_ε be a solution of the problem (2.1)–(2.3), and let U_N be a finite portion of the asymptotic series (2.5). Then the following inequality holds:*

$$\|u_\varepsilon - U_N\|_{\mathcal{L}_2(\Omega)} \leq \widetilde{C}\varepsilon^{N+1},$$

where the constant $\widetilde{C}$ does not depend on ε and N.

The proof of this theorem repeats word for word that of Theorem 1.1 from [20].

3. Asymptotics of the Solution of Heat Equation for $m \in (1,2)$

In the domain $\Omega = (-1,1) \times (0,T)$, consider the initial boundary value problem for the heat equation of the kind

$$\Big(1 + \varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big) u_t = u_{xx}, \tag{3.1}$$
$$u(-1,1) = u(1,t) = 0, \tag{3.2}$$
$$u(x,0) = u_0(x), \tag{3.3}$$

where $\varepsilon \in (0,1)$, and the functions χ and u_0 are the same as in [20] and Section 2. Let the rational number $m \in (1,2)$. We can represent it in the form $m = 1 + \frac{l}{p}$, where l, p are positive integers, and $l < p$.

As above, under a solution of the problem (3.1)–(3.3) will be understood a function u_ε which satisfies the equation (3.1) for $x \neq \pm\varepsilon$ and the conditions (3.2) and (3.3), and at the points $x = \pm\varepsilon$ it satisfies the "sewing" conditions

$$\begin{aligned} u(\varepsilon+0,t) &= u(\varepsilon-0,t), \quad \frac{\partial u}{\partial x}(\varepsilon+0,t) = \frac{\partial u}{\partial x}(\varepsilon-0,t), \\ u(-\varepsilon+0,t) &= u(-\varepsilon-0,t), \quad \frac{\partial u}{\partial x}(-\varepsilon+0,t) = \frac{\partial u}{\partial x}(-\varepsilon-0,t). \end{aligned} \tag{3.4}$$

According to [18], the problem (3.1)–(3.3) is uniquely solvable as in [20] and Section 2. Introduce $\delta = \varepsilon^{\frac{1}{p}}$ and construct an asymptotic expansion of the function u_ε in powers of δ as $\delta \to 0$.

The asymptotic expansion will be sought in the form

$$u_\varepsilon(x,t) \sim \begin{cases} \sum\limits_{i=0}^{\infty} \delta^i v_i^{\pm}(x,t), & |x| > \varepsilon, \\ \sum\limits_{i=0}^{\infty} \delta^i w_i\Big(\dfrac{x}{\varepsilon}, t\Big), & |x| < \varepsilon, \end{cases} \tag{3.5}$$

where v_i^+ and v_i^- are defined for $x > \varepsilon$ and $x < -\varepsilon$, respectively. In what follows, instead of $v_i^{\pm}$ we will write simply v_i.

Similarly as in [20], the functions v_i and w_i satisfy the following conditions:

$$v_0(x,0) = u_0, \quad v_i(x,0) = 0, \quad i \geq 1, \tag{3.6}$$

$$w_{ip}(\xi,0) = \frac{\xi^i}{i!}\frac{d^i}{dx^i}u_0(0), \quad i = 0,1,2,\ldots, \quad \xi = \frac{x}{\varepsilon}, \tag{3.7}$$

$$w_j(\xi,0) = 0, \quad j \neq pk, \quad k = 0,1,2,\ldots, \tag{3.8}$$

$$v_i(-1,t) = v_i(1,t) = 0, \quad i \geq 0. \tag{3.9}$$

Substituting the formal expansion (3.5) into the equation (3.1), we obtain

$$A) \quad \frac{\partial}{\partial t}\sum_{i=0}^{\infty}\delta^i v_i(x,t) - \frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\delta^i v_i(x,t) \sim 0 \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty}\delta^i\Big(\frac{\partial}{\partial t}v_i(x,t) - \frac{\partial^2}{\partial x^2}v_i(x,t)\Big) \sim 0;$$

$$B) \quad \Big(1 + \varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)\frac{\partial}{\partial t}\sum_{i=0}^{\infty}\delta^i w_i(\xi,t) - \frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\delta^i w_i(\xi,t) \sim 0 \Rightarrow$$

$$\Rightarrow \Big(1 + \delta^{-p-l}\chi(\xi)\Big)\sum_{i=0}^{\infty}\delta^i\frac{\partial}{\partial t}w_i(\xi,t) - \sum_{i=0}^{\infty}\delta^{i-2p}\frac{\partial^2}{\xi^2}w_i(\xi,t) \sim 0 \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty} \delta^{i-2p}\Big(\frac{\partial}{\partial t}w_{i-2p} + \chi(\xi)\frac{\partial}{\partial t}w_{i-p+l} - \frac{\partial^2}{\partial \xi^2}w_i\Big) \sim 0.$$

This implies that

$$\frac{\partial}{\partial t}v_i(x,t) - \frac{\partial^2}{\partial x^2}v_i(x,t) = 0, \quad |x| > \varepsilon, \quad i \geq 0, \tag{3.10}$$

$$\frac{\partial^2}{\partial \xi^2}w_i(\xi,t) = \frac{\partial}{\partial t}w_{i-2p}(\xi,t) + \chi(\xi)\frac{\partial}{\partial t}w_{i-p+l}(\xi,t), \quad |\xi| < 1, \tag{3.11}$$

where the terms with negative indices are absent.

The "sewing" conditions (3.4) and the formal expansion (3.5) similarly to [20] yield

$$w_i(\pm 1,t) = \sum_{S=0}^{[\frac{i}{p}]} \frac{(\pm)^S}{S!}\frac{\partial^S}{\partial x^S}v_{i-pS}(\pm 0,t) = 0, \tag{3.12}$$

$$\frac{\partial}{\partial \xi}w_i(\pm 1,t) = \sum_{S=0}^{[\frac{i}{p}]-1} \frac{\partial^{S+1}}{\partial x^{S+1}}v_{i-p-pS}(\pm 0,t). \tag{3.13}$$

Let us now show how one can construct successively all the functions v_i and w_i.

I. First step. Consider the equation (3.11) and the conditions (3.13) for w_0. We have

$$\frac{\partial^2}{\partial \xi^2}w_0(\xi,t) = 0, \quad \frac{\partial}{\partial \xi}w_0(\pm 1,t) = 0,$$

whence $w_0(\xi,t) = C_0(t)$. It is easily seen that the equations $w_1(\xi,t) = C_1(t), \dots, w_{p-l-1}(\xi,t) = C_{p-l-1}(t)$ can be obtained analogously. For $i = p-l$ we find that

$$\frac{\partial^2}{\partial \xi^2}w_{p-l}(\xi,t) = \chi(\xi)\frac{\partial}{\partial t}w_0(\xi,t), \quad \frac{\partial}{\partial \xi}w_{p-l}(\pm 1,t) = 0.$$

This implies

$$\frac{\partial}{\partial \xi}w_{p-l}(\xi,t) = C_0'(t)\int_{\xi_0}^{\xi}\chi(S)dS + a_{p-l}(t).$$

Then we obtain

$$C_0'(t)\int_{\xi_0}^{1}\chi(S)dS + a_{p-l}(t) = 0, \quad C_0'(t)\int_{\xi_0}^{-1}\chi(S)dS + a_{p-l}(t) = 0.$$

Hence $\frac{\partial}{\partial \xi}w_{p-l}(\xi,t) = 0$, and so $w_{p-l}(\xi,t) = C_{p-l}(t)$ and $w_0(\xi,t) = C_0^{(t)} = \text{const}$. It follows from (3.7) that $w_0(\xi,t) = u_0(0)$. Analogously we obtain $C_1'(t) = C_2'(t) = \dots = C_{l-1}'(t) = 0$. Thus $w_1(\xi,t) = 0$, $w_2(\xi,t) = 0, \dots, w_{l-1}(\xi,t) = 0$ since $w_j(\xi,0) = 0$ for $j \neq pk$, and $w_l, w_{l+1}, \dots, w_{p-1}$ are defined as functions of t.

For the function v_0 we have the problem

$$\frac{\partial}{\partial t} v_0(x,t) = \frac{\partial^2}{\partial x^2} v_0(x,t), \quad x \neq 0,$$
$$v_0(-1,t) = 0, \quad v_0(-0,t) = u_0(0), \quad v_0(1,t) = 0, \quad v_0(x,0) = u_0(x).$$

Obviously, the problem is divided into two problems, one on the domain $(-1,0)\times(0,T)$ and the other in $(0,1)\times(0,T)$. Both problems are uniquely solvable.

For the functions $v_1, \dots, v_{l-1}$ we find from (3.12) that $v_i(\pm 0,t) = 0$. Then we can see that these functions are equal to zero.

Thus at the first step we have defined uniquely the functions $v_0, v_1, \dots, v_{l-1}$, $w_0, w_1, \dots, w_{l-1}$ while the functions $w_l, \dots, w_{p-1}$ have been defined as depending on t.

II. Second step. To define the function w_p, consider the problem

$$\frac{\partial^2}{\partial \xi^2} w_p(\xi,t) = \chi(\xi)\frac{\partial}{\partial t} w_l(\xi,t), \quad \frac{\partial}{\partial \xi} w_p(\pm 1,t) = \frac{\partial}{\partial x} v_0(\pm 0,t).$$

Since $w_l(\xi,t) = C_l(t)$, we have

$$\frac{\partial}{\partial \xi} w_p(\xi,t) = C_l'(t)\int_{\xi_0}^{\xi} \chi(S)dS + a_p(t),$$
$$\frac{\partial}{\partial \xi} w_p(1,t) = \frac{\partial}{\partial x} v_0(+0,t), \quad \frac{\partial}{\partial \xi} w_p(-1,t) = \frac{\partial}{\partial x} v_0(-0,t).$$

This implies

$$C_l'(t)\int_{\xi_0}^{1} \chi(\xi)d\xi + a_p(t) = \frac{\partial}{\partial x} v_0(+0,t),$$
$$C_l'(t)\int_{\xi_0}^{-1} \chi(\xi)d\xi + a_p(t) = \frac{\partial}{\partial x} v_0(-0,t).$$

From this system, C_l' and a_p are defined uniquely. Hence $\frac{\partial w_p}{\partial \xi}$ is defined exactly, and then the function w_p is defined to within a summand C_p which depends only on t. At the same step we find that the function w_l is defined to within a constant C_{p_0}. Thus C_{p_0} and hence the function w_l are defined uniquely from (3.7)–(3.8). It follows from (3.12) that $v_l(+0,t) = w_l(1,t)$, $v_l(-0,t) = w_l(-1,t)$.

To define the function v_l, we have two problems:

$$A)\ \frac{\partial}{\partial t} v_l(x,t) = \frac{\partial^2}{\partial x^2} v_l(x,t), \quad x \in (-1,0)$$
$$v_l(x,0) = 0, \quad v_l(-1,0) = 0, \quad v_p(0,t) = w_l(-1,t);$$
$$B)\ \frac{\partial}{\partial t} v_l(x,t) = \frac{\partial^2}{\partial x^2} v_l(x,t), \quad x \in (0,1),$$
$$v_l(x,0) = 0, \quad v_l(0,t) = w_l(1,t), \quad v_l(1,t) = 0.$$

Obviously, these problems are uniquely solvable. Thus at the second step we have defined uniquely the functions w_l and v_l, and the function w_p have been defined to within a summand C_p which depends only on t.

III. n-th step. Suppose the functions w_i and v_i are defined uniquely for all $i \leq n$, and the functions $w_{n+1}, \dots, w_{n+p-l}$ are defined to within summands $C_{n+1}, \dots, C_{n+p-l}$ depending only on t

Write out the problem for $w_{n+p-l+1}$:

$$\frac{\partial^2}{\partial \xi^2} w_{n+p+1-l}(\xi, t) = \chi(\xi) \frac{\partial}{\partial t} w_{n+1}(\xi, t),$$

$$\frac{\partial}{\partial \xi} w_{n+p+1-l}(\pm 1, t) = \sum_{S=0}^{[\frac{n+1-l}{p}]} \frac{(\pm 1)^S}{S!} \frac{\partial^{S+1}}{\partial x^{S+1}} v_{n+1-l-pS}(\pm 0, t).$$

It is easily seen that the right-hand side in the boundary conditions depends on the functions v_i for $i \leq n$, and hence is defined uniquely.

Thus we have obtained the problem

$$\frac{\partial^2}{\partial \xi^2} w_{n+p+1-l}(\xi, t) = C'_{n+1}(t) \chi(\xi)^+ f_{n+1}(\xi, t),$$

$$\frac{\partial}{\partial \xi} w_{n+p+1-l}(\pm 1, t) = h_{n+1}(\pm 0, t),$$

where the functions f_{n+1} and h_{n+1} are defined uniquely.

Hence we have

$$\frac{\partial}{\partial \xi} w_{n+p+1-l}(\xi, t) = C'_{n+1}(t) \int_{\xi_0}^{\xi} \chi(S) dS + \int_{\xi_0}^{\xi} f_{n+1}(S, t) dS + a_{n+p+1-l}(t).$$

To define the functions C_{n+1} and $a_{n+p+1-l}$, we obtain the system

$$C'_{n+1}(t) \int_{\xi_0}^{1} \chi(S) dS + a_{n+p+1-l}(t) = h_{n+1}(+0, t) - \int_{\xi_0}^{1} f_{n+1}(S, t) dS,$$

$$C'_{n+1}(t) \int_{\xi_0}^{-1} \chi(S) dS + a_{n+p+1-l}(t) = h_{n+1}(-0, t) - \int_{\xi_0}^{-1} f_{n+1}(S, t) dS$$

from which the functions C'_{n+1} and $a_{n+p+1-l}$ are defined uniquely. But then for the function $w_{n+p+1-l}$ we obtain $\frac{\partial}{\partial \xi} w_{n+p+1-l}(\xi, t) = \widehat{f}_{n+1}(\xi, t)$, where $\widehat{f}_{n+1}$ is a known function. This implies that the function $w_{n+p+1-l}$ is defined to within a summand $C_{n+p+1-l}$ which depends on t. The function C_{n+1} and hence w_{n+1} are now defined to within constant summands. Therefore from (3.7)–(3.8) one can already define w_{n+1} uniquely. By (3.12), in this case for the function v_{n+1} we get

$$w_{n+1}(\pm 1, t) - v_{n+1}(\pm 0, t) = \sum_{S=1}^{[\frac{n+1}{p}]} \frac{(\pm 1)^S}{S!} \frac{\partial^S}{\partial x^S} v_{n+1-pS}(\pm 0, t),$$

and $v_{n+1}(+0,t) = H^{+}_{n+1}(t)$, $v_{n+1}(-0,t) = H^{-}_{n+1}(t)$, where H^{+}_{n+1} and H^{-}_{n+1} are known functions.

To define the function v_{n+1}, we obtain two problems:

$$\begin{aligned}
&A)\ \frac{\partial}{\partial t}v_{n+1}(x,t) = \frac{\partial^2}{\partial x^2}v_{n+1}(x,t), \quad x \in (-1,0)\\
&\quad v_{n+1}(x,0) = 0, \quad v_{n+1}(-1,0) = 0, \quad v_{n+1}(0,t) = H^{-}_{n+1}(t);\\
&B)\ \frac{\partial}{\partial t}v_{n+1}(x,t) = \frac{\partial^2}{\partial x^2}v_{n+1}(x,t), \quad x \in (0,1),\\
&\quad v_{n+1}(x,0) = 0, \quad v_{n+1}(0,t) = H^{+}_{n+1}(t), \quad v_{n+1}(1,t) = 0,
\end{aligned}$$

which are uniquely solvable.

Thus we have defined uniquely the functions v_{n+1} and w_{n+1}, while the function $w_{n+p-l+1}$ has been defined to within a summand which depends on t.

By induction we conclude that the steps I, II and III enable one to construct the functions v_i and w_i for all i. Thus we have constructed the formal asymptotic expansion.

Introduce the notation

$$U_N(x,t) = \begin{cases} \sum\limits_{i=0}^{N} \delta^i v_i^{\pm}(x,t), & |x| > \varepsilon,\\ \sum\limits_{i=0}^{N} \delta^i w_i\Big(\frac{x}{\varepsilon},t\Big), & |x| < \varepsilon. \end{cases}$$

Theorem 3.1. *Let u_ε be a solution of the problem* (3.1)–(3.3), *and let u_N be a finite part of the series* (3.5). *Then the following inequality is valid:*

$$\|u_\varepsilon - U_N\|_{\mathcal{L}_2(\Omega)} \le \widetilde{C}\delta^{N+1},$$

where the constant $\widehat{C}$ does not depend on ε and N.

The proof of this theorem is the same as that of Theorem 1.1 from [20].

4. Asymptotics of the Solution of Heat Equation for $m = 2$

In the domain $\Omega = (-1,1) \times (0,T)$ consider the initial boundary value problem for the heat equation of the kind

$$\Big(1 + \varepsilon^{-2}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)u_t = u_{xx}, \tag{4.1}$$

$$u(-1,t) = u(1,t) = 0, \tag{4.2}$$

$$u(x,0) = u_0(x), \tag{4.3}$$

where the functions u_0 and χ are the same as in [20].

As before, under a solution of the problem (4.1)–(4.3) is understood a function u_ε which satisfies the equation (4.1) for $x \neq \pm\varepsilon$, the conditions

(4.2) and (4.3), and at the points of discontinuity the function χ satisfies the "sewing" conditions

$$\begin{aligned} u(\varepsilon+0,t)&=u(\varepsilon-0,t), & \frac{\partial u}{\partial x}(\varepsilon+0,t)&=\frac{\partial u}{\partial x}(\varepsilon-0,t),\\ u(-\varepsilon+0,t)&=u(-\varepsilon-0,t), & \frac{\partial u}{\partial x}(-\varepsilon+0,t)&=\frac{\partial u}{\partial x}(-\varepsilon-0,t). \end{aligned} \tag{4.4}$$

As it has been shown previously, the problem (4.1)–(4.3) is uniquely solvable according to [18].

Formal asymptotic expansion will be written in the form of the series

$$u_\varepsilon(x,t)\sim\begin{cases}\displaystyle\sum_{i=0}^{\infty}\varepsilon^i v_i(x,t), & |x|>\varepsilon,\\ \displaystyle\sum_{i=0}^{\infty}\varepsilon^i w_i(\frac{x}{\varepsilon},t), & |x|<\varepsilon.\end{cases} \tag{4.5}$$

As in the foregoing sections, we easily find that the functions v_i and w_i satisfy the following conditions:

$$v_0(x,0)=u_0(x), \quad v_i(x,0)=0, \quad i\geq 0, \tag{4.6}$$

$$v_i(\pm 1,t)=0, \quad i\geq 0, \tag{4.7}$$

$$w_i(\xi,0)=\frac{\xi^i}{i!}\frac{d^i}{dx^i}u_0(0), \quad i\geq 0, \quad \xi=\frac{x}{\varepsilon}. \tag{4.8}$$

Substituting the formal asymptotic series (4.5) into (4.1), we obtain

$$A)\quad \frac{\partial}{\partial t}\sum_{i=0}^{\infty}\varepsilon^i v_i(x,t)-\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\varepsilon^i v_i(x,t)\sim 0\Rightarrow$$

$$\Rightarrow\sum_{i=0}^{\infty}\varepsilon^i\Big[\frac{\partial}{\partial t}v_i(x,t)-\frac{\partial^2}{\partial x^2}v_i(x,t)\Big]\sim 0;$$

$$B)\quad \Big(1+\varepsilon^{-2}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)\frac{\partial}{\partial t}\sum_{i=0}^{\infty}\varepsilon^i w_i(\xi,t)-\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\varepsilon^i w_i(\xi,t)\sim 0\Rightarrow$$

$$\Rightarrow\sum_{i=0}^{\infty}\varepsilon^{i-2}\Big(\frac{\partial}{\partial t}w_{i-2}(\xi,t)+\chi(\xi)\frac{\partial}{\partial t}w_i(\xi,t)-\frac{\partial^2}{\partial \xi^2}w_i(\xi,t)\Big)\sim 0.$$

We can see that the functions v_i and w_i satisfy the equations

$$\frac{\partial}{\partial t}v_i(x,t)=\frac{\partial^2}{\partial x^2}v_i(x,t), \quad i\geq 0, \tag{4.9}$$

$$\chi(\xi)\frac{\partial}{\partial t}w_i(\xi,t)-\frac{\partial^2}{\partial \xi^2}w_i(\xi,t)=-\frac{\partial}{\partial t}w_{i-2}(\xi,t), \quad i\geq 0, \tag{4.10}$$

where the functions with negative indices are absent.

Decomposition of the function v_i and substitution of the formal asymptotic expansion (4.5) into the "sewing" conditions (4.4) result, as in Section 3, in

$$w_i(\pm 1,t) = \sum_{S=0}^{i} \frac{(\pm 1)^S}{S!}\frac{\partial^S}{\partial x^S} v_{i-S}(\pm 0,t), \quad i \geq 0, \tag{4.11}$$

$$\frac{\partial}{\partial \xi} w_0(\pm 1,t)=0,\ \frac{\partial}{\partial \xi} w_i(\pm 1,t)=\sum_{S=0}^{i-1} \frac{(\pm 1)^S}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}} v_{i-S-1}(\pm 0,t),\ i \geq 1. \tag{4.12}$$

Let us show how one can construct successively the functions v_i and w_i.

I. First step. To define the function w_0, we obtain from the equation (4.1) and the conditions (4.8) and (4.11) the following Neumann problem

$$\chi(\xi)\frac{\partial}{\partial t} w_0(\xi,t) = \frac{\partial^2}{\partial x^2} w_0(\xi,t),$$
$$w_0(\xi,0) = u_0(0), \quad \frac{\partial}{\partial \xi} w_0(-1,t) = 0, \quad \frac{\partial}{\partial \xi} w_0(1,t) = 0$$

for the heat equation whose solution $w_0(\xi,t) = u_0(0)$ is defined uniquely.

From the conditions (4.11) we find that $w_0(\pm 1,t) = v_0(\pm 0,t)$, whence

$$v_0(-0,t) = v_0(+0,t) = u_0(0). \tag{4.13}$$

To define the function v_0, we obtain from the (4.9) and the conditions (4.6)–(4.7) and (4.13) the following two problems:

$$A) \ \frac{\partial}{\partial t} v_0(x,t) = \frac{\partial^2}{\partial x^2} v_0(x,t), \quad x \in (-1,0)$$
$$v_0(x,0) = u_0(x), \quad v_0(-1,0) = 0, \quad v_0(0,t) = u_0(0);$$
$$B) \ \frac{\partial}{\partial t} v_0(x,t) = \frac{\partial^2}{\partial x^2} v_0(x,t), \quad x \in (0,1),$$
$$v_0(x,0) = u_0(x), \quad v_0(0,t) = u_0(0), \quad v_0(1,t) = 0$$

which are uniquely solvable. Thus this step enables one to define uniquely the functions v_i and w_i.

II. Second step. Assume that the functions v_i and w_i are defined uniquely for all $i \leq n$. Define the functions v_{n+1} and w_{n+1}.

To define the function w_{n+1}, we obtain from the equations (4.10) and conditions (4.8) and (4.12) the following Neumann problem

$$\chi(\xi)\frac{\partial}{\partial t} w_{n+1}(\xi,t) - \frac{\partial^2}{\partial \xi^2} w_{n+1}(\xi,t) = -\frac{\partial}{\partial t} w_{n-1}(\xi,t),$$
$$w_{n+1}(\xi,0) = \frac{\xi^{n+1}}{(n+1)!}\frac{\partial^{n+1}}{\partial x^{n+1}} u_0(0),$$
$$\frac{\partial}{\partial \xi} w_{n+1}(\pm 1,t) = \sum_{S=0}^{n} \frac{(\pm 1)^S}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}} v_{n-S}(\pm 0,t)$$

for the inhomogeneous heat equation. This problem is uniquely solvable.

The conditions (4.12) yield

$$w_{n+1}(\pm 1,t)=v_{n+1}(\pm 0,t)+\sum_{S=1}^{n+1}\frac{(\pm 1)^S}{S!}\frac{\partial^S}{\partial x^S}v_{n-S}(\pm 0,t). \qquad (4.14)$$

To define the function v_{n+1}, we obtain from the equations (4.9) and the conditions (4.6)–(4.7) and (4.14) the following two problems:

$$\begin{aligned}
A)\ & \frac{\partial}{\partial t}v_{n+1}(x,t)=\frac{\partial^2}{\partial x^2}v_{n+1}(x,t), \quad x\in(-1,0)\\
& v_{n+1}(x,0)=0, \quad v_{n+1}(-1,1)=0,\\
& v_{n+1}(0,t)=w_{n+1}(-1,t)-\sum_{S=1}^{n+1}\frac{(-1)^S}{S!}\frac{\partial^S}{\partial x^S}v_{n-S}(-0,t);\\
B)\ & \frac{\partial}{\partial t}v_{n+1}(x,t)=\frac{\partial^2}{\partial x^2}v_{n+1}(x,t), \quad x\in(0,1), \quad v_{n+1}(x,0)=0,\\
& v_{n+1}(0,t)=w_{n+1}(1,t)-\sum_{S=1}^{n+1}\frac{1}{S!}\frac{\partial^S}{\partial x^S}v_{n-S}(+0,t), \quad v_{n+1}(1,t)=0.
\end{aligned}$$

These problems are uniquely solvable. Consequently, we have defined the function v_{n+1}.

Thus assuming that the functions v_i and w_i are known for all $i\le n$, we can define them for $i=n+1$. Then by induction one can construct the functions v_i and w_i for all i. Hence we have constructed the formal asymptotic series.

Introduce the notation

$$U_N(x,t)=\begin{cases}\displaystyle\sum_{i=0}^{N}\varepsilon^i v_i(x,t), & |x|>\varepsilon,\\ \displaystyle\sum_{i=0}^{N}\varepsilon^i w_i\Big(\frac{x}{\varepsilon},t\Big), & |x|<\varepsilon.\end{cases}$$

Theorem 4.1. *Let u_ε be a solution of the problem (4.1)–(4.3), and let U_N be a finite part of the formal asymptotic series (4.5). Then the following inequality holds:*

$$\|u_\varepsilon-U_N\|_{\mathcal{L}_2(\Omega)}\le\widetilde{C}\varepsilon^{N+1},$$

where the constant $\widetilde{C}$ does not depend on ε and N.

The proof of this theorem is the same as that of Theorem 1.1 from [20].

5. Asymptotics of the Solution under Strong Perturbation of Density

In the domain $\Omega = (-1,1)\times(0,T)$ consider the initial boundary value problem for the heat equation of the kind

$$\Big(1+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\Big)u_t = u_{xx}, \tag{5.1}$$

$$u(-1,t) = u(1,t) = 0, \tag{5.2}$$

$$u(x,0) = \overline{u}_0(x), \tag{5.3}$$

where the functions χ and $\overline{u}_0$ are the same as in [20], $\varepsilon \in (0,1)$, and m is a rational number greater than 2. Let $m = 2+\frac{l}{p}$, where l and p are positive integers. As for the function χ, we additionally assume that in some neighborhood of the point $x=\varepsilon$ and in the right neighborhood of the point $x=-\varepsilon$ it can be expanded into Taylor series.

As before, under a solution of the problem (5.1)–(5.3) will be understood a function u which satisfies (5.1) for $x \neq \pm\varepsilon$ as well as the conditions (5.2) and (5.3) and the "sewing" conditions at the points $x=\pm\varepsilon$

$$\begin{aligned} u(\varepsilon+0,t) &= u(\varepsilon-0,t), \quad \frac{\partial u}{\partial x}(\varepsilon+0,t) = \frac{\partial u}{\partial x}(\varepsilon-0,t),\\ u(-\varepsilon+0,t) &= u(-\varepsilon-0,t), \quad \frac{\partial u}{\partial x}(-\varepsilon+0,t) = \frac{\partial u}{\partial x}(-\varepsilon-0,t). \end{aligned} \tag{5.4}$$

The problem (5.1)–(5.3) is uniquely solvable by [18]. Construct an asymptotic expansion of the solution u_ε of the problem (5.1)–(5.3) as $\varepsilon \to 0$.

We divide the segment $[-1,1]$ into several parts and introduce new variables.

Consider on $(-1,-\varepsilon)$ and $(\varepsilon,1)$ the function $u_\varepsilon(x,t) = u(x,t)$. It satisfies the equation

$$\frac{\partial}{\partial t}u^\pm(x,t) = \frac{\partial^2}{\partial x^2}u^\pm(x,t). \tag{5.5}$$

Introduce on $(-\varepsilon+\varepsilon^{1+\frac{l}{2p}}, \varepsilon-\varepsilon^{1+\frac{l}{2p}})$ a new independent variable $\xi = \frac{x}{\varepsilon}$ and consider the function $u_\varepsilon(x,t) = v(\xi,t)$. It satisfies the equation

$$\varepsilon^2\frac{\partial}{\partial t}v(\xi,t) + \varepsilon^{-\frac{l}{p}}\chi(\xi)\frac{\partial}{\partial t}v(\xi,t) = \frac{\partial^2}{\partial \xi^2}v(\xi,t). \tag{5.6}$$

On $(-\varepsilon,-\varepsilon+\varepsilon^{1+\frac{l}{2p}})$ and $(\varepsilon-\varepsilon^{1+\frac{l}{2p}},\varepsilon)$ we introduce new independent variables $\eta = (\pm1-\xi)\cdot\varepsilon^{-\frac{l}{2p}}$, respectively, and consider the function $u_\varepsilon(x,t) = w^\pm(\eta,t)$. It satisfies the equation

$$\varepsilon^{2+\frac{l}{p}}\frac{\partial}{\partial t}w^\pm(\eta,t) + \chi(\pm1-\varepsilon^{\frac{l}{2p}}\eta)\frac{\partial}{\partial t}w^\pm(\eta,t) = \frac{\partial^2}{\partial \eta^2}w^\pm(\eta,t). \tag{5.7}$$

In what follows, we will omit the superscripts $''\pm''$ and construct the functions u^+ and w^+. Obviously, the functions u^- and w^- can be constructed analogously.

Introduce $\delta=\varepsilon^{\frac{1}{2p}}$ and construct the formal asymptotic expansion of the solution u_ε in the form

$$u_\varepsilon(x,t)\sim\begin{cases}\sum\limits_{i=0}^{\infty}\delta^i u_i^{\pm}(x,t), & |x|>\varepsilon;\\ \sum\limits_{i=0}^{\infty}\delta^i v_i(\xi,t), & \xi=\frac{x}{\varepsilon},\quad |\xi|\le 1-\varepsilon^{\frac{l}{2p}};\\ \sum\limits_{i=0}^{\infty}\delta^i w_i^{\pm}(\eta,t), & \eta=\frac{\pm 1-\xi}{\varepsilon^{\frac{l}{2p}}},\quad \eta\in(0,1).\end{cases}\tag{5.8}$$

Substituting the formal series (5.8) into each equation (5.5)–(5.7), we have

$$A)\quad \frac{\partial}{\partial t}\sum_{i=0}^{\infty}\delta^i u_i(x,t)-\frac{\partial^2}{\partial x^2}\sum_{i=0}^{\infty}\delta^i u_i(x,t)\sim 0\Rightarrow$$
$$\Rightarrow\sum_{i=0}^{\infty}\delta^i\Big(\frac{\partial}{\partial t}u_i(x,t)-\frac{\partial^2}{\partial x^2}u_i(x,t)\Big)\sim 0;$$

$$B)\quad \varepsilon^2\frac{\partial}{\partial t}\sum_{i=0}^{\infty}\delta^i v_i(\xi,t)+\varepsilon^{-\frac{l}{p}}\chi(\xi)\frac{\partial}{\partial t}\sum_{i=0}^{\infty}\delta^i v_i(\xi,t)-\frac{\partial^2}{\partial\xi^2}\sum_{i=0}^{\infty}\delta^i v_i(\xi,t)\sim 0\Rightarrow$$
$$\Rightarrow\sum_{i=0}^{\infty}\Big(\delta^{i+4p}\frac{\partial}{\partial t}v_i(\xi,t)+\delta^{i-2l}\chi(\xi)\frac{\partial}{\partial t}v_i(\xi,t)-\delta^i\frac{\partial^2}{\partial\xi^2}v_i(\xi,t)\Big)\sim 0\Rightarrow$$
$$\Rightarrow\sum_{i=0}^{\infty}\delta^{i-2l}\Big(\chi(\xi)\frac{\partial}{\partial t}v_i(\xi,t)+\frac{\partial}{\partial t}v_{i-4p-2l}(\xi,t)-\frac{\partial^2}{\partial\xi^2}v_{i-4p}(\xi,t)\Big)\sim 0;$$

$$C)\quad \varepsilon^{2+\frac{l}{p}}\frac{\partial}{\partial t}\sum_{i=0}^{\infty}\delta^i w_i(\eta,t)+\chi\big(1-\varepsilon_\eta^{\frac{l}{2p}}\big)\frac{\partial}{\partial t}\sum_{i=0}^{\infty}\delta^i w_i(\eta,t)-$$
$$-\frac{\partial^2}{\partial\eta^2}\sum_{i=0}^{\infty}\delta^i w_i(\eta,t)\sim 0\Rightarrow$$
$$\Rightarrow\sum_{i=0}^{\infty}\Big(\delta^{i+4p+2l}\frac{\partial}{\partial t}w_i(\eta,t)+\sum_{S=0}^{\infty}\frac{(-1)^S}{S!}\varepsilon^{\frac{Sl}{2p}}\frac{\partial^S}{\partial\xi^S}\chi(1)\frac{\partial}{\partial t}w_i(\eta,t)-$$
$$-\frac{\partial^2}{\partial\eta^2}w_i(\eta,t)\Big)\sim 0\Rightarrow\sum_{i=0}^{\infty}\Big(\delta^{i+4p+2l}\frac{\partial}{\partial t}w_i+$$
$$+\delta^i\sum_{S=0}^{\infty}\frac{(-1)^S}{S!}\delta^{Sl}\eta^S\frac{\partial^S}{\partial\xi^S}\chi(1)\frac{\partial}{\partial t}w_i-\delta^i\frac{\partial^2}{\partial\eta^2}w_i\Big)\sim 0\Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty} \delta^i \Big(\frac{\partial}{\partial t} w_{i-4p-2l} + \sum_{S=0}^{[\frac{i}{l}]} \frac{(-\eta)^S}{S!} \frac{d^S}{d\xi^S} \chi(1) \frac{\partial}{\partial t} w_{i-lS} - \frac{\partial^2}{\partial \xi^2} w_i \Big) \sim 0.$$

We equate the coefficients to zero and find that the functions u_i, v_i and w_i satisfy the following equations:

$$\frac{\partial}{\partial t} u_i(x,t) = \frac{\partial^2}{\partial x^2} u_i(x,t), \quad i \geq 0; \tag{5.9}$$

$$\chi(\xi) \frac{\partial}{\partial t} v_i(\xi,t) = \frac{\partial^2}{\partial \xi^2} v_{i-4p}(\xi,t) - \frac{\partial}{\partial t} v_{i-4p-2l}, \quad i \geq 0, \tag{5.10}$$

$$\chi(1) \frac{\partial}{\partial t} w_i(\eta,t) - \frac{\partial^2}{\partial \eta^2} w_i(\eta,t) = -\frac{\partial}{\partial t} w_{i-4p-2l}(\eta,t) - \\ - \sum_{S=1}^{[\frac{i}{l}]} \frac{(-\eta)^S}{S!} \frac{d^S}{d\xi^S} \chi(1) \frac{\partial}{\partial t} w_{i-lS}, \quad i \geq 0. \tag{5.11}$$

Obviously, the terms in the equations (5.9)–(5.11) with negative indices are absent.

Determine now the initial and boundary conditions. The boundary conditions (5.2) and the formal expansion (5.8) result in

$$u_i(-1,t) = u_i(1,t) = 0. \tag{5.12}$$

The initial conditions (5.3) and the formal expansion (5.8) yield

$$A) \quad \sum_{i=0}^{\infty} \delta^i u_i(x,0) = \overline{u}_0(x) \Rightarrow u_0(x,0) = \overline{u}_0(x), \quad u_i(x,0) = 0, \quad i \geq 1; \tag{5.13}$$

$$B) \quad \sum_{i=0}^{\infty} \delta^i v_i(\xi,0) = \overline{u}_0(\varepsilon\xi) \Rightarrow \sum_{i=0}^{\infty} \delta^i v_i(\xi,0) = \sum_{i=0}^{\infty} \frac{\varepsilon^S \xi^S}{S!} \frac{d^S}{dx^S} \overline{u}_0(0) \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty} \delta^i v_i(\xi,0) = \sum_{S=0}^{\infty} \delta^{2pS} \frac{\xi^S}{S!} \frac{d^S}{dx^S} \overline{u}_0(0) \Rightarrow$$

$$\Rightarrow \begin{cases} v_{2pS}(\xi,0) = \dfrac{\xi^S}{S!} \dfrac{d^S}{dx^S} \overline{u}_0(0), \quad S = 0,1,2,\dots, \\ v_j(\xi,t) = 0, \quad j \neq 2\pi S; \end{cases} \tag{5.14}$$

$$C) \quad \sum_{i=0}^{\infty} \delta^i w_i(\eta,0) = \overline{u}_0(\varepsilon - \varepsilon^{1+\frac{l}{2p}} \eta) = \overline{u}_0(\delta^{2p} - \delta^{2p+l}\eta) \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty} \delta^i w_i(\eta,0) = \sum_{S=0}^{\infty} \frac{(-\eta)^S}{S!} \delta^{(2p+l)S} \frac{d^S}{dx^S} \overline{u}_0(\delta^{2p}) \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty} \delta^i w_i(\eta,0) = \sum_{S=0}^{\infty} \frac{(-\eta)^S}{S!} \delta^{(2p+l)S} \sum_{k=0}^{\infty} \frac{\delta^{2pk}}{k!} \frac{d^{S+k}}{dx^{S+k}} \overline{u}_0(0). \tag{5.15}$$

We easily see that $w_i(\eta,0)$ from (5.15) can be defined for any i after equating the coefficients at the same degrees of δ. In particular,

$$w_0(\eta,0)=\overline{u}_0(0),\quad w_1(\eta,0)=0,\ldots,w_{2p-1}(\eta,0)=0,\quad w_{2p}(\eta,0)=\frac{d}{dx}\overline{u}_0(0),$$
$$w_{2p+1}(\eta,0)=0,\ldots,w_{2p+l-1}(\eta,0)=0,\quad w_{2p+l}(\eta,0)=-\eta\frac{d}{dx}\overline{u}_0(0)\ldots.$$

It remains to find the conditions at the points $x=\varepsilon-\varepsilon^{1=\frac{l}{2p}}$ and $x=\varepsilon$. We will start from the conditions $v(1-\delta^l,t)=w(1,t)$, $u(\varepsilon,t)=w(0,t)$, $\frac{\partial u}{\partial x}(\varepsilon,t)=\frac{\partial}{\partial x}w(0,t)$. It follows from the condition $w(1,1)=v(1-\delta^l,1)$ that

$$\begin{aligned}
&\sum_{i=0}^{\infty}\delta^i w_i(1,t)=\sum_{i=0}^{\infty}\delta^i v_i(1,t)\Rightarrow\\
\Rightarrow&\sum_{i=0}^{\infty}\delta^i w_i(1,t)=\sum_{i=0}^{\infty}\delta^i\sum_{S=0}^{\infty}\frac{(-\delta)^{lS}}{S!}\frac{\partial^S}{\partial\xi^S}v_i(1,t)\Rightarrow\\
\Rightarrow&\sum_{i=0}^{\infty}\delta^i w_i(1,t)=\sum_{i=0}^{\infty}\delta^i\sum_{S=0}^{[\frac{i}{l}]}\frac{(-1)^S}{S!}\frac{\partial^S}{\partial\xi^S}v_{i-lS}(1,t)\Rightarrow\\
\Rightarrow&w_i(1,t)=v_i(1,t)+\sum_{S=1}^{[\frac{i}{l}]}\frac{(-1)^S}{S!}\frac{\partial^S}{\partial\xi^S}v_{i-lS}(1,t).
\end{aligned}\tag{5.16}$$

From the condition $u(\varepsilon,t)=w(0,t)$ we obtain

$$\begin{aligned}
&\sum_{i=0}^{\infty}\delta^i w_i(0,t)=\sum_{i=0}^{\infty}\delta^i u_i(\varepsilon,t)\Rightarrow\\
\Rightarrow&\sum_{i=0}^{\infty}\delta^i w_i(0,t)=\sum_{i=0}^{\infty}\delta^i\sum_{S=0}^{\infty}\frac{\varepsilon^S}{S!}\frac{\partial^S}{\partial x^S}u_i(0,t)\Rightarrow\\
\Rightarrow&\sum_{i=0}^{\infty}\delta^i w_i(0,t)=\sum_{i=0}^{\infty}\delta^i\sum_{S=0}^{\infty}\frac{\delta^{2pS}}{S!}\frac{\partial^S}{\partial x^S}u_i(0,t)\Rightarrow\\
\Rightarrow&\sum_{i=0}^{\infty}\delta^i w_i(0,t)=\sum_{i=0}^{\infty}\delta^i\sum_{S=0}^{[\frac{i}{l}]}\frac{(-1)^S}{S!}\frac{\partial^S}{\partial x^S}u_{i-2pS}(0,t)\Rightarrow\\
\Rightarrow&w_i(0,t)-u_i(0,t)=\sum_{S=1}^{[\frac{i}{2p}]}\frac{1}{S!}\frac{\partial^S}{\partial x^S}u_{i-2pS}(0,t),
\end{aligned}\tag{5.17}$$

while from the condition $\frac{\partial u}{\partial x}(\varepsilon,t)=\frac{\partial}{\partial x}w(0,t)$ we arrive at

$$\sum_{i=0}^{\infty}\frac{\partial}{\partial x}u_i(\varepsilon,t)=\sum_{i=0}^{\infty}\delta^i\frac{\partial}{\partial x}w_i(0,t)\Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty}\delta^i \sum_{S=1}^{\infty}\frac{\varepsilon^{S-1}}{(S-1)!}\frac{\partial^S}{\partial x^S}u_i(0,t) = \sum_{i=0}^{\infty}\delta^i \frac{\partial}{\partial \eta}w_i(0,t)\delta^{-(2p+l)} \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty}\delta^i \sum_{S=0}^{\infty}\frac{\delta^{2pS}}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}}u_i(0,t) = \sum_{i=0}^{\infty}\delta^{i-2p-l}\frac{\partial}{\partial \eta}w_i(0,t) \Rightarrow$$

$$\Rightarrow \sum_{i=0}^{\infty}\delta^i w \sum_{S=0}^{[\frac{i}{2p}]}\frac{1}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}}u_{i-2pS}(0,t) = \sum_{i=0}^{\infty}\delta^{i-2p-l}\frac{\partial}{\partial \eta}w_i(0,t) \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial \eta}w_i(0,t) = \sum_{S=0}^{[\frac{i-l}{2p}]}\frac{-1}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}}u_{i-2p-l-2pS}(0,t). \tag{5.18}$$

Obviously, the terms with negative indices in the formulas (5.16)–(5.18) are absent.

Let us now show how one can construct successively all the functions u_i, v_i and w_i.

I. First step. From the above-obtained relations we write out everything which concerns the functions v_0, w_0 and u_0.

For v_0 we obtain respectively the equation and the initial condition: $\frac{\partial}{\partial t}v_0(\xi,t) = 0$, $v_0(\xi,0) = \overline{u}_0(0)$. Hence $v_0(\xi,t) = \overline{u}_0(0)$.

For the function w_0 we obtain the problem

$$\chi(1)\frac{\partial}{\partial t}w_0(\eta,t) = \frac{\partial^2}{\partial \eta^2}w_0(\eta,t),$$
$$w_0(\eta,0) = u_0(0), \quad w_0(1,t) = v_0(1,t) = \overline{u}_0(0), \quad \frac{\partial}{\partial \eta}w_0(0,t) = 0.$$

This simple problem for the heat equation has the unique solution $w_0(\eta) = \overline{u}_0(0)$.

To define the function u_0, we get the problem

$$\frac{\partial}{\partial t}u_0(x,t) = \frac{\partial^2}{\partial x^2}u_0(x,t),$$
$$u_0(x,0) = \overline{u}_0(x), \quad u_0(1,t) = 0, \quad u_0(0,t) = w_0(0,t) = \overline{u}_0(0).$$

This problem is uniquely solvable.

Thus at the first step we have defined uniquely the functions v_0, w_0 and u_0.

II. Second step. Suppose the function u_i, v_i and w_i are given for all $i \le n$. Define the functions u_{n+1}, v_{n+1} and w_{n+1}.

To define the function v_{n+1}, we get the problem

$$\chi(\xi)\frac{\partial}{\partial t}v_{n+1}(\xi,t) = \frac{\partial^2}{\partial \xi^2}v_{n-4p+1}(\xi,t) - \frac{\partial}{\partial t}v_{n-4p-2l+1}(\xi,t),$$

$$\left[\begin{array}{lll} w_{n+1}(\xi,0) = \frac{\xi^S}{dx^S}\overline{u}_0(0), & \text{for} \quad n+1 = 2pS, & S \in \mathbb{Z}, \\ u_{n+1}(\xi,0) = 0, & \text{for} \quad n+1 = 2pS, & S \in \mathbb{Z}. \end{array}\right.$$

Obviously, this problem is uniquely solvable. To define the function w_{n+1}, we get the problem

$$\chi(1)\frac{\partial}{\partial t}w_{n+1}(\eta,t)-\frac{\partial^2}{\partial\eta^2}w_{n+1}(\eta,t)=-\frac{\partial}{\partial t}w_{n+1-4p-2l}(\eta,t)-$$
$$-\sum_{S=1}^{[\frac{n+1}{p}]}\frac{(-\eta)^S}{S!}\frac{\partial^S}{\partial\xi^S}\chi(1)\frac{\partial}{\partial t}w_{n+1-lS}(\eta,t),$$
$$w_{n+1}(\eta,0)=f_{n+1}(\eta),\quad w_{n+1}(1,t)=\sum_{S=0}^{[\frac{n+1}{l}]}\frac{(-1)^S}{S!}\frac{\partial^S}{\partial\xi^S}v_{n-lS+1}(1,t),$$
$$\frac{\partial}{\partial\eta}w_{n+1}(0,t)=\sum_{S=0}^{[\frac{n+1-l}{2p}]-1}\frac{1}{S!}\frac{\partial^{S+1}}{\partial x^{S+1}}u_{n+1-2p-l-2pS}(0,t),$$

where the function f_{n+1} is defined from (5.15).

The problem under consideration is an inhomogeneous initial boundary value problem for the inhomogeneous heat equation. This problem is uniquely solvable.

Thus, having supposed that the functions u_i, v_i and w_i are known for all $i\le n$, we have defined them for $i=n+1$. Then by induction one can construct the functions u_i, v_i and w_i for all i. Consequently, the formal asymptotic series (5.8) is constructed.

Introduce the notation

$$U_N(x,t)=\begin{cases}\sum_{i=0}^{N}\delta^i u_i^{\pm}(x,t), & |x|>\varepsilon,\\ \sum_{i=0}^{N}\delta^i v_i(\xi,t), \quad \xi=\frac{x}{\varepsilon}, & |\xi|<1-\varepsilon^{\frac{l}{2p}},\\ \sum_{i=0}^{N}\delta^i w_i^{\pm}(\eta,t), \quad \eta=\frac{\pm1-\xi}{\varepsilon^{\frac{l}{2p}}}, & 0<\eta<1.\end{cases}$$

We have to show that the above-constructed series (5.8) is in fact the asymptotic series for the function u_ε.

Estimate now the difference $U_N(\varepsilon+0,t)-U_N(\varepsilon-0,t)$. As is easily seen,

$$U_N(\varepsilon+0,t)=\sum_{i=0}^{N}\delta^i u_i(\varepsilon,t)=\sum_{i=0}^{N}\delta^i\sum_{S=0}^{\infty}\frac{\varepsilon^S}{S!}\frac{\partial^S}{\partial x^S}u_i(0,t)=$$
$$=\sum_{i=0}^{N}\delta^i\sum_{S=0}^{\infty}\frac{\delta^{2pS}}{S!}\frac{\partial^S}{\partial x^S}u_i(0,t)=\sum_{i=0}^{N}\delta^i\sum_{S=0}^{[\frac{i}{2p}]}\frac{1}{S!}\frac{\partial^S}{\partial x^S}u_{i-2pS}(0,t)+O(\delta^{N+1}),$$

$$U_N(\varepsilon-0,t)=\sum_{i=0}^{N}\delta^i w_i(0,t).$$

The conditions (5.17) imply $U_N(\varepsilon+0,t)-U_N(\varepsilon-0,t=O(\delta^{N+1})$.

Analogously it follows from (5.16)–(5.18) that the function U_N (as well as its derivative) may have at the points $x=-\varepsilon$, $x=\varepsilon-\varepsilon^{1+\frac{l}{2p}}$, $x=-\varepsilon+\varepsilon^{1+\frac{l}{2p}}$ discontinuities which are of order $O(\delta^{N+1})$ for the function and of order $O(\delta^N)$ for the derivative. Evidently, one can construct a function φ_N with discontinuities at the same points, but with opposte sign, where φ_N and φ_N' will be of order $O(\delta^{N+1})$ and $O(\delta^N)$, respectively, and $\varphi_N(-1)=\varphi_N(1)=0$.

Consider the function V_N defined by $V_N(x,t)=U_N(x,t)-\varphi_N(x)$. The function V_N is continuously differentiable, and $||V_N(x,t)-U_N(x,t)||=O(\delta^N)$. From the construction of the function V_N we can see that

$$\Big(\frac{\partial}{\partial t}V_N+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial t}V_N-\frac{\partial^2}{\partial x^2}V_N\Big)=O(\delta^{N-2pm}),$$
$$V_N(x,0)=u_0(x)+O(\delta N+1).$$

Then for the function $U_*=u_\varepsilon-V_N$ we obtain the initial boundary value problem of the kind

$$\frac{\partial}{\partial t}U_*+\varepsilon^{-m}\chi\Big(\frac{x}{\varepsilon}\Big)\frac{\partial}{\partial t}U_*-\frac{\partial^2}{\partial x^2}U_*=F_\delta,$$
$$U_*(-1,t)=U_*(1,t)=0,\quad U_*(x,0)=\varphi_\delta,$$

where $F_\delta=O(\delta^{N-2pm})$ and $\varphi_\delta=O(\delta^{N+1})$.

Then we multiply the equation (5.19) by U_* and integrate the obtained equality with respect to the domain $[-1,1]\times[0,\tau_0]$. Similarly to the proof of Theorem 1.1 we arrive at

$$\int_{-1}^{1}U_*^2(x,\tau_0)dx\le\int_{-1}^{1}\varphi_\delta^2(x)dx+\varepsilon^{-m}\int_{-\varepsilon}^{\varepsilon}\chi\Big(\frac{x}{\varepsilon}\Big)\varphi_\delta^2(x)dx+\Big|\int_{-1}^{1}\int_0^{\tau_0}F_\delta(x,t)U_*dxdt\Big|.$$

Hence we find

$$\int_{-1}^{1}\int_0^{T}U_*^2(x,t)dxdt\le C\delta^{2(N-2pm)}.$$

Thus for any N_1 we obtain $||U_\varepsilon-V_{N_1}||_{\mathcal{L}_2(\Omega)}\le\widehat{C}\delta^{N_1-2pm}$.

Let $N_1=N+2pm+1$. Then $||u_\varepsilon-V_{N_1}||_{\mathcal{L}_2(\Omega)}\le\widehat{C}\delta^{N_1}$. But $||V_{N_1}-U_{N_1}||\le\overline{C}\delta^{N_1}$, and we find $||u_\varepsilon-U_{N+1+2pm}||\le\widetilde{C}\delta^{N+1}$. It immediately follows that $||u_\varepsilon-U_N||\le\widetilde{M}\delta^{N+1}$. Thus we have proved the following

Theorem 5.1. *Let u_ε be a solution of the problem* (5.1)–(5.3), *and let U_N be a partial sum of the series* (5.8). *Then the inequality*

$$||u_\varepsilon-U_N||\le\widetilde{M}\delta^{N+1}$$

is valid, where the constant $\widetilde{M}$ does not depend on δ and N.

References

1. M. I. Vishik and L. A. Lyusternik, Regular degeneration and boundary layer of linear differential equations with a small parameter. (Russian) *Uspekhi Mat. Nauk* **12**(1957), No. 5, 3–122.

2. M. I. Vishik and L. A. Lyusternik, Asymptotic behaviour of solutions of linear differential equations with large and rapidly varying coefficients and boundary conditions. (Russian) *Uspekhi Mat. Nauk* **15**(1960), No. 4, 27–95.

3. E. Sanchez-Palencia, Non-homogeneous media and vibration theory. *Springer-Verlag, New York,* 1980.

4. N. S. Bakhvalov and G. P. Panasenko, Process averaging in periodic media. (Russian) *Nauka, Moscow,* 1984.

5. A. M. Il'yin, Concordance of asymptotic expansions of solutions of boundary value problems. *Nauka, Moscow,* 1989.

6. O. A. Oleĭnik, G. A. Iosifyan, and A. S. Shamaev, Mathematical problems of the theory of strongly inhomogeneous elastic media. *Moscow State University Press, Moscow,* 1990.

7. V. G. Maz'ya, S. A. Nazarov, and B. A. Plamenevskiĭ, Asymptotics of solutions of boundary value problems under singular perturbations of a domain. (Russain) *Tbilisi,* 1981.

8. S. A. Nazarov, Formalism in constructing and in justification of asymptotic expansions of solutions of elliptic boundary value problems with a parameter. (Russain) *Editorial Board of the journal "Vestnik LGU",* 1982.

9. A. S. Demidov, Asymptotics of solutions of boundary value problems for a linear second order elliptic equation having coefficients with a "splash". (Russian) *Trudy Moskov. Mat. Obshch.* **23**(1970), 77–112.

10. E. Sanchez-Palencia, Perturbation of eigenvalues in thermoelasticity and vibration of systems with concentrated masses. *Lecture Notes in Phys.* 1155, *Springer-Verlag,* 1984, 346–368.

11. E. Sanchez-Palencia and H. Tchatat, Vibration des systémes élastiques avec des masses concentrées. *Rend. Sem. Mat. Univ. Politec. Torino* **42**(1984), No. 3, 43–63.

12. Yu. D. Golovatyĭ, S. A. Nazarov, O. A. Oleĭnik, and T. S. Soboleva, On eigenoscillations of a string with additional mass. (Russian) *Sibirsk. Mat. Zh.* **29**(1988), No. 5, 71–91.

13. Yu. D. Golovatyĭ, S. A. Nazarov, and O. A. Oleĭnik, Asymptotics of eigenvalues and eigenfunctions in problems on oscillation of a medium with singular density perturbation. (Russian) *Uspekhi Mat. Nauk* **43**(1988), No. 51, 89–190.

14. Yu. D. Golovatyĭ, S. A. Nazarov, and O. A. Oleĭnik, Asymptotic expansion of eigenvalues and eigenfunctions of problems on oscillation of medium with concentrated perturbations. (Russian) *Trudy Mat. Inst. Steklov* **192**(1990), 42–60.

15. S. A. Nazarov, Asymptotic expansion of eigenvalues. (Russian) *Izd. LGU, Leningrad,* 1987.

16. O. A. Oleĭnik, On eigen oscillations of bodies with concentrated masses. *Nauka, Moscow,* 1988.

17. O. A. Oleĭnik, On frequencies of eigen oscillations of bodies with concentrated masses. (Russian) *Kiev, Naukova Dumka,* 1988.

18. O. A. Oleĭnik, Boundary value problems for linear differential equations of elliptic and parabolic type with discontinuous coefficients. (Russian) *Izv. AN SSSR Ser. Mat.* **25**(1961), No. 1 3–20.

19. A. G. Gagnidze, Asymptotics of solutions of boundary value problems for parabolic equations with coefficients having concentrated perturbation. *Rep. Enlarged Sess. Sem. I. Vekua Inst. Appl. Math.* **8**(1993), No.1.

20. A. G. Gagnidze, Asymptotic expansion of solutions of parabolic equations with a small parameter. *Georgian Math. J.* **5**(1998), No. 6, 501–512.

(Received 29.10.1997)

Authors' address:
Department of Mathematics and Mechanics
I. Javakhishvili Tbilisi State University
2, University St., Tbilisi 380043
Georgia