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ON EXISTENCE OF SOLUTIONS OF THE PERIODIC BOUNDARY VALUE PROBLEM FOR NONLINEAR SYSTEM OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Consider the ω-periodic problem for the nonlinear system

$$dx(t) = dA(t) \cdot f(t, x(t)), \tag{1}$$

where ω is a positive number, $A(t) = A^{(1)}(t) - A^{(2)}(t)$, $A(t) = (a_{ik}(t))_{i,k=1}^{n}$, $A^{(m)} = (a_{ik}^{(m)})_{i,k=1}^{n} \in \mathrm{BV}_{\omega}(\mathbb{R}, \mathbb{R}^{n\times n})$ $(m=1,2)$ are nondecreasing on $[0,\omega]$, $f(t,x)=(f_k(t,x))_{k=1}^{n}$, the functions $f_k : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ $(k=1,\ldots,n)$ are periodic with respect to the first variable by period ω, i.e., $f_k(t+\omega,x) \equiv f_k(t,x)$ $\quad(k=1,\ldots,n)$, and their restriction on $[0,\omega]$ belong to $K([0,\omega]\times\mathbb{R}^n,\mathbb{R};a_{ik})$ for every $i\in\{1,\ldots,n\}$.

The following notation and definitions will be used: $\mathbb{R} =]-\infty,+\infty[$, $\mathbb{R}_+ = [0,+\infty[$, $[a,b]$ $(a,b\in\mathbb{R})$ is a closed segment, $\mathbb{R}^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ik})_{i,k=1}^{n,m}$ with the norm

$$\|X\| = \max_{k=1,\ldots,m} \sum_{i=1}^{n} |x_{ik}|;$$

$$\mathbb{R}_+^{n\times m} = \{(x_{ik})_{i,k=1}^{n,m} : x_{ik} \geq 0 \ \ (i=1,\ldots,n;\ k=1,\ldots,m)\};$$

$\mathbb{R}^n = \mathbb{R}^{n\times 1}$ is the space of all real column n-vectors $x = (x_i)_{i=1}^{n}$; $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$; δ_{ij} is the Kroneker symbol, i.e., $\delta_{ij}=1$ if $i=j$, $\delta_{ij}=0$ if $i\neq j$.

$\mathrm{BV}([a,b],D)$, where $D\subset\mathbb{R}^{n\times m}$, is the set of all matrix-functions $X = (x_{ik})_{i,k=1}^{n,m} : [a,b]\to\mathbb{R}^{n\times m}$ such that every its component x_{ik} has bounded total variation on $[a,b]$.

$s_k : \mathrm{BV}([a,b],\mathbb{R}) \to \mathrm{BV}([a,b],\mathbb{R})$ $(k=0,1,2)$ are the operators defined by

$$s_1(x)(a) = s_2(x)(a) = 0, \quad s_1(x)(t) = \sum_{a<\tau\leq t} d_1x(\tau),$$

$$s_2(x)(t) = \sum_{a\leq\tau<t} d_2x(\tau) \quad \text{for} \quad t\in]a,b]; \quad s_0(x)(t) \equiv x(t) - s_1(x)(t) - s_2(x)(t).$$

$\mathrm{BV}_{\omega}(\mathbb{R},\mathbb{R}^{n\times m})$ is the set of all matrix-functions $X:\mathbb{R}\to\mathbb{R}^{n\times m}$ such that $X(t+\omega) = X(t)+X(\omega)$ for $t\in\mathbb{R}$, and its restriction on $[0,\omega]$ belongs to $\mathrm{BV}([0,\omega],\mathbb{R}^{n\times m})$; $X(t-)$ and $X(t+)$ are the left and the right limits of X at the point $t\in\mathbb{R}$; $d_1X(t) = X(t) - X(t-)$, $d_2X(t) = X(t+) - X(t)$.

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If $x \in \mathrm{BV}([0,\omega],\mathbb{R})$, $1+(-1)^j d_j x(t) \neq 0$ for $t \in [0,\omega]$ $(j=1,2)$ and $\lambda(x)(\omega) \neq 1$, where

$$\lambda(x)(t) = \exp\big(s_0(x)(t)\big) \prod_{0\leq\tau<t} \big(1+d_2x(\tau)\big) \Big/ \prod_{0<\tau\leq t} \big(1-d_1x(\tau)\big), \tag{2}$$

then

$$g_0(x)(t,\tau) = \begin{cases} (1-\lambda(x)(\omega))^{-1}\lambda(x)(\omega)\cdot\lambda(x)(t)[\lambda(x)(\tau)]^{-1} & \text{for } 0\leq t\leq\tau\leq\omega, \\ (1-\lambda(x)(\omega))^{-1}\cdot\lambda(x)(t)[\lambda(x)(\tau)]^{-1} & \text{for } 0\leq\tau<t\leq\omega; \end{cases} \tag{3}$$

$$g_j(x)(t,\tau) = (1+(-1)^j d_j x(\tau))^{-1} g_0(x)(t,\tau) \text{ for } t\neq\tau,\ \ t,\tau\in[0,\omega]\ \ (j=1,2) \tag{4}$$

and

$$g_j(x)(t,t) = (1+(-1)^j d_j x(t))^{-1}\cdot\lambda^{j-2}(x)(\omega) g_0(x)(t,t) \text{ for } t\in[0,\omega]\ (j=1,2). \tag{5}$$

If $g:[a,b]\to\mathbb{R}$ is a nondecreasing function, $x:[a,b]\to\mathbb{R}$ and $a\leq s<t\leq b$, then

$$\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)\,dg_1(\tau) - \int_{]s,t[} x(\tau)dg_2(\tau) + \sum_{s<\tau\leq t} x(\tau)d_1\,g(\tau) - \sum_{s\leq\tau<t} x(\tau)d_2\,g(\tau),$$

where $g_k:[a,b]\to\mathbb{R}$ $(k=1,2)$ are nondecreasing functions such that

$$g_1(t)-g_2(t)\equiv s_0(g)(t)$$

and $\int_{]s,t[} x(\tau)\,dg_k(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $]s,t[$ with respect to the measure $\mu(g_k)$, corresponding to the function g_k (if $s=t$, then $\int_s^t x(\tau)dg(\tau)=0$); $L([a,b],\mathbb{R};g)$ is the set of all $\mu(g_k)$-measurable (i.e., measurable with respect to the measures $\mu(g_1)$ and $\mu(g_2)$) functions $x:[a,b]\to\mathbb{R}$ such that $\int_a^b |x(t)|dg(t)<+\infty$.

A matrix-function is said to be nondecreasing if every its components are such.

If $G=(g_{ik})_{i,k=1}^{l,n}:[a,b]\to\mathbb{R}^{l\times n}$ is a nondecreasing matrix-function and $D\subset\mathbb{R}^{n\times m}$, then $L([a,b],D;G)$ is the set of all matrix-functions $X=(x_{kj})_{k,j=1}^{n,m}:[a,b]\to D$ such that $x_{kj}\in L([a,b],\mathbb{R};g_{ik})$ $(i=1,\dots,l;\ k=1,\dots,n;\ j=1,\dots,m)$;

$$\int_s^t dG(\tau)\cdot X(\tau) = \Big(\sum_{k=1}^n \int_s^t x_{kj}(\tau)\,dg_{ik}(\tau)\Big)_{i,j=1}^{l,m} \quad \text{for} \quad a\leq s\leq t\leq b.$$

If $D_1\subset\mathbb{R}^n$ and $D_2\subset\mathbb{R}^{n\times m}$, then $K\big([a,b]\times D_1,D_2;G\big)$ is the Caratheodory class, i.e., the set of all mappings $F=(f_{kj})_{k,j=1}^{n,m}:[a,b]\times D_1\to D_2$ such that for each $i\in\{1,\dots,l\}$, $j\in\{1,\dots,m\}$, and $k\in\{1,\dots,n\}$: (a) the function $f_{kj}(\cdot,x):[a,b]\to D_2$ is $\mu(g_{ik})$-measurable for every $x\in D_1$; (b) the function $f_{kj}(t,\cdot):D_1\to D_2$ is continuous for $\mu(g_{ik})$-almost all $t\in[a,b]$, and

$$\sup\big\{|f_{kj}(\cdot,x)| : x\in D_0\big\}\in L\big([a,b],\mathbb{R};g_{ik}\big)$$

for every compact $D_0\subset D_1$.

If $G_j:[a,b]\to\mathbb{R}^{l\times n}$ $(j=1,2)$ are nondecreasing matrix-functions, $G=G_1-G_2$ and $X:[a,b]\to\mathbb{R}^{n\times m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \int_s^t d\,G_1(\tau)\cdot X(\tau) - \int_s^t dG_2(\tau)\cdot X(\tau) \text{ for } a\leq s\leq t\leq b.$$

Inequalities between both vectors and matrices are understood componentwise.

A vector-function $x : \mathbb{R} \to \mathbb{R}^n$ is said to be a solution of the system (1) (of the system os generalized ordinary differential inequalities $dx(t) \leq dA(t) \cdot f(t, x(t))$ if its restriction on $[s, t]$ belongs to $\mathrm{BV}([s, t], \mathbb{R}^n)$) and

$$x(t) - x(s) - \int_s^t dA(\tau) \cdot f(\tau, x(t)) = 0 \ \ (\leq 0) \ \text{ for } \ s < t \ (s, t \in \mathbb{R}).$$

If $\sigma_i \in \{-1, 1\}$, then $t_i = \frac{1-\sigma_i}{2}\omega$, $j_i = \frac{3+\sigma_i}{2}$, and $I_\omega^{\sigma_i} = [0, \omega] \setminus \{t_i\}$.

Definition 1. Let $\sigma_i \in \{-1, 1\}$ $(i = 1, \ldots, n)$. We shall say that a matrix-function $C = (c_{il})_{i,l=1}^n : \mathbb{R} \to \mathbb{R}^{n \times n}$ belongs to the set $U_\omega^{\sigma_1, \ldots, \sigma_n}$ if $c_{il} \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R})$ $(i, l = 1, \ldots, n)$, the functions c_{il} $(i \neq l;\ i, l = 1, \ldots, n)$ are nondecreasing on $[0, \omega]$ and continuous at the point t_i,

$$d_j c_{ii}(t_i) \geq 0 \quad (j = 1, 2; \ \ i = 1, \ldots, n), \tag{6}$$

$$\|d_j C(t)\| < 1 \quad \text{for} \quad t \in [0, \omega] \ \ (j = 1, 2) \tag{7}$$

and the system

$$\sigma_i dy_i(t) \leq \sum_{l=1}^n y_l(t) dc_{il}(t) \quad (i = 1, \ldots, n)$$

has no nontrivial nonnegative ω-periodic solution.

Theorem 1. *Let the conditions*

$$(-1)^{m+1} \sigma_i f_k(t, x_1, \ldots, x_n) \operatorname{sign} x_i \leq \sum_{l=1}^n p_{mikl}(t)|x_l| + q_k(t)$$

$$\text{for} \quad \mu(a_{ik}^{(m)})\text{- almost all} \quad t \in I_\omega^{\sigma_i} \quad (i, k = 1, \ldots, n) \tag{8}$$

and

$$\Big[(-1)^{m+1} \sigma_i f_k(t_i, x_1, \ldots, x_n) \operatorname{sign} x_i - \sum_{l=1}^n \alpha_{mikjl}|x_l| - q_k(t_i)\Big] d_{j_i} a_{ik}^{(m)}(t_i) \leq 0$$

$$(j = 1, 2; \ \ i, k = 1, \ldots, n) \tag{9}$$

be fulfilled on $\mathbb{R}^n$ *for every* $m \in \{1, 2\}$, *where* $\sigma_i \in \{-1, 1\}$, $\alpha_{mikl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0, \omega], \mathbb{R}^{n \times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0, \omega], \mathbb{R}_+^n; A)$. *Let, moreover,*

$$\sum_{m=1}^2 \sum_{k=1}^n \int_s^t p_{mikl}(\tau) da_{ik}^{(m)}(\tau) \leq c_{il}(t) - c_{il}(s) \ \ \text{for } s < t;\ s, t \in I_\omega^{\sigma_i} \ (i, l = 1, \ldots, n) \tag{10}$$

and

$$\sum_{m=1}^2 \sum_{k=1}^n \alpha_{mikl} d_{j_i} a_{ik}^{(m)}(t_i) \leq \delta_{il} d_{j_i} c_{ii}(t_i) \ \ (i, l = 1, \ldots, n), \tag{11}$$

where $C = (c_{il})_{i,l=1}^n \in U_\omega^{\sigma_1, \ldots, \sigma_n}$. *Then the system* (1) *has at least one* ω*-periodic solution.*

Corollary 1. *Let the conditions* (6)–(11) *and*

$$\sigma_i \lambda(\sigma_i c_{ii})(\omega) < 1 \quad (i = 1, \dots, n)$$

be fulfilled on $\mathbb{R}^n$ *for every* $m \in \{1, 2\}$, *where* $\sigma_i \in \{-1, 1\}$, $\alpha_{mikl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0,\omega], \mathbb{R}^{n\times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0,\omega], \mathbb{R}_+^n; A)$, $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$, *the functions* c_{il} $(i \neq l)$ *are nondecreasing on* $[0,\omega]$ *and continuous at the point* t_i, *the function* $\lambda(\sigma_i c_{ii})$ *is defined by* (2). *Let, moreover, the module of every characteristic value of the matrix* $S = (s_{il})_{i,l=1}^n$ *be less than* 1, *where*

$$s_{ii} = 0, \quad s_{il} = \sup\left\{ \sum_{j=0}^{2} \int_0^\omega \sigma_i g_j(\sigma_i c_{ii})(t,\tau) ds_j(c_{il})(\tau) : t \in [0,\omega] \right\}$$
$$(i \neq l;\ i, l = 1, \dots, n),$$

g_j $(j = 0, 1, 2)$ *are operators defined by* (3)–(5). *Then the conclusion of Theorem* 1 *is true.*

Corollary 2. *Let the conditions* (8)–(11),

$$c_{il}(t) = \eta_{il}\alpha_i(t) \quad \text{for} \quad t \in \mathbb{R} \quad (i, l = 1, \dots, n)$$

and

$$\left(|\eta_{ii}| + \sum_{l\neq i, l=1}^{n} \eta_{il}\right) d_j\alpha_i(t) < 1 \quad \text{for} \quad t \in [0,\omega] \quad (j = 1, 2;\ i = 1, \dots, n)$$

be fulfilled on $\mathbb{R}^n$ *for every* $m \in \{1, 2\}$, *where* $\sigma_i \in \{-1, 1\}$, $\alpha_{mikl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0,\omega], \mathbb{R}^{n\times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0,\omega], \mathbb{R}_+^n; A)$, $\eta_{ii} \in \mathbb{R}$, $\eta_{il} \in \mathbb{R}_+$ $(i \neq l)$, $\alpha_i \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R})$ *is nondecreasing on* $[0,\omega]$ *and continuous at the point* t_i, $\alpha_i(\omega) \neq 0$. *Let, moreover, the real part of every characteristic value of the matrix* $\mathcal{H} = (\eta_{il})_{i,l=1}^n$ *be negative. Then the conclusion of Theorem* 1 *is true.*

Corollary 3. *Let the conditions* (6), (7), *and* (8)–(11) *be fulfilled on* $\mathbb{R}^n$ *for every* $m \in \{1, 2\}$, *where* $\sigma_1 = \cdots = \sigma_n = \sigma_0$, $\sigma_0 \in \{-1, 1\}$, $\alpha_{mikl} \in \mathbb{R}$, $(p_{mikl})_{k,l=1}^n \in L([0,\omega], \mathbb{R}^{n\times n}; A^{(m)})$, $q = (q_k)_{k=1}^n \in L([0,\omega], \mathbb{R}_+^n; A)$, $C = (c_{il})_{i,l=1}^n \in \mathrm{BV}_\omega(\mathbb{R}, \mathbb{R}^{n\times n})$, *the functions* c_{il} $(i \neq l)$ *are nondecreasing on* $[0,\omega]$ *and continuous at the point* t_i. *Let, moreover, the module of every multiplicator of the system* $dy(t) = dC_{\sigma_0}(t) \cdot y(t)$, *where* $C_{\sigma_0}(t) = \sigma_0 C(\sigma_0 t + \frac{1-\sigma_0}{2}\omega)$, *be less than 1. Then the conclusion of Theorem* 1 *is true.*

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