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SOLUTION OF SOME PLANE FILTRATION PROBLEMS WITH PARTIALLY UNKNOWN BOUNDARIES

Abstract. Plane problems of the stationary filtration theory with partially unknown boundaries are considered. The porous medium is assumed to be homogeneous, isotropic and non-deformable. The motion of the fluid obeys the Darcy law. The simply connected domain occupied by the moving fluid is bounded by a simple sectionally analytic contour consisting of unknown depression curves, line segments, half-lines and straight lines. The paper describes mathematical methods of finding the unknown parts of the boundary of the fluid motion domain, as well as of determining geometric, cinematic and physical characteristics of the moving fluid. In solving the corresponding mathematical problem, the use is made of the general solution of the non-linear Schwarz differential equation. The general solution is constructed in the paper.

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რეზიუმე. განხილულია სტაციონარული ფილტრაციის თეორიის ბრტყელი ამოცანები ნაწილობრივ უცნობი საზღვრებით. იგულისხმება, რომ ფოროვანი გარემო ერთგვაროვანი, იზოტროპული და არადეფორმირებადია. სითხის მოძრაობა ემორჩილება დარსის კანონს. მოძრავი სითხის მიერ დაკავებული ცალადბმული არე შემოსაზღვრულია უბან–უბან ანალიზური კონტურით, რომელიც შედგება უცნობი დეპრესიული წირების, წრფივი სეგმენტების, ნახევარწრფეებისა და წრფეებისაგან. ნაშრომში აღწერილია სითხის მოძრაობის არის საზღვრის უცნობი ნაწილების განსაზღვრის, მოძრავი სითხის გეომეტრიული, კინემატიკური და ფიზიკური მახასიათებლების მოძებნის მეთოდები. შესაბამისი მათემატიკური ამოცანის ამოხსნისას გამოყენებულია შვარცის არაწრფივი დიფერენციალური განტოლების ზოგადი ამონახსნი, რომელიც ამავე ნაშრომშია აგებული.

1. Introduction

In the present paper we consider some plane problems of the filtration theory for the stationary motion of an incompressible fluid in a porous medium obeying the Darcy law. The porous medium is assumed to be non-deformable, isotropic and homogeneous. The formulation and fundamental investigation of these problems belongs to P. Ya. Polubarinova-Kochina [1–5].

The plane of the motion of the fluid coincides with that of the complex variable $z = x + iy$. We introduce the complex potential $\omega(z) = \varphi(x, y) + i\psi(x, y)$, where $\varphi(x, y)$ and $\psi(x, y)$ are the velocity potential and the flow function, respectively. The functions $\varphi(x, y)$, $\psi(x, y)$ are connected by the Cauchy–Riemann conditions.

If the analytic function $\omega(z)$ is found, then by virtue of the equalities

$$\varphi(x, y) = -k(p/\gamma + y) + c, \quad \omega'(z) = u - iv, \quad u = \frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

we can find all characteristics of the filtration flow, i.e., the filtration velocity, the pressure, the stress, the discharge of the fluid upon filtration, etc. Here k is the filtration coefficient, c is an arbitrary constant, p is the hydrodynamic pressure, γ is the specific weight of the fluid, u, v are the components of the vector of filtration velocity, $\omega'(z)$ is the complex velocity.

The boundary of the domain of the flow involves unknown parts, the depression curves whose equations are to be found. Denote the simply connected domains of the flow of the fluid, of the complex potential and of the complex velocity respectively by $S(z)$, $S(\omega)$ and $S(w)$, and their boundaries respectively by $l(z)$, $l(\omega)$ and $l(w)$. Here $w = \omega'(z)$. Below the boundary $l(z)$ of the domain $S(z)$ will be assumed to be a simple, sectionally analytic, closed contour consisting of a finite number of unknown depression curves, line segments, half-lines and straight lines. The domain $S(z)$ may be bounded or unbounded. In the particular case where all parts of the boundary $l(z)$ are known, the domain $S(z)$ is a linear polygon.

In the domain $S(z)$, we seek for an analytic function $\omega(z) = \varphi(x, y) + i\psi(x, y)$ satisfying two linearly independent boundary conditions of the type [2]

$$a_{11}\varphi(x, y) + a_{12}\psi(x, y) + a_{13}x + a_{14}y = f_1, \quad (x, y) \in l(z), \tag{1.1}$$

$$a_{21}\varphi(x, y) + a_{22}\psi(x, y) + a_{23}x + a_{24}y = f_2, \quad (x, y) \in l(z), \tag{1.2}$$

where a_{ik}, f_i, $i = 1, 2$, $k = 1, 2, 3, 4$, are known piecewise-constant real functions, i.e., they are constant on every above-mentioned part of the boundary, and the rank of the matrix

$$a = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

is equal to two.

If a part of the boundary $l(z)$ of $S(z)$ is known, then in one of the conditions (1.1) or (1.2) the coefficients at the functions $\varphi(x, y)$, $\psi(x, y)$ for the known part of the boundary $l(z)$ turn out to be equal to zero.

There is a theory [1–6] which allows one to determine the boundary $l(w)$ of $S(w)$ and a part of the boundary $l(\omega)$ of $S(\omega)$ without solving the basic problem. Moreover, one can determine the coordinates of those vertices of the domain $S(w)$ to which there correspond angular points on the boundary $l(z)$ of $S(z)$. As for the vertices of the domain $S(w)$ (the cut ends with the angles 2π) to which there correspond ordinary (non-angular) points on the boundary $l(z)$ of $S(z)$, the coordinates of these vertices remain undetermined until the problem is solved completely.

In determining the boundary $l(w)$ of the domain $S(w)$, we have used some known results from the complex analysis [2, 21, 30, 31, 32].

Under the conditions imposed on the domains $S(z)$ and on the corresponding boundaries $l(z)$, one can claim that the function $\omega(z)$ is analytic in $S(z)$, continuous in the closed domain $S(z)$, satisfies $\omega'(z) \neq 0$ everywhere including the boundary $l(z)$ except its angular points, and is analytically continuable across any part of the boundary $l(z)$ not containing angular points.

As far as the functions $\omega(z)$ and $\omega'(z)$ map conformally the domain $S(z)$ and its boundary $l(z)$ (the conformity is violated at the angular points of $l(z)$) respectively onto the domains $S(\omega)$ and $S(w)$ with the boundaries $l(\omega)$ and $l(w)$, these functions are analytically continuable across the parts of the boundaries not containing angular points [30, Ch. II, §28–29].

In the sequel, for the complex-conjugate functions we will use the notation $f(z) = f_1(x, y) + if_2(x, y)$, $\overline{f(z)} = f_1(x, y) - if_2(x, y)$, while for the derivatives of functions and matrices, the notation $f'(z) = \frac{d}{dz} f(z)$.

Theorem. *If an analytic function $\omega(z)$ satisfies in the domain $S(z)$ two linearly independent boundary conditions (1.1)–(1.2), then the function $w(z) = \omega'(z)$ maps the boundary $l(z)$ of $S(z)$ into the boundary of the domain $S(w)$ consisting of a finite number of circular arcs, line segments, half-lines and straight lines, that is, to the domain $S(z)$ with the boundary $l(z)$ there corresponds a circular polygon on the plane $w(z)$.*

Proof. If we take arbitrarily a part of the boundary $l(z)$ of $S(z)$ and differentiate the conditions (1.1)-(1.2) along this part with respect to the real parameter s, then we obtain

$$(a_{11}u - a_{12}v + a_{13})\cos(x, s) + (a_{11}v + a_{12}u + a_{14})\cos(y, s) = 0, \quad (1.3)$$

$$(a_{21}u - a_{22}v + a_{23})\cos(x, s) + (a_{21}v + a_{22}u + a_{24})\cos(y, s) = 0, \quad (1.4)$$

where s is the arc length of the arbitrarily taken part of the boundary $S(z)$, $\cos(x, s) = dx/ds$, $\cos(y, s) = dy/ds$.

In order for the system of equations (1.3), (1.4) to have a nonzero solution with respect to $\cos(x, s)$ and $\cos(y, s)$, it is necessary and sufficient that the determinant of this system be equal to zero,

$$\Delta_0 = (a_{11}u - a_{12}v + a_{13})(a_{21}v + a_{22}u + a_{24}) - \\ -(a_{21}u - a_{22}v + a_{23})(a_{11}v + a_{12}u + a_{14}). \tag{1.5}$$

From (1.5) we obtain

$$A_0(u^2 + v^2) + B_1^* u + B_2^* v + D_0 = 0, \tag{1.6}$$

where

$$A_0 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_0 = \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}, \tag{1.7}$$

$$B_1^* = \begin{vmatrix} a_{11} & a_{14} \\ a_{21} & a_{24} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{12} \\ a_{23} & a_{22} \end{vmatrix}, \tag{1.8}$$

$$B_2^* = \begin{vmatrix} a_{14} & a_{12} \\ a_{24} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{11} \\ a_{23} & a_{21} \end{vmatrix}. \tag{1.9}$$

The second order curve decompose into two straight (real or imaginary) lines if and only if $A_0^* = -A_0\Delta/4 = 0$, where $\Delta = (B_1^*)^2 + (B_2^*)^2 - 4A_0D_0$.

If $A_0^* \neq 0$, $A_0^2 > 0$, and A_0^* and A_0 are of the same sign, then we have an imaginary circle; if $A_0^2 > 0$, A_0^* and A_0 are of different signs, we have a circle [7, 8].

The center coordinates (u_0, v_0) of the circle (1.6) and its radius R are defined by

$$u_0 = -B_1^*/[2a_0], \quad v_0 = -B_2^*/[2A_0], \quad R = \sqrt{\Delta}/[2A_0].$$

The circle (1.6) will be tangent to the axis of abscissas ou if $(B_1^*)^2 = 4A_0D_0$ and to the axis of ordinates ov if $(B_2^*)^2 = 4A_0D_0$.

In deducing (1.6), a part of the boundary of $S(z)$ has been taken arbitrarily. To some other parts of the boundary of $S(z)$, on w there correspond arcs of the circles, i.e., the domain $S(w)$ is a circular polygon. In the case where $A_0 = 0$ along the whole contour $l(z)$, we have a linear polygon.

The equation (1.6) can be written as follows:

$$i2A_0 w\overline{w} - B_0 w + \overline{B}_0 \overline{w} + i2D_0 = 0, \tag{1.10}$$

where

$$w = u - iv, \quad \overline{w} = u + iv, \quad B_0 = B_2^* - iB_1^*, \tag{1.11}$$

From (1.10) we find that $w = \frac{-\overline{B}_0\overline{w} - i2D_0}{i2A_0\overline{w} - B_0}$, where $\Delta = B_0\overline{B}_0 - 4A_0D_0 = (B_1^*)^2 + (B_1^*)^2 - 4A_0D_0 \neq 0$.

Note that in the general case the equality

$$\Delta = 4A_0^2R^2 = 1 \tag{1.12}$$

does not hold.

We will get back to the equality (1.12) later on.

Here we make the following remark. Using a linear-fractional transformation, one can always transform the domain $S(w)$ in a way that a part of the boundary $l(w)$ on the plane w will coincide with the abscissae axis along which $w = \overline{w}$, i.e., $v = 0$. This remark will be used later on.

Below we will come across the class of matrices G_j, $j = 1, 2, \dots, n, \dots$, satisfying the following conditions: $G_j \overline{G}_j = \overline{G}_j G_j = E$, $\det G_j = 1$, $G_j G_k \neq G_k G_j$, $k \neq j$, $(G_j G_k)(\overline{G_j G_k}) \neq E$, $k \neq j$ where $\overline{G}_j$ is a matrix which is complex conjugate to the matrix G_j, and E is the unit matrix. The properties of the matrices G_j, $j = 1, 2, \dots$ are very close to those of the complex-orthogonal ones [32].

The matrices G_j can be represented as

$$G_j = \begin{pmatrix} \overline{B}_j & -iD_j \\ iA_j & B_j \end{pmatrix}, \quad j = 1, 2, \dots ,$$

where A_j, D_j are real and B_j, $\overline{B}_j$ are complex-conjugate numbers.

Denoting the characteristic numbers of the matrix G_j by λ_{kj}, $k = 1, 2$, we obtain $\lambda_{1j} + \lambda_{2j} = \overline{B}_j + B_j$, $\lambda_{1j}\lambda_{2j} = 1$.

It follows from the property of the matrix G_j that $\lambda_{2j} = \overline{\lambda}_{1j}$. Therefore $\lambda_{1j}\overline{\lambda}_{1j} = 1$, $|\lambda_{1j}| = 1$ and hence $\lambda_{kj} = \exp[i2\pi\alpha_{kj}]$, where α_{kj} are real numbers.

If we take two arbitrary matrices G_j and G_k from the above-mentioned class and consider the matrix $g_{jk} = G_j G_k$, then we can see that the characteristic numbers μ_{kj} of the matrices g_{jk} satisfy the conditions $\mu_{kj} = \exp[i2\pi\beta_{kj}]$, where β_{kj} are real numbers.

2. Statement of the Boundary Value Problem

Let a moving fluid occupy a simply connected domain $S(z)$ with the boundary $l(z)$ consisting of a finite number of known and unknown simple analytic Jordan arcs.

An analytic function $\omega(z)$ maps conformally the domain $S(z)$ onto a domain $S(\omega)$, and its boundary $l(z)$ into the boundary $l(\omega)$ of $S(\omega)$. Note that a part of angular points of the boundary $l(z)$ is mapped by the function $\omega(z)$ into angular points of $l(\omega)$, while the remaining angular points are mapped into non-angular points of the boundary $l(\omega)$ [1–6].

Analogously, the analytic function $w(z) = \omega'(z) = u(x, y) - iv(x, y)$ maps conformally the domain $S(z)$ onto a domain $S(w)$, and its boundary $l(z)$ into the boundary $l(w)$ of $S(w)$. Moreover, the function $w(z)$ maps a part of angular points of the boundary $l(z)$ into those of $l(w)$, and the remaining angular points are mapped into ordinary non-angular points of the boundary $l(w)$. The function $w(z)$ can map some non-angular points of the boundary $l(z)$ into angular points of the boundary $l(w)$ with interior (with respect to the domain $S(w)$) angles 2π [1–6].

Below the points of the boundaries $l(z)$, $l(\omega)$, $l(w)$ are assumed to be singular if to these points on either of the boundaries $l(z)$, $l(\omega)$, $l(w)$ there correspond angular points.

Let us take arbitrarily a singular point on the boundary $l(z)$ of the domain $S(z)$, for example, $l(z, E_1)$. Let to a point $l(z, E_1)$ on the boundaries $l(\omega)$, $l(w)$ there correspond the points $l(\omega, E_1')$, $l(w, E_1'')$. When the domain $S(z)$ is went around in the positive direction starting from the point $l(z, E_1)$, then the boundaries $l(\omega)$, $l(w)$ are went aound in the positive direction starting from the points $l(\omega, E_1')$, $l(w, E_1'')$. We enumerate all the singular points on the boundaries $l(z)$, $l(\omega)$, $l(w)$ as follows: $l(z, E_k)$, $l(\omega, E_k')$, $l(w, E_k'')$, $k = 1, 2, \ldots, n, n+1$.

Of all singular points $l(z.E_k)$, $l(\omega, E_k')$, $k = 1, 2, \ldots, n, n+1$, we distinguish such ones to which on the boundary $l(w)$ of the domain $S(w)$ there correspond ordinary non-angular points. Such singular points are commonly called removable singularities. Let the number of such points be equal to m_1. When the boundary $l(z)$ is went around in the positive direction, we enumerate the removable singular points as $\varepsilon_1^*, \varepsilon_2^*, \ldots, \varepsilon_{m_1}^*, \varepsilon_{m_1}^*$. The interior angles on $l(z)$ and $l(\omega)$ at the removable singular points are equal to $\pi/2$ [1–6].

In tracing $l(\omega)$ we enumerate all angular points: $l(\omega, \omega_k)$, $k = 1, 2, \ldots, m_2+1$, while in tracing $l(w)$ we enumerate them as follows: $l(w, w_k) = b_k$, $k = 1, 2, \ldots, m, m+1$, where b_k, $k = 1, 2, \ldots, m, m+1$, are the complex coordinates of the vertices of the domain $S(w)$.

The equation (1.6) determines completely the circle. Two circles pass through every vertex of the domain $S(w)$ (two straight lines upon degeneration), and each of them forms four angles. We have to choose one of them. To this end we, first of all, use the equation (1.6) and then the value of the corresponding angles of the domains $S(z)$, $S(\omega)$. By means of these angles we can determine the angles at the vertices of the domains $S(w)$, $S(\omega)$ [1–6]. Despite the fact that some of the interior angles of $S(\omega)$ are unknown, we have the angle values for the corresponding vertices of the domain $S(w)$. By means of the latter we can determine the unknown interior angle at the vertex of the domain $S(\omega)$ [2]. This henceforth allows us to take for granted that all the interior angles and the coordinates of the vertices of $S(w)$, excluding the cut ends with interior angles 2π, are determined.

Denote the interior angles at the vertices b_j, $j = 1, 2, \ldots, m, m+1$, of the domain $S(w)$ by $\pi\nu_j$, $j = 1, 2, \ldots, m, m+1$, respectively.

Note that the two neighboring circles passing through the point b_k intersect at the point b_k' which in the general case is beyond the boundary $l(w)$. If these circles are tangent, then $b_k = b_k'$.

In general it is quite difficult to find an analytic function $\omega(z) = \varphi(x, y) + i\psi(x, y)$ by the boundary conditions (1.1)–(1.2). Therefore one introduces an auxiliary complex plane $\zeta = t + i\tau$. The half-plane $\mathrm{Im}(\zeta) > 0$ of this plane is mapped conformally onto the domains $S(z)$, $S(\omega)$, $S(w)$. Denote the domain $\mathrm{Im}(\zeta) > 0$ and its boundary respectively by $S(\zeta)$ and $l(\zeta)$.

In what follows, we will need the following result from the papers [21, 30, 32].

If D and D^* are simply connected domains whose boundaries consist of a finite number of analytic Jordan arcs, then there exists a unique conformal mapping $w = f(z)$ of the domain D onto the domain D^*, transferring three boundary points z_k, $k = 1, 2, 3$, of D into three boundary points W_k, $k = 1, 2, 3$ of D^*. The points z_k and w_k are given arbitrarily, their order in tracing the boundaries of the domains being preserved.

Let the analytic functions $z(\zeta)$, $\omega(\zeta)$, $w(\zeta) = \omega'(\zeta)/z'(\zeta)$ map conformally the domain $S(\zeta)$ ($\mathrm{Im}(\zeta) > 0$) onto the domains $S(z)$, $S(\omega)$, $S(w)$, respectively. Moreover, let the points of the boundary $l(\zeta)$ of $S(\zeta)$, that is the points of the real axis t of the plane ζ, $t = e_k$, $k = 1, 2, \dots, n, n+1$ $(-\infty < e_1 < e_2 < \dots < e_{n+1} = \infty)$, be respectively mapped into the points $l(z, E_k)$, $l(\omega, E'_k)$, $l(w, E''_k)$, $k = 1, 2, \dots, n, n+1$, of the boundaries $l(z)$, $l(\omega)$, $l(w)$ of the domains $S(z)$, $S(\omega)$, $S(w)$.

The boundary values of the functions $z(\zeta)$, $\omega(\zeta)$, $w(\zeta)$, as $\zeta \to t$, are denoted by $z(t) = x(t) + iy(t)$, $\omega(t) = \varphi(t) + i\psi(t)$, $w(t) = u(t) - iv(t)$, and by $\overline{z(t)}$, $\overline{\omega(t)}$, $\overline{w(t)}$ we denote the complex functions conjugate to the functions $z(t)$, $\omega(t)$, $w(t)$.

The boundary conditions (1.1)–(1.2) with respect to the analytic functions $\omega(\zeta)$ and $z(\zeta)$ can be written in the form [2]

$$\mathrm{Im}[m_{11}(t)\omega(t) + m_{12}(t)z(t)] = f_1(t), \quad -\infty < t < +\infty, \tag{2.1}$$

$$\mathrm{Im}[m_{21}(t)\omega(t) + m_{22}(t)z(t)] = f_2(t), \quad -\infty < t < +\infty, \tag{2.2}$$

where $m_{k1}(t) = a_{k2}(t) + ia_{k1}(t)$, $m_{k2}(t) = a_{k4}(t) + ia_{k3}(t)$, $f_k(t)$, $k = 1, 2$, are piecewise constant functions with the discontinuity points $t = e_k$, $k = 1, 2, \dots, n, n+1$.

In the domain $S(\zeta)$ we have to find analytic functions $z(\zeta)$, $\omega(\zeta)$ satisfying the boundary conditions (2.1)–(2.2). By means of these functions, the points $z(\zeta)$, $\omega(\zeta)$ are mapped respectively into the points $t = e_k$, $k = 1, 2, \dots, n, n+1$. Moreover, each part of the boundary must necessarily be mapped into the corresponding parts of the boundaries $l(z, E_k)$, $l(\omega, E'_k)$, $k = 1, 2, \dots, n+1$. The unknown parts of the boundaries $l(\zeta)$, $-\infty < t < e_1$, $e_k < t < e_{k+1}$, $k = 1, 2, \dots, n$ and the parameters $t = e_k$, $k = 1, 2, \dots, n$, are to be determined.

If we succeed in constructing analytic functions $z(\zeta)$, $\omega(\zeta)$, which map conformally the domain $S(\zeta)$ respectively onto the domains $S(z)$, $S(\omega)$, then the boundary values $z(t)$, $\omega(t)$ of these functions will satisfy the conditions (2.1)–(2.2). Moreover, if the functions $z(\zeta)$, $\omega(\zeta)$ are known, then we can construct the function $w(\zeta) = \omega'(\zeta)/z'(\zeta)$.

If one or several coefficients m_{kj}, $k = 1, 2$; $j = 1, 2$, are equal to zero, and $m_{11}(t)m_{22}(t) - m_{12}(t)m_{21}(t) \neq 0$, then by the conditions (2.1)–(2.2) the functions $\omega(\zeta)$, $z(\zeta)$ can be constructed by means of the Cauchy type integrals. There are particular cases where all $m_{kj}(t) \neq 0$, $k = 1, 2$, $j = 1, 2$, but nevertheless one manages to construct the functions $\omega(\zeta)$, $z(\zeta)$ in the elementary way [12].

As we will see below, in the general case we have managed to construct first the analytic function $w(\zeta)$. Then, by means of this function, we have constructed analytic functions $\omega'(\zeta)$, $z'(\zeta)$ and, finally, we have found the functions $\omega(\zeta)$ and $z(\zeta)$.

The notion of singular and removable singular points of the boundary $l(z)$ has been introduced above. As is said, to singular points of the boundary $l(z)$ there correspond singular points $t = e_k$, $k = 1, 2, \dots, n, n+1$, of the boundary $l(\zeta)$. They can be divided into two groups: removable and unremovable. We have enumerated the removable points by $t = \varepsilon_k$, $n = 1, 2, \dots, m_1$, and the unremovable ones by $t = a_k$, $k = 1, 2, \dots, m, m{+}1$. To the points $t = a_k$, $k = 1, 2, \dots, m+1$, on the boundary $l(w)$ there correspond the points $l(w, w_k) = b_k$ while to the points $t = \varepsilon_k$, $k = 1, 2, \dots, m$, there correspond the points $l(z, z_k) = \varepsilon^*_k$, $k = 1, 2, \dots, m_1$. By our choice, the point $t = e_{n+1} = a_{m+1} = \infty$ is a unremovable singular point. Among the points $t = a_k$, $k = 1, 2, \dots, m$, we select and fix arbitrarily two points, because one point $t = a_{m+1} = \infty$ is already fixed.

An investigation of the problem (2.1)–(2.2) from the point of view of the Riemann-Hilbert problem can be found in [17, 18].

Introduce an analytic vector $\Phi(\zeta)$ and a vector $f(t)$ as follows:

$$\Phi(\zeta) = [\omega(\zeta), z(\zeta)], \quad \mathrm{Im}(\zeta) > 0; \quad \overline{\Phi(\overline{\zeta})} = [\overline{\omega}(\overline{\zeta}), \overline{z(\overline{\zeta})}], \quad \mathrm{Im}(\zeta) < 0,$$
$$f(t) = [f_1(t), f_2(t)], \quad -\infty < t < +\infty.$$

The conditions (2.1)–(2.2) with respect to the vector $\Phi(\zeta)$ can be written as

$$\Phi(t) = A_*^{-1}(t)\overline{A}_*(t)\overline{\Phi(t)} + 2iA_*^{-1}(t)f(t), \quad -\infty < t < +\infty, \tag{2.3}$$

where

$$A_*(t) = \begin{pmatrix} m_{11}(t) & m_{12}(t) \\ m_{21}(t) & m_{22}(t) \end{pmatrix}, \quad -\infty < t < +\infty,$$

is a non-singular piecewise-constant matrix, A_*^{-1} is the inverse to A_*

$$A_*^{-1}(t) = \frac{1}{\det A_*(t)} \begin{pmatrix} m_{22}(t) & m_{12}(t) \\ m_{21}(t) & m_{11}(t) \end{pmatrix}, \quad -\infty < t < +\infty,$$

and $\overline{A_*(t)}$ is the complex conjugate to $A_*(t)$.

It can be easily verified that

$$A_*^{-1}(t)\overline{A_*(t)} = \frac{1}{\det A_*(t)} \begin{pmatrix} \overline{-B_0(t)} & -i2D_0(t) \\ i2A_0(t) & -B_0(t) \end{pmatrix}, \quad -\infty < t < +\infty,$$

where $A_0(t)$, $B_0(t)$, $D_0(t)$ are defined by (1.7)–(1.9) and (1.11).

We can directly verify that the equalities

$$\Delta(t) = B_0(t)\overline{B_0(t)} - 4A_0(t)D_0(t) = \det A_*(t) \cdot \det \overline{A_*(t)},$$
$$\det[A_*^{-1}(t) \cdot \overline{A_*(t)}] = [\det \overline{A_*(t)}]/[\det A_*(t)]$$

are also valid.

Differentiating the equality (2.3) along the t, u-axis and writing it in terms of projections, we obtain

$$\omega'(t) = [-\overline{B_0(t)}\,\overline{\omega'(t)} - i2D_0(t)\overline{z'(t)}]/\det A_*(t), \quad -\infty < t < +\infty, \tag{2.4}$$

$$z'(t) = [i2a_0(t)\overline{\omega'(t)} - B_0(t)\overline{z'(t)}]/\det A_*(t), \quad -\infty < t < +\infty. \tag{2.5}$$

After division, from (2.4) and (2.5) we get

$$\frac{\omega'(t)}{z'(t)} = \frac{-\overline{B_0(t)}\,\overline{\omega'(t)} - i2D_0(t)\overline{z'(t)}}{i2A_0(t)\overline{\omega'(t)} - B_0(t)\overline{z'(t)}}, \quad -\infty < t < +\infty. \tag{2.6}$$

The equality (2.6) can also be written as

$$w(t) = \frac{-\overline{B_0(t)}\,\overline{w(t)} - i2D_0(t)}{i2A_0\overline{w(t)} - \overline{B_0(t)}}, \quad -\infty < t < +\infty, \tag{2.7}$$

where $w(t) = \omega'(t)/z'(t)$.

As we will see below, by means of the solution of the well-known Schwarz differential equation we can find an analytic function satisfying (2.7) on the t-axis, provided the condition

$$\Delta(t) = B_0(t)\overline{B_0(t)} - 4A_0(t)D_0(t) = 1, \quad -\infty < t < +\infty, \tag{2.8}$$

is fulfilled.

However, the condition (2.8) may not be fulfilled. If we divide the numerator and the denominator in (2.7) by $\sqrt{\Delta(t)}$ and introduce the notation

$$B(t) = -B_0(t)/\sqrt{\Delta(t)}, \ \ \overline{B(t)} = -\overline{B_0(t)}/\sqrt{\Delta(t)}, \ \ -\infty < t < +\infty, \tag{2.9}$$

$$A(t) = 2A_0(t)/\sqrt{\Delta(t)}, \ \ \overline{D(t)} = 2D_0(t)/\sqrt{\Delta(t)}, \ \ -\infty < t < +\infty, \tag{2.10}$$

then the condition

$$\Delta_1(t) = \overline{B(t)}B(t) - A(t)D(t) = 1, \quad -\infty < t < +\infty, \tag{2.11}$$

will be fulfilled.

With regard for (2.9) and (2.10), we can rewrite (2.7) as

$$w(t) = \frac{\overline{B(t)}\,\overline{w(t)} - iD(t)}{iA(t)\overline{w(t)} + B(t)}, \quad -\infty < t < +\infty, \tag{2.12}$$

and (2.4) and (2.5) as

$$\omega'(t) = \sqrt{\det\overline{A_*(t)}/\det A_*(t)}\,\big[\,\overline{B(t)}\,\overline{\omega'(t)} - iD(t)\overline{z'(t)}\,\big], \ -\infty < t < +\infty, \tag{2.13}$$

$$z'(t) = \sqrt{\det\overline{A_*(t)}/\det A_*(t)}\,\big[\,iA(t)\overline{\omega'(t)} + B(t)\overline{z'(t)}\,\big], \ -\infty < t < +\infty. \tag{2.14}$$

A solution of the system (2.13)-(2.14) will be sought in the form

$$\omega'(t) = \gamma(t)\omega_1(t), \quad z'(t) = \gamma(t)z_1(t), \quad -\infty < t < +\infty, \tag{2.15}$$

where $\omega_1(t)$, $z_1(t)$ and $\gamma(t)$ must satisfy the boundary conditions

$$\omega_1(t) = \overline{B(t)}\,\overline{\omega_1(t)} - iD(t)\overline{z_1(t)}, \quad -\infty < t < +\infty, \tag{2.16}$$

$$z_1(t) = iA(t)\overline{\omega_1(t)} + B(t)\overline{z_1(t)}, \quad -\infty < t < +\infty, \tag{2.17}$$

$$\gamma(t) = \sqrt{\det \overline{A_*(t)} / \det A_*(t)}\; \overline{\gamma(t)}, \quad -\infty < t < +\infty. \tag{2.18}$$

Note that the value of the function $w(t) = \omega'(t)/z'(t)$ does not change after the representation (2.15), and hence so does (2.12). A little later we will prove that (2.12) implies (2.16)–(2.17) [13–16].

If we denote the values of the matrix $A_*(t)$ for the intervals $-\infty < t < e_1$, $e_j < t < e_{j+1}$, $j = 1, 2, \dots, n$, respectively by $A_{*(n+1)}$, A_{*j}, $j = 1, 2, \dots, n$, then we can write

$$\det A_{*j} = |\det A_{*j}| \exp[i\varphi_j], \quad \det \overline{A_{*j}} = |\det \overline{A_{*j}}| \exp[-i\varphi_j],$$

$$|\det \overline{A_{*j}}| / |\det A_{*j}| = 1, \quad j = 1, 2, \dots, n, n+1,$$

$$\sqrt{\det \overline{A_{*j}} / \det A_{*j}} = \sqrt{\exp[-i2\varphi_j]} = \exp[-i\varphi_j], \quad j = 1, 2, \dots, n+1. \tag{2.19}$$

Taking into account (2.19), we rewrite (2.18) as

$$\gamma(t) = \exp[i\varphi_0(t)]\overline{\gamma(t)}, \quad -\infty < t < +\infty, \tag{2.20}$$

where $\varphi_0(t)$ is a piecewise constant function defined by

$$\sqrt{\det \overline{A_*(t)} / \det A_*(t)} = \exp[-i\varphi_0(t)], \quad -\infty < t < +\infty.$$

After taking the logarithm of (2.20), we get

$$\ln \gamma(t) - \ln \overline{\gamma(t)} = -i\varphi_0(t), \quad -\infty < t < +\infty. \tag{2.21}$$

We will not introduce here the notion of index but will act formally and will find from (2.21) a particular solution belonging to some class, and then we will define more exactly which solution out of all possible solutions of (2.21) is just needed.

The particular solution of the boundary value problem (2.21) can be obtained by the formula [17]

$$\ln \gamma(\zeta) = \frac{(-1)}{2\pi} \int_{-\infty}^{+\infty} \frac{\zeta + i}{t + i} \frac{\varphi_0(t)dt}{t - \zeta}. \tag{2.22}$$

From (2.22) we find that

$$\gamma(\zeta) = \mathrm{const}(\zeta - e_1)^{\beta_1}(\zeta - e_2)^{\beta_2} \cdots (\zeta - e_n)^{\beta_n}, \tag{2.23}$$

where $\beta_1 = (\varphi_{n+1} - \varphi_1)/2\pi$, $\beta_j = (\varphi_{j-1} - \varphi_j)/2\pi$, $j = 2, 3, \dots, n$, and φ_j, $j = 1, 2, \dots, n+1$ are the values of the function $\varphi_0(t)$ on the intervals $e_j < t < e_{j+1}$, $j = 1, 2, \dots, n$, $-\infty < t < e_1$, respectively.

The numbers φ_j, $j = 1, 2, \dots, n+1$, in (2.23) will be chosen appropriately after finding the functions $\omega'(\zeta)$ and $z'(\zeta)$.

It follows from the above-said that to construct in the domain $S(\zeta)$ the analytic functions $\omega(\zeta)$ and $z(\zeta)$ satisfying the boundary conditions (2.1)–(2.2), it is necessary first to construct in the domain $S(\zeta)$ the functions $\omega_1(\zeta)$, $z_1(\zeta)$ satisfying the conditions (2.16)–(2.17). And, as we will see below, to construct the functions $\omega_1(\zeta)$ and $z_1(\zeta)$, it is necessary first to construct in the domain $S(\zeta)$ the function $w(\zeta) = \omega'(\zeta)/z'(\zeta) = \omega_1(\zeta)/z_1(\zeta)$ satisfying the boundary condition (2.12).

3. Investigation of the Problem (2.16)–(2.17)

We write the boundary value problem (2.16)–(2.17) in the vector form:

$$\Phi_1(t) = g(t)\overline{\Phi_1(t)}, \quad -\infty < t < +\infty, \tag{3.1}$$

where

$$\Phi_1(\zeta) = [\omega_1(\zeta), z_1(\zeta)],\ \operatorname{Im}(\zeta) > 0; \quad \overline{\Phi_1(\overline{\zeta})} = [\overline{\omega_1(\overline{\zeta})}, \overline{z_1(\overline{\zeta})}],\ \operatorname{Im}(\zeta) < 0,$$

$$g(t) = \begin{pmatrix} \overline{B(t)} & -i2D(t) \\ iA(t) & B(t) \end{pmatrix}, \quad -\infty < t < +\infty.$$

For the intervals $a_j < t < a_{j+1}$, $j = 1, 2, \dots, n$, $-\infty < t < a_1$, denote the values of the matrix $g(t)$ respectively by g_j, $j = 1, 2, \dots, n, n+1$. There is a close connection between the characteristic numbers of the matrices $g_j^{-1}g_{j-1}$, $j = 1, 2, \dots, n+1$, and the interior angles at the vertices of the circular polygon $S(\omega)$. Indeed, consider the characteristic equation for the point $t = e_j$ [2, 17, 18]:

$$\det(g_j^{-1}(t)g_{j-1}(t) - \lambda_j E) = 0, \tag{3.2}$$

where λ_j is a parameter and E is the unit matrix.

The equation (3.2) can be also written as $\det(g_{j-1}(t) - \lambda_j g_j(t)) = 0$. Hence, taking into account the fact that $\det g_j = 1$, $j = 1, 2, \dots, n+1$, we obtain $\lambda_j^2 - a_0\lambda_j + 1 = 0$, $a_{0j} = \overline{B}_{j-1}B_j + \overline{B}_j B_{j-1} - A_{j-1}D_j - A_j D_{j-1}$, which implies that $\lambda_{1j}\lambda_{2j} = 1$, $\lambda_{1j} + \lambda_{2j} = a_{0j}$, where λ_{1j} and λ_{2j} are the characteristic roots of (3.2).

Consider the numbers $\alpha_{kj} = \frac{1}{2\pi i}\ln \lambda_{kj}$, which are defined to within integer summands.

It has been proved in [2] that α_{kj} are real numbers satisfying $\alpha_{1j} - \alpha_{2j} = \nu_j$.

Let us get back to the removable singular points $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{m_1}$ For these points the neighboring matrices g_{j-1} and g_j are diagonal, and $t = e_j = \varepsilon_j$, $\lambda_{kj} = -1$, $k = 1, 2$, $\alpha_{1j} = -1/2$, $\alpha_{2j} = -1/2$ [1–5]. Moreover, $A_0(t) = 0$,

$B_1(t) = 0$, $B_2(t) \neq 0$, $D_0(t) = 0$, $v(t) = 0$, $u(t) \neq 0$ or $A_0(t) = 0$, $B_2(t) = 0$, $B_1(t) \neq 0$, $D_0(t) = 0$, $u(t) = 0$, $v(t) \neq 0$, where $t \in (a_{j-1}, a_{j+1})$.

Introduce a new sought for vector $\Phi_2(\zeta) = [\omega_2(\zeta), z_2(\zeta)]$ by

$$\Phi_1(\zeta) = \chi_1(\zeta)\Phi_2(\zeta), \tag{3.3}$$

where

$$\chi_1(t) = [(t-\varepsilon_1)(t-\varepsilon_2)\cdots(t-\varepsilon_{m_1})]^{-1/2}, \quad \chi_1(t) > 0, \quad t > \varepsilon_{m_1}. \tag{3.4}$$

The boundary condition (3.1) for $\Phi_2(\zeta)$ takes the form

$$\Phi_2(t) = G(t)\overline{\Phi_2(t)}, \quad -\infty < t < +\infty, \tag{3.5}$$

where $G(t) = [\chi_1(t)]^{-1} g(t)\overline{\chi_1(t)}$, $-\infty < t < +\infty$, also is a piecewise constant matrix with the discontinuity points $a_1, a_2, a_3, \ldots, a_m, a_{m+1} = \infty$.

The matrix $G(t)$ differs from the matrix $g(t)$ only by the fact that some matrices g_j are multiplied by -1 and the others remain unchanged.

If some elements of the matrix $G(t)$ are equal to zero and $\det G(t) \neq 0$, then the problem (3.5) is solved completely by the Cauchy type integrals, and the equations for the determination of the unknown parameters are derived [17]. Besides these cases, there are the ones where all elements of the matrix $G(t)$ differ from zero and the problem (3.5) is solved simply. Such cases involve circular polygons, when the boundary $S_1(w)$ consists of a finite number of arcs of concentric circles with the center $M(w_0)$ and straight cuts passing through $M(w_0)$ upon their extension. By means of the logarithmic function such domains $S_1(w)$ can be transformed into linear polygons. Moreover, there exist many domains $S_2(w)$ which by the linear-fractional transformation reduce to a set of domains $S_1(w)$. Hence, using the Christoffel–Schwarz formula [12], for the domains $S_1(w)$, $S_2(w)$ we construct the functions $w(\zeta)$.

We will now proceed to the solution of (3.5). If a circular polygon is bounded, then $0 \leq \nu_k \leq 2$. Below we will consider the case where one or several vertices of the domain $S(w)$ are at the point $w = \infty$. This may happen if two neighboring circular arcs degenerate to half-lines or straight lines. Moreover, if the sides of the corresponding angle are parallel, then the vertex of the interior angle is assumed to be equal to zero. If, however, the sides at the vertex $b_k = \infty$ diverge and intersect at a finite point b_k^* upon their extention, forming the angle $\pi\nu_k^*$ turned to the vertex b_k^*, then we will assume that $\pi\nu_k = -\pi\nu_k^*$; hence, ν_k may take the values $-2 \leq \nu_k \leq 2$.

It is known that the construction of the sought for function $w(\zeta)$ is reduced to the solution of the nonlinear Schwarz equation which in its turn reduces to a Fuchs class equation. Therefore for the domain $S(w)$ we construct the Fuchs class equation

$$V''(\zeta) + P_*(\zeta)V'(\zeta) + q_*(\zeta)V(\zeta) = 0, \tag{3.6}$$

where

$$P_*(\zeta) = \sum_{j=1}^{m} \frac{1-\nu_j}{\zeta - a_j}, \quad q_*(\zeta) = \sum_{j=1}^{m} \frac{c_j}{\zeta - a_j},$$

c_j, $j = 1, 2, \ldots, m$, are the unknown accessory real parameters which for the present satisfy the conditions

$$\sum_{j=1}^{m} c_j = 0, \quad \sum_{j=1}^{m} c_j a_j = \alpha_1 \alpha_2, \tag{3.7}$$

$$\sum_{j=1}^{m} \nu_j + \alpha_1 + \alpha_2 = m - 1, \quad \alpha_1 - \alpha_2 = \nu_{m+1}.$$

Denote by $V_1(\zeta)$, $V_2(\zeta)$ linearly independent solutions of the equation (3.6) and construct the function $w_1(\zeta) = V_1(\zeta)/V_2(\zeta)$. The function $w_1(\zeta)$ is a particular solution of the following Schwarz equation:

$$\frac{w'''(\zeta)}{w'(\zeta)} - \frac{3}{2}\Big(\frac{w''(\zeta)}{w'(\zeta)}\Big)^2 = 2q_*(\zeta) - P_*'(\zeta) - \frac{1}{2}[P_*(\zeta)]^2, \tag{3.8}$$

which is constructed with regard for the equation (3.6).

The general solution of the equation (3.8) is given by $w(\zeta) = \frac{pw_1(\zeta)+q}{rw_1(\zeta)+s}$, where p, q, r, s are constants (complex in general) of integration of (3.8) satisfying $ps - rq = 1$.

The equation (3.8) is invariant under linear-fractional transformations both of the function $w(\zeta)$ and of ζ. Note that the coefficients of the transformation of $w(\zeta)$ may be either complex or real, while those of the transformation of ζ may be only real. Moreover, the equation (3.6) is also invariant under the transformations of ζ with real coefficients [19–22].

In constructing a general solution of the equation (3.8), we have already used its invariance property with respect to $w(\zeta)$. Exploit now the invariance of the equation (3.6) with respect to ζ. Using this property, we choose arbitrarily and fix three of the parameters $t = a_k$, $k = 1, 2, \ldots, m+1$, while the remaining $(m-2)$ ones are to be defined. Moreover, the coefficients of the equation (3.6) involve the parameters c_j, $j = 1, 2, \ldots, m$ which for the present satisfy only two conditions (3.7), so one can define only two of them. The remaining $(m-2)$ parameters are also to be defined. Consequently, the coefficients of the equation (3.6) depend on $2(m-2)$ unknown parameters. The parameters p, q, r, s are also to be defined. Thus, to construct $w(\zeta)$ we must define only $2(m+1)$ parameters, while to construct the functions $\omega'(\zeta)$, $z'(\zeta)$ we must add the parameters connected with the removable singular points. Their number is m_1.

4. Solution of Equation (3.6)

Each of the Fuchs class equations (3.6) near every singular point $t = a_k$, $k = 1, 2, \dots, m+1$, and near any ordinary point, where $p_*(\zeta)$, $q_*(\zeta)$ are analytic, have two linearly independent local solutions. They are constructed by means of infinite series whose coefficients are defined in the well-known manner. The series converge in the circles with centers at the points for which they have been constructed. Radii of these circles are determined by the distances to the singular points nearest from the centers.

Denote by $V_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, 3, \dots, m+1$, linearly independent local solutions of the equation (3.6) for the singular points $\zeta = a_k$, $k = 1, 2, \dots, m+1$, and by $\varphi_{kj}(\zeta)$, $k = 1, 2$, $j = 1, 2, \dots, m-1$, the ones for the points $t = a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \dots, m-1$.

Assume $u_{1j}(\zeta) = pv_{1j}(\zeta) + qv_{2j}(\zeta)$, $u_{2j}(\zeta) = rv_{1j}(\zeta) + sv_{2j}(\zeta)$, where p, q, r, s are integration constants of (3.8).

The differential equation (3.6) can be written in the form of a system

$$\chi'(\zeta) = \chi(\zeta)\mathcal{P}(\zeta), \tag{4.1}$$

where

$$\chi(\zeta) = \begin{pmatrix} u_1(\zeta) & u_1'(\zeta) \\ u_2(\zeta) & u_2'(\zeta) \end{pmatrix}, \quad \mathcal{P}(\zeta) = \begin{pmatrix} 0 & -q_*(\zeta) \\ 1 & -p_*(\zeta) \end{pmatrix}, \tag{4.2}$$

$u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of (3.6).

First we find the solution of (4.1), that is, we construct the matrix $\chi(\zeta)$. Then by means of this matrix $\chi(\zeta)$ we seek for a solution of the boundary value problem (3.5).

It is known that if the matrix $\chi_*(\zeta)$ is a solution of (4.1), then the matrix $T\chi_*(\zeta)$ is also a solution of (4.1), where

$$T = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \det T = 1. \tag{4.3}$$

If we construct the local linearly independent solutions $v_{kj}(\zeta)$ and $\varphi_{kj}(\zeta)$ of the equation (3.6) for the points $\zeta = a_j$, $j = 1, 2, \dots, m+1$, and $\zeta = a_j^* = (a_j + a_{j+1})/2$, respectively, then the local fundamental matrices for (4.1) will take the form

$$\begin{aligned} \Theta_j(\zeta) &= \begin{pmatrix} v_{1j}(\zeta) & v_{1j}'(\zeta) \\ v_{2j}(\zeta) & v_{2j}'(\zeta) \end{pmatrix}, \quad j = 1, 2, \dots, m+1, \\ H_j(\zeta) &= \begin{pmatrix} \varphi_{1j}(\zeta) & \varphi_{1j}'(\zeta) \\ \varphi_{2j}(\zeta) & \varphi_{2j}'(\zeta) \end{pmatrix}, \quad j = 1, 2, \dots, m-1. \end{aligned} \tag{4.4}$$

Assume that the inequality $|a_m| > |a_1|$ holds. Then at the point $a_m^* = -|a_m|$ we construct the local series $\varphi_{*k}(\zeta)$, $k = 1, 2$, and the corresponding local matrix $H_*(\zeta)$. Radii of convergence of these series will be determined by the distance from the point $t = a_m^*$ to the singular point $t = a_1$. Analogously, if $|a_1| > |a_m|$, then at the point $a_1^* = |a_1|$ we construct local series $\varphi_k^*(\zeta)$,

$k = 1, 2$, and the matrix H. Radii of convergence of these series will be determined by the distance from the point a_1^* to the point $t = a_m$.

After this we can see that there exists a finite number of circles with the centers $\zeta = a_j$, $j = 1, 2, \ldots, m+1$, $\zeta = a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \ldots, m$, covering completely the abscissae axis. Note that by the circle with the center $\zeta = \infty$ will be meant the exterior of the circle $|\zeta| < r$, where r will be assumed to be equal to the greatest of the numbers $|a_1|$, $|a_m|$.

The equation (3.6) near the point $\zeta = a_j$ can be written as

$$(\zeta - a_j)^2 v''(\zeta) + (\zeta - a_j) p_j(\zeta) v'(\zeta) + q_j(\zeta) v(\zeta) = 0, \tag{4.5}$$

where

$$p_j(\zeta) = \sum_{k=0}^{\infty} p_{kj} (\zeta - a_j)^k, \quad q_j(\zeta) = \sum_{k=1}^{\infty} q_{kj} (\zeta - a_j)^k. \tag{4.6}$$

The solutions of the equations (4.5) and (4.6) for the point $\zeta = a_{m+1} = \infty$ by means of the transformation $\zeta = 1/\zeta_1$, can be written in the form [29, 27]

$$\zeta_1^2 v''(\zeta_1) + \zeta_1 [2 - \sum_{k=0}^{\infty} p_{k\infty} \zeta_1^k] v'(\zeta_1) + \left[\sum_{k=0}^{\infty} q_{k\infty} \zeta^k \right] v(\zeta_1) = 0, \tag{4.7}$$

where

$$p_*(1/\zeta_1) = \zeta_1 \sum_{k=0}^{\infty} p_{k\infty} \zeta_1^k, \quad q_*(1/\zeta_1) = \zeta_1^2 \sum_{k=0}^{\infty} q_{k\infty} \zeta_1^k. \tag{4.8}$$

The solutions of the equations (4.5) and (4.7) for the points $\zeta = a_j$, $j = 1, 2, \ldots, m$, $\zeta = \infty$ are sought respectively in the form [22, 27]

$$v_j(t) = (t - a_j)^{\alpha_j} \widetilde{v}_j(t), \quad \widetilde{v}_j(t) = \sum_{n=0}^{\infty} \gamma_{nj} (t - a_j)^n, \tag{4.9}$$

$$v_\infty(t) = t^{-\alpha_\infty} \widetilde{v}_\infty(t), \quad \widetilde{v}_\infty(t) = \sum_{n=0}^{\infty} \gamma_{nj} t^{-n}. \tag{4.10}$$

Substituting (4.9) in (4.5), we obtain

$$(\zeta - a_j)^{\alpha_j} \left[\sum_{k=0}^{\infty} M_{kj} (\zeta - a_j)^k \right] = 0,$$

whence there follows an infinite recursion system of equations to define γ_{nj}, $n = 1, 2, \ldots$,

$$M_{0j}(\alpha_j) = \gamma_{0j} f_{0j}(\alpha_j) = 0; \;\; f_{0j}(\alpha_j) = \alpha_j(\alpha_j - 1) + \alpha_j p_{0j} + q_{0j} = 0, \tag{4.11}$$

$$M_{1j}(\alpha_j) = \gamma_{1j}(\alpha_j) f_{0j}(\alpha_j + 1) + \gamma_{0j} f_{1j}(\alpha_j) = 0, \tag{4.12}$$

$$M_{2j}(\alpha_j) = \gamma_{2j}(\alpha_j) f_{0j}(\alpha_j + 2) + \gamma_{1j}(\alpha_j) f_{1j}(\alpha_j + 1) + \gamma_{0j} f_{2j}(\alpha_j) = 0, \tag{4.13}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned}M_{nj}(\alpha_j) = \gamma_{nj}(\alpha_j)f_{0j}(\alpha_j+n) + \gamma_{(n-1)j}(\alpha_j)f_{1j}(\alpha_j+n-1)+\cdots+\\ +\gamma_{[n-(k-2)]j}(\alpha_j)f_{(k-2)j}(\alpha_j+n-k+2)+\cdots+\\ +\gamma_{1j}(\alpha_j)f_{(n-1)j}(\alpha_j+1)+\gamma_{0j}f_{nj}(\alpha_j)=0,\end{aligned} \tag{4.14}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_{kj}(\alpha_j) = \alpha_j p_{kj} + q_{kj}, \quad q_{0j}=0. \tag{4.15}$$

If the determining equation (4.11) has the roots α_{1j}, α_{2j} $(\alpha_{1j} > \alpha_{2j})$ such that $\alpha_{1j}-\alpha_{2j} \neq n$, $n=0,1,2$, then by the formulas (4.12)–(4.14) we construct for the equation (4.5) two local linearly independent solutions of the form

$$v_{kj}(t) = (t-a_j)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}_{kj}(t), \quad \widetilde{v}_{kj}(t) = 1+\sum_{n=1}^{\infty}\gamma_{nj}^{k}(t-a_j)^n, \tag{4.16}$$
$$k=1,2, \quad j=1,2,\dots,m.$$

The proof for the convergence of the series (4.16) can be found in [19,22]. The convergence radii of the series $\widetilde{v}_{kj}(\zeta)$ are determined by the distance from the point $t=a_j$ to the nearest of the points $t=a_{j-1}$, $t=a_{j+1}$.

In the case where the equation (4.11) has the roots such that $\alpha_{1j} = \alpha_{2j}$, we differentiate it with respect to α_j, calculate and obtain the second solution,

$$v_{2j}(\zeta) = v_{1j}(\zeta)\ln(\zeta-a_j) + v_{2j}^*(\zeta),$$
$$v_{2j}^*(\zeta) = (\zeta-a_j)^{\alpha_{2j}}\gamma_{0j}\sum_{k=1}^{\infty}\frac{d}{d\alpha_j}[\gamma_{kj}(\alpha_j)]_{\alpha_j=\alpha_{2j}}(\zeta-a_j)^k.$$

The first one $v_{1j}(\zeta)$ is of the form (4.16).

Finally, if the equation (4.11) has the roots such that $\alpha_{1j}-\alpha_{2j}=s$, $s=1,2$, then the first solution in these cases can again be determined by (4.16), and the second one is sought in the form [23]

$$v_j(t) = \gamma_{0j}(t-a_j)^{\alpha_j}[\alpha_j - \alpha_{2j} + \sum_{n=1}^{\infty}\gamma_{nj}(\alpha_j)(t-a_j)^n]. \tag{4.17}$$

To calculate $\gamma_{nj}(\alpha_j)$, in (4.11)–(4.14) we substitute instead of γ_{0j} the product $\gamma_{0j}(\alpha_j-\alpha_{2j})$ and then define successively $\gamma_{nj}(\alpha_j)$, $n=1,2,\dots$. The defined in such a manner $\gamma_{nj}(\alpha_j)$ we substitute in (4.17), differentiate with respect to α_j and then calculate the limit as $\alpha_j \to \alpha_{2j}$. We obtain

$$v_{2j}(t) = \lim_{\alpha_j\to\alpha_{2j}}\gamma_{0j}\left\{(t-a_j)^{\alpha_j}\left[\alpha_j-\alpha_{2j}+\sum_{n=1}^{\infty}\gamma_{nj}(\alpha_j)(t-a_j)^n\right]\ln(t-a_j)+\right.$$

$$+(t-a_j)^{\alpha_j}\left[1+\sum_{n=1}^{\infty}\frac{d}{d\alpha_j}[\gamma_{nj}(\alpha_j)](t-a_j)^n\right]_{\alpha_j\to\alpha_{2j}}\Bigg\}.$$

To define $v_{k\infty}(t)$, $k=1,2$, we act as in defining $v_{kj}(\zeta)$ but in this case the use is made of the equation (4.7) and the representation (4.10).

Consider the case where the contour of the circular polygon contains a cut with the vertex b_j, where $\alpha_{1j}-\alpha_{2j}=2$. For this case, P.Ya. Polubarinova–Kochina has proved that the solution $v_{2j}(\zeta)$ must be with no logarithmic term.She has also obtained an equation connecting the parameters a_j, c_j [2].

A necessary and sufficient conditions for the absense of the logarithmic term in the solution $v_{2j}(\zeta)$ is of the form [9, 11]

$$\gamma_{1j}^1(\alpha_{2j})f_{1j}(\alpha_{2j}+1)+f_{2j}(\alpha_{2j})=0, \tag{4.18}$$

where $\gamma_{1j}^1(\alpha_{2j})$ is defined by (4.12), and $f_{1j}(\alpha_{2j}+1)$, $f_{2j}(\alpha_{2j})$ by (4.15).

After some transformations, (4.18) takes the form $q_{2j}+q_{1j}^2+q_{1j}p_{1j}=0$.

To construct $v_{2j}(\zeta)$ for the cut end, it suffices to calculate $\gamma_{2j}^2(\alpha_{2j})$. The other coefficients $\gamma_{nj}^2(\alpha_{2j})$, $n=1,3,4,5,\dots$ are calculated by the formulas (4.14). The equation (4.13) is fulfilled under the condition (4.18), since $f_{0j}(\alpha_{2j}+2)=f_{0j}(\alpha_{1j})=0$. To define $\gamma_{nj}^2(\alpha_{2j})$ uniquely, we have to solve (4.13) for any $\alpha_j\neq\alpha_{2j}$ with respect to $\gamma_{2j}(\alpha_j)$:

$$\gamma_{2j}(\alpha_j)=\frac{-[\gamma_{1j}(\alpha_j)f_{1j}(\alpha_j+1)+\gamma_{0j}f_{2j}(\alpha_j)]}{f_{0j}(\alpha_j+2)}. \tag{4.19}$$

In (4.19), the numerator and the denominator vanish as $\alpha_j\to\alpha_{2j}$. Hence there is an indeterminacy. Uncovering the indeterminacy by the L'Hospital rule, we get $\gamma_{2j}^2=-0,5[p_{1j}(p_{1j}+2q_{1j})+p_{2j}]$.

5. Local Matrices

From the set of branches of the functions $\exp[\alpha_{kj}\ln(t-a_j)]$ appearing in the local solutions $v_{kj}(\zeta)$ we choose as follows:

$$\exp[\alpha_{kj}\ln(t-a_j)]>0,\quad t>a_j,$$
$$[\exp[\alpha_{kj}\ln(t-a_j)]]^{\pm}=\exp[\pm i\pi\alpha_{kj}]\exp[\alpha_{kj}\ln(a_j-t)],\quad t<a_j;$$
$$\exp[-\alpha_{k\infty}\ln(-t)]^{\pm}>0,\quad -\infty<t<a_1;$$
$$[\exp[-\alpha_{k\infty}\ln(t-a_j)]]^{\pm}=\exp[\pm i\pi(-\alpha_{k\infty})]\exp[-\alpha_{k\infty}\ln t],\quad a_m<t<+\infty.$$

Besides the matrices (4.4), let us introduce the matrices

$$\Theta_j^*(t)=\begin{pmatrix} v_{1j^*(t)} & v_{1j}'^*(t)\\ v_{2j^*(t)} & v_{2j}'^*(t)\end{pmatrix},\quad a_{j-1}<t<a_j,$$

where

$$v_{kj}^*(t)=(a_j-t)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}_{kj}(t),\quad v_{kj}'^*(t)=-(a_j-t)^{\alpha_{kj}}\gamma_{0j}\widetilde{v}_{kj}'^*(t),$$

$$v'_{kj}(t) = \frac{d}{dt}[v_{kj}(t)], \quad \widetilde{v}'^{*}_{kj}(t) = \alpha_{kj} + \sum_{n=1}^{\infty} \gamma^k_{nj}(\alpha_{kj}+n)(t-a_j)^n.$$

Between the matrices $\Theta_j(t)$ and $\Theta^*_j(t)$ there is a connection:

$$\Theta^{\pm}_j(t) = \theta^{\pm}_j \Theta^*_j(t), \ a_{j-1} < t < a_j, \ \Theta^{\pm}_{\infty}(t) = \theta^{\pm}_{\infty}\Theta^*_{\infty}(t), \ a_m < t < +\infty.$$

Here the matrices $\theta^{\pm}_j$ for $\alpha_{1j} - \alpha_{2j} \neq s$, $s = 0, 1, 2$, are defined by

$$\theta^{\pm}_j = \begin{pmatrix} \exp(\pm i\pi\alpha_{1j}) & 0 \\ 0 & \exp(\pm i\pi\alpha_{2j}) \end{pmatrix},$$

while for $\alpha_{1j} - \alpha_{2j} = s$, $s = 0, 1, 2$, by

$$\theta^{\pm}_j = e^{\pm i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ \pm\pi i & 1 \end{pmatrix}.$$

For the cut end $w = b_j$, the matrices $\theta^{\pm}_j$ are defined as follows. If the characteristic numbers are of the form $\alpha_{1j} = 3/2$, $\alpha_{2j} = -1/2$, then $\theta^{\pm}_j = \mp iE$, where E is the unit matrix, and if $\alpha_{1j} = 2$, $\alpha_{2j} = 0$ then $\theta^{\pm}_j = E$.

The elements of the matrix $\theta^*_j(t)$ containing the logarithmic terms are defined by

$$v^*_{2j}(t) = \gamma_{0j}\Big\{(a_j - t)^{\alpha_{2j}}[(t-a_j)^s \widetilde{v}_{1j}(t)\ln(a_j - t) + \widetilde{v}^2_{2j}(t)],$$

$$v'^*_{2j}(t) = -\gamma_{0j}(a_j - t)^{\alpha_{2j}-1}[(a_j - t)^s e^{i\pi s}\widetilde{v}'_{2j}(t)\ln(a_j - t) + \widetilde{v}_{1j}(t)] + \widetilde{v}^2_{2j}(t)\Big\}.$$

In the local solutions $v_{kj}(\zeta)$, $\varphi_{kj}(\zeta)$ there appear the constants γ_{0j}, φ_{0j} which are defined by means of the Liouville formula

$$\gamma_{0j} = \Big\{\prod_{k=1, k\neq j}^{m} |\nu_j|\,|a_j - a_k|^{1-\nu-k}\Big\}^{-1/2}, \quad \varphi_{0j} = \Big\{\prod_{k=1}^{m} |a^*_j - a_k|^{1-\nu_k}\Big\}^{-1/2}.$$

If $\nu_j = 0$, then we take $|\nu_j| = 1$.

6. Construction of the Fundamental Matrix

Construct the matrix

$$\chi(\zeta) = \begin{pmatrix} u_1(\zeta) & u'_1(\zeta) \\ u_2(\zeta) & u'_2(\zeta) \end{pmatrix}, \tag{6.1}$$

where $u_1(\zeta)$ and $u_2(\zeta)$ are linearly independent solutions of (3.6).

The convergence domains of the matrices $\Theta_j(t)$ and $H_j(t)$ have always a common part in which we can write the equalities

$$\Theta^*_j(t) = T^*_j H_j(t), \quad H_j(t) = T_{0j}\Theta_{j-1}(t), \quad a_{j-1} < t < a_j, \tag{6.2}$$

$$\Theta^*_1(t) = T_{-m}H_{-m}(t), \quad H_{-m}(t) = T_{-\infty}\Theta_{\infty}(t), \quad -\infty < t < a_1, \tag{6.3}$$

$$\Theta^*_{\infty}(t) = T_{\infty}\Theta_m(t), \quad a_m < t < +\infty,$$

where T_j^*, T_{0j}, T_{-m}, $T_{-\infty}$, T_∞ are the constant real matrices defined from the equalities (6.2) and (6.3). Note that in these equalities t can be fixed arbitrarily in the domains where both local matrices occurring in the above-mentioned equalities converge.

Define the matrix (6.1) along the t-axis of the plane ζ as follows:

$$\begin{aligned}
&\chi^{\pm}(t) = T\Theta_m^{\pm}(t), \quad \Theta_m^{+}(t) = \Theta_m^{-}(t), \quad a_m < t < +\infty;\\
&\chi^{\pm}(t) = T\theta_m^{+}\Theta_m^{*}(t), \quad a_{m-1} < t < a_m;\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m\Theta_{m-1}(t), \quad T_m = T_m^{*}T_{0m}, \quad a_{m-1} < t < a_m;\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m\cdots T_1\theta_1^{\pm}\Theta_1^{*}(t), \quad -\infty < t < a_1;\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m\cdots\theta_1^{\pm}\Theta_{-m}T_{-\infty}\Theta_\infty(t), \quad -\infty < t < a_1;\\
&\chi^{\pm}(t) = T\theta_m^{\pm}T_m\cdots\theta_\infty^{\pm}T_\infty\Theta_\infty^{\pm}(t), \quad a_m < t < \infty.
\end{aligned}\tag{6.4}$$

The signs $(\pm)$ in the matrices (6.4) denote the limiting values of the matrix $\chi(\zeta)$ in the upper and in the lower half-planes, respectively. The matrix T is defined by (4.3).

7. Solution of the Boundary Value Problem

Direct checking shows that the matrices (6.4) satisfy the equation (4.1). Therefore, the parameters a_j, c_j, $j = 1, 2, \dots, m$, p, q, r, s being chosen appropriately, the same matrices must satisfy the boundary condition (3.5). Indeed, begin our proof with the interval $(a_m, +\infty)$. We have

$$T\Theta_m^{+}(t) = G_m T\Theta_m^{-}(t), \ \Theta_m^{+}(t) = \Theta_m^{-}(t), \ G_m = E, T = \overline{T}, \ a_m < t < +\infty. \tag{7.1}$$

For the interval (a_{m-1}, a_m) we obtain

$$T\theta_m^{+}\Theta_m^{*}(t) = G_{m-1}T\theta_m^{-}\Theta_m^{*}(t), \quad a_{m-1} < t < a_m. \tag{7.2}$$

The equations (7.1) and (7.2) result in

$$(\theta_m^{+})^2 = T^{-1}G_m^{-1}G_{m-1}T \tag{7.3}$$

from which we can see that the matrices $(\theta_m^{+})^2$, $G_m^{-1}G_{m-1}$ are similar.

In a similar way we find the matrix equations for the remaining points $\zeta = a_j$, $j = m-1, m-2, \dots, 2, 1$, $\zeta = \infty$. We have

$$T\theta_m^{+}T_m\theta_{m-1}^{+} = G_{m-2}T\theta_m^{-}T_m\theta_{m-1}^{-}, \tag{7.4}$$

$$T\theta_m^{+}T_m\theta_{m-1}T_{m-1}\theta_{m-2}^{+} = G_{m-3}T\theta_m^{-}T_m\theta_{m-1}^{-}T_{m-1}\theta_{m-2}^{-}, \tag{7.5}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\begin{aligned}
&T\theta_m^{+}T_m\theta_{m-1}^{+}T_{m-1}\theta_{m-2}^{+}T_{m-2}\dots T_1\theta_1^{+} =\\
&= G_{m+1}T\theta_m^{-}T_m\theta_{m-1}^{-}T_{m-1}\theta_{m-2}^{-}T_{m-2}\dots T_1\theta_1^{-},
\end{aligned}\tag{7.6}$$

$$T\theta_m^{+}T_m\theta_{m-1}^{+}T_{m-1}\dots T_{-\infty}\theta_m^{+} = G_m T\theta_m^{-}T_m\theta_{m-1}^{-}T_{m-1}\dots T_{-\infty}\theta_\infty^{-}. \tag{7.7}$$

The matrix equations (7.4)–(7.7) can be written as

$$(\theta_{m-1}^{+})^2 = T_m^{-1}(\theta_m^{+})^{-1}T^{-1}G_{m-1}^{-1}G_{m-2}T\theta_m^{-}T_m. \tag{7.8}$$

Here we make the following remark. In composing the matrix equations, we must take into account that two neighboring circular arcs forming a cut with the end $w = b_j$ belong to one circle.

We rewrite (7.3) as $T\theta_m^{+} = G_{m-1}T\theta_m^{-}$, whence

$$p\exp(i\pi\alpha_{1m}) = \overline{B}_{m-1}p\exp(-i\pi\alpha_{1m}) - iD_{m-1}r\exp(-i\pi\alpha_{1m}), \tag{7.9}$$
$$r\exp(i\pi\alpha_{1m}) = iA_{m-1}p\exp(-i\pi\alpha_{1m}) + B_{m-1}r\exp(-i\pi\alpha_{1m}), \tag{7.10}$$
$$q\exp(i\pi\alpha_{2m}) = \overline{B}_{m-1}q\exp(-i\pi\alpha_{2m}) - iD_{m-1}s\exp(-i\pi\alpha_{2m}), \tag{7.11}$$
$$s\exp(i\pi\alpha_{2m}) = iA_{m-1}q\exp(-i\pi\alpha_{2m}) + B_{m-1}s\exp(-i\pi\alpha_{2m}). \tag{7.12}$$

If we divide the corresponding parts of the equalities (7.9), (7.10) and (7.11), (7.12), then we can see that the ratios p/r and q/s in the interval (a_{m-1}, a_m) satisfy the boundary condition (3.5),

$$\frac{p}{r} = \frac{\overline{B}_{m-1}p/r - iD_{m-1}}{iA_{m-1}p/r + B_{m-1}}, \quad \frac{q}{s} = \frac{\overline{B}_{m-1}q/s - iD_{m-1}}{iA_{m-1}q/s + B_{m-1}}.$$

The coordinates of the points $w = b_m$ and $w = b_m'$ also satisfy this condition; hence

$$p/r = b_m, \quad q/s = b_m', \tag{7.13}$$

where b_m' is the second intersection point of the two neighboring circles.

Take advantage of the remark made at the beginning of Section 1. The origin on the plane w coincides with the point $w = b_m$. Therefore $b_m = 0$ and $b_m' = \infty$. Hence $p = 0$, $s = 0$.

Remind that by b_k, b_k', $k = 1, 2, \dots, m+1$ we have denoted the complex coordinates of the angular points of the circular polygon at which two neighboring circles may intersect, and b_k' is more often exterior to the contour $l(w)$.

Note that if (a_{m-1}, a_m), then for the interval $\nu_m \neq 0$ we can always suppose that

$$G_{m-1} = \begin{pmatrix} \overline{B}_{m-1} & 0 \\ 0 & B_{m-1} \end{pmatrix}.$$

Consider the matrix equation (7.4),

$$T^{+}_{*m\theta_{m-1}} = G_{m-2}\overline{T}_{*m}\theta_{m-1}^{-}, \quad T_{*m} = T\theta_m^{+}T_m. \tag{7.14}$$

From (7.14) there follows the following system of equations:

$$p_{*m}/r_{*m} = b_{m-1}, \quad q_{*m}/s_{*m} = b_{m-1}', \tag{7.15}$$

where p_{*m}, q_{*m}, r_{*m}, s_{*m} are the elements of the matrix T_{*m}.

Taking into account (7.14), we rewrite the equalities (7.15) as

$$\frac{p_* p_m + q_* r_m}{r_* p_m + s_* r_m} = b_{m-1}, \quad \frac{p_* q_m + q_* s_m}{r_* q_m + s_* s_m} = b'_{m-1}, \tag{7.16}$$

where p_*, q_*, r_*, s_* are the elements of the matrix $T_* = T\theta_m^+$.

Bearing in mind (7.13), the equalities (7.16) can be written as

$$\frac{r_* p_m b_m + s_* r_m b'_m}{r_* p_m + s_* r_m} = b_{m-1}, \quad \frac{r_* q_m b_m + s_* s_m b'_m}{r_* q_m + s_* s_m} = b'_{m-1}. \tag{7.17}$$

After simplification, the equalities (7.17) take the form

$$r_* p_m (b_m - b_{m-1}) + s_* r_m (b'_m - b_{m-1}) = 0, \tag{7.18}$$

$$r_* q_m (b_m - b'_{m-1}) + s_* s_m (b'_m - b'_{m-1}) = 0. \tag{7.19}$$

The condition of the compatibility of (7.18) and (7.19) with respect to r_*, s_* is of the form

$$\frac{p_m s_m}{r_m q_m} = \frac{b'_m - b_{m-1}}{b_m - b_{m-1}} \frac{b_m - b'_{m-1}}{b'_m - b'_{m-1}}. \tag{7.20}$$

From the matrix equation (7.5) we get the system of equations:

$$\begin{aligned} \frac{p_{*(m-1)} p_{m-1} + q_{*(m-1)} r_{m-1}}{r_{*(m-1)} p_{m-1} + s_{*(m-1)} r_{m-1}} &= b_{m-2}, \\ \frac{p_{*(m-1)} q_{m-1} + q_{*(m-1)} s_{m-1}}{r_{*(m-1)} q_{m-1} + s_{*(m-1)} s_{m-1}} &= b'_{m-2}, \end{aligned} \tag{7.21}$$

where $p_{*(m-1)}$, $q_{*(m-1)}$, $r_{*(m-1)}$, $s_{*(m-1)}$ are the elements of the matrix $T_{*(m-1)} = T\theta_m^+ T_m \theta_m \theta_{m-1}^+$.

After some transformations, (7.21) can be rewritten in the form

$$r_{*(m-1)} p_{m-1} (b_{m-1} - b_{m-2}) + s_{*(m-1)} r_{m-1} (b'_{m-1} - b_{m-2}) = 0,$$

$$.r_{*(m-1)} q_{m-1} (b_{m-1} - b'_{m-2}) + s'_{*(m-1)} s_{m-1} (b'_{m-1} - b'_{m-2}) = 0.$$

These equalities imply

$$\frac{p_{m-1} s_{m-1}}{r_{m-1} q_{m-1}} = \frac{b'_{m-1} - b_{m-2}}{b_{m-1} - b_{m-2}} \frac{b_{m-1} - b'_{m-2}}{b'_{m-1} - b'_{m-2}}. \tag{7.22}$$

All the matrix equations can be considered analogously.

The equations (7.20) and (7.22) are exactly the invariant anharmonic ratios of the four points of the circle at which it intersects two neighboring circles.

From the matrix equations one can obtain all the required equations with respect to a_k, c_k and to the integration constants p, q, r, s. For every point $\zeta = a_j$ we obtain a system of two equations which are homogeneous with respect to the elements of the matrix T_k. Their conditions of compatibility, for example, for the points $\zeta = a_m$, $\zeta = a_{m-1}$, are of the form (7.20) and

(7.22). These conditions have been obtained under the assumption that $\alpha_{1j}-\alpha_{2j}\neq s$, $s\neq 0,1,2$.

Consider now the case where $\alpha_{1j}-\alpha_{2j}=s$, $s=0,1,2$.

According to the representation (6.4), the unknown matrices $\chi^+(t)$, $\chi^-(t)$ for the interval (a_{j-1},a_j) must satisfy the boundary condition

$$\begin{pmatrix} p_{*j} & q_{*j} \\ r_{*j} & s_{*j} \end{pmatrix} e^{i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ \pi i & 1 \end{pmatrix} =$$
$$= \begin{pmatrix} \overline{B_{j-1}} & -iD_{j-1} \\ iA_{j-1} & B_{j-1} \end{pmatrix} \begin{pmatrix} \overline{p}_{*j} & \overline{q}_{*j} \\ \overline{r}_{*j} & \overline{s}_{*j} \end{pmatrix} e^{-i\pi\alpha_{2j}} \begin{pmatrix} 1 & 0 \\ -\pi i & 1 \end{pmatrix},$$

where p_{*j}, q_{*j}, r_{*j}, s_{*j} are defined by (6.4).

Reasoning as when deducing (7.1)–(7.8), we see that the ratios $\frac{p_{*j}+\pi i q_{*j}}{r_{*j}+\pi i s_{*j}}$, $\frac{q_{*j}}{s_{*j}}$ satisfy the boundary condition (3.5). But the coordinates of the point $w=b_j$ and the coordinates b_{j-1} and b'_{j-1} will also satisfy (3.5). Hence we obtain the system of equations

$$\frac{p_{*j}+\pi i q_{*j}}{r_{*j}+\pi i s_{*j}}=b_j, \quad \frac{q_{*j}}{s_{*j}}=b_j^*, \tag{7.23}$$

where b_j^* is equal either to b_{j-1} or to b'_{j-1}.

The system (7.23) is also homogeneous with respect to the elements of the corresponding matrices T_{*j}, but the compatibility condition this time fails to provide us with the ratios like (7.20) and (7.22).

As is mentioned above, the matrix equations similar to (7.1)–(7.7) can be obtained for all the points $\zeta=a_k$, with the exclusion of those $\zeta=a_n$ to which there correspond the cut ends $w=b_j$ whith $\nu_j=2$. For such points there are the conditions for the absense of logarithmic terms in the solutions $v_{2j}(\zeta)$, for example, the equation (4.18). This gives us one condition for one point; the second equation will be given below.

From the matrix representations $\chi^+(t)$ we define first $u_1^+(t)$, $u_2^+(t)$ and then construct the relation $w^+(t)=u_1^+(t)/u_2^+(t)$.

According to the representation (6.4), let the function (a_j,a_{j+1}) for the interval $w^+(t)$ be defined by $w^+(t)=\frac{A_j^*v_{1j}^+(t)+B_j^*v_{2j}^+(t)}{C_j^*v_{1j}^+(t)+D_j^*v_{2j}(t)}$. If, using this, we calculate the limit as $\zeta\to a_j$, then we get the equation

$$b_j=B_j^*/D_j^*. \tag{7.24}$$

The corresponding equations for the other points $\zeta=a_k$, $k=1,2,3,\ldots,$ $m,m+1$, can be obtained in a similar way.

Consequently, for every point $t=a_j$ we obtain two real equations homogeneous with respect to p_j, q_j, r_js_j, for example, (7.3)–(7.7). For $\nu_j\neq 0,1,2$, from the condition of compatibility of homogeneous equations there follow invariant anharmonic ratios for four points of a circle, for example, (7.20), (7.22). In the case where $\nu_j = 0, 1, 2$, the condition of compatibility of the two equations provides us with well-defined but not anharmonic ratios.

Finally, we can take from every system of two equations one equation and add one more equation of compatibility , i.e., we will have two equations for every point $\zeta = a_j$. The number of equations will be $2(m+1)$ and the number of unknown parameters a_k, c_k, p, q, r, s, $ps - rq = 1$, will be equal to $2m-1$. Hence the number of equations will be greater by three than that of the unknown parameters. This is connected with the fact that the going around the singular points $\zeta = a_k$, $k = 1, 2, \dots, n$ in the positive direction is equivalent to that of the point $\zeta = \infty$ in the negative direction. This provides us with one matrix equation. Therefore any three equations from the obtained system of equations are consequences of the remaining ones.

The appearance of the three additional equations can be explained as in the case of linear polygons.

Having found the system of equations to determine a_k, c_k, p, q, r, s, it is necessary to define the intervals of variation of the parameters c_k, to solve the system with respect to a_k, c_k and finally to determine p, q, r, s. Remind that p_j, q_j, r_j, s_j, $j = 1, 2, \dots, m+1$, depend implicitly on the parameters a_k, c_k, $k = 1, 2, \dots, m$ via the coefficients of the generalized hypergeometric series . The variation intervals of the parameters c_k, $k = 1, 2, \dots, m$, can be defined according to [16].

It is known that the series $v_{kj}(\zeta)$ and $j = 1, 2, \dots, m, m+1$, converge near the points $\zeta = a_j$, and $j = 1, 2, \dots, m, m+1$, respectively. The convergence radii $\varphi_{kj}(\zeta)$ of these series are determined by the distance $a_j^* = (a_j + a_{j+1})/2$ from the point $t = a_j$ (or from the point a_j^*) to the nearest points $\zeta = a_{j-1}$, $\zeta = a_{j+1}$.

The series v_{kj} are entire functions of the parameters c_j, $j = 1, 2, \dots, m$, and converge slowly with respect to ζ. This makes numerical calculations very difficult. As n grows, the coefficients sometimes strongly increase, though their multipliers $(\zeta - a_j)^n$, on the contrary, decrease. Computers are unable to multiply γ_{nj}^k by $(t - a_j)^n$ despite the fact that these series converge. To eliminate this defect, we suggest to write these series in the form of rapidly and uniformly converging functional series.

Consider the structure of recursive formulas (4.12)–(4.14). The sum of the first lower indices in the expressions $\gamma_{(k-n)j} f_{nj}(\alpha + k - n)$ is always equal to k, i.e., to the exponent $(t - a_j)^k$. Instead of the series (4.9), let us consider the functional series of the form

$$v_j(t) = (t - a_j)^{\alpha_j} \widetilde{v}_j(t - a_j), \quad \widetilde{v}_j(t) = \sum_{n=0}^{\infty} = \sum_{n=0}^{\infty} \gamma_{nj}(t - a_j), \quad (7.25)$$

where according to (4.12)–(4.14) γ_{nj} is defined via $\gamma_{1j}, \gamma_{2j}, \dots, \gamma_{(n-1)j}$ while the latters are defined via $f_{kj}(\alpha_j)$, where

$$f_{kj}(t - a_j, \alpha_j) = \alpha_j p_{kj}(t - a_j) + q_{kj}(t - a_j),$$

$$p_{nj}(t-a_j) = \sum_{k=1,k\neq j} (-1)^n(1-\nu_k)\Big(\frac{t-a_j}{a_j-a_k}\Big)^n, \quad p_{oj} = 1-\nu_k,$$

$$q_{nj}(t-a_j) = \sum_{k=1,k\neq j} (-1)^{n-1}c_k\Big(\frac{t-a_j}{a_j-a_k}\Big)^n, \quad q_{oj} = 0, \quad q_{1j} = c_j,$$

$$|t-a_j| < \min\{|a_j-a_{j-1}|, |a_j-a_{j+1}|\}, \tag{7.26}$$

$$\Big|\frac{t-a_j}{a_j-a_k}\Big| < 1, \quad k \neq j. \tag{7.27}$$

We can see from (7.27) that the functional series (7.25) converges in the domain (7.26) more rapidly in comparison with the series (4.8).

The functional series for the point $\zeta = a_{m+1} = \infty$ is constructed analogously. In all the above formulas instead of $v_{kj}(\zeta)$ we will have to substitute the functional series (7.25). It is obvious that the functional series for the ordinary points $t = a_j^*$, $a_j^* = (a_j + a_{j+1})/2$, $j = 1, 2, \ldots, m-1$, will also converge uniformly and rapidly.

8. On a Connection Between the Conditions (2.12) and (2.16)–(2.18)

We write the matrix $\chi(\zeta)$ defined by (4.2) as

$$\chi(\zeta) = T\chi_2(\zeta), \tag{8.1}$$

where the constant matrix T is defined by (4.3), and the matrix $\chi_2(\zeta)$ by

$$\chi_2(\zeta) = \begin{pmatrix} v_1(\zeta) & v_1'(\zeta) \\ v_2(\zeta) & v_2'(\zeta) \end{pmatrix},$$

$v_1(\zeta)$ and $v_2(\zeta)$ being the linearly independent solutions of (3.6) along the t-axis defined by (6.4).

The equality (8.1) implies

$$u_1(\zeta) = pv_1(\zeta) + qv_2(\zeta), \quad u_2(\zeta) = rv_1(\zeta) + sv_2(\zeta). \tag{8.2}$$

The functions $u_1(\zeta)$, $u_2(\zeta)$ are again linearly independent solutions of the equation (3.6) provided $ps - rq \neq 0$, where p, q, r, s are arbitrary complex numbers. Below we will assume that $ps - rq = 1$.

The functions $w_1(\zeta) = v_1(\zeta)/v_2(\zeta)$ and $w(\zeta) = u_1(\zeta)/u_2(\zeta)$ satisfy Schwarz's equation (3.8), where $w_1(\zeta)$ will be its partial and $w(\zeta)$ its general solutions.

Remind also that $w(\zeta) = \omega'(\zeta)/z'(\zeta) = \omega_1(\zeta)/z_1(\zeta) = \omega_2(\zeta)/z_2(\zeta)$, where $\omega_k(s)$, $z_k(\zeta)$, $k = 1, 2$, are defined by (2.15) and (3.3).

Now we present the proof of a theorem proven by us in [13–16]. It can be formulated as follows: if the equality (2.12) holds, then so do the equalities (2.16)–(2.17), and vice versa, (2.16)–(2.17) imply (2.12).

The second part of our theorem is evident, therefore we dwell on proving the first part.

The equality (2.12) with regard for $w(\zeta)=u_1(\zeta)/u_2(\zeta)$ can be rewritten as

$$\frac{u_1(t)}{u_2(t)}=\frac{\overline{B(t)}\,\overline{u_1(t)}-iD(t)\overline{u_2(t)}}{iA(t)\overline{u_1(t)}+B(t)\overline{u_2(t)}},\quad -\infty<t<+\infty.$$

Assume that

$$u_1(t)=\lambda(t)u_1^*(t),\quad u_2(t)=\lambda(t)u_2^*(t),\quad -\infty<t<+\infty, \tag{8.3}$$

where $u_1^*(t)=\overline{B(t)}\,\overline{u_1(t)}-iD(t)\overline{u_2(t)}$, $u_2^*(t)=iA(t)\overline{u_1(t)}+B(t)\overline{u_2(t)}$, $-\infty<t<+\infty$.

If we substitute (8.3) in (3.6), then we obtain

$$\lambda''(t)u_1^*(t)+\lambda'(t)[2[u_1^*(t)]'+p_*(t)u_1^*(t)]=0,\quad -\infty<t<+\infty, \tag{8.4}$$
$$\lambda''(t)u_2^*(t)+\lambda'(t)[2[u_2^*(t)]'+p_*(t)u_2^*(t)]=0,\quad -\infty<t<+\infty. \tag{8.5}$$

Multiplying (8.4) by $u_2^*(t)$ and (8.5) by $u_1^*(t)$ and then subtracting the second equality from the first one, one gets

$$2\lambda'(t)[[u_1^*(t)]'u_2^*(t)-[u_2^*(t)]'u_1^*(t)]=0. \tag{8.6}$$

In the braces of (8.6) there is the Wronskian $w^*[u_1^*(t),u_2^*(t)]\neq 0$, therefore (8.6) implies $\lambda'(t)=0$, $-\infty<t<+\infty$, which yields $\lambda(t)=\text{const}$, $t\in(a_j,a_{j+1})$.

Note that

$$w^*[u_1^*(t),u_2^*(t)]=w^*[\overline{u_1(t)},\overline{u_2(t)}]=w^*[u_1(t),u_2(t)], \tag{8.7}$$

since the equality (2.11) holds.

If for (8.3) we calculate the Wronskian with regard for (8.7), then we obtain $\lambda^2(t)=1$, $t\in(a_j,a_{j+1})$, which in its turn, implies $\lambda(t)=\pm1$, $t\in(a_j,a_{j+1})$.

But the functions $A(t)$, $B(t)$, $D(t)$ are defined uniquely from the conditions (1.1)–(1.2), hence $\lambda(t)$ is also defined uniquely.

9. Definition of the Functions $\omega(\zeta)$, $z(\zeta)$

The function $w^+(t)$ along the real t-axis is defined by $w^+(t)=u_1^+(t)/u_2^+(t)$, $-\infty<t<+\infty$, where $u_1^+(t)$, $u_2^+(t)$ are defined by (6.4).

Given $w^+(t)$, we can find $w(\zeta)$ for all $\mathrm{Im}(\zeta)>0$ by [24, 30]

$$w(\zeta)=\frac{1}{\pi}\int_{-\infty}^{+\infty}w^+(t)\frac{\eta dt}{(t-\xi)^2+\eta^2},\quad \zeta=\xi+i\eta.$$

Note that one can construct a canonical matrix for the problem (2.3) and solve the inhomogeneous boundary value problem (2.3) by using the Cauchy type integral. This has been done in our paper [13]. In the present work, we seek the solution of the inhomogeneous problem (2.3) we seek in a somewhat different way [2].

Multiply the functions $u_1^+(t)$, $u_2^+(t)$ by and $\chi_1^+(t)$, where $\gamma^+(t)$ is defined by (2.23) and $\chi_1^+(t)$ by (3.4).

The matrix $\chi(\zeta)$ defined by (6.1)–(6.10) satisfies the boundary condition (3.5) since the equalities (7.1)–(7.2) are assumed to be fulfilled. This means that the columns of the matrix $\chi(\zeta)$ defined by (6.1)–(6.10) satisfy the boundary condition (3.5).

In order to obtain a sought for solution $\Phi_2(\zeta)$ of the boundary value problem (3.5), we have to take the first column elements of the matrix $\chi(\zeta)$ and construct the vector $\Phi_2(\zeta) = [u_1(\zeta), u_2(\zeta)]$, $\operatorname{Im}(\zeta) \geq 0$.

We have taken the first column elements of the matrix $\chi(\zeta)$ because the ratio $w(\zeta) = u_1(\zeta)/u_2(\zeta)$ provides the general solution of the Schwarz differential equation (3.8), while the ratio $u_1'(\zeta)/u_2'(\zeta)$ does not satisfy Schwarz's equation. This implies that $\omega_2(\zeta) = u_1(\zeta)$, $z_2(\zeta) = u_2(\zeta)$.

The vector $\Phi_1(\zeta) = \chi_1(\zeta)\Phi_2(\zeta)$, where $\chi_1(\zeta)$ is defined by (3.4), satisfies the boundary condition (3.1), and the components of the vector $\Phi_1(\zeta)$ are defined as $\omega_1(\zeta) = \chi_1(\zeta)\omega_2(\zeta)$, $\quad z_1(\zeta) = \chi_1(\zeta)z_2(\zeta)$, $\quad \operatorname{Im}(\zeta) \geq 0$.

The vector $\Phi'(\zeta) = \gamma(\zeta)\Phi_1(\zeta)$, where $\Phi'(\zeta) = d\Phi(\zeta)/d\zeta$, satisfies the boundary condition $\Phi'(t) = A_*^{-1}(t)\overline{A_*(t)}\,\overline{\Phi}'(t)$, $\;-\infty < t < +\infty$, where $\gamma(\zeta)$ is defined by (2.23).

Hence, the components of the vector $\Phi'(\zeta)$, $\omega'(\zeta) = \gamma(\zeta)\chi_1(\zeta)u_1(\zeta)$, $z'(\zeta) = \gamma(\zeta)\chi_1(\zeta)u_2(\zeta)$, $\operatorname{Im}(\zeta) \geq 0$, satisfy the boundary conditions (2.4)–(2.5).

According to [2], we are aware of the behavior of the functions $\omega'(\zeta)$, $z'(\zeta)$ at all singular points $t = e_k$, $k = 1, 2, \ldots, n, n+1$. Therefore the choice of the arguments φ_j, $j = 1, 2, \ldots, n+1$, of the complex numbers $\det A_j(t)$ should be performed with regard for the behaviors of the functions $\omega'(\zeta)$, $z'(\zeta)$ at all singular points. In this way we construct uniquely the function $\omega'(\zeta)$, $z'(\zeta)$. Then we can write

$$d\omega^+(t) = u_1^+(t)\gamma^+(t)\chi_1^+(t)dt, \quad -\infty < t < +\infty, \tag{9.1}$$

$$dz^+(t) = u_2^+(t)\gamma^+(t)\chi_1^+(t)dt, \quad -\infty < t < +\infty. \tag{9.2}$$

Obviously, the functions (9.1) and (9.2) satisfy the boundary conditions (2.4)–(2.5).

Integrating the equalities (9.1)–(9.2) in the intervals $(-\infty, t)$, (e_j, t), $j = 1, 2, \ldots, n$, we obtain

$$\omega^+(t) = \int_{-\infty}^{t} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt + \omega^+(-\infty), \quad -\infty < t < e_1, \tag{9.3}$$

$$z^+(t) = \int_{-\infty}^{t} u_2^+(t)\gamma^+(t)\chi_1^+(t)dt + z^+(-\infty), \quad -\infty < t < e_1, \tag{9.4}$$

$$\omega^+(t) = \int_{e_j}^{t} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt + \omega_j^+(e_j), \; j = 1, 2, \ldots, n, \; e_j < t < e_{j+1}; \tag{9.5}$$

$$z^+(t)=\int_{e_j}^{t} u_2^+(t)\gamma^+(t)\chi_1^+(t)dt+z_j^+(e_j),\ j=1,2,\dots,n,\ e_j<t<e_{j+1},\ (9.6)$$

where $\omega^+(-\infty)$, $z^+(-\infty)$, $\omega^+(e_j)$, $z^+(e_j)$ are the right limits of the corresponding functions at the points $-\infty$, e_j, $j=1,2,\dots,n$.

It is also evident that the functions $\omega^+(t)$, $z^+(t)$ defined by (9.3)–(9.6) satisfy the boundary conditions (2.1)–(2.2).

In (9.3)–(9.6) we can separate the real and the imaginary parts and obtain the expression for the functions $\varphi(t)$, $\psi(t)$, $x(t)$, $y(t)$.

Moreover, taking $t=e_1$ in (9.3)–(9.4) and $t=e_{j+1}$ in (9.5) and (9.6), we get

$$\omega^+(e_1)=\int_{-\infty}^{e_1} u_1^+(t)\gamma^+(t)\chi^+(t)dt+\omega^+(-\infty), \tag{9.7}$$

$$z^+(e_1)=\int_{-\infty}^{e_1} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt+z^+(-\infty), \tag{9.8}$$

$$\omega^+(e_{j+1})=\int_{e_j}^{e_{j+1}} u_1^+(t)\gamma^+(t)\chi_1^+(t)dt+\omega^+(e_j),\quad j=1,2,\dots,n, \tag{9.9}$$

$$z^+(e_{j+1})=\int_{e_j}^{e_{j+1}} u_2^+(t)\gamma^+(t)\chi_1^+(t)dt+z^+(e_j),\quad j=1,2,\dots,n, \tag{9.10}$$

where $\omega^+(e_{j+1})$, $z^+(e_{j+1})$ are the left limits of the function $\omega^+(t)$, $z^+(t)$ at the point $t=e_{j+1}$.

In (9.3)–(9.6) the integrands are supposed to be integrable at the left ends of the intervals. If it is not the case, then we can take as the lower limits either the right end or an interior point of the corresponding interval.

For the unknown parameters a_j, c_j appearing in (3.6), we have obtained a system of higher transcendental equations, e.g., the equation (7.24). As to the parameters $t=e_j$ not coinciding with the parameters $t=a_j$ and which the functions $\gamma(\zeta)$ and $\chi_1(\zeta)$ depend on, and the parameter Q connected with the discharge of the fluid, we have obtained the system (9.7)–(9.10) for their determination.

Having found all the unknown parameters which the functions $u_1^+(t)$, $u_2^+(t)$, $\gamma^+(t)$, $\chi_1^+(t)$ depend on, by (9.3)–(9.6) we can determine the equations of the unknown parts of the boundary of the domains $s(z)$, $s(\omega)$, $s(w)$ as well as other geometric and mechanical parameters of the flow of the fluid.

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