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TWO-DIMENSIONAL PROBLEMS OF ELASTICITY OF ANISOTROPIC BODIES

Dedicated to the memory of my parents Natasha and Onofre Basheleishvili, my wife Lili Basheleishvili and my dear teacher Victor Kupradze

Abstract. The paper is devoted to systematic application of the potential method and the theory of integral equations to the two-dimensional problems of statics of an anisotropic elastic body. Chapter I deals with the Fredholm integral equations of second kind which are composed for all basic boundary value problems (except the mixed ones). Theorems on the existence and uniqueness of solution are proved. For the mixed problem, a singular integral equation with discontinuous coefficients is obtained; its investigation is carried out by the same method as in the isotropic case.

Chapter II is wholly concerned with applications of integral equations constructed in Chapter I. Effective solutions of integral equations for concrete regions are given. By means of these solutions, the solutions of boundary value problems are presented by absolutely and uniformly convergent series or in quadratures. Solutions of many specific problems which have not been solved by other methods, are given here.

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რეზიუმე. ნაშრომში გადმოცემულია პოტენციალთა მეთოდისა და ინტეგრალურ განტოლებათა თეორიის სისტემატური გამოყენება ანიზოტროპული დრეკადი ტანის სტატიკის ორგანზომილებიანი ამოცანებისათვის. პირველ თავში ყველა სასაზღვრო ამოცანისათვის (შერეულის გარდა) შედგენილია ფრედჰოლმის მეორე გვარის ინტეგრალური განტოლებები და მოცემულია ამონახსნთა არსებობისა და ერთადერთობის თეორემების დამტკიცება. შერეული ამოცანებისათვის მიღებულია წყვეტილკოეფიციენტებიანი სინგულარული ინტეგრალური განტოლება, რომლის გამოკვლევა ხდება იზოტროპული დრეკადი ტანის შემთხვევის მსგავსად.

ნაშრომის მეორე თავი მთლიანად ეძღვნება პირველ თავში აგებულ ინტეგრალურ განტოლებათა გამოყენებას. აქ მოცემულია ინტეგრალურ განტოლებათა ეფექტური ამონახსნები მრავალრიცხოვანი კონკრეტული არეებისათვის და მათი გამოყენებით სასაზღვრო ამოცანების ამონახსნები ჩაწერილია აბსოლუტურად და თანაბრად კრებადი მწკრივების ან კვადრატურების სახით. მოცემულია მრავალი ისეთი კონკრეტული ამოცანის ამონახსნები, რომელიც ჯერჯერობით არ არის ამოხსნილი სხვა მეთოდებით.

Introduction

In recent years, the classical method of potential theory and the theory of integral equations are extensively developed and widely used in various directions of the elasticity theory.

The first fundamental works concerned with the potential method and the theory of integral equations and aimed at the investigation of the basic boundary value problems of statics of an isotropic elastic body appeared at the beginning of our century. In this direction the works of Fredholm, Lauricella, etc. are well-known. The detailed survey of the above-mentined method in the problems of elasticity can be found in the work of Burchuladze and Gegelia [19].

In the present monograph, the potential method and the theory of integral equations are systematically developed and applied to the investigation of two-dimensional problems of statics of an anisotropic elastic body.

A great merit in the study of two-dimensional boundary value problems of an anisotropic elastic body is due to the excellent works of Lekhnitskiĭ [30], [31], Mikhlin [38], Savin [43], Sherman [44], [46], [47] and others. They, in general, generalized and applied the methods of the theory of functions of a complex variable developed by Kolosov and Muskhelishvili to an isotropic elastic body (see [40]).

This work is based, with a few exceptions, on the results of the author's investigations. It generalizes the results obtained by Kupradze and Basheleishvili [25], [26] for two-dimensional boundary value problems of statics of an orthotropic elastic body.

We consider all basic and some two-dimensional boundary value problems for the equations of statics of an anisotropic elastic body. For all the above-mentioned problems (except the mixed ones), we construct Fredholm type integral equations. Using these equations and the potential method, we prove the existence and uniqueness of a solution for boundary value problems both for simply and multiply connected regions. For the mixed problem we obtained a singular integral equation of normal type with discontinuous coefficients; its investigation is carried out just in the same way as for an isotropic elastic body.

In Chapter I we introduce four new constants which are expressed through the roots of the characteristic equation of the fourth degree and Hooke's coefficients. These constants, very important in the elasticity theory of an anisotropic body, enable one to state in a comparatively simple form general questions dealing with the existence and uniqueness of solution of all the above-mentioned problems. Moreover, these constants make it possible to obtain explicit solutions in the most compact form, as well as to find explicit solutions for those regions for which they were not available by other methods.

The four above-mentioned constants allow one to formulate the generalized Cauchy–Riemann conditions. This approach was unknown even in the case of an isotropic elastic body. Using these constants, we can write formulas for a general representation of the displacement and stress vectors. In the present monograph, these representations are given for non-homogeneous equations of statics of an anisotropic elastic body.

The results presented in Chapters I and II are valid both for a thin anisotropic elastic plate and for a piecewise homogeneous anisotropic elastic body. In view of the limitedness of space, we do not set forth these results here.

CHAPTER I
INVESTIGATION OF TWO-DIMENSIONAL PROBLEMS OF STATICS OF AN ANISOTROPIC ELASTIC BODY

In this chapter, the fundamental and some other matrices of singular solutions are constructed for the equations of statics of an anisotropic elastic body. Using these matrices, the potentials are composed and their fundamental properties near the boundary are studied. All two-dimensional problems (except the mixed ones) are reduced to Fredholm integral equations of second kind and the proofs of the existence and uniqueness of their solutions are given. For the mixed problem, we obtained a singular integral equation with discontinuous coefficients; its investigation is carried out by the same methods as for an isotropic elastic body.

Four constants which are of great importance in considering many problems of the theory of anisotropic elastic body are introduced. In particular, using these constants, the generalized Cauchy–Riemann conditions are written down, some new boundary value problems are formulated, relations between the roots of a characteristic equation of the fourth degree and Hooke's coefficients are found. The last point is significant for establishing some inequalities which guarantee positive definiteness of the so-called pseudo-energy, which in its turn plays an important part in investigating the uniqueness of solution of boundary value problems.

1. Basic Equations and Boundary Value Problems

We say that a body is subject to a plane deformation parallel to the plane $0x_1x_2$, if the third component of the displacement vector $u = (u_1, u_2, u_3)$ equals zero, and the components u_1, u_2 depend only on x_1, x_2 but not on x_3. In this case, Hooke's law for an anisotropic elastic body can be written as follows (see [30], [9]):

$$\begin{aligned} \sigma_{x_1} &= A_{11}\varepsilon_{x_1} + A_{12}\varepsilon_{x_2} + A_{13}\varepsilon_{x_1x_2}, \\ \sigma_{x_2} &= A_{12}\varepsilon_{x_1} + A_{22}\varepsilon_{x_2} + A_{23}\varepsilon_{x_1x_2}, \\ \tau_{x_1x_2} &= A_{13}\varepsilon_{x_1} + A_{23}\varepsilon_{x_2} + A_{33}\varepsilon_{x_1x_2}, \end{aligned} \tag{1.1}$$

where $A_{11}, A_{12}, \ldots, A_{33}$ are Hooke's coefficients, σ_{x_1}, σ_{x_2}, $\tau_{x_1x_2}$ are the components of the stress tensor and ε_{x_1}, ε_{x_2}, $\varepsilon_{x_1x_2}$ are the components of the deformation tensor. Between the components of the deformation tensor and the displacement vector $u = (u_1, u_2)$ there exist the relations

$$\varepsilon_{x_1} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{x_2} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{x_1x_2} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}. \tag{1.2}$$

The basic equations of statics of an anisotropic elastic body in the case of plane deformation have the form

$$\frac{\partial \sigma_{x_1}}{\partial x_1} + \frac{\partial \tau_{x_1x_2}}{\partial x_2} + F_1 = 0, \quad \frac{\partial \tau_{x_1x_2}}{\partial x_1} + \frac{\partial \sigma_{x_2}}{\partial x_2} + F_2 = 0, \tag{1.3}$$

where $F=(F_1,F_2)$ is the body force. If body forces are absent, then the basic homogeneous equations of statics take the form

$$\frac{\partial\sigma_{x_1}}{\partial x_1}+\frac{\partial\tau_{x_1x_2}}{\partial x_2}=0,\quad \frac{\partial\tau_{x_1x_2}}{\partial x_1}+\frac{\partial\sigma_{x_2}}{\partial x_2}=0. \tag{1.4}$$

Taking into account (1.1) and (1.2), the equations (1.3) can be written in terms of the components of the displacement vector as follows:

$$C_{11}(\partial x)u_1+C_{12}(\partial x)u_2+F_1=0,\ \ C_{12}(\partial x)u_1+C_{22}(\partial x)u_2+F_2=0, \tag{1.5}$$

where $C_{11}(\partial x)$, $C_{12}(\partial x)$, $C_{22}(\partial x)$ are the components of the matrix

$$C(\partial x)=\begin{Vmatrix} C_{11}(\partial x), & C_{12}(\partial x)\\ C_{12}(\partial x), & C_{22}(\partial x)\end{Vmatrix}; \tag{1.6}$$

they are of the form

$$\begin{aligned} C_{11}(\partial x)&=A_{11}\frac{\partial^2}{\partial x_1^2}+2A_{13}\frac{\partial^2}{\partial x_1\partial x_2}+A_{33}\frac{\partial^2}{\partial x_2^2},\\ C_{12}(\partial x)&=A_{13}\frac{\partial^2}{\partial x_1^2}+(A_{12}+A_{33})\frac{\partial^2}{\partial x_1\partial x_2}+A_{23}\frac{\partial^2}{\partial x_2^2},\\ C_{22}(\partial x)&=A_{33}\frac{\partial^2}{\partial x_1^2}+2A_{23}\frac{\partial^2}{\partial x_1\partial x_2}+A_{22}\frac{\partial^2}{\partial x_2^2}.\end{aligned} \tag{1.7}$$

The equations (1.5) can be written in the matrix-vector form:

$$C(\partial x)u(x)+F(x)=0, \tag{1.8}$$

and the corresponding homogeneous equation is

$$C(\partial x)u(x)=0. \tag{1.9}$$

The equation (1.8) is obtained in the case where any point of the body possesses a plane with the property that any two directions symmetrical with respect to that plane are equivalent with regard to elastic properties. In this case the number of independent elastic constants is equal to 13, and the above introduced coefficients $A_{kj}=A_{jk}$ $(k,j=1,2,3)$ are expressed via these constants [30].

If the body possesses three planes of elastic symmetry, then

$$A_{13}=A_{23}=0. \tag{1.10}$$

In the case of plane symmetry, i.e., when the body is isotropic, we have

$$A_{11}=A_{22}=\lambda+2\mu,\quad A_{12}=\lambda,\quad A_{33}=\mu,\quad A_{13}=A_{23}=0, \tag{1.11}$$

where λ and μ are the Lamé constants.

Let us now introduce the functions

$$\begin{aligned} \sigma_{11} &= \sigma_{x_1} + \varkappa \frac{\partial u_2}{\partial x_2}, \quad \sigma_{12} = \tau_{x_1x_2} - \varkappa \frac{\partial u_2}{\partial x_1}, \\ \sigma_{21} &= \tau_{x_1x_2} - \varkappa \frac{\partial u_1}{\partial x_2}, \quad \sigma_{22} = \sigma_{x_2} + \varkappa \frac{\partial u_1}{\partial x_1}, \end{aligned} \tag{1.12}$$

where σ_{x_1}, σ_{x_2}, $\tau_{x_1x_2}$ are the stress tensor components and $\varkappa$ is an arbitrary real number. It is easily seen that the equations (1.3) are equivalent to the system of equations

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + F_1 = 0, \quad \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + F_2 = 0. \tag{1.13}$$

The equations (1.4) can be rewritten analogously.

Remark. (1.12) involves only one arbitrary constant $\varkappa$. Earlier (see [11], [24]) for this purpose we used four arbitrary constants.

It should be noted that the idea of introducing an arbitrary constant in the problems of the theory of elasticity is due to the author [1].

Introduce the notation

$$\Delta = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}. \tag{1.14}$$

Since $\Delta \neq 0$, from (1.1) we obtain

$$\begin{aligned} \varepsilon_{x_1} &= a_{11}\sigma_{x_1} + a_{12}\sigma_{x_2} + a_{13}\tau_{x_1x_2}, \\ \varepsilon_{x_2} &= a_{12}\sigma_{x_1} + a_{22}\sigma_{x_2} + a_{23}\tau_{x_1x_2}, \\ \varepsilon_{x_1x_2} &= a_{13}\sigma_{x_1} + a_{23}\sigma_{x_2} + a_{33}\tau_{x_1x_2}. \end{aligned} \tag{1.15}$$

Between the coefficients A_{kj} and a_{kj} $(k, j = 1, 2, 3)$ there are the relations:

$$\begin{aligned} a_{11} &= \frac{A_{22}A_{33} - A_{23}^2}{\Delta}, \quad a_{22} = \frac{A_{11}A_{33} - A_{13}^2}{\Delta}, \\ a_{33} &= \frac{A_{11}A_{22} - A_{12}^2}{\Delta}, \quad a_{12} = \frac{A_{13}A_{23} - A_{12}A_{33}}{\Delta}, \\ a_{13} &= \frac{A_{12}A_{23} - A_{13}A_{22}}{\Delta}, \quad a_{23} = \frac{A_{12}A_{13} - A_{11}A_{23}}{\Delta}. \end{aligned} \tag{1.16}$$

For an orthotropic elastic body we have

$$\begin{aligned} a_{11} &= \frac{A_{22}}{A_{11}A_{22} - A_{12}^2}, \quad a_{22} = \frac{A_{11}}{A_{11}A_{22} - A_{12}^2}, \quad a_{33} = \frac{1}{A_{33}}, \\ a_{12} &= -\frac{A_{12}}{A_{11}A_{22} - A_{12}^2}, \quad a_{13} = a_{23} = 0, \end{aligned} \tag{1.17}$$

and for an isotropic elastic body it holds

$$\begin{aligned} &a_{11} = a_{22} = \frac{\lambda + 2\mu}{4\mu(\lambda+\mu)}, \quad a_{12} = -\frac{\lambda}{4\mu(\lambda+\mu)} \\ &a_{33} = \frac{1}{\mu}, \quad a_{13} = a_{23} = 0. \end{aligned} \tag{1.18}$$

Write now the expressions for the components of the stress vector which acts on an element of the arc with the normal $n = (n_1, n_2)$; denoting the stress vector by Tu, we obtain

$$(Tu)_1 = \sigma_{x_1} n_1 + \tau_{x_1x_2} n_2, \quad (Tu)_2 = \tau_{x_1x_2} n_1 + \sigma_{x_2} n_2, \tag{1.19}$$

where

$$\begin{aligned} T \equiv{}& T(\partial_x, n) = \|T_{kj}(\partial_x, n)\|_{2\times 2}, \\ T_{11}(\partial_x, n) ={}& n_1\Big(A_{11}\frac{\partial}{\partial x_1} + A_{13}\frac{\partial}{\partial x_2}\Big) + n_2\Big(A_{13}\frac{\partial}{\partial x_1} + A_{33}\frac{\partial}{\partial x_2}\Big), \\ T_{12}(\partial_x, n) ={}& n_1\Big(A_{13}\frac{\partial}{\partial x_1} + A_{12}\frac{\partial}{\partial x_2}\Big) + n_2\Big(A_{33}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\Big), \\ T_{21}(\partial_x, n) ={}& n_1\Big(A_{13}\frac{\partial}{\partial x_1} + A_{33}\frac{\partial}{\partial x_2}\Big) + n_2\Big(A_{12}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\Big), \\ T_{22}(\partial_x, n) ={}& n_1\Big(A_{33}\frac{\partial}{\partial x_1} + A_{23}\frac{\partial}{\partial x_2}\Big) + n_2\Big(A_{23}\frac{\partial}{\partial x_1} + A_{22}\frac{\partial}{\partial x_2}\Big). \end{aligned} \tag{1.20}$$

Analogously, using the formulas (1.12), we can write the components of the generalized stress vector $\overset{\varkappa}{T}u$ [4]:

$$(\overset{\varkappa}{T}u)_1 = \sigma_{11} n_1 + \sigma_{12} n_2, \quad (\overset{\varkappa}{T}u)_2 = \sigma_{21} n_1 + \sigma_{22} n_2, \tag{1.21}$$

where

$$\begin{aligned} &\overset{\varkappa}{T} \equiv \overset{\varkappa}{T}(\partial_x, n) = \|\overset{\varkappa}{T}_{kj}(\partial_x, n)\|_{2\times 2}, \\ &\overset{\varkappa}{T}_{kj}(\partial_x, n) = T_{kj}(\partial_x, n) + \varkappa(\delta_{1k}\delta_{2j} - \delta_{1j}\delta_{2k})\frac{\partial}{\partial s(x)}, \\ &\frac{\partial}{\partial s(x)} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1}, \quad \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \end{aligned} \tag{1.22}$$

It is obvious from (1.21) and (1.22) that the generalized stress vector can be written by means of the stress vector as

$$\overset{\varkappa}{T}u = Tu + \varkappa \begin{Vmatrix} 0, & 1 \\ -1, & 0 \end{Vmatrix} \frac{\partial u}{\partial s}. \tag{1.23}$$

This implies that the generalized stress vector and the stress vector coincide for $\varkappa = 0$. Introduction of the generalized stress vector makes easier the investigation of many problems of the theory of elasticity of an anisotropic body.

Let D^+ (D^-) be a finite (an infinite) two-dimensional region bounded by then contour S. Suppose that $S \in C^{1,\beta}$, $0 < \beta \le 1$ (see [28]), i.e., S is a Lyapunov curve.

Introduce the definition of a regular vector function. A vector function u defined in the region D^+ (D^-) is called regular, if it has integrable in D^+ (D^-) continuous second derivatives, and $u(x)$ itself and its first derivatives are continuously extendable at every point of the boundary of D^+ (D^-).

For the region D^-, the conditions at infinity are added:

$$u(x) = O(1), \quad |x|^2 \frac{\partial u}{\partial x_k} = O(1), \quad k = 1, 2, \tag{1.24}$$

where $O(1)$ denotes a bounded function and $|x|^2 = x_1^2 + x_2^2$.

For the equation (1.9), we pose the following boundary value problems. Find a regular vector function $u = (u_1, u_2)$ satisfying in D^+ (D^-) the equation (1.9), if on the boundary S one of the following conditions is given:

Problem 1. The displacement vector u is given in the form

$$(u)^\pm = f,$$

where $(\ )^\pm$ denotes the limiting value from $D^\pm$, and $f = (f_1, f_2)$ is a given vector.

Problem 2. The stress vector Tu is given in the form

$$(Tu)^\pm = f.$$

Problem 3. The normal component of the displacement vector and the tangential component of the stress vector are given in the form

$$(nu)^\pm = f_1, \quad (sTu)^\pm = f_2,$$

where $nu = n_1 u_1 + n_2 u_2$ is the scalar product of the vectors n and u, and $sTu = -n_2 (Tu)_1 + n_1 (Tu)_2$.

Problem 4. The normal component of the stress vector and the tangential component of the displacement vector are given in the form

$$(nTu)^\pm = f_1, \quad (su)^\pm = f_2.$$

Problem 5. Let the boundary S of the region D^+ (D^-) be divided into non-intersecting segments S_1, S_2, S_3, S_4 ($S = S_1 \cup S_2 \cup S_3 \cup S_4$), and let the conditions of Problems 1, 2, 3, and 4 be given on S_1, S_2, S_3, and S_4, respectively.

In the sequel, besides the above-mentioned problems we will also consider some other ones.

In the above-formulated problems, the generalized stress vector can appear instead of the stress vector.

Throughout the paper, under the positive direction of the normal will be understood the outer (with respect to D^+) one.

It should be noted that the basic equations of statics of an anisotropic elastic body both for plane deformation and for generalized plane stressed state [30] coincide completely, and therefore the both cases can be treated by the same methods. In the present work, we will consider only the plane deformation.

2. Generalized Green's Formulas

Let u and v be two regular vector functions in D^+. Multiply the first equation of (1.13) by v_1 and the second one by v_2, where v_1 and v_2 are the projections of v; integration of the results with respect to D^+ and then their addition gives

$$0=\int_{D^+}\sum_{k=1}^{2}v_k\Big(\frac{\partial\sigma_{k1}}{\partial y_1}+\frac{\partial\sigma_{k2}}{\partial y_2}\Big)d\sigma+\int_{D^+}vF\,d\sigma,\quad d\sigma=dy_1dy_2,$$

where $vF=v_1F_1+v_2F_2$ is the scalar product of the two vectors.

Integration by part results in

$$\int_{D^+}\sum_{k=1}^{2}\Big(\sigma_{k1}\frac{\partial v_k}{\partial y_1}+\sigma_{k2}\frac{\partial v_k}{\partial y_2}\Big)d\sigma-\int_{D^+}vF\,d\sigma=\int_S v\overset{\varkappa}{T}u\,dS,\qquad(2.1)$$

where $\overset{\varkappa}{T}u$ is specified in (1.23).

Introducing the notation

$$\overset{\varkappa}{E}(u,v)=\sum_{k=1}^{2}\Big(\sigma_{k1}\frac{\partial v_k}{\partial y_1}+\sigma_{k2}\frac{\partial v_k}{\partial y_2}\Big),\qquad(2.2)$$

we can rewrite (2.1) in the form

$$\int_{D^+}\overset{\varkappa}{E}(u,v)d\sigma-\int_{D^+}vF\,d\sigma=\int_S v\overset{\varkappa}{T}u\,dS.\qquad(2.3)$$

Taking (1.1), (1.2), and (1.12) into account (2.2) can be written as

$$\begin{aligned}\overset{\varkappa}{E}(u,v)&=A_{11}\frac{\partial u_1}{\partial y_1}\frac{\partial v_1}{\partial y_1}+(A_{12}+\varkappa)\Big(\frac{\partial u_1}{\partial y_1}\frac{\partial v_2}{\partial y_2}+\frac{\partial u_2}{\partial y_2}\frac{\partial v_1}{\partial y_1}\Big)+\\ &A_{22}\frac{\partial u_2}{\partial y_2}\frac{\partial v_2}{\partial y_2}+A_{13}\Big[\frac{\partial v_1}{\partial y_1}\Big(\frac{\partial u_1}{\partial y_2}+\frac{\partial u_2}{\partial y_1}\Big)+\frac{\partial u_1}{\partial y_1}\Big(\frac{\partial v_1}{\partial y_2}+\frac{\partial v_2}{\partial y_1}\Big)\Big]+\\ &+A_{23}\Big[\frac{\partial v_2}{\partial y_2}\Big(\frac{\partial u_1}{\partial y_2}+\frac{\partial u_2}{\partial y_1}\Big)+\frac{\partial u_2}{\partial y_2}\Big(\frac{\partial v_1}{\partial y_2}+\frac{\partial v_2}{\partial y_1}\Big)\Big]+\\ &+A_{33}\Big(\frac{\partial u_2}{\partial y_1}+\frac{\partial u_1}{\partial y_2}\Big)\Big(\frac{\partial v_2}{\partial y_1}+\frac{\partial v_1}{\partial y_2}\Big)-\varkappa\Big(\frac{\partial u_2}{\partial y_1}\frac{\partial v_1}{\partial y_2}+\frac{\partial v_2}{\partial y_1}\frac{\partial u_1}{\partial y_2}\Big).\end{aligned}\qquad(2.4)$$

Using now (1.8), the formula (2.3) takes the form

$$\int_{D^+}\overset{\varkappa}{E}(u,v)d\sigma+\int_{D^+}vC(\partial y)u(y)d\sigma=\int_S v\overset{\varkappa}{T}u\,dS.\qquad(2.5)$$

Analogously we obtain

$$\int_{D+} \overset{\varkappa}{E}(v,u)d\sigma + \int_{D+} uC(\partial y)v(y)d\sigma = \int_S u\overset{\varkappa}{T}vdS. \tag{2.6}$$

If we take into consideration the fact that $\overset{\varkappa}{E}(u,v) \equiv \overset{\varkappa}{E}(v,u)$, then from (2.5) and (2.6) we will have

$$\int_{D+} [uC(\partial y)v(y) - vC(\partial y)u(y)]d\sigma = \int_S (u\overset{\varkappa}{T}v - v\overset{\varkappa}{T}u)dS. \tag{2.7}$$

If in (2.5) we put $v = u$, then

$$\int_{D+} \overset{\varkappa}{E}(u,u)d\sigma + \int_{D+} uC(\partial y)u(y)d\sigma = \int_S u\overset{\varkappa}{T}udS, \tag{2.8}$$

where

$$\begin{aligned}\overset{\varkappa}{E}(u,u) &= A_{11}\varepsilon_{y_1}^2 + 2(A_{12}+\varkappa)\varepsilon_{y_1}\varepsilon_{y_2} + A_{22}\varepsilon_{y_2}^2 + \\ &+2(A_{13}\varepsilon_{y_1} + A_{23}\varepsilon_{y_2})\varepsilon_{y_1y_2} + \Big(A_{33} - \frac{\varkappa}{2}\Big)\varepsilon_{y_1y_2}^2 + \frac{\varkappa}{2} l_{y_1y_2}^2,\end{aligned} \tag{2.9}$$

ε_{y_1}, ε_{y_2}, $\varepsilon_{y_1y_2}$ are defined in (1.2), and

$$l_{y_1y_2} = \frac{\partial u_2}{\partial y_1} - \frac{\partial u_1}{\partial y_2}. \tag{2.10}$$

If u is a solution of (1.9), then (2.8) takes the form

$$\int_{D+} \overset{\varkappa}{E}(u,u)d\sigma = \int_S u\overset{\varkappa}{T}udS. \tag{2.11}$$

$\overset{\varkappa}{E}(u,u)$ is the doubled generalized potential energy which, on the other hand, is a quadratic form with respect to the variables ε_{y_1}, ε_{y_2}, $\varepsilon_{y_1y_2}$ and $l_{y_1y_2}$. For the positive definiteness of the generalized potential energy with respect to the above-mentioned variables the inequalities

$$\begin{gathered} A_{11} > 0, \quad \begin{vmatrix} A_{11} & A_{12}+\varkappa \\ A_{12}+\varkappa & A_{22} \end{vmatrix} > 0, \\ \begin{vmatrix} A_{11} & A_{12}+\varkappa & A_{13} \\ A_{12}+\varkappa & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33}-\frac{\varkappa}{2} \end{vmatrix} > 0, \quad \varkappa > 0, \end{gathered} \tag{2.12}$$

are necessary and sufficient.

For $\varkappa = 0$ from (2.8) and (2.11) we have

$$\int_{D+} E(u,u)d\sigma + \int_{D+} uC(\partial y)u(y)d\sigma = \int_S uTudS, \tag{2.13}$$

$$\int_{D+} E(u,u)d\sigma = \int_S uTudS, \tag{2.14}$$

where

$$E(u,u) = A_{11}\varepsilon_{y_1}^2 + 2A_{12}\varepsilon_{y_1}\varepsilon_{y_2} + A_{22}\varepsilon_{y_2}^2 + \\ + A_{33}\varepsilon_{y_1y_2}^2 + 2(A_{13}\varepsilon_{y_1} + A_{23}\varepsilon_{y_2})\varepsilon_{y_1y_2} \tag{2.15}$$

is the doubled potential energy. For the positive definiteness of the potential energy the inequalities

$$A_{11}>0, \quad \begin{vmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{vmatrix} > 0, \quad \Delta = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{vmatrix} > 0 \tag{2.16}$$

are necessary and sufficient.

Taking into accout (1.16) and the implied by it formulas

$$\begin{gathered} A_{11} = (a_{22}a_{33} - a_{23}^2)\Delta, \quad A_{22} = (a_{11}a_{33} - a_{13}^2)\Delta, \\ A_{33} = (a_{11}a_{22} - a_{12}^2)\Delta, \quad A_{11} = (a_{13}a_{23} - a_{12}a_{33})\Delta, \\ A_{13} = (a_{12}a_{23} - a_{13}a_{22})\Delta, \quad A_{23} = (a_{11}a_{13} - a_{11}a_{23})\Delta, \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \Delta^{-1}, \end{gathered} \tag{2.17}$$

we can easily see that the inequalities $A_{22} > 0$, $A_{33} > 0$, $a_{11} > 0$, $a_{22} > 0$ follow from (2.16).

Thus, the necessary and sufficient conditions for the positive definiteness of the potential energy can be written as

$$A_{11}>0, \quad A_{22}>0, \quad A_{33}>0, \quad a_{11}>0, \quad a_{22}>0, \quad a_{33}>0, \quad \Delta>0. \tag{2.18}$$

Similar arguments can be applied to the conditions (2.12). Indeed, if we introduce the notation

$$f(\varkappa) = 2\begin{vmatrix} A_{11} & A_{12}+\varkappa & A_{13} \\ A_{12}+\varkappa & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33}-\frac{\varkappa}{2} \end{vmatrix} \equiv \varkappa^3 + \\ +2(A_{12} - A_{33})\varkappa^2 - \Delta(a_{33} - 4a_{12})\varkappa + 2\Delta, \tag{2.19}$$

$$\begin{gathered} a_{11}(\varkappa) = 2\Delta a_{11} - A_{22}\varkappa, \quad a_{22}(\varkappa) = 2\Delta a_{22} - A_{11}\varkappa, \\ a_{33}(\varkappa) = 2(\Delta a_{33} - 2A_{12}\varkappa - \varkappa^2), \\ a_{12}(\varkappa) = 2\Delta a_{12} + (A_{12} - 2A_{33})\varkappa + \varkappa^2, \\ a_{13}(\varkappa) = 2(\Delta a_{13} + A_{23}\varkappa), \quad a_{23}(\varkappa) = 2(\Delta a_{23} + A_{13}\varkappa), \end{gathered} \tag{2.20}$$

then (2.12) can be rewritten in the form

$$A_{11} > 0, \quad a_{33}(\varkappa) > 0, \quad f(\varkappa) > 0, \quad \varkappa > 0. \tag{2.21}$$

Taking now into account the following easily verifiable identities

$$\begin{aligned} 2A_{11}f(\varkappa) &= a_{22}(\varkappa)a_{33}(\varkappa) - a_{23}^2(\varkappa),\\ 2A_{22}f(\varkappa) &= a_{11}(\varkappa)a_{33}(\varkappa) - a_{13}^2(\varkappa),\\ (2A_{33}-\varkappa)f(\varkappa) &= a_{11}(\varkappa)a_{22}(\varkappa) - a_{12}^2(\varkappa),\\ 2(A_{12}+\varkappa)f(\varkappa) &= a_{13}(\varkappa)a_{23}(\varkappa) - a_{12}(\varkappa)a_{33}(\varkappa),\\ 2A_{13}f(\varkappa) &= a_{12}(\varkappa)a_{23}(\varkappa) - a_{13}(\varkappa)a_{22}(\varkappa),\\ 2A_{23}f(\varkappa) &= a_{12}(\varkappa)a_{13}(\varkappa) - a_{11}(\varkappa)a_{23}(\varkappa), \end{aligned} \tag{2.22}$$

we find from (2.20) and (2.21) that the inequalities

$$a_{11}(\varkappa) > 0, \quad a_{22}(\varkappa) > 0, \quad A_{22} > 0, \quad 2A_{33} - \varkappa > 0 \tag{2.23}$$

are valid.

Thus, for the positive definiteness of the generalized potential energy it is necessary and sufficient that the conditions

$$\begin{aligned} &A_{11} > 0, \ \ A_{22} > 0, \ \ 2A_{33} - \varkappa > 0, \ \ f(\varkappa) > 0,\\ &a_{11}(\varkappa) > 0, \ \ a_{22}(\varkappa) > 0, \ \ a_{33}(\varkappa) > 0, \ \ \varkappa > 0 \end{aligned} \tag{2.24}$$

be fulfilled.

One can generalize the formulas (2.7), (2.8), and (2.11) for an infinite region D^-, provided the conditions

$$\begin{aligned} &\lim_{R\to\infty} \int_{S(O,R)} u\overset{\varkappa}{T} v\,dS = 0, \quad \lim_{R\to\infty} \int_{S(O,R)} v\overset{\varkappa}{T} u\,dS = 0,\\ &\lim_{R\to\infty} \int_{S(O,R)} u\overset{\varkappa}{T} u\,dS = 0 \end{aligned} \tag{2.25}$$

are fulfilled, where $S(0,R)$ is the circumference of radius R with center at the point 0 lying inside D^+. The radius R is taken so large that the region D^+ lies entirely inside the circumference $S(0,R)$.

Obviously, the conditions (2.25) are fulfilled if the vector function u and v satisfy the conditions (1.24).

If (2.25) are fulfilled, then Green's formulas (2.7), (2.8), and (2.11) for the region D^- take the form

$$\int_{D^-} \big[uC(\partial y)v(y) - vC(\partial y)u(y)\big]\,d\sigma = \int_S (v\overset{\varkappa}{T} u - u\overset{\varkappa}{T} v)dS, \tag{2.26}$$

$$\int_{D^-} \overset{\varkappa}{E}(u,u)d\sigma + \int_{D^-} uC(\partial y)u(y)d\sigma = -\int_S u\overset{\varkappa}{T} u\,dS, \tag{2.27}$$

$$\int_{D^-} \overset{\varkappa}{E}(u,u)d\sigma = -\int_S u\overset{\varkappa}{T} u\,dS. \tag{2.28}$$

Note that the formulas (2.7), (2.8), and (2.11) remain valid even in the case where D^+ is a finite multiply connected region bounded by the contours $S_1, S_2, \dots, S_m, S_{m+1}$, the last of which contains all the others. In this case, the boundary of D^+ is $S = \bigcup\limits_{k=1}^{m+1} S_k$; note that the contours S_k $(k \leq m)$ are oriented clockwise, while S_{m+1} is oriented counterclockwise.

3. Characteristic Equation

The fundamental equations of statics of an anisotropic elastic body (see (1.8) and (1.9)) form an elliptic system. Consequently, the roots of the characteristic equation

$$\begin{vmatrix} A_{33}\alpha^2 + 2A_{13}\alpha + A_{11}, & A_{23}\alpha^2 + (A_{12}+A_{33})\alpha + A_{13} \\ A_{23}\alpha^2 + (A_{12}+A_{33})\alpha + A_{13}, & A_{22}\alpha^2 + 2A_{23}\alpha + A_{33} \end{vmatrix} \equiv$$
$$\equiv \Delta[a_{11}\alpha^4 - 2a_{13}\alpha^3 + (2a_{12}+a_{33})\alpha^2 - 2a_{23}\alpha + a_{22}] = 0 \qquad (3.1)$$

are complex (self-conjugate) numbers

$$\alpha_k = a_k + ib_k, \quad \overline{\alpha}_k = a_k - ib_k, \quad k = 1, 2. \qquad (3.2)$$

Without loss of generality we assume that

$$b_1 > 0, \quad b_2 > 0. \qquad (3.3)$$

In the case of an orthotropic elastic body, where

$$A_{13} = A_{23} = 0, \quad a_{13} = a_{23} = 0,$$

the characteristic equation (3.1) is biquadratic with the roots

$$\alpha_1 = a + ib, \;\; \alpha_2 = -a + ib, \;\; \overline{\alpha}_1 = a - ib, \;\; \overline{\alpha}_2 = -a - ib, \qquad (3.4)$$

or

$$\alpha_k = ib_k, \quad \overline{\alpha}_k = -ib_k, \quad k = 1, 2, \qquad (3.5)$$

where $a > 0$, $b > 0$, $b_1 > 0$, $b_2 > 0$.

Note that (3.4) and (3.5) take place when $(A_{12} + 2A_{33})^2 - A_{11}A_{22} > 0$ and $A_{11}A_{22} - (A_{12} + 2A_{33})^2 > 0$, respectively; moreover,

$$\begin{gathered}
2\sqrt{A_{22}A_{33}}\, a = \sqrt{A_{12} + 2A_{33} - \sqrt{A_{11}A_{22}}} \cdot \sqrt{\sqrt{A_{11}A_{22}} + A_{12}}, \\
2\sqrt{A_{22}A_{33}}\, b = \sqrt{A_{12} + 2A_{33} + \sqrt{A_{11}A_{22}}} \cdot \sqrt{\sqrt{A_{11}A_{22}} - A_{12}}, \\
2b_1\sqrt{A_{22}A_{33}} = \gamma + \delta, \quad 2b_2\sqrt{A_{22}A_{33}} = \gamma - \delta, \\
\gamma = \sqrt{\sqrt{A_{11}A_{22}} + A_{12} + 2A_{33}} \cdot \sqrt{\sqrt{A_{11}A_{22}} - A_{12}}, \\
\delta = \sqrt{\sqrt{A_{11}A_{22}} - A_{12} - 2A_{33}} \cdot \sqrt{\sqrt{A_{11}A_{22}} + A_{12}}.
\end{gathered}$$

If $A_{11}A_{22} - (A_{12} + 2A_{33})^2 = 0$, then $a = 0$, $b_1 = b_2 = b = \sqrt[4]{\frac{A_{11}}{A_{22}}}$.

The characteristic equation for an isotropic elastic body has double roots

$$\alpha_1 = i, \quad \overline{\alpha}_1 = -i. \qquad (3.6)$$

In the general case, no necessary and sufficient conditions (written in terms of inequalities) ensuring only complex roots for the characteristic equation are available as yet.

However, S. G. Lekhnitskiĭ [31, Ch. 3] pointed out a sufficient condition and proved the theorem: the equation (3.1) fails to have real roots if the potential energy per unit volume is a positive quantity for any strain components.

Here we give a very simple proof of the above theorem. We rewrite (3.1) as

$$(a_{11}\alpha^2 - a_{13}\alpha + a_{12})^2 + \frac{1}{\Delta A_{22}}(A_{22}\alpha + A_{23})^2 + \frac{a_{11}}{A_{22}} = 0 \qquad (3.7)$$

and then, taking into account (2.18), we immediately conclude that the left-hand side never equals zero for real α.

4. Matrix of Fundamental Solutions

The basic problem in applying the potential method is the construction of fundamental solutions of corresponding differential equations. The ways of constructing the fundamental solutions are different. At present, the most wide-spread method (which for two-dimensional regions is comparatively difficult) consists in finding fundamental solutions by using the Fourier transform. A more simple and efficient tool for constructing fundamental solutions in the case of two independent variables was suggested at the beginning of our century by Italian mathematician E. Levi [32]. The method will be used below for the system (1.9).

Let

$$C^*(\partial x) = \begin{Vmatrix} C_{22}(\partial x), & -C_{12}(\partial x) \\ -C_{12}(\partial x), & C_{11}(\partial x) \end{Vmatrix}, \qquad (4.1)$$

where $C_{kj}(\partial x)$, $k, j = 1, 2$, are given by (1.7). Supposing

$$u(x) = C^*(\partial x)\Psi(x), \qquad (4.2)$$

where $\Psi = (\Psi_1, \Psi_2)$ is a four times differentiable vector function, we can write the equation (1.9) as

$$\mathcal{D}(\partial x)\Psi(x) = 0, \qquad (4.3)$$

where

$$\begin{aligned}\mathcal{D}(\partial x) \equiv & a_{22}\frac{\partial^4}{\partial x_1^4} - 2a_{23}\frac{\partial^4}{\partial x_1^3\partial x_2} + (2a_{12} + a_{33})\frac{\partial^4}{\partial x_1^2\partial x_2^2} - \\ & - 2a_{13}\frac{\partial^4}{\partial x_1\partial x_2^3} + a_{11}\frac{\partial^4}{\partial x_2^4}. \end{aligned} \qquad (4.4)$$

The equation (4.3) is sometimes called a generalized biharmonic equation. This equation in the case of an isotropic elastic body (see (1.11) and (1.18)) is really the biharmonic equation

$$\Delta(\partial x)\Delta(\partial x)\Psi(x) = 0, \tag{4.5}$$

where $\Delta(\partial x) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the two-dimensional Laplace operator.

The fundamental matrix for the equation (4.3) can be written as

$$\Psi(x) = \begin{Vmatrix} 1, & 0 \\ 0, & 1 \end{Vmatrix} \Psi_0(x), \tag{4.6}$$

where $\Psi_0(x)$ is the fundamental solution of the scalar equation

$$\mathcal{D}(\partial x)\Psi_0(x) = 0. \tag{4.7}$$

Consider the Vandermonde determinant composed by the roots α_1, $\overline{\alpha}_1$, α_2, $\overline{\alpha}_2$ of the characteristic equation (3.1)

$$d = \begin{vmatrix} 1 & \alpha_1 & \alpha_1^2 & \alpha_1^3 \\ 1 & \overline{\alpha}_1 & \overline{\alpha}_1^2 & \overline{\alpha}_1^3 \\ 1 & \alpha_2 & \alpha_2^2 & \alpha_2^3 \\ 1 & \overline{\alpha}_2 & \overline{\alpha}_2^2 & \overline{\alpha}_2^3 \end{vmatrix} .$$

Denote by d_k the algebraic complement of the element α_k^3 divided by d. A simple calculation gives

$$\begin{aligned} d_1 &= \frac{1}{(\alpha_1 - \overline{\alpha}_1)(\alpha_1 - \alpha_2)(\alpha_1 - \overline{\alpha}_2)}, \\ d_2 &= \frac{1}{(\alpha_2 - \overline{\alpha}_1)(\alpha_2 - \alpha_1)(\alpha_2 - \overline{\alpha}_2)}. \end{aligned} \tag{4.8}$$

According to Levi's results [32], the fundamental solution of the equation (4.7) is of the form

$$\Psi_0(x) = -\frac{1}{\Delta a_{11}} \operatorname{Im} \sum_{k=1}^{2} d_k z_k^2 \ln z_k, \tag{4.9}$$

where $\Delta a_{11} = A_{22}A_{33} - A_{23}^2$, $z_k = x_1 + \alpha_k x_2$, $x = (x_1, x_2)$. The imaginary part is denoted by the symbol Im.

From (4.6) and (4.9) it follows that

$$\Psi(x-y) = -\begin{Vmatrix} 1, & 0 \\ 0, & 1 \end{Vmatrix} \frac{1}{\Delta a_{11}} \operatorname{Im} \sum_{k=1}^{2} d_k \sigma_k^2 \ln \sigma_k, \tag{4.10}$$

where

$$\sigma_k = x_1 - y_1 + \alpha_k(x_2 - y_2), \quad y = (y_1, y_2). \tag{4.11}$$

Substituting (4.10) in (4.2), after some calculations we obtain the fundamental matrix of the equation (1.9) which is denoted by $\Gamma(x,y)$:

$$\Gamma(x,y) = \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \ln \sigma_k, \tag{4.12}$$

where

$$\begin{aligned} A_k &= -\frac{2}{\Delta a_{11}} \left(A_{22}\alpha_k^2 + 2A_{23}\alpha_k + A_{33} \right) d_k, \\ B_k &= \frac{2}{\Delta a_{11}} \left[A_{23}\alpha_k^2 + (A_{12} + A_{33})\alpha_k + A_{13} \right] d_k, \\ C_k &= -\frac{2}{\Delta a_{11}} \left(A_{33}\alpha_k^2 + 2A_{13}\alpha_k + A_{11} \right) d_k. \end{aligned} \tag{4.13}$$

In what follows, we will assume that $\alpha_1 \neq \alpha_2$. Note that the case where $\alpha_2 = \alpha_1$ is obtained when passing to limit as $\alpha_2 \to \alpha_1$.

In a physical sense, the matrix (4.12) represents displacements caused by the concentrated force at the point $y = (y_1, y_2)$. It is evident that the matrix $\Gamma(x,y)$ has a logarithmic singularity as $|x-y| \to 0$.

Taking into consideration (4.8), after obvious calculations we obtain

$$\sum_{k=1}^{2} d_k = -\frac{i}{2}\, C, \quad \sum_{k=1}^{2} d_k\alpha_k = -\frac{i}{2}\, A, \quad \sum_{k=1}^{2} d_k\alpha_k^2 = -\frac{i}{2}\, B, \tag{4.14}$$

where

$$\begin{gathered} C = (b_1 + b_2)(BC - A^2), \quad A = (a_1 b_2 + a_2 b_1)(BC - A^2), \\ B = [b_1(a_2^2 + b_2^2) + b_2(a_1^2 + b_1^2)](BC - A^2), \\ BC - A^2 = \frac{1}{b_1 b_2[(a_1 - a_2)^2 + (b_1 + b_2)^2]} > 0. \end{gathered} \tag{4.15}$$

The coefficients A, B, C have been introduced by Basheleishvili [4]. As it will be seen below, they substantially simplify the exposition of the matter.

From (4.14), after some operations we find that

$$\begin{gathered} d_k = \frac{i}{2}\,(Ar_k - Cs_k), \quad \alpha_k d_k = \frac{i}{2}\,(Cq_k - Ap_k), \\ \alpha_k^2 d_k = \frac{i}{2}\,(Aq_k - Bp_k), \quad k = 1, 2, \end{gathered} \tag{4.16}$$

where

$$Cq_k - Ap_k = Br_k - As_k, \tag{4.17}$$

$$\begin{aligned} p_k &= \frac{(-1)^{k-1}\alpha_k}{\alpha_1 - \alpha_2}, \quad q_k = \frac{(-1)^{k-1}\alpha_1\alpha_2}{\alpha_1 - \alpha_2}, \\ r_k &= \frac{(-1)^k}{\alpha_1 - \alpha_2}, \quad s_k = \frac{(-1)^k \alpha_1\alpha_2}{\alpha_k(\alpha_1 - \alpha_2)}. \end{aligned} \tag{4.18}$$

Noticing that

$$\mathrm{Re}\sum_{k=1}^{2} d_k = 0, \quad \mathrm{Re}\sum_{k=1}^{2} \alpha_k d_k = 0, \quad \mathrm{Re}\sum_{k=1}^{2} \alpha_k^2 d_k = 0,$$

which follow from (4.14), we can easily conclude that the elements of the matrix $\Gamma(x,y)$ are one-valued functions defined on the whole plane ($x \neq y$). Here and in the sequel, Re denotes the real part.

If in (4.12) we pass to limit as $\alpha_2 \to \alpha_1 = i$, then after simple calculations we get (to within the constant additive matrix)

$$\Gamma(x,y) = \mathrm{Re}\left[nE \ln\sigma - \frac{m}{2} \begin{Vmatrix} 1, & i \\ i, & -1 \end{Vmatrix} \frac{\overline{\sigma}}{\sigma} \right], \tag{4.19}$$

where E is the unit matrix, and

$$\begin{gathered} n = \frac{\lambda + 3\mu}{2\mu(\lambda + 2\mu)}, \quad m = \frac{\lambda + \mu}{2\mu(\lambda + 2\mu)}, \\ \sigma = (x_1 - y_1) + i(x_2 - y_2), \quad \overline{\sigma} = (x_1 - y_1) - i(x_2 - y_2). \end{gathered} \tag{4.20}$$

It is seen from the above-given formulas that the matrix $\Gamma(x,y)$ in the case of an isotropic elastic body coincides with the Kelvin–Somiglian matrix (see [33]).

5. Matrix of Singular Solutions

In solving boundary value problems by the method of potential theory, not only the fundamental matrix is of a great importance but also some other matrices of singular solutions which will be constructed explicitly in the present section with the help of elementary functions.

Let us calculate the result of acting of the operator $T(\partial x, n)$ (see (1.20)) on the matrix Γ

$$T(\partial x, n)\Gamma(x,y) = \begin{Vmatrix} T_{11}\Gamma_{11} + T_{12}\Gamma_{12}, & T_{11}\Gamma_{12} + T_{12}\Gamma_{22} \\ T_{21}\Gamma_{11} + T_{22}\Gamma_{12}, & T_{21}\Gamma_{12} + T_{22}\Gamma_{22} \end{Vmatrix}. \tag{5.1}$$

Obviously

$$\begin{gathered} T_{11}\Gamma_{11} + T_{12}\Gamma_{12} = \mathrm{Im}\sum_{k=1}^{2} \big\{ [n_1(A_{11} + A_{13}\alpha_k) + n_2(A_{13} + A_{33}\alpha_k)A_k + \\ + [n_1(A_{13} + A_{12}\alpha_k) + n_2(A_{33} + A_{23}\alpha_k)]B_k \big\} \frac{1}{\sigma_k}. \end{gathered}$$

Taking into account the equality $A_k C_k - B_k^2 = 0$, $k = 1, 2$, it is not difficult to see that the identity

$$\begin{gathered} (A_{13}\alpha_k + A_{11})A_k + (A_{12}\alpha_k + A_{13})B_k = \\ = -\alpha_k[(A_{33}\alpha_k + A_{13})A_k + (A_{23}\alpha_k + A_{33})B_k] \end{gathered}$$

holds.

On account of the above formulas we have

$$T_{11}\Gamma_{11}+T_{12}\Gamma_{12}=\operatorname{Im}\sum_{k=1}^{2}\big[(A_{33}\alpha_k+A_{13})A_k+(A_{23}\alpha_k+A_{33})B_k\big]\,\frac{(n_2-\alpha_k n_1)}{\sigma_k}.$$

If we notice (see (1.22)) that $\frac{\partial}{\partial s(x)}\ln\sigma_k=\frac{\alpha_k n_1-n_2}{\sigma_k}$, then we obtain

$$T_{11}\Gamma_{11}+T_{12}\Gamma_{12}=-\operatorname{Im}\sum_{k=1}^{2}\Big[(A_{33}\alpha_k+A_{13})A_k+(A_{23}\alpha_k+A_{33})B_k\Big]\,\frac{\partial\ln\sigma_k}{\partial s(x)}.$$

The other elements of the matrix (5.1) can be calculated analogously, and finally we have

$$T(\partial x,n)\Gamma(x,y)=\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & L_k\\ M_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(x)},\tag{5.2}$$

where

$$\begin{aligned}
N_k&=-(A_{33}\alpha_k+A_{13})A_k-(A_{23}\alpha_k+A_{33})B_k,\\
L_k&=-(A_{33}\alpha_k+A_{13})B_k-(A_{23}\alpha_k+A_{33})C_k,\\
M_k&=-(A_{23}\alpha_k+A_{12})A_k-(A_{22}\alpha_k+A_{23})B_k,\\
R_k&=-(A_{23}\alpha_k+A_{12})B_k-(A_{22}\alpha_k+A_{23})C_k.
\end{aligned}\tag{5.3}$$

Using the formulas (4.13) and (1.16), the above identities can be rewritten as

$$\begin{aligned}
N_k&=\frac{2}{a_{11}}(a_{11}\alpha_k^3-a_{13}\alpha_k^2+a_{12}\alpha_k)d_k,\\
L_k&=\frac{2}{a_{11}}(a_{12}\alpha_k^2-a_{23}\alpha_k+a_{22})d_k,\\
M_k&=-\frac{2}{a_{11}}(a_{11}\alpha_k^2-a_{13}\alpha_k+a_{12})d_k,\\
R_k&=-\frac{2}{a_{11}\alpha_k}(a_{12}\alpha_k^2-a_{23}\alpha_k+a_{22})d_k.
\end{aligned}\tag{5.4}$$

Finally, making use of Vieta's theorem for the characteristic equation (see (3.1)), we get

$$\begin{aligned}
N_k&=p_k+2\omega\alpha_k d_k, & M_k&=r_k-2\omega d_k\equiv -N_k\cdot\alpha_k^{-1},\\
L_k&=q_k+2\omega\alpha_k^2 d_k, & R_k&=s_k-2\omega\alpha_k d_k\equiv -L_k\cdot\alpha_k^{-1},
\end{aligned}\tag{5.5}$$

where p_k,q_p,r_k,s_k are given by (4.18), and

$$\omega=b_1b_2-a_1a_2+a_{12}/a_{11}.\tag{5.6}$$

The coefficient ω has been introduced by Basheleishvili [4]. Note once more that the coefficient ω (see (4.15)) along with the coefficients A,B,C

will significantly simplify the discussion and formulation of many important problems of elasticity of anisotropic bodies.

Taking (4.16) into consideration, the formulas (5.5) can be written in the form

$$\begin{aligned} N_k &= (1-i\omega A)p_k + i\omega C q_k, \quad M_k = (1-i\omega A)r_k + i\omega C s_k,\\ L_k &= (1+i\omega A)q_k - i\omega B p_k, \quad R_k = (1+i\omega A)s_k - i\omega B r_k. \end{aligned} \tag{5.7}$$

Next, we establish an important corelation between A_k, B_k, C_k and M_k, R_k.

For $\alpha=\alpha_k$, the characteristic equation (see (3.1)) may be of three types:

$$\Delta(a_{11}\alpha_k^2 - a_{13}\alpha_k + a_{12})^2 = -(A_{22}\alpha_k^2 + 2A_{23}\alpha_k + A_{33}),$$

$$\Delta\Big(a_{12}\alpha_k - a_{23} + \frac{a_{22}}{\alpha_k}\Big)^2 = -(A_{33}\alpha_k^2 + 2A_{13}\alpha_k + A_{11}),$$

$$\Delta(a_{11}\alpha_k^2 - a_{13}\alpha_k + a_{12})\Big(a_{12}\alpha_k - a_{23} + \frac{a_{22}}{\alpha_k}\Big) = \\ = [A_{23}\alpha_k^2 + (A_{12}+A_{33})\alpha_k + A_{13}],$$

whence, taking into account (4.13) and (5.4), we obtain

$$A_k = \frac{a_{11}}{2}\frac{M_k^2}{d_k}, \quad B_k = \frac{a_{11}}{2}\frac{M_k R_k}{d_k}, \quad C_k = \frac{a_{11}}{2}\frac{R_k^2}{d_k}. \tag{5.8}$$

These relations will be frequently used in the sequel.

Based on (4.14), (4.18), (5.4), and (5.7), after elementary calculations we find that

$$\sum_{k=1}^{2} M_k = \frac{i}{a_{11}}(a_{11}B + a_{12}C - a_{13}A) = i\omega C,$$

$$\sum_{k=1}^{2} L_k = -\frac{i}{a_{11}}(a_{12}B + a_{22}C - a_{23}A) = -i\omega B,$$

$$\sum_{k=1}^{2}(N_k - R_k) \equiv \frac{2}{a_{11}}\sum_{k=1}^{2}(a_{13}\alpha_k^2 - a_{33}\alpha_k + a_{23})d_k = \\ = -\frac{i}{a_{11}}\big(a_{13}B + a_{23}C - a_{33}A\big) = -2i\omega A,$$

whence

$$\begin{gathered} a_{11}B + a_{12}C - a_{13}A = \omega a_{11} C, \quad a_{12}B + a_{22}C - a_{23}A = \omega a_{11} B,\\ a_{13}B + a_{23}C - a_{33}A = 2\omega a_{11} A. \end{gathered} \tag{5.9}$$

The identities

$$\sum_{i=1}^{2} a_{ki}A_{ij} = \delta_{kj} = \begin{cases} 1, & k=j,\\ 0, & k\neq j, \end{cases}$$

derived from the formulas (1.16) and (2.17) result in

$$\begin{aligned} A_{11}C + A_{12}B + 2A_{13}A &= (\omega a_{11})^{-1}B, \\ A_{12}C + A_{22}B + 2A_{23}A &= (\omega a_{11})^{-1}C, \\ A_{13}C + A_{23}B + 2A_{33}A &= -(\omega a_{11})^{-1}A. \end{aligned} \tag{5.10}$$

We multiply the first equality from (5.10) by C, the second one by B and the third one by $2A$ and then add:

$$2(BC-A^2)(\omega a_{11})^{-1} = A_{11}C^2 + A_{22}B^2 + 4A_{33}A^2 + 4A_{13}AC + 4A_{23}AB + 2A_{12}BC.$$

Substituting in (2.15) $\varepsilon_{x_1} = C$, $\varepsilon_{x_2} = B$, $\varepsilon_{x_1x_2} = 2A$ and then comparing with the last equality, we can easily conclude that $2(BC - A^2)(\omega a_{11})^{-1} > 0$ by virtue of the positive definiteness of the potential energy. Further, taking into account (4.15) and the inequality $a_{11} > 0$ (see (2.18)), we obtain

$$\omega > 0. \tag{5.11}$$

From (4.16) we have

$$\begin{aligned} r_k &= \frac{2i(A - C\alpha_k)}{BC - A^2} d_k, \quad s_k = \frac{2i(B - A\alpha_k)}{BC - A^2} d_k, \\ p_k &= \frac{2i(C\alpha_k^2 - A\alpha_k)}{BC - A^2} d_k, \quad q_k = \frac{2i(A\alpha_k^2 - B\alpha_k)}{BC - A^2} d_k. \end{aligned} \tag{5.12}$$

Applying these formulas to (5.5), we obtain

$$\frac{M_k}{d_k} = \frac{2i(A - C\alpha_k)}{BC - A^2} - 2\omega, \quad \frac{R_k}{d_k} = \frac{2i(B - A\alpha_k)}{BC - A^2} - 2\omega\alpha_k. \tag{5.13}$$

If we take into consideration (5.5), after certain simplifications we find from (5.8) that

$$\begin{aligned} A_k &= a_{11}M_k\Big[\frac{i(A - C\alpha_k)}{BC - A^2} - \omega\Big] = \frac{ia_{11}(CN_k + AM_k)}{BC - A^2} - \omega a_{11}M_k, \\ B_k &= a_{11}R_k\Big[\frac{i(A - C\alpha_k)}{BC - A^2} - \omega\Big] = \frac{ia_{11}(CL_k + AR_k)}{BC - A^2} - \omega a_{11}R_k, \\ B_k &= a_{11}M_k\Big[\frac{i(B - A\alpha_k)}{BC - A^2} - \omega\alpha_k\Big] = \frac{ia_{11}(AN_k + BM_k)}{BC - A^2} + \omega a_{11}N_k, \\ C_k &= a_{11}R_k\Big[\frac{i(B - A\alpha_k)}{BC - A^2} - \omega\alpha_k\Big] = \frac{ia_{11}(AL_k + BR_k)}{BC - A^2} + \omega a_{11}L_k. \end{aligned}$$

We rewrite these formulas in the matrix form

$$\begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} = \begin{Vmatrix} \frac{ia_{11}C}{BC-A^2}, & \frac{ia_{11}A}{BC-A^2} - \omega a_{11} \\ \frac{ia_{11}A}{BC-A^2} + \omega a_{11}, & \frac{ia_{11}B}{BC-A^2} \end{Vmatrix} \begin{Vmatrix} N_k, & L_k \\ M_k, & R_k \end{Vmatrix} \tag{5.14}$$

and after transposition we obtain

$$\begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} = \begin{Vmatrix} N_k, & M_k \\ L_k, & R_k \end{Vmatrix} \begin{Vmatrix} \frac{ia_{11}C}{BC-A^2}, & \frac{ia_{11}A}{BC-A^2} + \omega a_{11} \\ \frac{ia_{11}A}{BC-A^2} - \omega a_{11}, & \frac{ia_{11}B}{BC-A^2} \end{Vmatrix}. \tag{5.15}$$

Taking into account (5.7), we see from (5.14) that

$$\sum_{k=1}^{2} A_k = \frac{imC}{BC-A^2}, \quad \sum_{k=1}^{2} B_k = \frac{imA}{BC-A^2}, \quad \sum_{k=1}^{2} C_k = \frac{imB}{BC-A^2}, \tag{5.16}$$

where

$$m = a_{11}[1-\omega^2(BC-A^2)]. \tag{5.17}$$

Let us now prove that $m>0$. Using the notation (4.14) and also the formulas (4.13) and (5.16), we easily deduce

$$\begin{gathered} A_{11}C + A_{33}B + 2A_{13}A = \Delta a_{11} mB(BC-A^2)^{-1} \\ A_{33}C + A_{22}B + 2A_{23}A = \Delta a_{11} mC(BC-A^2)^{-1}, \\ A_{13}C + A_{23}B + (A_{12}+A_{33})A = -\Delta a_{11} mA(BC-A^2)^{-1}. \end{gathered} \tag{5.18}$$

The second identity can be rewritten as

$$\begin{gathered} A_{22}\Delta a_{11} C(BC-A^2)^{-1} m = \\ = B\Big(A_{22} + A_{23}\frac{A}{B}\Big)^2 + \Delta a_{11} C + A_{23}^2 \frac{(BC-A^2)}{B}, \end{gathered} \tag{5.19}$$

whence we get

$$m > \frac{BC-A^2}{A_{22}} > 0. \tag{5.20}$$

From (5.15) we easily obtain

$$\begin{gathered} \left\| \begin{matrix} N_k, & M_k \\ L_k, & R_k \end{matrix} \right\| = \\ = \left\| \begin{matrix} A_k, & B_k \\ B_k, & C_k \end{matrix} \right\| \left[\frac{(-i)}{m} \left\| \begin{matrix} B, & -A \\ -A, & C \end{matrix} \right\| + \frac{\omega(BC-A^2)}{m} \left\| \begin{matrix} 0, & 1 \\ -1, & 0 \end{matrix} \right\| \right]. \end{gathered} \tag{5.21}$$

Suppose now that

$$x_1 - y_1 = r\cos\theta, \quad x_2 - y_2 = r\sin\theta, \tag{5.22}$$

where

$$r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}, \quad \theta = \operatorname{arctg} \frac{y_2 - x_2}{y_1 - x_1}. \tag{5.23}$$

Then

$$\sigma_k = r(\cos\theta + \alpha_k \sin\theta), \quad \frac{\partial \ln \sigma_k}{\partial S(x)} = \frac{\partial \ln r}{\partial S(x)} + \frac{\alpha_k \cos\theta - \sin\theta}{\cos\theta + \alpha_k \sin\theta} \, \frac{\partial \theta}{\partial S(x)}. \tag{5.24}$$

It is well-known (see, e.g., [39], [40]) that in the case of a Lyapunov curve ($S \in C^{1,\beta}$) the function $\frac{\partial \theta}{\partial S(x)} = \frac{\partial \ln r}{\partial n(x)}$ for $x, y \in S$ has in the neighbourhood of the point $x=y$ a weak singularity of the type $O(|x-y|^{\beta-1})$, and $\frac{\partial \ln r}{\partial S(x)}$ is integrable in the sense of the principal Cauchy value (i.e., is a singular kernel on S). Consequently, $\frac{\partial \ln \sigma_k}{\partial S(x)}$ is a singular kernel on S which is integrable in the sense of the principal Cauchy value.

Substituting (5.24) in (5.2) and taking into account (5.7), we have

$$T(\partial x, n)\Gamma(x,y) = \omega \begin{Vmatrix} -A, & -B \\ C, & A \end{Vmatrix} \frac{\partial \ln r}{\partial S(x)} +$$
$$+\operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} N_k, & L_k \\ M_k, & R_k \end{Vmatrix} \frac{\alpha_k \cos\theta - \sin\theta}{\cos\theta + \alpha_k \sin\theta} \frac{\partial \theta}{\partial S(x)}. \tag{5.25}$$

A simple analysis shows that the matrix $T(\partial x, n)\Gamma(x,y)$ always contains a singular part which is integrable in the sense of the principal Cauchy value.

The expressions (1.23) and (5.2) yield

$$\overset{\varkappa}{T}(\partial x, n)\Gamma(x,y) = \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} N_k + \varkappa B_k, & L_k + \varkappa C_k \\ M_k - \varkappa A_k, & R_k - \varkappa B_k \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial S(x)}. \tag{5.26}$$

Consider now the matrix

$$\overset{\varkappa}{T}(y,x) = \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} N_k + \varkappa B_k, & M_k - \varkappa A_k \\ L_k + \varkappa C_k, & R_k - \varkappa B_k \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial S(y)}, \tag{5.27}$$

which is obtained from (5.26) by transposition of the columns and rows and of the variables x and y. We can easily prove that every column of the matrix (5.27) for any value of $\varkappa$ is a solution of the system (1.9) with respect to the point x, if $x \neq y$.

On the basis of the formulas (5.7) and (5.16), we get

$$\begin{aligned}
&\sum_{k=1}^{2}(N_k + \varkappa B_k) = 1 - iA\Big(\omega - \frac{\varkappa m}{BC - A^2}\Big),\\
&\sum_{k=1}^{2}(M_k - \varkappa A_k) = iC\Big(\omega - \frac{\varkappa m}{BC - A^2}\Big),\\
&\sum_{k=1}^{2}(L_k + \varkappa C_k) = -iB\Big(\omega - \frac{\varkappa m}{BC - A^2}\Big),\\
&\sum_{k=1}^{2}(R_k - \varkappa B_k) = 1 + iA\Big(\omega - \frac{\varkappa m}{BC - A^2}\Big).
\end{aligned} \tag{5.28}$$

Taking into account (5.25), the formula (5.27) takes the form

$$\overset{\varkappa}{T}(y,x) = \Big(\omega - \frac{\varkappa m}{BC - A^2}\Big) \begin{Vmatrix} -A, & C \\ -B, & A \end{Vmatrix} \frac{\partial \ln r}{\partial S(y)} +$$
$$+\operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} N_k + \varkappa B_k, & M_k - \varkappa A_k \\ L_k + \varkappa C_k, & R_k - \varkappa B_k \end{Vmatrix} \frac{\alpha_k \cos\theta - \sin\theta}{\cos\theta + \alpha_k \sin\theta} \frac{\partial \theta}{\partial S(y)}. \tag{5.29}$$

For every term of the latter matrix to contain an integrable (in usual sense) singularity for $x = y$, it is necessary and sufficient that

$$\varkappa = \varkappa_N = \frac{\omega(BC - A^2)}{m}. \tag{5.30}$$

The operator which will be obtained from $\overset{\varkappa}{T}$ for $\varkappa$ defined by (5.30) will be called the operator N, and the vector Nu will be called the pseudo-stress vector. Obviously,

$$Nu = Tu + \varkappa_N \begin{Vmatrix} 0, & 1 \\ -1, & 0 \end{Vmatrix} \frac{\partial u}{\partial S}. \tag{5.31}$$

In the case of an orthotropic elastic body the operator N has been constructed by Kupradze and Basheleishvili ([25], [26]), and for an anisotropic elastic body by Basheleishvili [4].

Note that the pseudo-stress operator has been used by G. Lauricella [29] in the case of an isotropic elastic body. He also succeeded in obtaining the Fredholm integral equation of the second kind for the first boundary value problem.

Substituting in (5.27) instead of $\varkappa$ the value found from (5.30), we obtain

$$N(y, x) = \operatorname{Im} \sum_{k=1}^{n} \begin{Vmatrix} E_k, & F_k \\ G_k & H_k \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial S(y)}, \tag{5.32}$$

where

$$\begin{aligned} E_k &= N_k + \varkappa_N B_k, \quad & F_k &= M_k - \varkappa_N A_k \\ G_k &= L_k + \varkappa_N C_k, \quad & H_k &= R_k - \varkappa_N B_k. \end{aligned} \tag{5.33}$$

Taking now into account (5.21), we have

$$\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} = -\frac{i}{m} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}. \tag{5.34}$$

It is easy to see that the equations

$$\sum_{k=1}^{2} E_k = 1, \quad \sum_{k=1}^{2} F_k = 0, \quad \sum_{k=1}^{2} G_k = 0, \quad \sum_{k=1}^{2} H_k = 1 \tag{5.35}$$

follow from (5.34) with regard for (5.16).

Finally, we first express the values of the coefficients of the matrix $N(y, x)$ in terms of the coefficients p_k, q_k, r_k, s_k (see (4.18)) and then substitute the values N_k, M_k, L_k, R_k from (5.7) into (5.15):

$$\begin{aligned} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} &= \left[\frac{ia_{11}}{BC - A^2} \begin{Vmatrix} p_k, & r_k \\ q_k, & s_k \end{Vmatrix} - ia_{11}\omega^2 \begin{Vmatrix} s_k, & -r_k \\ -q_k, & p_k \end{Vmatrix} \right] \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} + \\ &+ a_{11}\omega \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} \begin{Vmatrix} p_k - s_k, & 2q_k \\ 2r_k, & s_k - p_k \end{Vmatrix}. \end{aligned} \tag{5.36}$$

We substitute the above obtained values A_k, B_k, C_k in (5.34) and after elementary transformations find that

$$\begin{Vmatrix} E_k, & F_k \\ G_k & H_k \end{Vmatrix} = \begin{Vmatrix} p_k, & r_k \\ q_k & s_k \end{Vmatrix} + \frac{a_{11}\omega}{m}\left[\omega(BC-A^2)\begin{Vmatrix} 1, & 0 \\ 0, & 1 \end{Vmatrix} + \right.$$
$$\left. +i\begin{Vmatrix} -A, & C \\ -B, & A \end{Vmatrix}\right]\begin{Vmatrix} p_k - s_k, & 2r_k \\ 2q_k, & s_k - p_k \end{Vmatrix}. \tag{5.37}$$

In the case of an isotropic elastic body the equation (5.32) takes the form (see (4.20))

$$N(y,x) = \begin{Vmatrix} 1+\frac{m}{n}\cos 2\theta, & \frac{m}{n}\sin 2\theta \\ \frac{m}{n}\sin 2\theta, & 1-\frac{m}{n}\cos 2\theta \end{Vmatrix} \frac{\partial\theta}{\partial S(y)}. \tag{5.38}$$

6. The Matrix $\overset{\varkappa}{M}(x,y)$

In this section we construct the matrix $\overset{\varkappa}{M}(x,y)$ which reduces the second generalized plane boundary value problem (see Section 1) to the solution of the Fredholm integral equation of second kind. A similar matrix for the second boundary value problem has been constructed by Basheleishvili [4]. It has the form

$$M(x,y) = \operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix}\ln\sigma_k, \tag{6.1}$$

where

$$\begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} = \frac{ia_{11}}{BC-A^2}\begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} + \omega a_{11}\begin{Vmatrix} -r_k, & -s_k \\ p_k, & q_k \end{Vmatrix}, \tag{6.2}$$

p_k, q_k, r_k, s_k are given by formula (4.18) and $\sigma_k = x_1 - y_1 + \alpha_k(x_2 - y_2)$.

Let

$$\overset{\varkappa}{M}(x,y) = \operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} H\ln\sigma_k, \tag{6.3}$$

where H is a constant complex matrix to be defined below.

It is not difficult to notice that every column of the matrix $\overset{\varkappa}{M}(x,y)$, considered as a column vector, for any constant matrix H satisfies the system (1.9) with respect to the point x if $x \neq y$.

After simple calculations, from (6.1) we find

$$\overset{\varkappa}{T}(\partial x,n)\overset{\varkappa}{M}(x,y) = \operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k + \varkappa C'_k, & q_k + \varkappa D'_k \\ r_k - \varkappa A'_k, & s_k - \varkappa B'_k \end{Vmatrix} H\frac{\ln\sigma_k}{\partial s(x)}. \tag{6.4}$$

For every element of the matrix $\overset{\varkappa}{T}(\partial x, n)\overset{\varkappa}{M}(x, y)$ to contain only a weak singularity, it is sufficient that the conditions

$$\sum_{k=1}^{2} \begin{Vmatrix} p_k + \varkappa C'_k, & q_k + \varkappa D'_k \\ r_k - \varkappa A'_k, & s_k - \varkappa B'_k \end{Vmatrix} H = E \tag{6.5}$$

be fulfilled, where E is the unit matrix.

Taking (4.18) and (6.2) into consideration, we can rewrite the foregoing formula as

$$\left[(1 + \varkappa\omega a_{11})E + \frac{i\varkappa a_{11}}{BC - A^2} \begin{Vmatrix} A, & B \\ -C, & -A \end{Vmatrix}\right] H = E.$$

whence we have

$$H = \frac{1}{\Delta(\varkappa)} \left[(1 + \varkappa\omega a_{11})E + \frac{i\varkappa a_{11}}{BC - A^2} \begin{Vmatrix} -A, & -B \\ C, & A \end{Vmatrix}\right], \tag{6.6}$$

where

$$\Delta(\varkappa) = (1 + \varkappa\omega a_{11})^2 - \frac{(\varkappa a_{11})^2}{BC - A^2}. \tag{6.7}$$

Obviously, the construction of the matrix H is possible if and only if

$$\Delta(\varkappa) \neq 0. \tag{6.8}$$

But if $\Delta(\varkappa) = 0$, then for the roots we obtain the values

$$\begin{aligned} \varkappa_1 = \delta &= \frac{\sqrt{BC - A^2}}{a_{11}(1 - \omega\sqrt{BC - A^2})}, \\ \varkappa_2 = \delta_1 &= -\frac{\sqrt{BC - A^2}}{a_{11}(1 - \omega\sqrt{BC - A^2})}. \end{aligned} \tag{6.9}$$

For $\varkappa = \delta$ and $\varkappa = \delta_1$, the generalized stress vector has been earlier applied by the author (see [7], [8]) to the third and fourth boundary value problems to obtain Fredholm integral equations of second kind.

Substituting (6.6) in (6.3), we get

$$\overset{\varkappa}{M}(x, y) = \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A'_k(\varkappa), & B'_k(\varkappa) \\ C'_k(\varkappa), & D'_k(\varkappa) \end{Vmatrix} \ln \sigma_k, \tag{6.10}$$

where

$$\begin{Vmatrix} A'_k(\varkappa), & B'_k(\varkappa) \\ C'_k(\varkappa), & D'_k(\varkappa) \end{Vmatrix} =$$
$$= \frac{1}{\Delta(\varkappa)} \begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} \left[(1 + \varkappa\omega a_{11})E + \frac{i\varkappa a_{11}}{BC - A^2} \begin{Vmatrix} -A, & -B \\ C, & A \end{Vmatrix}\right]. \tag{6.11}$$

It is easy to see that the matrix (6.11) coincides with the matrix (6.2) for $\varkappa = 0$, but if $\varkappa = -\frac{1}{\omega a_{11}}$, then

$$\begin{Vmatrix} A'_k(\varkappa), & B'_k(\varkappa) \\ C'_k(\varkappa), & D'_k(\varkappa) \end{Vmatrix} = i\omega \begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} \begin{Vmatrix} -A, & -B \\ C, & A \end{Vmatrix}. \tag{6.12}$$

Comparing (5.7) and (6.2), we obtain the identity

$$\begin{Vmatrix} N_k, & M_k \\ L_k, & R_k \end{Vmatrix} = \frac{(-i)}{a_{11}} \begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}. \tag{6.13}$$

The matrix function $\overset{\varkappa}{M}(x,y)$ is, in general, many-valued. Indeed, if we notice that from (6.11) it follows

$$\sum_{k=1}^{2} \begin{Vmatrix} A'_k(\varkappa), & B'_k(\varkappa) \\ C'_k(\varkappa), & D'_k(\varkappa) \end{Vmatrix} =$$

$$= \frac{1}{\Delta(\varkappa)} \left[\frac{ia_{11}}{BC - A^2} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} + \left(\omega a_{11} - \frac{\varkappa a_{11} m}{BC - A^2} \right) \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} \right], \tag{6.14}$$

then the increment of $\overset{\varkappa}{M}(x,y)$, as the result of going of the point y around the point x (or vice versa), will be equal to

$$\frac{2\pi a_{11}}{\Delta(\varkappa)} \left(\omega - \frac{\varkappa m}{BC - A^2} \right) \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix}.$$

Obviously, if $\varkappa \neq \frac{\omega(BC-A^2)}{m} = \varkappa_N$, then the matrix function $\overset{\varkappa}{M}(x,y)$ is many-valued. Below we will see how one can get rid of the many-valuedness of that matrix.

After elementary simplifications, substituting (6.6) in (6.4) we obtain

$$\overset{\varkappa}{T}(\partial x, n)\overset{\varkappa}{M}(x,y) = \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} p_k(\varkappa), & q_k(\varkappa) \\ r_k(\varkappa), & s_k(\varkappa) \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial s(x)}, \tag{6.15}$$

where

$$\begin{Vmatrix} p_k(\varkappa), & q_k(\varkappa) \\ r_k(\varkappa), & s_k(\varkappa) \end{Vmatrix} = a_0 \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} + (1 - a_0) \begin{Vmatrix} s_k, & -q_k \\ -r_k, & p_k \end{Vmatrix} +$$

$$+ib_0 \begin{Vmatrix} Cq_k + Br_k, & B(s_k - p_k) + 2Aq_k \\ C(s_k - p_k) - 2Ar_k, & -Cq_k - Br_k \end{Vmatrix}, \tag{6.16}$$

$$a_0 = \frac{(1 + \varkappa\omega a_{11})^2}{\Delta(\varkappa)}, \quad b_0 = \frac{\varkappa a_{11}(1 + \varkappa\omega a_{11})}{\Delta(\varkappa)(BC - A^2)}, \quad P_k(\varkappa) + S_k(\varkappa) = 1. \tag{6.17}$$

For $\varkappa = 0$, from (6.15) we have

$$T(\partial_x, n)M(x,y) = \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial S(x)} =$$

$$= \frac{1}{(BC - A^2)f(\theta)} \begin{Vmatrix} \cos^2\theta, & \sin\theta\cos\theta \\ \sin\theta\cos\theta, & \sin^2\theta \end{Vmatrix} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} \frac{\partial\theta}{\partial S(x)}, \quad (6.18)$$

where

$$f(\theta) = f_1(\theta)f_2(\theta), \quad f_k(\theta) = \cos^2\theta + 2a_k\sin\theta\cos\theta + (a_k^2 + b_k^2)\sin^2\theta. \quad (6.19)$$

Analogously, for $\varkappa = -\frac{1}{\omega a_{11}}$ we have

$$\operatorname{Im}\sum_{k=1}^{2} \begin{Vmatrix} p_k(\varkappa), & q_k(\varkappa) \\ r_k(\varkappa), & s_k(\varkappa) \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial S(x)} = \operatorname{Im}\sum_{k=1}^{2} \begin{Vmatrix} s_k, & -q_k \\ -r_k, & p_k \end{Vmatrix} \frac{\partial \ln\sigma_k}{\partial S(x)} =$$

$$= \frac{1}{(BC - A^2)f(\theta)} \begin{Vmatrix} \sin^2\theta, & -\sin\theta\cos\theta \\ -\sin\theta\cos\theta, & \cos^2\theta \end{Vmatrix} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} \frac{\partial\theta}{\partial S(x)}. \quad (6.20)$$

For an isotropic elastic body we can rewrite (6.18) as

$$T(\partial x, n(x))M(x,y) = \begin{Vmatrix} 1 + \cos 2\theta, & \sin 2\theta \\ \sin 2\theta, & 1 - \cos 2\theta \end{Vmatrix} \frac{\partial\theta}{\partial s(x)}.$$

7. Correlations Between Coefficients

As is obvious from Section 3, the characteristic equation has only the complex roots

$$\alpha_k = a_k + ib_k, \quad \overline{\alpha}_k = a_k - ib_k, \quad b_k > 0, \quad k = 1, 2. \quad (7.1)$$

Our desire now is to express these roots through the roots of a cubic resolvent of the characteristic equation. Keeping this in mind, we rewrite the characteristic equation as

$$(a_{11}\alpha^2 - a_{13}\alpha + a_{12} + \xi)^2 = \Big(2a_{11}\xi - \frac{A_{22}}{\Delta}\Big)\alpha^2 -$$
$$-2\Big(a_{13}\xi + \frac{a_{23}}{\Delta}\Big)\alpha + \xi^2 + 2a_{12}\xi - \frac{A_{33}}{\Delta},$$

where ξ is for the time being an arbitrary parameter. The right-hand side of the above equation is a full square if the parameter ξ satisfies the equation

$$\Big(2a_{11}\xi - \frac{A_{22}}{\Delta}\Big)\Big(\xi^2 + 2a_{12}\xi - \frac{A_{33}}{\Delta}\Big) - \Big(a_{13}\xi + \frac{A_{23}}{\Delta}\Big)^2 = 0,$$

or, what is the same,

$$2\xi^3 - (a_{33} - 4a_{12})\xi^2 + 2\frac{(A_{12} - A_{33})}{\Delta}\xi + \frac{1}{\Delta} = 0. \quad (7.2)$$

The equation (7.2) is a cubic resolvent of the characteristic equation (3.1).

To investigate the roots of the equation (7.2), we consider the function

$$F(\xi) = \begin{vmatrix} a_{11} & a_{12} + \xi & a_{13} \\ a_{12} + \xi & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} - 2\xi \end{vmatrix} \equiv$$

$$\equiv 2\xi^3 - (a_{33} - 4a_{12})\xi^2 + 2\frac{(A_{12} - A_{33})}{\Delta}\xi + \frac{1}{\Delta}. \tag{7.3}$$

It is evident that the roots of the function F and those of the cubic resolvent coincide.

Vieta's theorem for the characteristic equation (taking into account that the roots are of the form (7.1)) can be written as

$$\begin{gathered}\frac{a_{13}}{a_{11}} = a_1 + a_2, \quad \frac{2a_{12} + a_{33}}{a_{11}} = (a_1 + a_2)^2 + (b_1 + b_2)^2 - 2(b_1 b_2 - a_1 a_2),\\ \frac{a_{23}}{a_{11}} = -(a_1 + a_2)(b_1 b_2 - a_1 a_2) + (b_1 + b_2)(a_1 b_2 + a_2 b_1),\\ \frac{a_{22}}{a_{11}} = (b_1 b_2 - a_1 a_2)^2 + (a_1 b_2 + a_2 b_1)^2,\end{gathered} \tag{7.4}$$

whence on the basis of the notation (5.5) we find that

$$\begin{aligned}(b_1 + b_2)^2 &= \frac{A_{22}}{\Delta a_{11}^2} + 2\omega,\\ (b_1 + b_2)(a_1 b_2 + a_2 b_1) &= -\frac{A_{23}}{\Delta a_{11}^2} + \frac{a_{13}}{a_{11}}\omega,\\ (a_1 b_2 + a_2 b_1)^2 &= \frac{A_{33}}{\Delta a_{11}^2} + 2\frac{a_{12}}{a_{11}}\omega - \omega^2.\end{aligned} \tag{7.5}$$

Now we can easily determine that $\xi = \xi_1 = -\omega a_{11}$ is a root of the equation (7.2). Thus

$$F(-\omega a_{11}) = 0. \tag{7.6}$$

As it has been proved (see (5.11)),

$$\omega a_{11} > 0. \tag{7.7}$$

To determine properties of the remaining roots of (7.2), we write the identities

$$\begin{aligned}a_{11}F(\xi) &= \Big(\xi^2 + 2a_{12}\xi - \frac{A_{33}}{\Delta}\Big)\Big(2a_{11}\xi - \frac{A_{22}}{\Delta}\Big) - \Big(a_{13}\xi + \frac{A_{23}}{\Delta}\Big)^2,\\ a_{22}F(\xi) &= \Big(\xi^2 + 2a_{12}\xi - \frac{A_{33}}{\Delta}\Big)\Big(2a_{22}\xi - \frac{A_{11}}{\Delta}\Big) - \Big(a_{23}\xi + \frac{A_{13}}{\Delta}\Big)^2.\end{aligned}$$

Since $a_{11} > 0$ and $a_{22} > 0$, the above identities result in

$$\begin{gathered}F\Big(\frac{A_{11}}{2\Delta a_{22}}\Big) < 0, \quad F\Big(\frac{A_{22}}{2\Delta a_{11}}\Big) < 0,\\ F(\sqrt{a_{11}a_{22}} - a_{12}) < 0, \quad F(-\sqrt{a_{11}a_{22}} - a_{12}) < 0.\end{gathered}$$

Taking also into account $F(0) = \frac{1}{\Delta} > 0$ and $\lim\limits_{\xi\to\omega\infty} F(\xi) = \pm\infty$, we easily conclude that the equation (7.2) has one negative and two positive roots.

Represent $F(\xi)$ in the form

$$F(\xi) = 2(\xi + \omega a_{11})\Big[\xi^2 - \Big(\frac{a_{33}}{2} - 2a_{12} + \omega a_{11}\Big)\xi + \frac{1}{2\Delta\omega a_{11}}\Big].$$

For the roots of the equation (7.2) we have

$$\begin{gathered} \xi_1 = -\omega a_{11} \quad \xi_2 = \frac{a_{33}}{4} - a_{12} + \frac{\omega a_{11}}{2} - \sqrt{D}, \\ \xi_3 = \frac{a_{33}}{4} - a_{12} + \frac{\omega a_{11}}{2} + \sqrt{D}, \end{gathered} \tag{7.8}$$

where

$$D = \Big(\frac{a_{33}}{4} - a_{12} + \frac{\omega a_{11}}{2}\Big)^2 - \frac{1}{2\Delta\omega a_{11}}.$$

Since ξ_2 and ξ_3 are positive, $a_{33} - 4a_{12} + 2\omega a_{11} > 0$.

We obtain

$$\begin{gathered} \xi_2 < \frac{A_{11}}{2\Delta a_{22}} < \xi_3, \quad \xi_2 < \frac{A_{22}}{2\Delta a_{11}} < \xi_3, \\ \xi_2 < \sqrt{a_{11}a_{22}} - a_{12} < \xi_3, \quad \omega a_{11} < \sqrt{a_{11}a_{22}} + a_{12}. \end{gathered} \tag{7.9}$$

If we take into account the fact that $\xi_1 = -\omega a_{11}$ satisfies (7.2), then we have

$$\Delta a_{33}\omega^2 a_{11}^2 + 2A_{12}\omega a_{11} - 1 = 2\omega\Delta a_{11}^3(a_1b_2 + a_2b_1)^2 > 0,$$

and hence

$$\omega a_{11} > \frac{1}{\sqrt{A_{11}A_{22}} + A_{12}}.$$

Thus for ωa_{11}, by virtue of (7.9), we obtain

$$\frac{1}{\sqrt{A_{11}A_{22}} + A_{12}} < \omega a_{11} < \sqrt{a_{11}a_{22}} + a_{12}. \tag{7.10}$$

In the case $a_1b_2 + a_2b_1 = 0$, in particular for an orthotropic and isotropic body, the inequalities (7.10) can be replaced by the equalities.

Introduce now the quantity

$$\omega_0 = 2b_1b_2 - \omega \tag{7.11}$$

and prove that $\omega_0 > 0$. To this end, we rewrite the identities (7.5) as

$$\begin{gathered} (b_1 - b_2)^2 = \frac{A_{22}}{\Delta a_{11}^2} - 2\omega_0, \\ (b_1 - b_2)(a_1b_2 - a_2b_1) = \frac{A_{23}}{\Delta a_{11}^2} + \frac{a_{13}}{a_{11}}\omega_0, \\ (a_1b_2 - a_2b_1)^2 = \frac{A_{33}}{\Delta a_{11}^2} - 2\frac{a_{12}}{a_{11}}\omega_0 - \omega_0^2. \end{gathered} \tag{7.12}$$

We can easily see that $a_{11}\omega_0$ is a root of (7.2). But this equation has three roots which are defined by (7.7). If, for example, $a_{11}\omega_0 = \xi_1 = -\omega a_{11}$, then we obtain $b_1b_2 = 0$ which is impossible. Consequently, $a_{11}\omega_0$ coincides either with ξ_2 or with ξ_3. From the first relation of (7.12) we have $a_{11}\omega_0 < \frac{A_{22}}{2\Delta a_{11}}$, and taking into account (7.9) we conclude that

$$a_{11}\omega_0 = \xi_2. \tag{7.13}$$

Thus

$$2b_1b_2 - \omega = \frac{a_{33}}{4a_{11}} - \frac{a_{12}}{a_{11}} + \frac{\omega}{2} - \sqrt{\left(\frac{a_{33}}{4a_{11}} - \frac{a_{12}}{a_{11}} + \frac{\omega}{2}\right)^2 - \frac{1}{2\Delta\omega a_{11}^3}} > 0. \tag{7.14}$$

Express now the roots of the characteristic equation (7.1) through the roots of the cubic resolvent, ξ_1, ξ_2, and ξ_3.

From (7.5) and (7.12) we have

$$b_1 + b_2 = \sqrt{\frac{A_{22}}{\Delta a_{11}^2} + 2\omega}, \quad b_1 - b_2 = \pm\sqrt{\frac{A_{22}}{\Delta a_{11}^2} - 2\omega_0}.$$

Without loss of generality we will assume that $b_1 < b_2$. Then

$$2b_k = \sqrt{\frac{A_{22}}{\Delta a_{11}^2} + 2\omega} - (-1)^k\sqrt{\frac{A_{22}}{\Delta a_{11}^2} - 2\omega_0}, \quad k = 1, 2. \tag{7.15}$$

Next we determine a_1 and a_2. As is known, $\xi_1 = -\omega a_{11}$ and $\xi_2 = a_{11}\omega_0$ are the roots of the cubic resolvent. To find the third root, we use Vieta's theorem for the equation (7.2). It is not difficult to see that

$$\xi_3 = \frac{a_{33}}{2} - 2a_{11}a_1a_2 = \frac{a_{33}}{4} - a_{12} + \frac{\omega a_{11}}{2} + \\ + \sqrt{\left(\frac{a_{33}}{4} - a_{12} + \frac{\omega a_{11}}{2}\right)^2 - \frac{1}{2\Delta\omega a_{11}}}.$$

We have two conditions for finding a_1 and a_2:

$$a_1 + a_2 = \frac{a_{13}}{a_{11}}, \quad 2a_1a_2 = \frac{a_{33}}{2a_{11}} - \frac{\xi_3}{a_{11}},$$

whence

$$2a_k = \frac{a_{13}}{a_{11}} - (-1)^k\sqrt{2\frac{\xi_3}{a_{11}} - \frac{A_{22}}{\Delta a_{11}^2}}, \quad k = 1, 2; \tag{7.16}$$

here the radicand is positive because of (7.9).

Thus, the formulas (7.15) and (7.16) express the roots of the characteristic equation by the roots of the cubic resolvent.

In Section 2 we have studied the question of positive definiteness of the generalized potential energy and have written necessary and sufficient conditions in terms of the inequalities (2.12) or (2.21).

Among generalized potential energies, the most important is (except the case where $\varkappa = 0$) the energy which corresponds to the operator N, i.e., the case where $\varkappa$ is defined by (5.30).

Let us prove that the conditions (2.21) are valid for the values of the parameter $\varkappa$ defined by (5.30). It is evident that the first and the fourth inequalities from (2.21) are fulfilled. It remains to prove the second and the third ones.

Taking into account (2.19) and (7.3), it is not difficult to notice that

$$f\Big(\frac{1}{\xi}\Big) = \frac{\Delta F(\xi)}{\xi^3}. \tag{7.17}$$

This means that if ξ_0 is a root of the equation (7.2), then since $\xi_0 \neq 0$, $\frac{1}{\xi_0}$ will be a root of the equation $f(\varkappa) = 0$.

Thus

$$\varkappa_1 = \frac{1}{\xi_1} = -\frac{1}{\omega a_{11}}, \quad \varkappa_2 = \frac{1}{\xi_3}, \quad \varkappa_3 = \frac{1}{\xi_2}. \tag{7.18}$$

For the roots ξ_2 and ξ_3 one can obtain other expressions as well. Indeed, since $-\omega a_{11}$ is a root of (7.2),

$$2(\omega a_{11})^3 + (a_{33} - 4a_{12})(\omega a_{11})^2 + 2\frac{(A_{12} - A_{23})}{\Delta}\omega a_{11} - \frac{1}{\Delta} = 0,$$

whence it follows that

$$\Big(\frac{a_{33}}{4} - a_{12} + \frac{\omega a_{11}}{2}\Big)(\omega a_{11})^2 = \frac{1 + 2\omega a_{11}(A_{33} - A_{12})}{4\Delta}$$

and

$$\frac{a_{33}}{4} - a_{12} + \frac{\omega a_{11}}{2} = \frac{1 + 2(A_{33} - A_{12})\omega a_{11}}{4\Delta(\omega a_{11})^2}.$$

Substituting these expressions in (7.8), we obtain

$$2\Delta\omega a_{11}\xi_{2,3} = A_{33} - A_{12} + \frac{1}{2\omega a_{11}} \mp \sqrt{\Big(A_{33} - A_{12} + \frac{1}{2\omega a_{11}}\Big)^2 - 2\Delta\omega a_{11}},$$

and by (7.18) we easily find that

$$\begin{aligned} \varkappa_2 &= A_{33} - A_{12} + \frac{1}{2\omega a_{11}} - \sqrt{\Big(A_{33} - A_{12} + \frac{1}{2\omega a_{11}}\Big)^2 - 2\Delta\omega a_{11}}, \\ \varkappa_3 &= A_{33} - A_{12} + \frac{1}{2\omega a_{11}} + \sqrt{\Big(A_{33} - A_{12} + \frac{1}{2\omega a_{11}}\Big)^2 - 2\Delta\omega a_{11}}. \end{aligned} \tag{7.19}$$

Taking into account the identities

$$\begin{aligned} A_{11}f(\varkappa) &= (2\Delta a_{22} - A_{11}\varkappa)(\Delta a_{33} - 2A_{12}\varkappa - \varkappa^2) - 2(\Delta a_{23} + A_{13}\varkappa)^2, \\ A_{22}f(\varkappa) &= (2\Delta a_{11} - A_{22}\varkappa)(\Delta a_{33} - 2A_{12}\varkappa - \varkappa^2) - 2(\Delta a_{13} + A_{23}\varkappa)^2, \end{aligned}$$

we have

$$f\big(-\sqrt{A_{11}A_{22}} - A_{12}\big) < 0, \quad f\Big(\frac{2\Delta a_{11}}{A_{22}}\Big) < 0,$$

$$f\Big(\frac{2\Delta a_{22}}{A_{11}}\Big) < 0, \quad f\big(\sqrt{A_{11}A_{22}} - A_{12}\big) < 0.$$

Thus we conclude that

$$\begin{gathered} \varkappa_2 < \sqrt{A_{11}A_{22}} - A_{12} < \varkappa_3, \\ \varkappa_2 < \frac{2\Delta a_{11}}{A_{22}} < \varkappa_3, \quad \varkappa_2 < \frac{2\Delta a_{22}}{A_{11}} < \varkappa_3. \end{gathered} \tag{7.20}$$

Rewrite $f(\varkappa)$ as

$$f(\varkappa) = \Big(\varkappa + \frac{1}{\omega a_{11}}\Big)\Big[\varkappa^2 - 2\Big(A_{33} - A_{12} + \frac{1}{2\omega a_{11}}\Big)\varkappa + 2\Delta\omega a_{11}\Big]. \tag{7.21}$$

From (5.10) and (5.18) we obtain

$$A_{33} - A_{12} + \frac{1}{\omega a_{11}} = \frac{\Delta a_{11} m}{BC - A^2}. \tag{7.22}$$

Using the above formula in (7.21), after direct calculation we get

$$f(\varkappa_N) = \frac{BC - A^2}{\omega m^3} > 0. \tag{7.23}$$

Thus it remains to prove the second inequality from (2.21).

If we notice that $2(A_{33} - A_{12} + \frac{1}{2\omega a_{11}}) = \varkappa_2 + \varkappa_3$, then we find $A_{33} - A_{12} = \frac{\varkappa_2 + \varkappa_3}{2} - \frac{1}{2\omega a_{11}}$.

Substituting this equality in (7.22) and taking into account (5.30), we obtain

$$\frac{\varkappa_2 + \varkappa_3}{2} + \frac{1}{2\omega a_{11}} = \frac{\Delta\omega a_{11}}{\varkappa_N}.$$

Since $\varkappa_2\varkappa_3 = 2\Delta\omega a_{11}$, we have

$$\frac{1}{\varkappa_N} = \frac{1}{\varkappa_2} + \frac{1}{\varkappa_3} + \frac{1}{\omega a_{11}\varkappa_2\varkappa_3}. \tag{7.24}$$

From (7.24) we find $\frac{1}{\varkappa_N} > \frac{1}{\varkappa_2}$, or, taking into account (7.20), $\varkappa_N < \varkappa_2 < \sqrt{A_{11}A_{22}} - A_{12}$.

Using the above inequality, we obtain

$$\Delta a_{33} - 2A_{12}\varkappa_N - \varkappa_N^2 = \big(\sqrt{A_{11}A_{22}} + A_{12} + \varkappa_N\big)\big(\sqrt{A_{11}A_{22}} - A_{12} - \varkappa_N\big) > 0.$$

Thus for $\varkappa$ given by (5.30), the inequalities (2.21) are valid.

The contents of this section can be found in the author's work Basheleishvili [12].

8. Generalized Cauchy–Riemann Conditions

The vector functions (U,V), where $U=(U_1,U_2)$ and $V=(V_1,V_2)$, are said to be conjugate, if they satisfy the generalized Cauchy–Riemann conditions

$$\begin{aligned}
\tau_{11}(U)&\equiv\sigma_{x_1}(U)+\varkappa_N\frac{\partial U_2}{\partial x_2}=\frac{1}{m}\Big(B\frac{\partial V_1}{\partial x_2}-A\frac{\partial V_2}{\partial x_2}\Big),\\
\tau_{22}(U)&\equiv\sigma_{x_2}(U)+\varkappa_N\frac{\partial U_1}{\partial x_1}=\frac{1}{m}\Big(A\frac{\partial V_1}{\partial x_1}-C\frac{\partial V_2}{\partial x_1}\Big),\\
\tau_{12}(U)&\equiv\tau_{x_1x_2}(U)-\varkappa_N\frac{\partial U_2}{\partial x_1}=\frac{1}{m}\Big(-B\frac{\partial V_1}{\partial x_1}+A\frac{\partial V_2}{\partial x_1}\Big),\\
\tau_{21}(U)&\equiv\tau_{x_1x_2}(U)-\varkappa_N\frac{\partial U_1}{\partial x_2}=\frac{1}{m}\Big(-A\frac{\partial V_1}{\partial x_2}+C\frac{\partial V_2}{\partial x_2}\Big),
\end{aligned}\tag{8.1}$$

where $\sigma_{x_1}(U)$, $\sigma_{x_2}(U)$, and $\tau_{x_1x_2}(U)$ are the stress vector components defined by (1.1) and (1.2), $\varkappa_N$ is given by (5.30), and the coefficients A,B,C, and m are defined respectively by (4.15) and (5.17).

From (8.1) by elementary calculations we obtain

$$\begin{aligned}
\frac{\partial\tau_{11}(U)}{\partial x_1}+\frac{\partial\tau_{12}(U)}{\partial x_2}&\equiv\big(C(\partial x)U\big)_1=0,\\
\frac{\partial\tau_{21}(U)}{\partial x_1}+\frac{\partial\tau_{22}(U)}{\partial x_2}&\equiv\big(C(\partial x)U\big)_2=0,
\end{aligned}$$

where $C(\partial x)$ is given by (1.6).

Thus we have proved that if the vector functions U and V satisfy the conditions (8.1), then U for any twice continuously differentiable V is a solution of the homogeneous system of equations of statics of an anisotropic elastic body.

Introduce the vector functions:

$$\varphi=\Big(\frac{\partial U_1}{\partial x_1},\frac{\partial U_2}{\partial x_2},\frac{\partial U_1}{\partial x_2},\frac{\partial U_2}{\partial x_1}\Big),\quad \psi=\Big(\frac{\partial V_1}{\partial x_1},\frac{\partial V_2}{\partial x_2},\frac{\partial V_1}{\partial x_2},\frac{\partial V_2}{\partial x_1}\Big).\tag{8.2}$$

Then (8.1) can be rewritten (in a matrix form) as

$$a\varphi=b\psi,\tag{8.3}$$

where

$$a=\begin{Vmatrix}A_{11}, & A_{12}+\varkappa_N, & A_{13}, & A_{13}\\ A_{12}+\varkappa_N, & A_{22}, & A_{23}, & A_{23}\\ A_{13}, & A_{23}, & A_{33}, & A_{33}-\varkappa_N\\ A_{13}, & A_{23}, & A_{33}-\varkappa_N, & A_{33}\end{Vmatrix},\tag{8.4}$$

$$b=\frac{1}{m}\begin{Vmatrix}0, & -A, & B, & 0\\ A, & 0, & 0, & -C\\ -B, & 0, & 0, & A\\ 0, & C, & -A, & 0\end{Vmatrix}.\tag{8.5}$$

From (8.3) we have

$$\psi = b^{-1} a \varphi, \tag{8.6}$$

where

$$b^{-1} = \frac{m}{BC - A^2} \begin{Vmatrix} 0, & -A, & -C, & 0 \\ A, & 0, & 0, & B \\ C, & 0, & 0, & A \\ 0, & -B, & -A, & 0 \end{Vmatrix}. \tag{8.7}$$

Application of (8.6) yields

$$a\psi = ab^{-1} a\varphi \equiv d\varphi. \tag{8.8}$$

Our aim is to simplify the matrix d. Taking into account (7.22) and the resulting identities

$$\Big(\frac{1}{\omega a_{11}} + A_{33} - A_{12}\Big)\varkappa_N - \Delta\omega a_{11} = 0,$$

$$\varkappa_N^2 - \Big[\frac{1}{\omega a_{11}} + 2(A_{33} - A_{12})\Big]\varkappa_N + 2\Delta\omega a_{11} = \frac{BC - A^2}{m^2},$$

$$\varkappa_N^2 + (A_{12} - A_{33})\varkappa_N + \Delta\omega a_{11} = \frac{BC - A^2}{m},$$

after long but elementary transformations we find that $d = -b$, and (8.8) takes the form $a\psi = -b\varphi$, that is,

$$\begin{aligned} \tau_{11}(V) &\equiv \sigma_{x_1}(V) + \varkappa_N \frac{\partial V_2}{\partial x_2} = -\frac{1}{m}\Big(B\frac{\partial U_1}{\partial x_2} - A\frac{\partial U_2}{\partial x_2}\Big), \\ \tau_{22}(V) &\equiv \sigma_{x_2}(V) + \varkappa_N \frac{\partial V_1}{\partial x_1} = -\frac{1}{m}\Big(A\frac{\partial U_1}{\partial x_1} - C\frac{\partial U_2}{\partial x_1}\Big), \\ \tau_{12}(V) &\equiv \tau_{x_1x_2}(V) - \varkappa_N \frac{\partial V_2}{\partial x_1} = -\frac{1}{m}\Big(-B\frac{\partial U_1}{\partial x_1} + A\frac{\partial U_2}{\partial x_1}\Big), \\ \tau_{21}(V) &\equiv \tau_{x_1x_2}(V) - \varkappa_N \frac{\partial V_1}{\partial x_2} = -\frac{1}{m}\Big(-A\frac{\partial U_1}{\partial x_2} + C\frac{\partial U_2}{\partial x_2}\Big). \end{aligned} \tag{8.9}$$

As above, by virtue of (8.9) we can prove that for any twice continuously differentiable U the vector function V is a solution of the equation (1.9).

On the other hand, we have proved that if U and V are conjugate vectors, then such are the vectors V and $-U$.

We call the conditions (8.1) the generalized Cauchy–Riemann conditions for equations of statics of an anisotropic elastic body.

In the case of an isotropic elastic body, the conditions (8.1) become simpler and take the form

$$\begin{aligned} &\frac{1+a}{2a}\frac{\partial U_1}{\partial x_1} + \frac{b}{2a}\frac{\partial U_2}{\partial x_2} = \frac{\partial V_1}{\partial x_2}, \quad \frac{b}{2a}\frac{\partial U_1}{\partial x_1} + \frac{1+a}{2a}\frac{\partial U_2}{\partial x_2} = -\frac{\partial V_2}{\partial x_1}, \\ &\frac{1+a}{2}\frac{\partial U_1}{\partial x_1} + \frac{b}{2}\frac{\partial U_2}{\partial x_1} = -\frac{\partial V_1}{\partial x_1}, \quad \frac{b}{2}\frac{\partial U_1}{\partial x_2} + \frac{1+a}{2}\frac{\partial U_2}{\partial x_1} = \frac{\partial V_2}{\partial x_2}, \end{aligned} \tag{8.10}$$

where

$$a = \mu(\lambda + 2\mu)^{-1}, \quad b = (\lambda + \mu)(\lambda + 2\mu)^{-1}, \quad a + b = 1. \tag{8.11}$$

If we formally suppose that $\lambda + \mu = 0$, $\mu = 1$, i.e., $a = 1$, $b = 0$, then (8.10) will take the form

$$\frac{\partial U_1}{\partial x_1} = \frac{\partial V_1}{\partial x_2}, \quad \frac{\partial U_2}{\partial x_2} = -\frac{\partial V_2}{\partial x_1}, \quad \frac{\partial U_1}{\partial x_2} = -\frac{\partial V_1}{\partial x_1}, \quad \frac{\partial U_2}{\partial x_1} = \frac{\partial V_2}{\partial x_2}, \tag{8.12}$$

that is, we obtain the usual Cauchy–Riemann conditions for harmonic vector functions U and V.

Multiplying in (8.1) the first equality by n_1, the third one by n_2 and then summing up, we obtain

$$(Nu)_1 = \frac{1}{m}\Big(B\frac{\partial V_1}{\partial S} - A\frac{\partial V_2}{\partial S}\Big).$$

Analogously, multiplying in (8.1) the second equality by n_2 the fourth one by n_1, then summing up, we arrive at

$$(Nu)_2 = \frac{1}{m}\Big(- A\frac{\partial V_1}{\partial S} + C\frac{\partial V_2}{\partial S}\Big).$$

Here N is the operator defined by (5.31). We rewrite the last two formulas as

$$NU = \frac{1}{m}\begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}\frac{\partial V}{\partial S}. \tag{8.13}$$

Using now (8.9), we can easily see that along with (8.13), the following formula is also valid:

$$NV = -\frac{1}{m}\begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}\frac{\partial U}{\partial S}. \tag{8.14}$$

Hence the conjugate vector functions U and V satisfy (8.13) and (8.14).

Introduce now the notation

$$W = U + iV. \tag{8.15}$$

Then (8.13) and (8.14) can be written in the form of one vector relation

$$NW = -\frac{i}{m}\begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}\frac{\partial W}{\partial S}. \tag{8.16}$$

Taking into account (1.23) and (5.31), we obtain

$$\overset{\varkappa_0}{T}W = NW + (\varkappa_0 - \varkappa_N)\begin{Vmatrix} 0, & 1 \\ -1, & 0 \end{Vmatrix}\frac{\partial W}{\partial S}$$

for $\varkappa = \varkappa_0$. Substituting in it the expression NW from (8.16), we find that

$$\overset{\varkappa_0}{T}W = \left[-\frac{i}{m}\begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix} + (\varkappa_0 - \varkappa_N)\begin{Vmatrix} 0, & 1 \\ -1, & 0 \end{Vmatrix}\right]\frac{\partial W}{\partial S}, \tag{8.17}$$

whence

$$\frac{\partial W}{\partial S}=\frac{\left[-\frac{i}{m}\begin{Vmatrix} C, & A\\ A, & B\end{Vmatrix}+(\varkappa_0-\varkappa_N)\begin{Vmatrix} 0, & -1\\ 1, & 0\end{Vmatrix}\right]}{(\varkappa_0-\varkappa_N)^2-\frac{BC-A^2}{m^2}}\overset{\varkappa_0}{T}W. \tag{8.18}$$

First we write (8.17) for $\varkappa_0=\varkappa$ and then insert in it the value $\frac{\partial W}{\partial S}$ from (8.18). We have

$$\overset{\varkappa}{T}W=\frac{\frac{i(\varkappa_0-\varkappa)}{m}\begin{Vmatrix} A, & B\\ -C, & -A\end{Vmatrix}+\left[(\varkappa_0-\varkappa_N)(\varkappa_0-\varkappa_N)-\frac{BC-A^2}{m^2}\right]E}{(\varkappa_0-\varkappa_N)^2-\frac{BC-A^2}{m^2}}\overset{\varkappa_0}{T}W,$$

where E is the unit matrix. The last formula has a comparatively simple form if

$$(\varkappa-\varkappa_N)(\varkappa_0-\varkappa_N)=\frac{BC-A^2}{m^2}. \tag{8.19}$$

In this case, after certain simplifications we obtain

$$\overset{\varkappa}{T}W=\frac{i\begin{Vmatrix} A, & B\\ -C, & -A\end{Vmatrix}}{m\varkappa_0-\omega(BC-A^2)}\overset{\varkappa_0}{T}W. \tag{8.20}$$

The equation (8.19) yields

$$\varkappa a_{11}=\frac{(1+\omega a_{11}\varkappa_0)(BC-A^2)}{m\varkappa_0-\omega(BC-A^2)}. \tag{8.21}$$

Now we introduce the following definition: $\overset{\varkappa}{T}$ and $\overset{\varkappa_0}{T}$ are said to be conjugate operators, if the condition (8.19) is fulfilled.

The formula (8.20) is always valid if $m\varkappa_0-\omega(BC-A^2)\neq 0$, i.e., $\varkappa_0\neq\varkappa_N$. This allows one to conclude that the operator N has no conjugate operator of the type $\overset{\varkappa}{T}$.

In this exceptional case, by definition we assume that the conjugate to the operator N is $\frac{\partial}{\partial S}$, which is in agreement with the equatily (8.16).

Denote by L the operator obtained from $\overset{\varkappa_0}{T}$ when $\varkappa_0=-\frac{1}{\omega a_{11}}$. Then from (8.21) we find that $\varkappa=0$, i.e., $\overset{\varkappa}{T}\equiv T$, and (8.20) takes the form

$$TW=i\omega\begin{Vmatrix} -A, & -B\\ C, & A\end{Vmatrix}LW,$$

i.e.,

$$TU=\omega\begin{Vmatrix} A, & B\\ -C, & -A\end{Vmatrix}LV,\quad TV=\omega\begin{Vmatrix} -A, & -B\\ C, & A\end{Vmatrix}LU. \tag{8.22}$$

As is known (see Section 2), the potential energy corresponding to the operator T is positive definite. It is proved that the energy for the operator L is not, in general, positive definite. But the formulas (8.22) make it possible to investigate the problems with the operator L by means of the operator T. The formula (8.20) gives infinitely many such possibilities. Thus, if $0 \leq \varkappa_0 < \varkappa_N$ or $\varkappa_N < \varkappa_0 \leq \varkappa_2$, then the corresponding generalized potential energy is positive definite. In the case $-\infty < \varkappa \leq -\frac{1}{\omega a_{11}}$ or $\varkappa_3 \leq \varkappa < \infty$, the energy is not positive definite. But one can study problems for the operator $\overset{\varkappa}{T}$ with the help of the operator $\overset{\varkappa_0}{T}$, if (8.21) takes place.

The results of this section have been published by the author in [14].

9. General Representation of the Solution

First let us prove the following

Theorem 1. *If $S \in C^{1,\beta}$, $0 < \beta \leq 1$, and V is a regular solution of the equation* (1.8) *in the domain D^+, then*

$$V(x) = \frac{1}{2\pi}\int_S \Big[\overset{\varkappa}{T}(y,x)(V)^+ - \Gamma(x,y)(\overset{\varkappa}{T}V)^+\Big]dS - \\ - \frac{1}{2\pi}\int_{D^+}\Gamma(x,y)F(y)d\sigma, \quad x \in D^+, \tag{9.1}$$

where $\Gamma(x,y)$ is the fundamental matrix defined by (4.12) *and $\overset{\varkappa}{T}(y,x)$ is given by* (5.27).

Proof. Consider the circumference $S(x,\varepsilon)$ of radius ε with the center $x \in D^+$ and suppose that the circle $K(x,\varepsilon)$ lies completely in D^+. Consider the domain $D_\varepsilon = D^+ \setminus \overline{K(x,\varepsilon)}$. It is evident that $V(y) = \Gamma^{(j)}(x,y)$ (the j-th column of the matrix $\Gamma(x,y)$ is denoted by $\Gamma^{(j)}(x,y)$) is a regular solution of the equation (1.9) in D_ε. Using (2.7) for the domain D_ε and taking into account the equalities $C(\partial y)\Gamma^{(j)}(x,y) = 0$, $C(\partial y)V(y) = -F(y)$, we obtain

$$\int_{D_\varepsilon}\Gamma^{(j)}(y-x)F(y)d\sigma = \\ = \int_S\Big[(V(y))^+\overset{\varkappa}{T}(\partial y,n(y))\Gamma^{(j)}(y,x) - \Gamma^{(j)}(y,x)(\overset{\varkappa}{T}V)^+\Big]dS + \\ + \int_{S(x,\varepsilon)}\Big[(V(y))\overset{\varkappa}{T}(\partial y,n(y))\Gamma^{(j)}(y,x) - \Gamma^{(j)}(y,x)\overset{\varkappa}{T}V\Big]dS. \tag{9.2}$$

In what follows, we will need the value of the integral

$$I_k(x) = \int_S \frac{\partial \ln \sigma_k}{\partial S(y)}dS, \tag{9.3}$$

where $\sigma_k = x_1 - y_1 + \alpha_k(x_2 - y_2)$, $k = 1,2$.

Using for the domain D_ε the identities

$$0 = \int_{D^+}\Big(\frac{\partial^2 U}{\partial y_1\partial y_2} - \frac{\partial^2 U}{\partial y_2 \partial y_1}\Big)d\sigma = \int_S \frac{\partial U}{\partial S}dS,$$

we obtain

$$\int_S \frac{\partial \ln \sigma_k}{\partial S(y)}\, dS + \int_{S(x,\varepsilon)} \frac{\partial \ln \sigma_k}{\partial S(y)}\, dS = 0. \tag{9.4}$$

Obviously, for $y \in S(x,\varepsilon)$ we have

$$y_1 - x_1 = \varepsilon \cos\varphi, \quad y_2 - x_2 = \varepsilon \sin\varphi, \quad d_y S = \varepsilon\, d\varphi,$$
$$n_1(y) = -\cos\varphi, \quad n_2(y) = -\sin\varphi,$$

and after certain simplifications we arrive at

$$\frac{\partial \ln \sigma_k}{\partial S(y)}\, dS = \frac{1}{i}\left[1 + 2\sum_{n=1}^{\infty} (-1)^n \lambda_k^n e^{-2in\varphi}\right] d\varphi, \tag{9.5}$$

where

$$\lambda_k = \frac{1 + i\alpha_k}{1 - i\alpha_k}, \quad |\lambda_k| < 1, \quad k = 1, 2.$$

Making use of the last formula, we obtain

$$\int_{S(x,\varepsilon)} \frac{\partial \ln \sigma_k}{\partial S(y)}\, dS = -2\pi i, \tag{9.6}$$

and from (9.4) we have

$$\int_S \frac{\partial \ln \sigma_k}{\partial S(y)}\, dS = 2\pi i, \quad x \in D^+. \tag{9.7}$$

Analogously,

$$\int_S \frac{\partial \ln \sigma_k}{\partial S(y)}\, dS = \begin{cases} \pi i, & x \in S, \\ 0, & x \in D^-. \end{cases} \tag{9.8}$$

Taking into consideration (9.6), after some transformations we prove that

$$\lim_{\varepsilon \to 0} \int_{S(x,\varepsilon)} V(y) \overset{\varkappa}{T}(\partial y, n(y)) \Gamma^{(j)}(y,x)\, dS = 2V_j(x).$$

Obviously,

$$\lim_{\varepsilon \to 0} \int_{S(x,\varepsilon)} \Gamma^{(j)}(y,x) \overset{\varkappa}{T} V(y) dS = 0.$$

The equation (9.1) now follows from (9.2).

If $F = 0$, then (9.1) takes the form

$$V(x) = \frac{1}{2\pi} \int_S \left[\overset{\varkappa}{T}(y,x)(V)^+ - \Gamma(y,x)(\overset{\varkappa}{T}V)^+\right] dS, \quad x \in D^+. \tag{9.9}$$

By virtue of (9.8), we have

$$0 = \frac{1}{2\pi} \int_S \left[\overset{\varkappa}{T}(y,x)(V)^+ - \Gamma(y,x)(\overset{\varkappa}{T}V)^+\right] dS, \quad x \in D^-. \tag{9.10}$$

Introduce the notation

$$V^{(0)}(x) = -\frac{1}{2\pi} \int_{D^+} \Gamma(x,y) F(y)\, d\sigma, \quad x \in D^+, \tag{9.11}$$

and prove that $V^{(0)}(x)$ is a particular solution of the equation (1.8) if F is Hölder continuous.

We write (see Section 4) that

$$\Gamma(x,y) = \begin{Vmatrix} C_{22}(\partial x), & -C_{12}(\partial x) \\ -C_{12}(\partial x), & C_{11}(\partial x) \end{Vmatrix} \Big(-\frac{1}{\Delta a_{11}} \Big) \operatorname{Im} \sum_{k=1}^{2} d_k \sigma_k^2 \ln \sigma_k.$$

Then

$$C(\partial x) V^{(0)}(x) = \frac{1}{2\pi a_{11}} \mathcal{D}(\partial x) \int_{D^+} \operatorname{Im} \sum_{k=1}^{2} d_k \sigma_k^2 \ln \sigma_k F(y)\, d\sigma, \tag{9.12}$$

where $\mathcal{D}(\partial x)$ is given by (4.4).

It is easy to note that $\mathcal{D}(\partial x)$ can be represented as

$$\mathcal{D}(\partial x) = a_{11} \Delta_1(\partial x) \Delta_2(\partial x),$$

where

$$\Delta_k(\partial x) = \frac{\partial^2}{\partial x_2^2} - 2a_k \frac{\partial^2}{\partial x_1 \partial x_2} + (a_k^2 + b_k^2) \frac{\partial^2}{\partial x_1^2}, \quad k = 1, 2.$$

Making use of the expression $\mathcal{D}(\partial x)$ from (9.12), we obtain

$$C(\partial x) V^{(0)}(x) = -\frac{1}{2\pi b_1} \Delta_1(\partial x) \int_{D^+} \ln r_1 F(y)\, d\sigma, \tag{9.13}$$

where $r_1 = \sqrt{[x_1 - y_1 + a_1(x_2 - y_2)]^2 + b_1^2 (x_2 - y_2)^2}$.

Applying now a result obtained by Vekua [51] for a plane body potential in the right-hand side of (9.13), we conclude that the above proposition is valid for the potential (9.11).

The representation (9.9) (as well as (9.10)) holds for every $\varkappa$. We take $\varkappa = \varkappa_N$ and apply the relation (8.14). In view of the fact that the elements of the matrix $\Gamma(y,x)$ and the components of the vector U are one-valued functions, integration by parts gives

$$V(x) = \frac{1}{2\pi} \int_S \big[N(y,x)(V)^+ - \widetilde{N}(y,x)(U)^+ \big]\, dS, \tag{9.14}$$

where

$$\widetilde{N}(y,x) = \operatorname{Re} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial S(y)}.$$

Finally, taking (5.32) into account, we can rewrite (9.14) as

$$V(x)=\frac{1}{2\pi}\int_S \operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial S(y)}[(V)^+-i(U)^+]dS,\ x\in D^+. \quad (9.15)$$

Analogously, for the conjugate vector function U we have

$$U(x)=\frac{1}{2\pi}\int_S \operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial S(y)}[(V)^+-i(U)^+]dS,\ x\in D^+. \quad (9.16)$$

Bearing in mind the notation (8.15), with regard for (9.10) we combine (9.15) and (9.16) and obtain

$$W(x)=\frac{1}{2\pi i}\int_S \sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial S(y)}[W(y)]^+dS,\quad x\in D^+, \quad (9.17)$$

$$0=\frac{1}{2\pi i}\int_S \sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial S(y)}[W(y)]^+dS,\quad x\in D^-. \quad (9.18)$$

In a similar way, for the infinite domain D^- we have

$$W(x)=W(\infty)-\frac{1}{2\pi i}\int_S \sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial S(y)}[W(y)]^-dS,\ x\in D^-, \quad (9.19)$$

$$W(x)=W(\infty),\quad x\in D^+, \quad (9.20)$$

where $W(\infty)=\lim\limits_{|x|\to+\infty}W(x)$.

The formulas (9.17), (9.18) and (9.19), (9.20) are the generalized Cauchy integral formulas for the domains D^+ and D^-.

Consequently, for any regular solution of the equation (1.8) we obtain in the domain D^+ a general representation in the form (9.1). Similar representation takes place in the domain D^- as well. ■

10. Generalized Cauchy Formulas

In the previous section we have obtained generalized Cauchy formulas (9.17)–(9.20). In order for these formulas to be valid, it is necessary that the vectors U and V, that is W, be regular in the corresponding domain. To prove the above formulas, we will need some relations and identities between the coefficients.

Substituting in (5.33) the values of the coefficients N_k, M_k, L_k, R_k from (5.3) and taking into account (5.30), we will have

$$-\begin{Vmatrix} E_k, & G_k\\ F_k, & H_k\end{Vmatrix}=$$

$$=\begin{Vmatrix} A_{13}+A_{33}\alpha_k, & A_{33}-\varkappa_N+A_{23}\alpha_k\\ A_{12}+\varkappa_N+A_{23}\alpha_k, & A_{23}+A_{22}\alpha_k\end{Vmatrix}\begin{Vmatrix} A_k, & B_k\\ B_k, & C_k\end{Vmatrix}. \quad (10.1)$$

Bearing in mind the identities $A_kC_k - B_k^2 = 0$, $k = 1, 2$, after certain transformations we find from (10.1) that

$$\begin{Vmatrix} E_k, & G_k \\ F_k, & H_k \end{Vmatrix} \alpha_k =$$

$$= \begin{Vmatrix} A_{11}+A_{13}\alpha_k, & A_{13}+(A_{12}+\varkappa_N)\alpha_k \\ A_{13}+(A_{33}-\varkappa_N)\alpha_k, & A_{33}+A_{23}\alpha_k \end{Vmatrix} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix}. \tag{10.2}$$

Introduce (for the time being) the notation

$$\sum_{k=1}^{2} \alpha_k A_k = X_0, \quad \sum_{k=1}^{2} \alpha_k B_k = Y_0, \quad \sum_{k=1}^{2} \alpha_k C_k = Z_0. \tag{10.3}$$

Taking now (5.16) and (5.35) into consideration, from (10.1) we get

$$-E = \frac{im}{BC - A^2} \begin{Vmatrix} A_{13}, & A_{33} - \varkappa_N \\ A_{12} + \varkappa_N, & A_{23} \end{Vmatrix} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} +$$

$$+ \begin{Vmatrix} A_{33}, & A_{23} \\ A_{23}, & A_{22} \end{Vmatrix} \begin{Vmatrix} X_0, & Y_0 \\ Y_0, & Z_0 \end{Vmatrix}.$$

We multiply the last expression on the left-hand side by the matrix

$$\frac{1}{\Delta A_{11}} \begin{Vmatrix} A_{22}, & -A_{23} \\ -A_{23}, & A_{33} \end{Vmatrix}$$

and obtain

$$\begin{Vmatrix} X_0, & Y_0 \\ Y_0, & Z_0 \end{Vmatrix} = -\frac{1}{\Delta a_{11}} \begin{Vmatrix} A_{22}, & -A_{23} \\ -A_{23}, & A_{33} \end{Vmatrix} + \frac{im}{\Delta a_{11}(BC - A^2)} \times$$

$$\times \begin{Vmatrix} \Delta a_{13} + A_{23}\varkappa_N, & A_{22}\varkappa_N - \Delta a_{11} \\ \Delta a_{12} - A_{33}\varkappa_N, & -A_{23}\varkappa_N \end{Vmatrix} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix}. \tag{10.4}$$

Multiplying now the right-hand side of (10.4) by the matrix

$$-\frac{i}{m} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix},$$

with regard for (5.34) we obtain

$$Q \equiv \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k & H_k \end{Vmatrix} \alpha_k = \frac{1}{\Delta a_{11}} \times$$

$$\times \begin{Vmatrix} \Delta a_{13}+A_{23}\varkappa_N+\frac{i}{m}(A_{22}B+A_{23}A), & A_{22}\varkappa_N-\Delta a_{11}-\frac{i}{m}(A_{22}A+A_{23}C) \\ \Delta a_{12}-A_{33}\varkappa_N-\frac{i}{m}(A_{23}B+A_{33}A), & -A_{23}\varkappa_N+\frac{i}{m}(A_{23}A+A_{33}C) \end{Vmatrix}. \tag{10.5}$$

In a similar manner, (10.2) yields

$$\mathcal{P} \equiv \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k & H_k \end{Vmatrix} \frac{1}{\alpha_k} = \frac{1}{\Delta a_{22}} \times$$

$$\times \left\| \begin{matrix} -A_{13}\varkappa_N - \frac{i}{m}(A_{33}B + A_{13}A), & \Delta a_{12} - A_{33}\varkappa_N + \frac{i}{m}(A_{33}A + A_{13}C) \\ -\Delta a_{22} + A_{11}\varkappa_N + \frac{i}{m}(A_{13}B + A_{11}A), & \Delta a_{23} + A_{13}\varkappa_N - \frac{i}{m}(A_{13}A + A_{11}C) \end{matrix} \right\|. \tag{10.6}$$

From (10.5) and (10.6), by virtue of (5.35) we have

$$Q\mathcal{P} = \mathcal{P}Q = E, \tag{10.7}$$

i.e.,

$$\begin{aligned} \mathcal{P} = Q^{-1} &= \frac{1}{\alpha_1\alpha_2} \sum_{k=1}^{2} \left\| \begin{matrix} H_k\alpha_k, & -F_k\alpha_k \\ -G_k\alpha_k, & E_k\alpha_k \end{matrix} \right\|, \\ Q = \mathcal{P}^{-1} &= \alpha_1\alpha_2 \sum_{k=1}^{2} \left\| \begin{matrix} H_k, & -F_k \\ -G_k, & E_k \end{matrix} \right\| \frac{1}{\alpha_k}. \end{aligned} \tag{10.8}$$

It should be finally noted that the following easily verifiable identities are valid:

$$\begin{aligned} &\left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\| \alpha_k = \left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\| Q = Q \left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\|, \\ &\left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\| = \left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\| \alpha_k\mathcal{P} = \mathcal{P} \left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\| \alpha_k. \end{aligned} \tag{10.9}$$

Multiply now (8.9) by i and then add to (8.1). Taking into account the notation (8.15) and the formulas (10.5) and (10.6), after cumbersome calculations we will have

$$\frac{\partial W}{\partial x_2} = Q\frac{\partial W}{\partial x_1}, \quad \frac{\partial W}{\partial x_1} = \mathcal{P}\frac{\partial W}{\partial x_2}. \tag{10.10}$$

The latter are the generalized Cauchy–Riemann conditions in the complex form.

Let us now prove the following

Theorem 2. *If the vector function W has integrable first derivatives in the domain D^+ and W itself is Hölder continuous up to the boundary, then* (9.17) *and* (9.18) *are valid.*

Proof. Introduce the notation

$$H(y,x) = \sum_{k=1}^{2} \left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\| \frac{\partial \ln\sigma_k}{\partial S(y)} W(y). \tag{10.11}$$

Obviously, $H(y,x) = n_1H_2(y,x) - n_2H_1(y,x)$, where

$$H_j(y,x) = \sum_{k=1}^{2} \left\| \begin{matrix} E_k, & F_k \\ G_k, & H_k \end{matrix} \right\| \frac{\partial \ln\sigma_k}{\partial y_j} W(y), \quad j = 1,2. \tag{10.12}$$

The vector function $H(y,x)$ possesses a singularity at the point $y=x$. Therefore

$$\int_{D_\varepsilon}\Big(\frac{\partial H_2}{\partial y_1}-\frac{\partial H_1}{\partial y_2}\Big)d\sigma=\int_S H(y,x)dS+\int_{S(x,\varepsilon)}H(y,x)dS, \quad (10.13)$$

where D_ε and $S(x,\varepsilon)$ are the same as in the previous section.

From (10.12) we find that

$$\frac{\partial H_2}{\partial y_1}-\frac{\partial H_1}{\partial y_2}=\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\Big(\frac{\partial^2\ln\sigma_k}{\partial y_1\partial y_2}W+\frac{\partial\ln\sigma_k}{\partial y_2}\frac{\partial W}{\partial y_1}-\frac{\partial^2\ln\sigma_k}{\partial y_2\partial y_1}W-$$
$$-\frac{\partial\ln\sigma_k}{\partial y_1}\frac{\partial W}{\partial y_2}\Big)=\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{1}{\sigma_k}\Big(\frac{\partial W}{\partial y_2}-\alpha_k\frac{\partial W}{\partial y_1}\Big).$$

Taking into account (10.9) and (10.10), we have

$$\frac{\partial H_2}{\partial y_1}-\frac{\partial H_1}{\partial y_2}=\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{1}{\sigma_k}\Big(\frac{\partial W}{\partial y_2}-Q\frac{\partial W}{\partial y_1}\Big)\equiv 0,\ \ y\in D_\varepsilon.$$

Now we can easily show that calculations performed in Section 9 result in

$$\int_S H(y,x)\,dS=2\pi i W(x),\ \ x\in D^+,\qquad \int_S H(y,x)\,dS=0,\quad x\in D^-.$$

Thus we have proved that the formulas (9.17) and (9.18) are valid. The validity of the formulas (9.19) and (9.20) is proved analogously if the vector function W besides the above-mentioned conditions satisfies the following conditions at infinity:

$$W(x)=W(\infty)+O(\rho^{-1}),\quad \frac{\partial W}{\partial x_j}=O(\rho^{-2}),\quad j=1,2,\qquad (10.14)$$

where $\rho^2=x_1^2+x_2^2$. ■

11. Potentials and Their Properties

First of all, we introduce the following definitions:

Definition 11.1. The vector function defined by the equality

$$u(x)=\frac{1}{\pi}\int_S\Gamma(x,y)g(y)\,dS,\qquad (11.1)$$

where $\Gamma(x,y)$ is the fundamental matrix (see (4.12)), g is a continuous (or Hölder continuous) vector function and S is a closed Lyapunov curve, will be called a simple layer potential.

Definition 11.2. The vector defined by the equality

$$u(x) = \frac{1}{\pi} \int_S N(y,x) g(y)\, dS, \tag{11.2}$$

where $N(y,x)$ is given by (5.32), will be called double layer potential.

Definition 11.3. The vector defined by the equality

$$u(x) = \frac{1}{\pi} \int_S \overset{\varkappa}{M}(x,y) g(y)\, dS, \tag{11.3}$$

where $\overset{\varkappa}{M}(x,y)$ is given by (6.10), will be called generalized simple layer potential.

These potentials are solutions of the system (1.9) both in the domains D^+ and D^-. In the case where the point x tends to any point of the boundary $z(z_1, z_2)$ (from inside or from outside), the above-defined potentials possess some discontinuity or continuity properties.

As is mentioned above, the direction of the outer (with respect to D^+) normal is assumed to be positive. We choose the direction of the tangent so that the normal and the tangent be oriented as the coordinate system x_1, x_2.

To derive jump formulas for the double layer potential, we rewrite (11.2) as

$$u(x) = \frac{1}{\pi} \int_S \frac{\partial \theta}{\partial s} g(y) dS + \frac{1}{\pi} \int_S \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \frac{\partial}{\partial s(y)} \frac{\sigma_k}{\sigma} g(y) dS, \tag{11.4}$$

where the first term on the right-hand side is the harmonic double layer potential. Bearing calculations performed in Section 9 in mind, we can see that

$$\int_S \frac{\partial}{\partial s(y)} \ln \frac{\sigma_k}{\sigma} dS = 0, \quad x \in \mathbb{R}^2, \tag{11.5}$$

where $\mathbb{R}^2$ is the two-dimensional plane. Now the expression (11.5) can be rewritten as

$$u(x) = \frac{1}{\pi} \int_S \frac{\partial \theta}{\partial s} g(y) dS +$$
$$+ \frac{1}{\pi} \int_S \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \frac{\partial}{\partial s(y)} \ln \frac{\sigma_k}{\sigma} \Big[g(y) - g(z) \Big] dS, \tag{11.6}$$

where the second term on the right-hand side is a continuous vector function in $\mathbb{R}^2$. Thus, when the point x tends to the point $z \in S$, the potential (11.2) has just the same discontinuity as the harmonic double layer potential. Using this remark, from (11.6) we obtain

$$\big(u(z)\big)^+ = g(z) + u(z), \quad \big(u(z)\big)^- = -g(z) + u(z). \tag{11.7}$$

Rewrite now (11.2) in a somewhat different form. Using (5.22), after some calculations we have

$$u(x)=\frac{1}{\pi}\int_S \frac{\partial\theta}{\partial s}g(y)dS+\frac{2}{\pi}\operatorname{Re}\sum_{n=1}^{\infty}(-1)^n Q^{(n)}\int_S e^{-2in\theta}g(y)\frac{\partial\theta}{\partial s}dS, \quad (11.8)$$

where

$$Q^{(n)}=\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\lambda_k^n, \quad n=1,2,\ldots, \quad \lambda_k=\frac{1+i\alpha_k}{1-i\alpha_k}. \quad (11.9)$$

Now, passing in (11.8) to limit as $\lambda_2=\lambda_1\to 0$, we obtain

$$u(x)=\frac{1}{\pi}\int_S N(y,x)g\,dS,$$

where $N(y,x)$ is defined by (5.38). Hence in the case of an isotropic elastic body the series in (11.8) breaks and there remains only one term.

The simple layer potential (11.1) can be rewritten as

$$u(x)=\frac{1}{\pi}\frac{m}{BC-A^2}\begin{Vmatrix}C, & A\\ A, & B\end{Vmatrix}\int_S \ln r\, g(y)dS+$$
$$+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\int_S \ln\frac{\sigma_k}{\sigma}g(y)dS,$$

where the second term on the right-hand side is a one-valued continuous vector function, and the first term, being the harmonic simple layer potential, is continuous in the entire plane.

Let us now investigate properties of the operation $\overset{\varkappa}{T}$ acting on a simple layer potential. Elementary calculations yield

$$\overset{\varkappa}{T}(\partial_x,n)u(x)=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}N_k(\varkappa), & L_k(\varkappa)\\ M_k(\varkappa), & R_k(\varkappa)\end{Vmatrix}\int_S \frac{\ln\sigma_k}{\partial s(x)}g(y)dS, \quad (11.10)$$

where

$$\begin{aligned} N_k(\varkappa)&=N_k+\varkappa B_k, \quad L_k(\varkappa)=L_k+\varkappa C_k,\\ M_k(\varkappa)&=M_k-\varkappa A_k, \quad R_k(\varkappa)=R_k-\varkappa B_k.\end{aligned} \quad (11.11)$$

Rewrite (11.10) in the form

$$\overset{\varkappa}{T}(\partial_x,n)u(x)=-\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}N_k(\varkappa), & L_k(\varkappa)\\ M_k(\varkappa), & R_k(\varkappa)\end{Vmatrix}\int_S \frac{\partial\ln\sigma_k}{\partial s(y)}g(y)dS+$$
$$+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}N_k(\varkappa), & L_k(\varkappa)\\ M_k(\varkappa), & R_k(\varkappa)\end{Vmatrix}\times$$
$$\times\int_S \frac{\alpha_k[n_1(x)-n_1(y)]-[n_2(x)-n_2(y)]}{x_1-y_1+\alpha_k(x_2-y_2)}g(y)dS. \quad (11.12)$$

It is easy to see that if g is Hölder continuous, then the second summand on the right-hand side is also continuous, if the point x passes through the boundary, and the first term has just the same discontinuities as the double layer potential (11.2). Thus from (11.10) we have the following discontinuity formulas:

$$\begin{aligned}(\overset{\varkappa}{T}u(z))^{+} &= -g(z) + \overset{\varkappa}{T}u(z), \quad z \in S,\\ (\overset{\varkappa}{T}u(z))^{-} &= g(z) + \overset{\varkappa}{T}u(z), \quad z \in S,\end{aligned} \tag{11.13}$$

where under $\overset{\varkappa}{T}u(z)$ we mean $\overset{\varkappa}{T}(\partial z, n(z))u(z)$.

Finally, we study properties of the generalized simple layer potential.

As we have already noticed (see Section 6), if $\varkappa \neq \varkappa_N$, then the kernel of that potential is a many-valued matrix function and hence the generalized potential itself is a many-valued vector function. To eliminate the many-valuedness, it suffices to rewrite (11.3) for a finite domain in the form

$$u(x) = \frac{1}{\pi}\int_S \left[\overset{\varkappa}{M}(x,y) - \overset{\varkappa}{M}(0,y)\right] g(y)dS, \tag{11.14}$$

the origin being assumed to be in the finite domain D^+. For the infinite domain D^- we write the potential (11.3) in the form

$$u(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} A_k'(\varkappa), & B_k'(\varkappa)\\ C_k'(\varkappa), & D_k'(\varkappa)\end{Vmatrix}\int_S \ln\Big(1-\frac{\zeta_k}{z_k}\Big) g(y)\,dS, \tag{11.15}$$

where $z_k = x_1 + \alpha_k x_2$, $\zeta_k = y_1 + \alpha_k y_2$. In the infinite domain this potential is a one-valued vector function which vanishes at infinity.

Calculate $\overset{\varkappa_0}{T}$ from (11.3), where $\varkappa_0$ is an arbitrary real number. We have

$$\overset{\varkappa_0}{T}(\partial x, n)u(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k+\varkappa_0 C_k', & q_k+\varkappa_0 D_k'\\ r_k-\varkappa_0 A_k', & s_k-\varkappa_0 B_k'\end{Vmatrix} H\int_S \frac{\partial \ln \sigma_k}{\partial S(x)} g(y)dS, \tag{11.16}$$

where H is defined by (6.6). If g is Hölder continuous, then, as above, from (11.16) we can easily obtain

$$\begin{aligned}\big(\overset{\varkappa_0}{T}(\partial z, n)u(z)\big)^{+} &= -A(\varkappa_0,\varkappa)g(z) + \overset{\varkappa_0}{T}(\partial z, n)u(z), \quad z \in S,\\ \big(\overset{\varkappa_0}{T}(\partial z, n)u(z)\big)^{-} &= A(\varkappa_0,\varkappa)g(z) + \overset{\varkappa_0}{T}(\partial z, n)u(z), \quad z \in S,\end{aligned} \tag{11.17}$$

where

$$A(\varkappa_0,\varkappa) = \frac{1}{\Delta(\varkappa)}\Big[1 + \omega a_{11}(\varkappa+\varkappa_0) - \frac{\varkappa\varkappa_0 a_{11} m}{BC-A^2}\Big]; \tag{11.18}$$

$\Delta(\varkappa)$ is defined from (6.7).

For $\varkappa_0 = \varkappa$, from (11.17) we get

$$\begin{aligned} \big(\overset{\varkappa}{T}(\partial z, n)u(z)\big)^+ &= -g(z) + \overset{\varkappa}{T}(\partial z, n)u(z), \quad z \in S, \\ \big(\overset{\varkappa}{T}(\partial z, n)u(z)\big)^- &= g(z) + \overset{\varkappa}{T}(\partial z, n)u(z), \quad z \in S. \end{aligned} \tag{11.19}$$

If $A(\varkappa_0, \varkappa) = 0$, then

$$\varkappa = \frac{(\varkappa_0 + \frac{1}{\omega a_{11}})\varkappa_N}{\varkappa_0 - \varkappa_N}, \quad \varkappa_0 = \frac{(\varkappa_0 + \frac{1}{\omega a_{11}})\varkappa_N}{\varkappa - \varkappa_N}, \tag{11.20}$$

and the formulas (11.17) yield

$$\big(\overset{\varkappa_0}{T}(\partial z, n)u(z)\big)^+ = \big(\overset{\varkappa_0}{T}(\partial z, n)u(z)\big)^-, \quad z \in S. \tag{11.21}$$

Note that for the validity of the formulas (11.19) and (11.21), it is sufficient that g be continuous.

12. Inversion Formulas

Let g be any vector function satisfying on S the Hölder condition. Consider the vector function

$$\frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y)ds = \Psi(x), \tag{12.1}$$

where $\sigma_k = x_1 - y_1 + \alpha_k(x_2 - y_2)$, and $x = (x_1, x_2)$ is an arbitrary point in D^+.

The vector function $\Psi(x)$ is analytic in D^+ and continuous on S up to the boundary, if we put

$$\Psi^+(z) = g(z) + \frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y)ds \quad \text{for} \quad z \in S. \tag{12.2}$$

Taking into account (9.18), we have

$$\frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} \Psi^+(y)ds \quad \text{for} \quad z \in D^-.$$

Passing to the limit as $x \to z \in S$, we obtain

$$-\Psi^+(z) + \frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} \Psi^+(y)ds = 0.$$

Substituting in the above equation the values of $\Psi^+(z)$ from (12.2), we have

$$-g(z) - \frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y)ds + \frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \times$$

$$\times \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} \left[g(y) + \frac{1}{\pi i} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \tau_k}{\partial s(\xi)} g(\xi) ds \right] ds = 0,$$

where $\tau_k = y_1 - \xi_1 + \alpha_k(y_2 - \xi_2)$, whence we arrive at the relation

$$\frac{1}{\pi i} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \times$$
$$\times \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} \left[\frac{1}{\pi i} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \tau_k}{\partial s(\xi)} g(\xi) ds \right] ds = g(z), \quad (12.3)$$

which is valid for the arbitrary vector function g satisfying on S the Hölder condition. Formula (12.3) implies inversion formulas. Indeed, let

$$\frac{1}{\pi i} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y) ds = h(z). \quad (12.4)$$

Then from (12.3) we have

$$\frac{1}{\pi i} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} h(y) ds = g(z). \quad (12.5)$$

Thus (12.4) implies (12.5), and analogously, (12.5) implies (12.4). Hence, (12.5) is a unique solution of the equation (12.4); in other words, each of the relations

$$\begin{aligned} \frac{1}{\pi i} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y) ds = h(z), \\ \frac{1}{\pi i} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} h(y) ds = g(z) \end{aligned} \quad (12.6)$$

is a consequence of the other, i.e., these relations invert each other.

The inversion formulas (12.6) for the isotropic elastic body take the form

$$\begin{aligned} \frac{1}{\pi i} \int_S \left[E \frac{\partial \ln \sigma}{\partial s(y)} + \frac{i(\lambda+\mu)}{2(\lambda+3\mu)} \begin{Vmatrix} i, & -1 \\ -1, & -i \end{Vmatrix} \frac{\partial}{\partial s(y)} \frac{\overline{\sigma}}{\sigma} \right] g(y) ds = h(z), \\ \frac{1}{\pi i} \int_S \left[E \frac{\partial \ln \sigma}{\partial s(y)} + \frac{i(\lambda+\mu)}{2(\lambda+3\mu)} \begin{Vmatrix} i, & -1 \\ -1, & -i \end{Vmatrix} \frac{\partial}{\partial s(y)} \frac{\overline{\sigma}}{\sigma} \right] h(y) ds = g(z), \end{aligned} \quad (12.7)$$

where $\sigma = z_1 - y_1 + i(z_2 - y_2)$, $\overline{\sigma} = z_1 - y_1 - i(z_2 - y_2)$, and E is the unit matrix.

Formally, from (12.7) we can obtain inversion formulas for the Laplace equation ($\lambda + \mu = 0$, $\mu = 1$):

$$\frac{1}{\pi i} \int_S \frac{\partial \ln \sigma}{\partial s(y)} g(y) ds = h(z), \quad \frac{1}{\pi i} \int_S \frac{\partial \ln \sigma}{\partial s(y)} h(y) ds = g(z). \quad (12.8)$$

Three methods of proving the last inversion formulas have been suggested by N. I. Muskhelishvili [39]. We will use the second method to prove the inversion formulas (12.6).

13. Statement of New Problems

Having introduced the analytic vector $W(x)$ (see Section 8), we are able to formulate some new problems.

The Generalized Riemann–Hilbert Problem: Find a generalized vector function $W = U + iV$, analytic in the finite domain D^+ and continuous on S, satisfying the boundary condition

$$\mathrm{Re}[(\lambda - i\mu)(W)^+] \equiv \lambda(U)^+ + \mu(V)^+ = f \quad \text{on} \quad S, \tag{13.1}$$

where λ and μ are real continuous matrices given on S, and f is a given continuous vector function.

For the infinite domain D^- bounded by the same closed line S, the generalized Riemann–Hilbert problem is formulated analogously, but in this case, a smoothness condition, besides the condition at infinity, is required to be fulfilled.

The Generalized Problem of Linear Conjugation: Find a vector function W analytic in the domains D^+ and D^- with the common boundary S, having the canonical order at infinity and satisfying the boundary condition

$$[W(z)]^+ = G(z)[W(z)]^- + f(z), \quad z \in S, \tag{13.2}$$

where G is a given matrix function on S, $\det G \neq 0$ and $f(z)$ is a given Hölder continuous vector function on S.

The domains D^+ and D^- (for the sake of definiteness, the domain D^+ is finite and D^- is infinite), where the vector function $W(x)$ is sought, are assumed to be filled by an anisotropic material having the same Hooke's coefficients. The general case, where Hooke's coefficients are different for the domains D^+ and D^-, can also be considered.

14. The Uniqueness Theorems

If the boundary value problems formulated in Section 1 have regular solutions, then sometimes they occur to be unique. In this section we investigate the question of the uniqueness of solutions of the above-mentioned problems.

1. Let the first boundary value problem (with a displacement vector on the boundary) have in the domain D^+ two regular solutions $u^{(1)}$ and $u^{(2)}$. Write $u = u^{(1)} - u^{(2)}$. Evidently, the vector function u satisfies (1.9) and the boundary condition $(u)^+ = 0$ on S. Using (2.11) for $\varkappa = \varkappa_N$ and taking into account the fact that the pseudopotential energy is positive definite, we conclude that $u(x) = C$, $x \in D^+$, where C is the a constant vector. Since $(u)^+ = 0$, we have $C = 0$ and $u(x) = 0$, $x \in D^+$.

Thus the first boundary value problem has in the finite domain D^+ at most one regular solution.

The vectors $u^{(1)}$ and $u^{(2)}$ in the domain D^- must satisfy the condition (2.1). In this case the regular vector $u = u^{(1)} - u^{(2)}$ satisfies (2.1), and thus the formula (2.28) is valid. Using this formula for $\varkappa = \varkappa_N$, we obtain $u(x) = C$, $x \in D^-$, where C is again a constant vector. But u on the boundary satisfies $(u)^- = 0$ which implies that $C = 0$ and $u(x) = 0$, $x \in D^-$.

Thus the first boundary value problem has in the domain D^- at most one regular solution.

2. Let $(Tu)^+ = 0$ on S. Then applying (2.14) to a regular solution we have $u_1(x) = C_1 - \varepsilon x_2$, $u_2(x) = C_2 + \varepsilon x_1$, $x = (x_1, x_2) \in D^+$, where C_1, C_2 are arbitrary real constants.

Consequently, a regular solution of the second boundary value problem is not unique in the domain D^+. Two regular solutions may differ by a rigid displacement.

For the exterior second homogeneous boundary value problem the vector function u must satisfy (1.24). In this case, the formula (2.28) is valid for a regular u for any $\varkappa$. Using this formula for u and $\varkappa = 0$, we obtain $u_1(x) = C_1 - \varepsilon x_2$, $u_2(x) = C_2 + \varepsilon x_1$, $x \in D^-$.

Bearing in mind (1.24), we have $\varepsilon = 0$, and $u_1(x) = C_1$, $u_2(x) = C_2$, $x \in D^-$.

Thus in the domain D^-, two regular solutions of the second boundary value problem may differ by a constant vector.

3. Let now $(u_n)^+ = 0$ and $(T_s u)^+ = 0$ on S, where u_n is a normal component of the displacement vector and $T_s u$ is a tangential component of the stress vector. Rewrite the scalar product in (2.14) as

$$(uTu)^+ = (u_n)^+ (T_n u)^+ + (u_s)^+ (Tu)^+.$$

Taking into account the boundary condition for the third boundary value problem, we conclude that $(uTu)^+ = 0$ on S. Using now (2.14), we have

$$u_1(x) = C_1 - \varepsilon x_2, \quad u_2(x) = C_2 + \varepsilon x_1, \quad x \in D^+, \tag{14.1}$$

from which it follows that

$$u_n = n_1 C_1 + n_2 C_2 + \varepsilon(x_1 n_2 - n_1 x_2)$$

or

$$0 = (u_n)^+ = \frac{d}{ds}\Big[C_1 x_2 - C_2 x_1 - \frac{\varepsilon}{2}(x_1^2 + x_2^2)\Big] \quad \text{on} \quad S.$$

Integration with respect to S results in

$$\Big(x_1 + \frac{C_2}{\varepsilon}\Big)^2 + \Big(x_2 - \frac{C_1}{\varepsilon}\Big)^2 = 2\frac{C_3}{\varepsilon} + \frac{C_1^2 + C_2^2}{\varepsilon^2} \equiv R^2, \tag{14.2}$$

where C_3 is the integration constant.

Thus, if S is not a circumference, then $C_1 = C_2 = \varepsilon = 0$, and from (14.1) we obtain $u(x) = 0$, $x \in D^+$. If S is a circumference of radius R, with the center at the point $(-\frac{C_2}{\varepsilon}, \frac{C_1}{\varepsilon})$, then C_1, C_2, ε are arbitrary constants, and center of the two regular solutions differ by a rigid displacement. But if the center of the circumference lies at the point $0(0,0)$, then $C_1 = C_2 = 0$, and ε is an arbitrary constant. Now (14.1) takes the form

$$u_1(x) = -\varepsilon x_2, \quad u_2(x) = \varepsilon x_1. \tag{14.3}$$

In this case two regular solutions of the third boundary value problem differ by a rigid rotation.

Consider now the exterior boundary value problem, where $(u_n)^- = 0$ and $(T_s u)^- = 0$ on S. Using (2.28) for $\varkappa = 0$, we obtain

$$u_1(x) = C_1 - \varepsilon x_2, \quad u_2(x) = C_2 + \varepsilon x_1, \quad x \in D^-.$$

By virtue of (1.24), we have $\varepsilon = 0$, and

$$u_1(x) = C_1, \quad u_2(x) = C_2, \quad x \in D^-. \tag{14.4}$$

Owing to the homogeneous boundary condition,

$$0 = (u_n)^- = \frac{d}{ds}\Big(- C_2 x_1 + C_1 x_2 \Big) \quad \text{on} \quad S,$$

and after the integration we find $-C_2 x_1 + C_1 x_2 = C_3$, where C_3 is an arbitrary constant. Since S is not a straight line, $C_1 = C_2 = C_3 = 0$, and it follows from (14.4) that the third external boundary value problem has a unique regular solution.

4. In the case of the fourth boundary value problem, we have $(u_s)^+ = 0$, $(T_n u)^+ = 0$ on S, where u_s is the tangential component of the displacement vector, and $T_n u$ is the normal component of the stress vector. $(uTu)^+ = 0$, and using (2.14), we get $u_1(x) = C_1 - \varepsilon x_2$, $u_2(x) = C_2 + \varepsilon x_1$, $x \in D^+$. In this case

$$0 = (u_s)^+ = n_1(C_2 + \varepsilon x_1) + n_2(-C_1 + \varepsilon x_2), \tag{14.5}$$

whence

$$0 = \int_S (u_s)^+ ds = 2\varepsilon \int_{D^+} d\sigma$$

and $\varepsilon = 0$. From (14.5) we now have $0 = \frac{d}{ds}(C_1 x_1 + C_2 x_2)$, and hence $C_1 x_1 + C_2 x_2 = C_3$.

Thus, since S is not a straight line, the fourth boundary value problem in the domain D^+ has a unique regular solution.

In a similar manner we consider the problem, where $(u_s)^- = 0$ and $(T_n u)^- = 0$. Here $\varepsilon = 0$ by virtue of (1.24). Since S is not a straight line, $u(x) = 0$, $x \in D^-$.

5. In this case, $S = S_1 \cup S_2 \cup S_3 \cup S_4$, $(u)^+ = 0$ on S_1, $(Tu)^+ = 0$ on S_2, $(u_n)^+ = 0$ and $(T_s u)^+ = 0$ on S_3, and $(u_s)^+ = 0$ and $(T_n u)^+ = 0$ on

S_4. Using (2.14), we conclude that $u_1(x) = C_1 - \varepsilon x_2$, $u_2(x) = C_2 + \varepsilon x_1$, $x \in D^+$.

As far as $(u)^+ = 0$ on S_1, by the continuity of u on S we obtain $u(x) = 0$, $x \in D^+$.

Consequently, the general mixed boundary value problem has a unique regular solution in the domain D^+.

The uniqueness of solution of the general mixed boundary value problem for the domain D^- is proved analogously.

It should be noted that the formulas (2.14) and (2.28) will be used in the sequel even in the case where the conditions for elementary deducing of these formulas are not satisfied. But in all the cases we will have to come across these conditions will be satisfied for a domain D' obtained from D^+ (or from D^-) by removing infinitesimal portions cut out by circumferences of infinitesimal radii with centers at a finite number of points of S. Therefore we will be able to apply the formulas (2.14) and (2.28) to the domain D' and then pass to limit as radii of the above-mentioned circumferences tend to zero. Note that in all the cases we will have to do with, the integrals on the right-hand side, extended to the arcs of the circumferences contained in S, will tend to zero, and the formulas (2.14) and (2.28) will turn out to be valid for the domains D^+ and D^-.

Finally, the uniqueness theorems can be generalized in the case where the generalized potential energy is not positive definite for the operator $\overset{\varkappa_0}{T}$ but is positive definite for the adjoint operator $\overset{\varkappa}{T}$. A general reasoning on this point has been given at the end of Section 8. The same reasoning will be used in solving the second generalized boundary value problem.

15. Solution of the First Boundary Value Problem

First let us prove the existence of solution of the first boundary value problem in the domain D^+. A solution is sought in the form of the double layer potential

$$u(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix}\int_S \frac{\partial \ln \sigma_k}{\partial s(y)}\, g(y)dS. \tag{15.1}$$

Then for determining the unknown real vector function g, we obtain the following Fredholm integral equation of the second kind:

$$g(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix}\int_S \frac{\partial \ln \sigma_k}{\partial s(y)}\, g(y)ds = f(z), \quad z \in S, \tag{15.2}$$

where f is a given continuous vector function.

Let us prove that the equation (15.2) is solvable for any continuous right-hand side.

Consider the associated to (15.2) homogeneous equation

$$h(z)+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & G_k\\ F_k, & H_k\end{Vmatrix}\int_S\frac{\partial\ln\sigma_k}{\partial s(z)}h(y)dS=0 \tag{15.3}$$

and prove that it has only the trivial solution. Assume the contrary and denote by $h_0(z)$ a nenzero solution of (15.3). Compose the simple layer potential

$$V_0(x)=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\int_S\ln\sigma_k h_0(y)dS.$$

It is obvious from (15.3) that

$$[NV_0(z)]^-=0,\quad \int_S h_0(y)dS=0.$$

Using the formula (2.28) for $\varkappa=\varkappa_N$ in D^-, we obtain $V_0(x)=0$, $x\in D^-$.

Taking now into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we will have $V_0(x)=0$, $x\in D^+$.

Note that $2h_0(z)=[NV_0(z)]^- - [NV_0(z)]^+$. Since $[NV_0(z)]^+=0$ and $[NV_0(z)]^-=0$, we have $h_0(z)=0$, and hence the equation (15.3) has only the trivial solution. This implies that the associated to (15.3) homogeneous equation also has only the trivial solution, and the equation (15.2) is solvable (according to the first Fredholm theorem) for any continuous right-hand side.

So far we have assumed that $S\in C^{1,\beta}$ $(0<\beta\le 1)$. In order to prove the regularity of the double layer potential in the domain D^+, it is sufficient to assume that $S\in C^{2,\beta}$ $(0<\beta\le 1)$ and $\frac{\partial f}{\partial s}$ is Hölder continuous ($f\in C^{1,\alpha}(S)$, $0<\alpha<\beta$).

Consider now the first boundary value problem in the domain D^-. Its solution is sought in the form

$$u(x)=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_S\Big(\frac{\partial\ln\sigma_k}{\partial s(y)}+\frac{1}{2}\frac{\partial\ln\zeta_k}{\partial s(y)}\Big)g(y)dS. \tag{15.4}$$

Then for determining the unknown vector function g we obtain the following Fredholm integral equation of second kind

$$-g(z)+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_S\Big(\frac{\partial\ln\sigma_k}{\partial s(y)}+\frac{1}{2}\frac{\partial\ln\zeta_k}{\partial s(y)}\Big)g(y)dS=f(z), \tag{15.5}$$

where f is a given continuous vector function.

Prove that the equation (15.5) is solvable for an arbitrary right-hand side. To this end, we consider the associated to (15.5) homogeneous equation

$$-h(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & G_k \\ F_k, & H_k \end{Vmatrix}\int_S \frac{\partial \ln \sigma_k}{\partial s(z)} h(y)dS +$$

$$+\frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & G_k \\ F_k, & H_k \end{Vmatrix}\frac{\partial \ln z_{k0}}{\partial s(z)}\int_S h dS = 0, \tag{15.6}$$

where $z_{k0} = z_1 + \alpha_k z_2$, $z = (z_1, z_2) \in S$.

Let us prove that equation (15.6) has only the trivial solution. Suppose that it has a nonzero solution h_0. From (15.6) by integration we obtain

$$\int_S h_0(y)dS = 0. \tag{15.7}$$

In this case, the equation (15.6) corresponds to the boundary condition

$$[NV_0(z)]^+ = 0, \tag{15.8}$$

where

$$V_0(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix}\int_S \ln \sigma_k h_0(y)dS. \tag{15.9}$$

Using the formula (2.11) for $\varkappa = \varkappa_N$, we find that $V_0(x) = C$, $x \in D^+$, where C is a constant vector.

Taking into account the equation (15.7) and the fact that the simple layer potential is continuous while passing through the boundary and, using (2.28) for $\varkappa = \varkappa_N$, we obtain $V_0(x) = 0$, $x \in D^-$.

Since $2h_0 = [NV_0(z)]^- - [NV_0(z)]^+$ and $[NV_0(z)]^- = [NV_0(z)]^+ = 0$, then $h_0(z) = 0$.

Thus we conclude that the associated to (15.6) homogeneous equation has only the trivial solution, and the equation (15.5) is solvable for any continuous right-hand side.

To prove the regularity of the potential defined by (15.4) in the domain D^-, it is sufficient to assume that $S \in C^{2,\beta}$ $(0 < \beta \le 1)$ and $f \in C^{1,\alpha}(S)$, $0 < \alpha < \beta$.

So far we have assumed that the principal vector $\int_S F dS$, the stress components $\sigma_x^{(\infty)}$, $\sigma_y^{(\infty)}$, $\tau_{xy}^{(\infty)}$, and the rotation $\varepsilon^{(\infty)}$ are equal to zero at infinity.

Consider now the general case. Let the principal vector $\int_S F dS$, the stress components $\sigma_x^{(\infty)}$, $\sigma_y^{(\infty)}$, $\tau_{xy}^{(\infty)}$, and the rotation $\varepsilon^{(\infty)}$ be given at infinity.

In this case a solution is sought in the form

$$u(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix}\int_S \Big(\frac{\partial \ln \sigma_k}{\partial s(y)} + \frac{1}{2}\frac{\partial \ln \zeta_k}{\partial s(y)}\Big) g(y)dS +$$

$$+\frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\left\|\begin{matrix}A_k, & B_k\\ B_k, & C_k\end{matrix}\right\|\ln z_k\int_S F\,dS + A^{(\infty)}x_1 + B^{(\infty)}x_2, \quad (15.10)$$

where $A^{(\infty)}(A_1^{(\infty)}, A_2^{(\infty)})$ and $B^{(\infty)}(B_1^{(\infty)}, B_2^{(\infty)})$ are given vectors of the form

$$\begin{aligned}
A_1^{(\infty)} &= a_{11}\sigma_x^{(\infty)} + a_{12}\sigma_y^{(\infty)} + a_{13}\tau_{xy}^{(\infty)},\\
B_1^{(\infty)} &= -\varepsilon^{(\infty)} + \frac{1}{2}(a_{13}\sigma_x^{(\infty)} + a_{23}\sigma_y^{(\infty)} + a_{33}\tau_{xy}^{(\infty)}),\\
A_2^{(\infty)} &= \varepsilon^{(\infty)} + \frac{1}{2}(a_{13}\sigma_x^{(\infty)} + a_{23}\sigma_y^{(\infty)} + a_{33}\tau_{xy}^{(\infty)}),\\
B_2^{(\infty)} &= a_{12}\sigma_x^{(\infty)} + a_{22}\sigma_y^{(\infty)} + a_{23}\tau_{xy}^{(\infty)}.
\end{aligned}$$

For determining the unknown vector g we obtain the equation

$$-g(z)+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\left\|\begin{matrix}E_k, & F_k\\ G_k, & H_k\end{matrix}\right\|\int_S\Big(\frac{\partial\ln\sigma_k}{\partial s(y)}+\frac{1}{2}\frac{\partial\ln\zeta_k}{\partial s(y)}\Big)g(y)dS = f_0(z), \quad (15.11)$$

where

$$f_0(z) = f(z) - \frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\left\|\begin{matrix}A_k, & B_k\\ B_k, & C_k\end{matrix}\right\|\ln z_{k0}\int_S F\,dS - A^{(\infty)}z_1 - B^{(\infty)}z_2.$$

The equation (15.11) like (15.5) is always solvable.

Thus we have proved the solvability of the first boundary value problem in the domains D^+ and D^-.

Let now D^+ be a finite multiply connected domain (see Section 2). Consider the case where the domain has the contour S_{m+1}. A solution of the first boundary value problem in the domain D^+ is sought in terms of the potential

$$\begin{aligned}
V(x) = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\left\|\begin{matrix}E_k, & F_k\\ G_k, & H_k\end{matrix}\right\|\int_S\frac{\partial\ln\sigma_k}{\partial s(y)}g(y)dS +\\
+\frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\sum_{j=1}^{m}\left\|\begin{matrix}A_k, & B_k\\ B_k, & C_k\end{matrix}\right\|\ln(z_k - z_{kj})M_j,
\end{aligned}\quad (15.12)$$

where

$$M_j = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\left\|\begin{matrix}E_k, & F_k\\ G_k, & H_k\end{matrix}\right\|\int_S\frac{\partial}{\partial s(y)}\ln(z_{kj}-\zeta_k)g(y)dS = u(x_j), \quad (15.13)$$

$$u(x) = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\left\|\begin{matrix}E_k, & F_k\\ G_k, & H_k\end{matrix}\right\|\int_S\frac{\partial\ln\sigma_k}{\partial s(y)}g(y)dS, \quad (15.14)$$

$x^{(j)}(x_{1j}, x_{2j}) \in D_j$, D_j $(j = \overline{1, m})$ being a finite domain bounded by the contour S_j, $z_{kj} = x_{1j} + \alpha_k x_{2j}$, $z_k = x_1 + \alpha_k x_2$, $x(x_1, x_2) \in D^+$, A_k, B_k, C_k are the coefficients of the fundamental matrix (see (4.13)).

Using the properties of the double layer potential as $x \to \xi(\xi_1, \xi_2) \in S$, we obtain for the unknown vector function g the following Fredholm integral equation:

$$g(\xi) + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y) dS +$$
$$+ \frac{1}{2\pi} \operatorname{Im} \sum_{k=1}^{2} \sum_{j=1}^{m} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \ln(\xi_{k0} - z_{kj}) M_j = f(\xi), \tag{15.15}$$

where f is a given continuous vector function, and $\xi_{k0} = \xi_1 + \alpha_k \xi_2$.

Let us prove that the equation (15.15) is solvable for any continuous right-hand side. To this end, we consider the associated to (15.15) homogeneous equation

$$h(\xi) + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & G_k \\ F_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(\xi)} h(y) dS +$$
$$+ \frac{1}{2\pi} \operatorname{Im} \sum_{k=1}^{2} \sum_{j=1}^{m} \begin{Vmatrix} E_k, & G_k \\ F_k, & H_k \end{Vmatrix} \frac{\partial}{\partial s(\xi)} \ln(\xi_{k0} - z_{kj}) \varepsilon_j = 0, \tag{15.16}$$

where

$$\varepsilon_j = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \int_S \ln(z_{kj} - \zeta_k) h(y) dS. \tag{15.17}$$

Suppose that (15.16) has a nontrivial solution $h_0(z)$. Then from (15.16) by integration we obtain

$$\varepsilon_j = 0, \quad j = \overline{1, m} \quad \text{and} \quad \int_S h_0(y) dS = 0. \tag{15.18}$$

In this case, the equation (15.16) corresponds to the boundary condition $[Nu_0(\xi)]^- = 0$ for the domain D', where D' is the set of domains complementing $D^+ \cup S$ to the whole plane, and

$$u_0(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \int_S \ln \sigma_k h_0(y) dS. \tag{15.19}$$

Using the uniqueness theorem for the domain D', we obtain $u_0(x) = C_k$, $x \in D_k$, $k = \overline{1, m+1}$, whence, according to (15.18), $C_k = 0$, $k = \overline{1, m+1}$, and $u_0(x) = 0$. Since $u_0(x)$ is the simple layer potential, we can easily conclude that $u_0(x) = 0$, $x \in D^+$, i.e., $[Nu_0]^+ = 0$. Thus $2h_0 = [Nu_0]^- - [Nu_0]^+ = 0$, and hence we have proved that the equation (15.16) has only the trivial solution. This implies that the associated to (15.16) homogeneous equation also has only the likewise trivial solution, and the equation (15.15) is solvable for any continuous vector function f. For the obtained in such a manner vector function V (see (15.12)) to be regular in D^+, it is sufficient that $S \in C^{2,\beta}$ $(0 < \beta \leq 1)$ and $\frac{\partial f}{\partial s}$ satisfy the Hölder condition ($f \in C^{1,\alpha}(S)$, $0 < \alpha < \beta$).

Note that if a solution of the first boundary value problem in a multiply connected domain D^+ is sought in the form

$$u(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \frac{\sigma_k}{\zeta_k}}{\partial s(y)} g(y) dS -$$
$$-\frac{1}{\pi} \int_S g(y) dS - \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \int_S \ln \Big(1 - \frac{z_k}{\zeta_k}\Big) g dS, \qquad (15.20)$$

then for the unknown vector function g we will always obtain the solvable Fredholm integral equation

$$g(z) + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial}{\partial s(y)} \ln \frac{\sigma_k}{\zeta_k} g(y) dS -$$
$$-\frac{1}{\pi} \int_S g(y) dS - \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \int_S \ln \Big(1 - \frac{z_{k0}}{\zeta_k}\Big) g dS = f(z). \quad (15.21)$$

The vector function u will be regular, if the above-mentioned sufficient conditions concerning S and f are satisfied.

The solution of the first boundary value problem for a simply connected domain D^+ and for an infinite domain D^- is given by the author in his earlier work (see [4]). The solution of this problem for multiply connected domains can also be found in author's work (see [6]). Therein we use Fredholm integral equations of second kind as well as singular normal type integral equations of second kind for which Fredholm theorems are valid.

16. Solution of the Second Boundary Value Problem

We begin this section with the solving the second boundary value problem in a simply connected finite domain D^+. A solution will be sought in the form

$$u(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A'_k, & B'_k \\ C'_k & D'_k \end{Vmatrix} \int_S \ln \big(1 - \frac{z_k}{\zeta_k}\big) g(y) dS, \qquad (16.1)$$

where $z_k = x_1 + \alpha_k x_2$, $\zeta_k = y_1 + \alpha_k y_2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, $x \in D^+$, $y \in S$, $g(y)$ is an unknown continuous vector function, A'_k, B'_k, C'_k, D'_k are the coefficients defined by (6.2), and $S \in C^{1,\beta}$ $(0 < \beta \leq 1)$ is a closed Lyapunov curve.

Using (11.17) for $\varkappa = \varkappa_0 = 0$ and taking into account the condition of the second boundary value problem, for determining of the real vector function g we obtain the Fredholm integral equation of second kind.

$$-g(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\int_S \frac{\partial \ln \sigma_k}{\partial s(z)} g(y)dS = f(z), \tag{16.2}$$

where $\sigma_k = z_{k0} - \zeta_k$, $z_{k0} = z_1 + \alpha_k z_2$, $z = (z_1, z_2) \in S$, and f is a given continuous vector function on S.

It is expedient to modify the preceding equation by adding to the left-hand side the expression

$$\frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\frac{\partial \ln z_{k_0}}{\partial s(z)}\int_S g dS - \frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{pmatrix} q_k \\ s_k \end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0}}\cdot M,$$

where

$$M = \Big(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\Big)_{x_1=0,\ x_2=0} =$$
$$= -\frac{a_{11}}{\pi(BC - A^2)}\,\mathrm{Re}\sum_{k=1}^{2}(A - C\alpha_k)\int_S \frac{(g_1 + \frac{\alpha_1\alpha_2}{\alpha_k} g_2)}{\zeta_k} dS, \tag{16.3}$$

the origin being assumed to be in the domain D^+.

Thus we obtain the equation

$$-g(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\int_S \frac{\partial \ln \sigma_k}{\partial s(z)} g(y)dS + \frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k & s_k \end{Vmatrix} \times$$
$$\times \frac{\partial \ln z_{k_0}}{\partial s(z)}\int_S g(y)dS - \frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{pmatrix} q_k \\ s_k \end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0}}\cdot M = f(z). \tag{16.4}$$

Let us now show that if provided (16.4) has a solution, then

$$\int_S g dS = 0 \tag{16.5}$$

and

$$M = 0, \tag{16.6}$$

provided the principal vector and the principal moment of external stresses are equal to zero.

Indeed, performing elementary calculations, from (16.4) we get

$$\int_S g dS = \int_S f dS, \quad M = \int_S (y_1 f_2 - y_2 f_1) dS. \tag{16.7}$$

If the principal vector $\int_S f dS$ and the principal moment $\int_S (y_1 f_2 - y_2 f_1) dS$ are equal to zero, then $\int_S g dS = 0$ and $M = 0$, which was to be demonstrated.

Thus if the principal vector and the principal moment of external stresses are equal to zero, then every solution g of the equation (16.4) is, at the same time, a solution of the initial equation (16.2).

Let us prove that the equation (16.4) is always solvable. To this end, we consider the homogeneous equation obtained from (16.4) for $f = 0$ and prove that it has no nontrivial solution. Let $g_0(y)$ be any solution of that equation. Since $f = 0$, it is obvious that (16.5) and (16.6) are fulfilled for g_0. The obtained in this case homogeneous equation corresponds to the boundary condition

$$[Tu_0(z)]^+ = 0, \tag{16.8}$$

where u_0 is obtained from (16.1), if instead of g we take g_0. Taking into consideration (16.8) and using the uniqueness theorem for the second problem for D^+, we obtain

$$u_0(x) = A + \varepsilon \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad x \in D^+, \tag{16.9}$$

where A is an arbitrary constant vector, and ε is an arbitrary constant number.

Noticing that $M_0 = 0$ (see (16.3)) and $u_0(0) = 0$, the equation (16.9) yields $u_0(x) = 0$, $x \in D^+$, whence

$$0 = Nu_0(x) = -\frac{\begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}}{m} \frac{\partial V_0(x)}{\partial s(x)}, \quad x \in D^+, \tag{16.10}$$

where

$$V_0(x) = \frac{1}{\pi} \operatorname{Re} \sum_{k=1}^{2} \begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} \ln \sigma_k g_0(y) dS. \tag{16.11}$$

The equation (16.10) results in

$$V_0(x) = C, \quad x \in D^+, \tag{16.12}$$

where C is a constant vector.

As while proving the formula (11.21), we can easily establish that if g_0 is continuous, then $[TV_0(z)]^+ - [TV_0(z)]^- = 0$.

But since $[TV_0(z)]^+ = 0$, from the last formula we obtain

$$[TV_0(z)]^- = 0. \tag{16.13}$$

By virtue of (16.5), the vector function V_0 is one-valued on the entire plane and of order ρ^{-1} at infinity. Using formula (2.28) for $\varkappa = 0$ and taking into account (16.13), we have

$$V_0(x) = 0, \quad x \in D^-. \tag{16.14}$$

The formulas (16.12) and (16.14) yield

$$[LV_0(z)]^+ = 0, \quad [LV_0(z)]^- = 0, \tag{16.15}$$

where the operator L is obtained from $\overset{\varkappa}{T}$ for $\varkappa = -(\omega a_{11})^{-1}$ (see Section 8). On the other hand, upon calculating the boundary values of the result of acting of L on the vector function V_0, we find

$$[LV_0(z)]^+ - [LV_0(z)]^- = \frac{2\left\|\begin{matrix} 0, & -1 \\ 1, & 0 \end{matrix}\right\| \left\|\begin{matrix} C, & A \\ A, & B \end{matrix}\right\|}{\omega(BC - A^2)} g_0(z). \tag{16.16}$$

By virtue of (16.15), we have $g_0(z) = 0$.

Thus the homogeneous equation corresponding to (16.4) has no nontrivial solution. Consequently, the equation (16.4) has a unique solution g. Substituting g in (16.1), we get a solution of the second boundary value problem, provided the principal vector and the principal moment of external stresses are equal to zero.

Let us now prove the existence of solution of the second boundary value problem in the domain D^-. The solution is sought in the form

$$W(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \left\|\begin{matrix} A'_k, & B'_k \\ C'_k, & D'_k \end{matrix}\right\| \int_S \ln \sigma_k g(y) dS - \frac{\omega}{2\pi} \operatorname{Re} \sum_{k=1}^{2} \begin{pmatrix} B'_k \\ D'_k \end{pmatrix} \frac{1}{z_k} \cdot M, \tag{16.17}$$

where

$$M = \Big(-\frac{\partial V_2}{\partial x_1} + \frac{\partial V_1}{\partial x_2} \Big)_{\substack{x_1=0 \\ x_2=0}} =$$

$$= \frac{a_{11}}{\pi(BC - A^2)} \operatorname{Im} \sum_{k=1}^{2} \int_S \big[(C\alpha_k - A)g_1 + (A\alpha_k - B)g_2\big] \frac{dS}{\zeta_k}, \tag{16.18}$$

$$V(x) = \frac{1}{\pi} \operatorname{Re} \sum_{k=1}^{2} \left\|\begin{matrix} A'_k, & B'_k \\ C'_k, & D'_k \end{matrix}\right\| \int_S \ln \sigma_k g(y) dS;$$

here g is an unknown real vector function, continuous on S. For the vector function W to be single-valued and bounded at infinity, we assume that the condition

$$\int_S g dS = 0 \tag{16.19}$$

is fulfilled.

Reasoning as in Section 11, it is not difficult to calculate boundary values of the operation T on the vector W. Then for determining of g we obtain the following Fredholm integral equation of second kind:

$$g(z)+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_S\frac{\partial\ln\sigma_k}{\partial s(z)}g(y)dS-$$

$$-\frac{\omega}{2\pi}\operatorname{Re}\sum_{k=1}^{2}\begin{pmatrix}q_k\\ s_k\end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0}}\cdot M=f(z), \tag{16.20}$$

where $f(z)$ is a given continuous vector function, $z_{k0}=z_1+\alpha_k z_2$.

By integration, from (16.20) we obtain

$$2\int_S gdS=\int_S fdS. \tag{16.21}$$

For (16.19) to be fulfilled, it is necessary that the principal vector of external stresses be equal to zero. In what follows, the principal vector will be assumed to be equal to zero.

Now we will establish that the equation (16.20) is always solvable. To with this end, we consider the homogeneous equation obtained from (16.20) for $f=0$. Prove that this equation has no nontrivial solutions. Assume the contrary and denote by g_0 any solution of the homogeneous equation. Since $f=0$, from (16.21) we have

$$\int_S g_0(y)dS=0. \tag{16.22}$$

Note that the homogeneous equation corresponds now to the boundary condition $[TW_0(z)]^-=0$.

Taking into account the uniqueness theorem for the second boundary value problem in the domain D^-, we obtain $W_0(x)=0$, $x\in D^-$.

This allows one to conclude that there exists boundary value of the result of acting of L on W_0, and we can write

$$0=[LW_0(z)]^-=\frac{1}{\pi\omega(BC-A^2)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}-A, & -B\\ C, & A\end{Vmatrix}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_S g_0(y)\frac{\partial\ln\sigma_k}{\partial s(z)}dS+$$

$$+\frac{1}{2\pi(BC-A^2)}\operatorname{Im}\sum_{k=1}^{2}\begin{pmatrix}-Aq_k-Bs_k\\ As_k+Cq_k\end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0}}\cdot M_0=0.$$

This expression after elementary calculations yields

$$0=\int_S\{(Cz_1+Az_2)[LW_0]_1^-+(Az_1+Bz_2)[LW_0]_2^-\}dS=M_0. \tag{16.23}$$

Since $M_0 = 0$, from (16.17) we obtain

$$W_0(x) \equiv u_0(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A'_k, & B'_k \\ C'_k, & S'_k \end{Vmatrix} \ln \sigma_k g(y) dS = 0, \quad x \in D^-.$$

Using now (8.14), we arrive at

$$0 = N u_0(x) = -\frac{\begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}}{m} \frac{\partial V_0(x)}{\partial s(x)},$$

whence $V_0(x) = C$, $x \in D^-$, where C is an arbitrary constant vector. But according to (16.22), $V_0(\infty) = 0$, and therefore $C = 0$. Thus

$$V_0(x) = 0, \quad x \in D^-. \tag{16.24}$$

This implies that $[TV_0(z)]^- = 0$.

Reasoning as in Section 11, we can prove that g is continuous. Then $[TV_0(z)]^+ - [TV_0(z)]^- = 0$, and therefore $[TV_0(z)]^+ = 0$.

Using now the uniqueness theorem for the solution of the second boundary value problem in the domain D^+, we obtain $V_0(x) = A + \varepsilon \left(\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix} \right)$, $x \in D^+$, where A and ε are arbitrary constants. Using (16.18) and (16.23), we have $-2\varepsilon = M_0 = 0$.

Thus

$$V_0(x) = A, \quad x \in D^+. \tag{16.25}$$

Using (16.16) and taking into account (16.24) and (16.25), we obtain $g_0(z) = 0$.

Consequently, the homogeneous equation corresponding to (16.20) has no nontrivial solution. This means that (16.20) has a unique solution, provided the principal vector of external stresses is equal to zero.

So far we have assumed that the principal vector $\int_S f dS$, the stress components $\sigma_x^{(\infty)}$, $\sigma_y^{(\infty)}$, $\tau_{xy}^{(\infty)}$, and rotation $\varepsilon^{(\infty)}$ were equal to zero at infinity.

In the general case where the principal vector and the stress components at infinity are given beforehand, a solution of the second boundary value problem in the domain D^- is sought in the form

$$W(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} \int_S \ln \sigma_k g(y) dS - \frac{\omega}{2\pi} \operatorname{Re} \sum_{k=1}^{2} \begin{pmatrix} B'_k \\ D'_k \end{pmatrix} \frac{1}{z_k} \cdot M +$$
$$+ \frac{1}{2\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \ln z_k \int_S f dS + A^{(\infty)} x_1 + B^{(\infty)} x_2,$$

where

$$A_1^{(\infty)} = a_{11}\sigma_x^{(\infty)} + a_{12}\sigma_y^{(\infty)} + a_{13}\tau_{xy}^{(\infty)}, \quad B_2^{(\infty)} = a_{12}\sigma_x^{(\infty)} + a_{22}\sigma_y^{(\infty)} + a_{23}\tau_{xy}^{(\infty)},$$
$$A_2^{(\infty)} = B_1^{(\infty)} = \frac{1}{2}\left(a_{13}\sigma_x^{(\infty)} + a_{23}\sigma_y^{(\infty)} + a_{33}\tau_{xy}^{(\infty)} \right).$$

We assume that the rotation at infinity $\varepsilon^{(\infty)} = 0$, since it does not influence stresses.

For determining the vector function g we obtain the integral equation

$$g(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\int_S \frac{\ln\sigma_k}{\partial s(z)}g(y)dS -$$

$$-\frac{\omega}{2\pi}\,\mathrm{Re}\sum_{k=1}^{2}\begin{pmatrix} q_k \\ s_k \end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0}}\cdot M = f_0(z),$$

where

$$f_0(z) = f - \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} N_k, & L_k \\ M_k, & R_k \end{Vmatrix}\frac{\partial \ln z_{k_0}}{\partial s(z)}\int_S f dS - \begin{pmatrix} \sigma_x^{(\infty)} \\ \tau_{xy}^{(\infty)} \end{pmatrix} n_1 - \begin{pmatrix} \tau_{xy}^{(\infty)} \\ \sigma_y^{(\infty)} \end{pmatrix} n_2.$$

This equation like (16.20) is always solvable because the condition of solvability

$$\int_S f_0(y)dS = 0$$

is fulfilled for it.

Thus the existence of the solution of the second boundary value problem in the domain D^- is proved.

The above-given proof of the existence of the solution of the second boundary value problem for domains D^+ and D^- can be found in author's work [4]. In the same work, the second boundary value problem is solved with the help of singular integral equations.

Consider now the second boundary value problem in a finite multiply connected domain D^+. The solution is sought in the form

$$W(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix}\int_S \ln\sigma_k g(y)dS -$$

$$-\frac{\omega}{2\pi}\,\mathrm{Re}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{pmatrix} B'_k \\ D'_k \end{pmatrix}\frac{M_j}{z_k - z_{kj}} + F_0(x), \tag{16.26}$$

where

$$M_j = \Big(\frac{\partial V_1}{\partial x_2} - \frac{\partial V_2}{\partial x_1}\Big)_{\substack{x_1 = x_{1j} \\ x_2 = x_{2j}}} =$$

$$= \frac{a_{11}}{\pi(BC - A^2)}\,\mathrm{Im}\sum_{k=1}^{2}\int_S \big[(C\alpha_k - A)g_1 + (A\alpha_k - B)g_2\big]\frac{dS}{\zeta_k - z_{kj}}, \tag{16.27}$$

$$V(x) = \frac{1}{\pi}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix}\int_S \ln\sigma_k g(y)dS, \tag{16.28}$$

$$F_0(x) = -\frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln(z_k - z_{kj})\int_{S_j} f dS +$$

$$+\frac{1}{4\pi}\,\mathrm{Im}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{pmatrix}B'_k - iA'_k\\ D'_k - iC'_k\end{pmatrix}\frac{(1-i\alpha_k)}{z_k - z_{kj}}L_j, \tag{16.29}$$

$$L_j = \int_{S_j}\big[(y_1 - x_{1j})f_2 - (y_2 - x_{2j})f_1\big]dS,\ \ j = \overline{1,m}, \tag{16.30}$$

$$x^{(j)} = (x_{1j}, x_{2j}) \in D_j, \quad z_{kj} = z_{1j} + \alpha_k x_{2j}.$$

It is assumed that the principal vector and the principal moment of external stresses are equal to zero on every contour S_j $(j = \overline{1,m})$.

Taking into account the properties of the vector $T(\partial x, n)W(x)$, for the determination of the unknown density g we obtain the following Fredholm integral equation of second kind:

$$-g(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_S \frac{\partial \ln\sigma_k}{\partial s(z)}g(y)dS -$$

$$-\frac{\omega}{2\pi}\,\mathrm{Re}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{pmatrix}q_k\\ s_k\end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0} - z_{kj}}\cdot M_j = F(z), \tag{16.31}$$

where

$$F(z) = f + \frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{Vmatrix}N_k, & L_k\\ M_k, & R_k\end{Vmatrix}\frac{\partial}{\partial s(z)}\ln(z_{k_0} - z_{kj})\int_{S_j} f dS -$$

$$-\frac{1}{4\pi}\,\mathrm{Im}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{pmatrix}q_k - ip_k\\ s_k - ir_k\end{pmatrix}(1-i\alpha_k)\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0} - z_{kj}}\cdot L_j, \tag{16.32}$$

f is a given continuous vector function, $z_{k0} = z_1 + \alpha_k z_2$, $\sigma_k = z_{k0} - \zeta_k$.

Instead of (16.31) we consider the equation

$$-g(z) + \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_S \frac{\partial \ln\sigma_k}{\partial s(z)}g(y)dS -$$

$$-\frac{\omega}{2\pi}\,\mathrm{Re}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{pmatrix}q_k\\ s_k\end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0} - z_{kj}}M_j + \frac{1}{2\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\frac{\partial \ln z_{k_0}}{\partial s(z)}\int_S g dS -$$

$$-\frac{1}{4\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{pmatrix}-\alpha_k\\ 1\end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0}}\cdot M = F(z), \tag{16.33}$$

where

$$M=\Big(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\Big)_{\substack{x_1=0\\x_2=0}}=$$

$$=\frac{a_{11}}{\pi(BC-A^2)}\operatorname{Re}\sum_{k=1}^{2}(A-C\alpha_k)\int_S\frac{(g_1+\frac{\alpha_1\alpha_2}{\alpha_k}g_2)}{\zeta_k}dS, \tag{16.34}$$

$$u(x)=\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}A'_k, & B'_k\\ C'_k, & D'_k\end{Vmatrix}\int_S\ln\sigma_k g(y)dS. \tag{16.35}$$

From (16.33) we have

$$\begin{gathered}\int_S g\,dS=\int_S F\,dS=\int_S f\,dS,\\ M=\int_S(y_1F_2-y_2F_1)dS=\int_S(y_1f_2-y_2f_1)dS,\end{gathered} \tag{16.36}$$

which implies that if the principal vector and the principal moment of external stresses are equal to zero, then any solution of (16.33) coincides with a solution of (16.31).

Let us prove that the equation (16.33) is always solvable. To this end, it is sufficient to show that the homogeneous equation corresponding to (16.33) has only the trivial solution. Assume the contrary and denote by g_0 a nonzero solution of the above-mentioned homogeneous equation. Obviously, the conditions

$$\int_S g_0 dS=0,\quad M_0=0 \tag{16.37}$$

are fulfilled. In this case, the homogeneous equation corresponding to (16.33) coincides with (16.31) for $F=0$. This homogeneous equation corresponds to the boundary condition $[Tu_1^{(0)}(z)]^+=0$, where

$$u_1^{(0)}(x)=u_0(x)-\frac{\omega}{2\pi}\operatorname{Re}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{pmatrix}B'_k\\ D'_k\end{pmatrix}\frac{1}{z_k-z_{kj}}\cdot M_j^{(0)}. \tag{16.38}$$

According to the existence theorem for the second boundary value problem in the domain D^+, we have

$$u_1^{(0)}=A+\varepsilon\begin{pmatrix}-x_2\\ x_1\end{pmatrix},\quad x\in D^+, \tag{16.39}$$

whence

$$[Lu_1^{(0)}(z)]^+=\frac{1}{\pi\omega(BC-A^2)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}-A, & -B\\ C, & A\end{Vmatrix}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_S\frac{\partial\ln\sigma_k}{\partial s(z)}g_0(y)dS+$$

$$+\frac{1}{2\pi(BC-A^2)}\,\mathrm{Im}\sum_{k=1}^{2}\sum_{j=1}^{m}\begin{pmatrix}-Aq_k-Bs_k\\ As_k+Cq_k\end{pmatrix}\frac{\partial}{\partial s(z)}\frac{1}{z_{k_0}-z_{kj}}\cdot M_j^{(0)}=$$
$$=\frac{\varepsilon}{a_{11}\omega}\begin{pmatrix}n_2(z)\\ -n_1(z)\end{pmatrix},\quad z\in S,$$

and

$$0=\int_{S_j}\Big\{(Cz_1+Az_2)\big[Lu_1^{(0)}(z)\big]_1+(Az_1+Bz_2)\big[Lu_1^{(0)}(z)\big]_2^+\Big\}dS=$$
$$=M_j^{(0)},\quad j=\overline{1,m}. \tag{16.40}$$

Since $M_j^{(0)}=0$, we have $u_1^{(0)}\equiv u_0(x)=A+\varepsilon\binom{-x_2}{x_1}$. If we notice that

$$\Big(\frac{\partial u_{02}}{\partial x_1}-\frac{\partial u_{01}}{\partial x_2}\Big)_{\substack{x_1=0\\ x_2=0}}=M_0=0,$$

then we obtain $u_0(x)=A$, $x\in D^+$, which implies that

$$0=Nu_0(x)=-\frac{\left\|\begin{smallmatrix}B, & -A\\ -A, & C\end{smallmatrix}\right\|}{m}\frac{\partial V_0(x)}{\partial s(x)}\quad\text{and}\quad V_0(x)=C,\quad x\in D^+,$$

where C is a constant vector.

By elementary reasoning we can prove that the vector function TV_0 is continuous in a neighbourhood of the boundary S. Therefore, since $[TV_0(z)]^+=0$, we have $[TV_0(x)]^-=0$.

Using the uniqueness theorem for the second boundary value problem in the domain D' (see Section 15) and taking into account (16.40), we find $V_0(x)=C_j$, $x\in D_j$, $j=\overline{1,m+1}$.

Now, on the basis of formula (16.16) and the equality $[LV_0(z)]^+=[LV_0(z)]^-=0$ we obtain $g_0(z)=0$.

Thus the equation (16.33) is solvable for any continuous right-hand side. We substitute the above-obtained g in (16.26) and obtain a solution of the second boundary value problem in a multiply connected domain D^+, provided the principal vector and the principal moment of external stresses are equal to zero.

In a similar way we can consider the second boundary value problem in the domain D' and prove the existence of the solution.

The proof of the existence of the solution of the second boundary value problem for multiply connected domains can be found in author's work [6].

The problem under consideration has been solved with the help of the method of the theory of functions of a complex variable and the theory of Fredholm integral equations by D. I. Sherman ([44], [46], [47]). He investigates the integral equation under the condition of $C^{2,\beta}$-smoothness of the countour S $(0<\beta\leq 1)$.

D. I. Sherman noted [50] that his integral equation for the second boundary value problem and that obtained by Kupradze and Basheleishvili ([25], [26]) for a simply connected domain (being transformed to the same form) essentially coincide.

Consider this point in a more detail. Using our notation, write D. I. Sherman's integral equation (see [47]) (can be written as using our)

$$h(z)+\frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_S\frac{\partial\ln\sigma_k}{\partial s(y)}h(y)dS=\int_0^{s(z)}fdS+C,\qquad(16.41)$$

where C is an arbitrary constant vector. To facilitate the comparison, we write out the equation (16.2):

$$-g(z)+\frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_S\frac{\partial\ln\sigma_k}{\partial s(z)}g(y)dS=f(z).\qquad(16.2)$$

Let $g=-\frac{\partial h}{\partial S}$ in (16.2). After integration and obvious transformations we obtain (16.4) from (16.2). Thus we can always pass from our equation to the Sherman equation.

To pass from (16.41) to (16.2), it is necessary to require additionally that the curvature of the curve S belongs to the Hölder class. If the curvature of the curve S does not belong to the Hölder class, then the passage from Sherman's equation to (16.2) is impossible. Thus Sherman's equation and equation (16.2) are mutually equivalent if and only if the curvature of the curve S belongs to the Hölder class.

It should be noted that the heightened smoothness requirement imposed on the contour S ($S\in C^{2,\beta}$, $0<\beta\le 1$) is due to the fact that the conditions of the second boundary value problem are taken by Sherman not in a usual but in an integral form.

17. Solution of the Second Generalized Boundary Value Problem

In this section we study the following boundary value problem: Find a regular solution of the equation (1.9), satisfying the boundary condition

$$[\overset{\varkappa_0}{T}u(z)]^{\pm}=f(z),\quad z\in S,\qquad(17.1)$$

where f is a given continuous vector function and $\varkappa_0$ is for the time being an arbitrary real constant.

If $\varkappa_0=0$ in (17.1), then $\overset{\varkappa_0}{T}=T$ by (1.23), and the generalized stress vector coincides with the stress vector. For $\varkappa_0=0$, the problem (17.1) has been investigated in the previous section.

In Section 11, for the second boundary value problem we have constructed the generalized simple layer potential which reduces this problem to a Fredholm integral equation of second kind. When constructing the matrix $\overset{\varkappa_0}{M}(x,y)$, the kernel of the generalized simple layer potential, we have excluded two values of the parameter $\varkappa_0$:

$$\varkappa_0=\delta=\frac{\sqrt{BC-A^2}}{a_{11}(1-\omega\sqrt{BC-A^2})},\quad \varkappa_0=\delta_1=-\frac{\sqrt{BC-A^2}}{a_{11}(1+\omega\sqrt{BC-A^2})}.$$

These values, as is easily seen (see [9]), are contained in the intervals $\varkappa_2<\delta<\varkappa_3$, $-\frac{1}{\omega a_{11}}<\delta_1<0$, where $-\frac{1}{\omega a_{11}}$, $\varkappa_2$, and $\varkappa_3$ are the roots of the equation $f(\varkappa)=0$, $f(\varkappa)$ being given by (2.19).

In Section 8 we have defined the adjoint operators $\overset{\varkappa}{T}$ and $\overset{\varkappa_0}{T}$ for $\varkappa$ and $\varkappa_0$ satisfying the condition

$$\varkappa=\frac{(\varkappa_0+\frac{1}{\omega a_{11}})\varkappa_N}{\varkappa_0-\varkappa_N}. \tag{17.2}$$

Hence we easily see that

$$-\infty<\varkappa\le-\frac{1}{\omega a_{11}}\ \text{ for }\ 0\le\varkappa_0<\varkappa_N,\quad \varkappa_3<\varkappa<\infty\ \text{ for }\ \varkappa_N<\varkappa_0<\varkappa_2,$$
$$-\frac{1}{\omega a_{11}}<\varkappa<0\ \text{ for }\ -\frac{1}{\omega a_{11}}<\varkappa_0<0,\quad \varkappa_2\le\varkappa\le\varkappa_3\ \text{ for }\ \varkappa_2\le\varkappa_0\le\varkappa_3.$$

The intervals $]\varkappa_2,\varkappa_3[$ and $]-\frac{1}{\omega a_{11}},0[$ contain respectively the values δ and δ_1 for which the matrix function $\overset{\varkappa_0}{M}$ has not been defined.

Here we intend to show that the second generalized boundary value problem is solvable for $-\infty<\varkappa_0\le-\frac{1}{\omega a_{11}}$, $0\le\varkappa_0<\varkappa_2$, and $\varkappa_3<\varkappa_0<\infty$. For the case $0\le\varkappa_0<\varkappa_2$ the use will be made of the operator $\overset{\varkappa_0}{T}$, because the corresponding generalized potential energy is positive definite, and for the cases $-\infty<\varkappa_0\le-\frac{1}{\omega a_{11}}$ and $\varkappa_3<\varkappa_0<\infty$ we will use the adjoint operator $\overset{\varkappa}{T}$ as the one having the positive definete generalized potential energy.

Note that if a vector function u is defined in a doman, then the adjoint vector function v can be determined to within a constant vector from the generalized Cauchy–Riemann conditions (see (8.1)). A similar statement is true for the vector v.

Now a solution of the second boundary value problem in the domain D^+ is sought as

$$u(x)=\frac{1}{\pi}\int_S\big[\overset{\varkappa_0}{M}(x,y)-\overset{\varkappa_0}{M}(0,y)\big]g\,dS+\frac{1}{\pi}\int_S g\,dS, \tag{17.3}$$

where g is an unknown vector function, and $\overset{\varkappa_0}{M}(x,y)$ is defined by (6.10) and (6.11).

Taking into account the boundary condition (17.1) and the formula (11.19), for determining g we obtain the Fredholm integral equation of second kind

$$-g(z)+\frac{1}{\pi}\int_S \overset{\varkappa_0}{T}(\partial_z,n)\overset{\varkappa_0}{M}(z,y)g\,dS=f(z),\quad z\in S, \tag{17.4}$$

where the kernel $\overset{\varkappa_0}{T}(\partial_z,n)\overset{\varkappa_0}{M}(z,y)$ is defined from (6.15) by putting $\varkappa=\varkappa_0$ and $x=z$.

Using the arguments of the previous sections (see Sections 15 and 16), we can easily show that the equation (17.4) has a solution if the vector $f(z)$ satisfies the condition

$$\int_S f(z)\,dS=0.$$

In this case, the displacement vector (see (17.3)) is defined to within a constant vector, and the vector function $\overset{\varkappa_0}{T}u$ is defined uniquely.

A solution of the second generalized boundary value problem in the domain D^- will be sought in the form

$$u(x)=\frac{1}{\pi}\int_S \overset{\varkappa_0}{M}(x,y)g\,dS. \tag{17.5}$$

For determining g, we take into account the boundary properties of the vector function $\overset{\varkappa_0}{T}u$ (see Section 11, formula (11.13)) and obtain the following integral equation of second kind:

$$g(z)+\frac{1}{\pi}\int_S \overset{\varkappa_0}{T}(\partial_z,n)\overset{\varkappa_0}{M}(z,y)g(y)\,dS=f(z). \tag{17.6}$$

Integration of the above equation leads to

$$2\int_S g\,dS=\int_S f\,dS. \tag{17.7}$$

For the vector function u, defined by (17.5) to be homogeneous and vanishing at infinity, it is sufficient that

$$\int_S g\,dS=0.$$

By (17.7), the latter condition will be fulfilled if

$$\int_S f\,dS=0. \tag{17.8}$$

As above, we can prove that the second generalized boundary value problem has a solution which is bounded at infinity, provided besides continuity f satisfies (17.8).

The case where $\int_S f\,dS\neq 0$ can be considered as in Section 16.

Thus we have proved that the second generalized boundary value problem is solvable in the domains D^+ and D^- if $0 \le \varkappa_0 < \varkappa_2$.

Consider now the cases

$$-\infty < \varkappa_0 \le -\frac{1}{\omega a_{11}}, \quad \varkappa_3 < \varkappa_0 < \infty. \tag{17.9}$$

Using (17.2), we have

$$0 \le \varkappa < \varkappa_N, \quad \varkappa_N < \varkappa < \varkappa_2. \tag{17.10}$$

On the basis of (8.20), we can write

$$\overset{\varkappa}{T}v = \frac{\begin{Vmatrix} A, & B \\ -C, & -A \end{Vmatrix}}{m\varkappa_0 - \omega(BC - A^2)} \overset{\varkappa_0}{T}u, \tag{17.11}$$

where u and v are adjoint vectors (see Section 8), and $\varkappa_0$ and $\varkappa$ satisfy (17.2).

Consequently, if $\varkappa_0$ and $\varkappa$ are in the intervals (17.9) and (17.10), respectively, then instead of the problem (17.1) we consider the problem

$$(\overset{\varkappa}{T}v(z))^{\pm} = F(z), \quad z \in S. \tag{17.12}$$

The generalized potential energy for the operator $\overset{\varkappa}{T}$ is positive definite, and hence the problem (17.12) can be investigated in a complete analogy with the problem (17.1). In this case, the vector V is defined both in D^+ and in D^- to within a constant vector.

Now from the Cauchy–Riemann conditions (see (8.9)) we easily find u which is also defined to within a constant vector.

Thus, using the Cauchy–Riemann conditions, we can state that the problem (17.1) is solvable if

$$-\infty < \varkappa_0 \le -\frac{1}{\omega a_{11}}, \quad 0 \le \varkappa_0 < \varkappa_2, \quad \varkappa_3 < \varkappa_0 < \infty. \tag{17.13}$$

A detailed exposition of the point can be found in author's work [13].

18. Solution of the Third Boundary Value Problem

The third boundary value problem has been formulated in Section 1. This problem for equations of statics of an isotropic elastic body in the case of finite simply and multiply connected domains has been solved by D. I. Sherman ([48], [49]).

Fitst of all, derive some formulas which we will need below.

The stress vector (see formula (1.19)) can be represented as

$$Tu = Lu + \frac{1}{\omega a_{11}} \begin{Vmatrix} 0, & 1 \\ -1, & 0 \end{Vmatrix} \frac{\partial u}{\partial s}, \tag{18.1}$$

where Lu is the vector function obtained from (1.21) for $\varkappa = -\frac{1}{\omega a_{11}}$, and $\frac{\partial}{\partial s} = n_1 \frac{\partial}{\partial z_2} - n_2 \frac{\partial}{\partial z_1}$ is the derivative with respect to the tangent; $n = (n_1, n_2)$ and $s = (-n_2, n_1)$ denote the unit normal and tangent vectors, respectively, at the point $z \in S$.

From (18.1) we have

$$(sTu) = (sLu) - \frac{1}{\omega a_{11}}\Big(n\frac{\partial u}{\partial s}\Big), \quad (nTu) = (nLu) + \frac{1}{\omega a_{11}}\Big(s\frac{\partial u}{\partial s}\Big). \tag{18.2}$$

Using the well-known elementary relations $\frac{\partial n(z)}{\partial s} = \frac{s(z)}{\rho}$, $\frac{\partial s(z)}{\partial s} = -\frac{n(z)}{\rho}$, where $\frac{1}{\rho}$ is the curvature of the curve S at the point z, we obtain

$$\Big(n\frac{\partial u}{\partial s}\Big) = \frac{\partial (nu)}{\partial s} - \frac{(su)}{\rho}, \quad \Big(s\frac{\partial u}{\partial s}\Big) = \frac{\partial (su)}{\partial s} + \frac{(nu)}{\rho}.$$

Taking now into account the last formulas, from (18.2) we get

$$\begin{aligned}(sTu) &= (sLu) - \frac{1}{\omega a_{11}}\Big[\frac{\partial (nu)}{\partial s} - \frac{(su)}{\rho}\Big],\\ (nTu) &= (nLu) + \frac{1}{\omega a_{11}}\Big[\frac{\partial (su)}{\partial s} + \frac{(nu)}{\rho}\Big].\end{aligned} \tag{18.3}$$

Introduce the vector functions

$$\widetilde{s} = \frac{1}{\alpha(z)}\big(-Bn_2 - An_1, Cn_1 + An_2\big), \ \widetilde{n} = \frac{1}{\gamma(z)}\big(Bn_1 - An_2, Cn_2 - An_1\big), \tag{18.4}$$

where

$$\alpha(z) = Cn_1^2 + Bn_2^2 + 2An_1n_2 > 0, \quad \gamma(z) = Bn_1^2 + Cn_2^2 - 2An_1n_2 > 0. \tag{18.5}$$

Simple calculations result in

$$\widetilde{s} = s + n\frac{\beta(z)}{\alpha(z)}, \quad \widetilde{n} = n + s\frac{\beta(z)}{\gamma(z)}, \tag{18.6}$$

where

$$\beta(z) = (C - B)n_1n_2 + A(n_2^2 - n_1^2), \quad (s\widetilde{s}) = 1, \quad (n\widetilde{n}) = 1. \tag{18.7}$$

It is obvious from (18.6) that

$$(su) = (\widetilde{s}u) - (nu)\frac{\beta(z)}{\alpha(z)}, \quad (nu) = (\widetilde{n}u) - (su)\frac{\beta(z)}{\gamma(z)},$$

and (18.3) can be written in the form

$$\begin{aligned}(sTu) + \frac{1}{\omega a_{11}}\Big[\frac{\partial (nu)}{\partial s} + \frac{(nu)}{\rho}\frac{\beta(z)}{\alpha(z)}\Big] &= (sLu) + \frac{(\widetilde{s}u)}{\rho\omega a_{11}},\\ (nTu) - \frac{1}{\omega a_{11}}\Big[\frac{\partial (su)}{\partial s} - \frac{(su)}{\rho}\frac{\beta(z)}{\alpha(z)}\Big] &= (nLu) + \frac{(\widetilde{n}u)}{\rho\omega a_{11}}.\end{aligned} \tag{18.8}$$

Note that the above-given formulas take a more simple form if $A = 0$ and $B = C$. In this case, $\beta(z) = 0$, $\alpha(z) = \gamma(z) = C$, $\widetilde{n} = n$, $\widetilde{s} = s$.

Write now the conditions of the third boundary value problem for the domain D^+:

$$(nu)^+ = f_1(z), \quad (sTu)^+ = f_2(z), \quad z \in S, \tag{18.9}$$

where $f_1(z)$ and $f_2(z)$ are given functions, and $f_1 \in C^{1,\alpha}(S)$, $f_2 \in C^{0,\alpha}(S)$, $S \in C^{2,\beta}$, $0 < \alpha < \beta \le 1$.

A solution of this problem is sought in the domain D^+ as

$$u(x) = \frac{(BC - A^2)}{\pi a_{11}} \int_S \operatorname{Re} \sum_{k=1}^{2} \left\{ \begin{Vmatrix} -B'_k, & A'_k \\ -D'_k & C'_k \end{Vmatrix} \frac{\partial}{\partial s(y)} \ln \frac{\sigma_k}{(-\zeta_k)} \frac{s(y)}{\alpha(y)} g_1(y) + \right.$$
$$\left. + \begin{Vmatrix} A_k, & B_k \\ B_k & C_k \end{Vmatrix} \left[\frac{\widetilde{s}(y) g_1(y)}{\rho(y)\alpha(y)} - \frac{a_{11} n(y) g_2(y)}{\omega(BC - A^2)\alpha(y)} \right] \ln \frac{\sigma_k}{(-\zeta_k)} \right\} dS + \delta, \tag{18.10}$$

where $g_1(y)$ and $g_2(y)$ are the unknown functions, $\widetilde{s}(y)$ and $\alpha(y)$ are defined by (18.4) and (18.5), A_k, B_k,C_k are the coefficients of the fundamental matrix (see (4.12)), A'_k, B'_k, C'_k, D'_k are given by (6.2), $\frac{1}{\rho(y)}$ is the curvature of the curve S at the point y, $\sigma_k = z_k - \zeta_k$, and

$$\delta = \frac{1}{\pi} \int_S \left[\frac{\widetilde{s}(y) g_1(y)}{\rho(y)\alpha(y)} - \frac{a_{11} n(y) g_2(y)}{\omega(BC - A^2)\alpha(y)} \right] dS. \tag{18.11}$$

It follows from (18.10) that

$$Tu(x) = \frac{(BC - A^2)}{\pi a_{11}} \int_S \operatorname{Re} \sum_{k=1}^{2} \left\{ \begin{Vmatrix} -q_k, & p_k \\ -s_k & r_k \end{Vmatrix} \frac{\partial^2 \ln \sigma_k}{\partial s(x) \partial s(y)} \frac{s(y)}{\alpha(y)} g_1(y) + \right.$$
$$\left. + \begin{Vmatrix} N_k, & L_k \\ M_k & R_k \end{Vmatrix} \left[\frac{\widetilde{s}(y) g_1(y)}{\rho(y)\alpha(y)} - \frac{a_{11} n(y) g_2(y)}{\omega(BC - A^2)\alpha(y)} \right] \frac{\partial \ln \sigma_k}{\partial s(x)} \right\} dS, \tag{18.12}$$

where p_k, q_k, r_k, s_k are given by (4.18), and

$$\begin{Vmatrix} N_k, & L_k \\ M_k, & R_k \end{Vmatrix} = \frac{i}{a_{11}} \begin{Vmatrix} A, & B \\ -C, & -A \end{Vmatrix} \begin{Vmatrix} B'_k, & D'_k \\ -A'_k, & -C'_k \end{Vmatrix}. \tag{18.13}$$

Taking into account the conditions of the third boundary value problem and boundary properties of potentials appearing in (18.10) and (18.12), after certain obvious transformations we obtain the following system of integral equations for determining g_1 and g_2:

$$-g_1(z) + \frac{1}{\pi} \int_S \left\{ n(z) \frac{(BC - A^2)}{a_{11}} \operatorname{Re} \sum_{k=1}^{2} \left[\begin{Vmatrix} -B'_k, & A'_k \\ -D'_k, & C'_k \end{Vmatrix} \frac{\partial}{\partial s(y)} \frac{\sigma_k}{(-\zeta_k)} \frac{s(y)}{\alpha(y)} + \right. \right.$$
$$\left. + \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \frac{\widetilde{s}(y)}{\rho(y)\alpha(y)} \ln \frac{\sigma_k}{(-\zeta_k)} \right] g_1(y) -$$

$$-\frac{n(z)}{\omega}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{(-\zeta_k)}\frac{n(y)}{\alpha(y)}g_2(y)\Big\}dS+(n(z)\delta)=f_1(z),$$

$$-g_2(z)+\frac{1}{\pi}\int_S\Big\{s(z)\frac{(BC-A^2)}{a_{11}}\,\mathrm{Re}\sum_{k=1}^{2}\Big[\begin{Vmatrix}-q_k, & p_k\\ -s_k, & r_k\end{Vmatrix}\frac{\partial^2\ln\sigma_k}{\partial s(z)\partial s(y)}\,s(y)+$$

$$+\begin{Vmatrix}N_k, & L_k\\ M_k & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(z)}\frac{\widetilde{s}(y)}{\rho(y)}\Big]\frac{g_1(y)}{\alpha(y)}-$$

$$-\frac{s(z)}{\omega}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & L_k\\ M_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(z)}\frac{n(y)}{\alpha(y)}g_2(y)\Big\}dS=f_2(z),\quad z\in S.$$

We rewrite this system in the vector form

$$-g(z)+\frac{1}{\pi}\int_S K(z,y)g(y)dS=f(z), \tag{18.14}$$

where $f=(f_1,f_2)$, $g=(g_1,g_2)$, and

$$K(z,y)=\begin{Vmatrix}K_{11}, & K_{12}\\ K_{21}, & K_{22}\end{Vmatrix},$$

$$K_{11}=\frac{(BC-A^2)}{a_{11}}n(z)\,\mathrm{Re}\sum_{k=1}^{2}\Big[\begin{Vmatrix}-B'_k, & A'_k\\ -D'_k, & C'_k\end{Vmatrix}\frac{\partial}{\partial s(y)}\ln\frac{\sigma_k}{(-\zeta_k)}+$$

$$+\begin{Vmatrix}A_k, & B_k\\ B_k & C_k\end{Vmatrix}\frac{\widetilde{s}(y)}{\rho(y)\alpha(y)}\ln\frac{\sigma_k}{(-\zeta_k)}\Big]+\frac{n(z)\widetilde{s}(y)}{\rho(y)\alpha(y)},$$

$$K_{12}=-\frac{n(z)}{\omega}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{(-\zeta_k)}-\frac{a_{11}n(z)n(y)}{\omega(BC-A^2)\alpha(y)},$$

$$K_{21}=\frac{(BC-A^2)}{a_{11}}s(z)\,\mathrm{Re}\sum_{k=1}^{2}\Big[\begin{Vmatrix}-q_k, & p_k\\ -s_k, & r_k\end{Vmatrix}\frac{\partial^2\ln\sigma_k}{\partial s(z)\partial s(y)}\frac{s(y)}{\alpha(y)}+$$

$$+\begin{Vmatrix}N_k, & L_k\\ M_k & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(z)}\frac{\widetilde{s}(y)}{\rho(y)\alpha(y)}\Big],$$

$$K_{22}=-\frac{s(z)}{\omega}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & L_k\\ M_k & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(z)}\frac{n(y)}{\alpha(y)}.$$

It can be proved that the equation (18.14) is a Fredholm integral equation.

Let us investigate now the question of solvability of this equation. To this end, we suppose that the homogeneous integral equation

$$-g(z)+\frac{1}{\pi}\int_S K(z,y)g(y)ds=0 \tag{18.15}$$

has a nontrivial solution which we denote by $g_0=(g_{10},g_{20})$. Differentiating the first equation from (18.15) with respect to the arc and taking into account the fact that the curvature of the curve S is Hölder continuous, we obtain $g_{10} \in C^{1,\alpha}(s)$, $g_{20} \in C^{0,\alpha}(s)$, $0 < \alpha < \beta \leq 1$, where g_{10} and g_{20} are projections of the vector g_0. In these conditions, the displacement vector u_0 and the stress vector Tu_0 are continuous up to the boundary S. Since (18.15) corresponds to the boundary condition $(nu_0)^+ = 0$, $(sTu_0)^+ = 0$, we have $(u_0 T u_0)^+ = 0$, and from the uniqueness theorem (assuming that S is not a circumference) we obtain

$$u_0(x) = 0, \quad x \in D^+. \tag{18.16}$$

This and (18.10) yield

$$\delta_0 = \int_S \Big[\frac{\widetilde{s} g_{10}}{\rho\alpha} - \frac{a_{11} n g_{20}}{\omega(BC - A^2)\alpha}\Big] ds = 0, \tag{18.17}$$

since the point $0(0,0)$ is in the domain D^+. Moreover, using (8.13), we obtain

$$0 = N u_0 = \frac{1}{m} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix} \frac{\partial V_0}{\partial s}, \tag{18.18}$$

where

$$V_0(x) = \frac{(BC - A^2)}{\pi a_{11}} \int_S \operatorname{Im} \sum_{k=1}^{2} \Big\{ \begin{Vmatrix} -B_k', & A_k' \\ -D_k', & C_k' \end{Vmatrix} \frac{\partial}{\partial s(y)} \ln \frac{\sigma_k}{(-\zeta_k)} \frac{s(y)}{\alpha(y)} g_{10}(y) + \\ + \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \Big[\frac{\widetilde{s} g_{10}(y)}{\rho(y)\alpha(y)} - \frac{a_{11} n(y) g_{20}(y)}{\omega(BC - A^2)\alpha(y)}\Big] \ln \frac{\sigma_k}{(-\zeta_k)} \Big\} dS. \tag{18.19}$$

From (18.18) follows $V_0(x) = C_0$, $x \in D^+$, where C_0 is an arbitrary constant vector. But since $V_0(0) = 0$, we have $C_0 = 0$, and

$$V_0(x) = 0, \quad x \in D^+. \tag{18.20}$$

Using now (18.19), we find that

$$\begin{aligned} \big(V_0(z)\big)^+ - \big(V_0(z)\big)^- &= 2\omega(BC - A^2) \frac{s(z)}{\alpha(z)} g_{10}(z), \\ \big(TV_0(z)\big)^+ - \big(TV_0(z)\big)^- &= \frac{2}{\omega} \frac{n(z)}{\alpha(z)} g_{20}(z) - 2\frac{(BC - A^2)}{a_{11}} \Big[\frac{\widetilde{s}(z) g_{10}}{\rho(z)\alpha(z)} - \\ &\quad - \frac{\partial}{\partial s(z)} \Big(\frac{n(z) g_{10}}{\alpha(z)}\Big)\Big], \end{aligned}$$

whence

$$\begin{aligned}\big(V_0(z)\big)^- &= -2\omega(BC-A^2)\frac{s(z)}{\alpha(z)}g_{10}(z),\\ \big(TV_0(z)\big)^- &= -\frac{2}{\omega}\frac{n(z)}{\alpha(z)}g_{20}(z)+2\frac{(BC-A^2)}{a_{11}}\Big[\frac{\widetilde{s}(z)g_{10}}{\rho(z)\alpha(z)}- \\ &\qquad -\frac{\partial}{\partial s(z)}\Big(\frac{n(z)g_{10}}{\alpha(z)}\Big)\Big]\end{aligned} \tag{18.21}$$

since $(V_0(z))^+ = 0$ and $(TV_0(z))^+ = 0$.

The relations (18.21) result in

$$\big(V_0(z)TV_0(z)\big)^- = 0. \tag{18.22}$$

By virtue of (18.17), the vector $V_0(x)$ satisfies (1.24) and is single-valued in the entire plane. Using now the uniqueness theorem in the domain D^- and taking into account (18.22), we obtain $V_0(x) = 0$, $x \in D^-$. This implies $\big(sV_0(z)\big)^- = 0$, $\big(nTV_0(z)\big)^- = 0$.

Taking (18.21) into consideration, we have $g_{10} = 0$, $g_{20} = 0$.

Thus we have proved that when S is not a circumference, the homogeneous integral equation (18.15) has only the trivial solution. This, according to the first Fredholm theorem, means that the nonhomogeneous integral equation (18.14) is solvable for an arbitrary right-hand side.

It remains to consider the case where S is a circumference. For this purpose, we write the associated to (18.14) Fredholm integral equation

$$-h(z)+\frac{1}{\pi}\int_S [K(z,y)]'h(y)ds = F(z), \tag{18.23}$$

where $h = (h_1, h_2)$ is an unknown vector, F is a given vector, and

$$K(y,z) = \begin{Vmatrix} K_{11}(y,z), & K_{21}(y,z) \\ K_{12}(y,z), & K_{22}(y,z)\end{Vmatrix},$$

$$\begin{aligned}K_{11}(y,z) &= \frac{(BC-A^2)}{a_{11}}\Big[\frac{s(z)}{\alpha(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}-B_k', & -D_k' \\ A_k', & C_k'\end{Vmatrix}\frac{\partial}{\partial s(z)}\frac{\sigma_k}{z_{k0}}n(y)+ \\ &\quad +\frac{\widetilde{s}(z)}{\rho(z)\alpha(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k \\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{z_{k0}}n(y)\Big]+\frac{\widetilde{s}(z)n(y)}{\rho(z)\alpha(z)}\\ K_{21}(y,z) &= \frac{(BC-A^2)}{a_{11}}\Big[\frac{s(z)}{\alpha(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}-q_k, & -s_k \\ p_k, & r_k\end{Vmatrix}\frac{\partial^2\sigma_k}{\partial s(z)\partial s(y)}s(y)+ \\ &\quad +\frac{\widetilde{s}(z)}{\rho(z)\alpha(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & M_k \\ L_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(y)}s(y)\Big],\\ K_{12}(y,z) &= \frac{n(z)}{\omega\alpha(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k \\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{z_{k0}}n(y)+\frac{a_{11}n(z)n(y)}{\omega(BC-A^2)\alpha(z)},\end{aligned}$$

$$K_{22}(y,z)=\frac{n(z)}{\omega}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & M_k\\ L_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(y)}s(y),\quad z_{k0}=z_1+\alpha_k z_2.$$

Elementary calculations show that the equation (18.23) is obtained when looking for a solution of the boundary value problem

$$\frac{\omega(BC-A^2)}{\alpha(z)}\Big[(sLu)^-+\frac{1}{\omega a_{11}}\frac{(\widetilde{s}u)^-}{\rho(z)}\Big]=F_1(z),\quad \frac{(nu)^-}{\omega\alpha(z)}=F_2(z)\quad(18.24)$$

in the domain D^- as

$$u(x)=\frac{1}{\pi}\int_S\operatorname{Re}\sum_{k=1}^{2}\Big[\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{z_k}n(y)h_1(y)+$$
$$+\begin{Vmatrix}N_k, & M_k\\ L_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(y)}s(y)h_2(y)\Big]\,ds+\frac{a_{11}}{\pi(BC-A^2)}\int_S n(y)h_1(y)ds.\ (18.25)$$

The comparison of the first condition of (18.8) with (18.24) shows that (18.24) is by itself the third boundary value problem with modified (equivalent) conditions.

When S is a circumference, one is immediately convinced that the vector $h=(0,1)$ satisfies the homogeneous equation corresponding to (18.23). This means that the equation (18.23) may have no solution for some (continuous) right-hand side. Consequently, the nonhomogeneous integral equation (18.14) also may be unsolvable for some right-hand sides.

In the case of the circumference, instead of (18.14) we consider the following integral equation:

$$-g(z)+\frac{1}{\pi}\int_S K(z,y)g(y)ds-$$
$$-\frac{R}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{pmatrix}(B_k n_1(z)+C_k n_2(z))\frac{1}{z_{k0}}\\ R_k(n_1+\alpha_k n_2)\frac{\partial}{\partial s(z)}\frac{1}{z_{k0}}\end{pmatrix}\cdot M=f(z),\qquad(18.26)$$

where R is the radius of the circle, $z_{k0}=z_1+\alpha_k z_2$, and the origin is in the center of the circle,

$$M=\Big(\frac{\partial u_2}{\partial x_1}\Big)_{x_1=0,\,x_2=0}=$$
$$=-\frac{1}{\pi}\int_S\Big[\frac{(BC-A^2)}{a_{11}}\operatorname{Re}\sum_{k=1}^{2}\Big\{\frac{C_k' n_1(y)+D_k' n_2(y)}{\alpha(y)}\frac{\partial}{\partial s(y)}\frac{1}{\zeta_k}+$$
$$+\frac{[(CC_k-AB_k)n_1(y)+(AC_k-BB_k)n_2(y)]}{\zeta_k\rho(y)\alpha^2(y)}\Big\}g_1(y)-$$
$$-\frac{1}{\omega}\operatorname{Re}\sum_{k=1}^{2}\frac{B_k n_1+C_k n_2}{\alpha(y)\zeta_k}g_2(y)\Big]\,ds.\qquad(18.27)$$

On the basis of (18.26), the elementary calculation gives

$$M=\int_S f_2 ds. \tag{18.28}$$

Let us now prove that the equation (18.26) is always solvable. For this purpose it suffices to prove that the corresponding homogeneous equation has only the trivial solution. Suppose the contrary and denote a nontrivial solution of the equation

$$\begin{aligned}-g(z)+\frac{1}{\pi}\int_S K(z,y)g(y)ds-\\ -\frac{R}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{pmatrix}(B_kn_1(z)+C_kn_2(z))\frac{1}{z_{k0}}\\ R_k(n_1+\alpha_kn_2)\frac{\partial}{\partial s(z)}\frac{1}{z_{k0}}\end{pmatrix}\cdot M=0\end{aligned} \tag{18.29}$$

by g_0. Since $f=0$, from (18.28) we have

$$M=0, \tag{18.30}$$

and the equation (18.29) coincides with the equation (18.15). Applying the uniqueness theorem to the third boundary value problem (when S is a circumference), we obtain

$$u_0(x)=\varepsilon\begin{pmatrix}-x_2\\ x_1\end{pmatrix},\quad x\in D^+.$$

Using now (18.27) and (18.30), we arrive at $u_0(x)=0$, $x\in D^+$.

Repeating the above reasoning, we get $g_0=0$.

Consequently, the equation (18.26) has always a unique solution g. Substituting this solution in (18.10), we obtain a solution of the third boundary value problem for a circumference, provided the principal moment of external forces is equal to zero.

Thus the third boundary value problem for the domain D^+ is solved completely.

To solve the third boundary value problem in the domain D^-, we write the associated to (18.26) integral equation

$$-h(z)+\frac{1}{\pi}\int_S[K(z,y)]'h(y)ds-\frac{R}{2\pi}\begin{pmatrix}H_1(z)\\ H_2(z)\end{pmatrix}L=F(z), \tag{18.31}$$

where

$$\begin{aligned}H_1(z)=&\frac{\omega(BC-A^2)}{\alpha(z)a_{11}}\operatorname{Re}\sum_{k=1}^{2}\Big[(C_k'n_1+D_k'n_2)\frac{\partial}{\partial s(z)}\frac{1}{z_{k0}}+\\ &+\frac{(CC_k-AB_k)n_1+(AC_k-BB_k)n_2}{z_{k0}\rho\alpha^2(z)}\Big],\\ H_2(z)=&\frac{1}{\omega\alpha(z)}\operatorname{Re}\sum_{k=1}^{2}(B_kn_1+C_kn_2)\frac{1}{z_{k0}},\end{aligned}$$

$$L=-\frac{1}{\pi}\int_S \operatorname{Im}\sum_{k=1}^{2}\Big[(B_k n_1+C_k n_2)\frac{h_1}{\zeta_k}+R_k(n_1+\alpha n_2)\frac{\partial}{\partial s(y)}\frac{1}{\zeta_k}\,h_2\Big]\,ds=$$
$$=\Big(\frac{\partial u_2}{\partial x_1}\Big)_{x_1=0,\ x_2=0}.$$

The direct checking shows that one can obtain the equation (18.31) if a solution of the third modified boundary value problem

$$\frac{\omega(BC-A^2)}{\alpha(z)}\Big[(sLV)^- + \frac{1}{\omega a_{11}}\frac{(\widetilde{s}V)^-}{\rho(z)}\Big] = F_1(z), \quad \frac{(nV)^-}{\omega\alpha(z)} = F_2(z)$$

in the domain D^- is sought in the form

$$V(x) = u(x) + \frac{1}{\pi}\operatorname{Re}\sum_{k=1}^{2}\begin{pmatrix}B_k\\ C_k\end{pmatrix}\frac{1}{z_k}\cdot L, \quad x\in D^-, \tag{18.32}$$

where $u(x)$ is defined by (18.25). If $(nv)^- = f_1$ and $(sTv)^- = f_2$, then taking into account (18.8), we have

$$F_1(z)=\frac{\omega(BC-A^2)}{\alpha(z)}\Big[f_2+\frac{1}{\omega a_{11}}\Big(\frac{\partial f_1}{\partial s}+\frac{f_1}{\rho}\frac{\beta(z)}{\alpha(z)}\Big)\Big],\quad F_2(z)=\frac{f_1}{\omega\alpha(z)},\quad z\in S.$$

The equation (18.31), just as the equation (18.26), is always solvable. Thus we have proved that the third boundary value problem in the domain D^- always has a unique regular solution which is representable in the form (18.32).

It should be noted that if S is not a circumference, then we have to take $L=0$.

19. Solution of the Fourth Boundary Value Problem

The conditions for the fourth boundary value problem in the domain D^+ are of the form (see Section 1)

$$(su)^+ = f_1(z), \quad (nTu)^+ = f_2(z), \quad z\in S, \tag{19.1}$$

where $f_1\in C^{1,\alpha}(S)$ and $f_2\in C^{0,\alpha}(S)$ are given functions, $S\in C^{2,\beta}$, $0<\alpha<\beta\le 1$.

A solution of the fourth boundary value problem in the domain D^+ is sought in the form

$$u(x) = \frac{(BC-A^2)}{\pi a_{11}}\int_S \operatorname{Re}\sum_{k=1}^{2}\Big\{\begin{Vmatrix}-B_k', & A_k'\\ -D_k', & C_k'\end{Vmatrix}\frac{\partial}{\partial s(y)}\frac{\sigma_k}{(-\zeta_k)}\frac{n(y)}{\gamma(y)}g_1(y)+$$
$$+\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{(-\zeta_k)}\Big[\frac{\widetilde{n}(y)}{\rho(y)\gamma(y)}g_1(y)-\frac{a_{11}s(y)g_2(y)}{\omega(BC-A^2)\gamma(y)}\Big]\Big\}\,ds+\delta, \tag{19.2}$$

$x \in D^+$, where g_1 and g_2 are unknown functions, $\widetilde{n}(y)$ and $\gamma(y)$ are given by (18.4) and (18.5),

$$\delta = \frac{1}{\pi}\int_S \Big[\frac{\widetilde{n}(y)}{\rho(y)\gamma(y)}g_1(y) - \frac{a_{11}s(y)g_2(y)}{\omega(BC-A^2)\gamma(y)}\Big]ds, \tag{19.3}$$

and the remaining coefficients and functions appearing in (19.2) have been defined in the previous section.

From (19.2) we get

$$Tu(x) = \frac{(BC-A^2)}{\pi a_{11}}\int_S \operatorname{Re}\sum_{k=1}^{2}\Bigg\{\begin{Vmatrix} -q_k, & p_k \\ -s_k, & r_k\end{Vmatrix}\frac{\partial^2 \ln\sigma_k}{\partial s(x)\partial s(y)}\frac{n(y)}{\gamma(y)}g_1(y)+$$
$$+\begin{Vmatrix} N_k, & L_k \\ M_k, & R_k\end{Vmatrix}\frac{\partial \ln\sigma_k}{\partial s(x)}\Big[\frac{\widetilde{n}(y)}{\rho(y)\gamma(y)}g_1(y) - \frac{a_{11}s(y)g_2(y)}{\omega(BC-A^2)\gamma(y)}\Big]\Bigg\}ds. \tag{19.4}$$

Taking into account the boundary conditions of the fourth boundary value problem and the boundary properties of potentials appearing in (19.2) and (19.4), after elementary transformations for determining the unknown vector $g = (g_1, g_2)$ we obtain the following Fredholm integral equation of second kind:

$$g(z) + \frac{1}{\pi}\int_S H(z,y)g(y)ds = f(z), \quad z \in S, \tag{19.5}$$

where

$$H(z,y) = \begin{Vmatrix} H_{11}, & H_{12} \\ H_{21}, & H_{22}\end{Vmatrix},$$

$$H_{11} = \frac{(BC-A^2)}{a_{11}}s(z)\operatorname{Re}\sum_{k=1}^{2}\Bigg[\begin{Vmatrix} -B_k', & A_k' \\ -D_k', & C_k'\end{Vmatrix}\frac{\partial}{\partial s(y)}\ln\frac{\sigma_k}{(-\zeta_k)}\frac{n(y)}{\gamma(y)}+$$
$$+\begin{Vmatrix} A_k, & B_k \\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{(-\zeta_k)}\frac{\widetilde{n}(y)}{\rho(y)\gamma(y)}\Bigg] + \frac{\widetilde{n}(y)s(z)}{\rho(y)\gamma(y)},$$

$$H_{12} = -\frac{s(z)}{\omega}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix} A_k, & B_k \\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{(-\zeta_k)}\frac{s(y)}{\gamma(y)} - \frac{a_{11}}{\omega(BC-A^2)}\frac{s(z)s(y)}{\gamma(y)},$$

$$H_{21} = \frac{(BC-A^2)}{a_{11}}n(z)\operatorname{Re}\sum_{k=1}^{2}\Bigg[\begin{Vmatrix} -q_k, & p_k \\ -s_k, & r_k\end{Vmatrix}\frac{\partial^2 \ln\sigma_k}{\partial s(z)\partial s(y)}\frac{n(y)}{\gamma(y)}+$$
$$+\begin{Vmatrix} N_k, & L_k \\ M_k, & R_k\end{Vmatrix}\frac{\partial \ln\sigma_k}{\partial s(z)}\frac{\widetilde{n}(y)}{\rho(y)\gamma(y)}\Bigg],$$

$$H_{22} = -\frac{n(z)}{\omega}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix} N_k, & L_k \\ M_k, & R_k\end{Vmatrix}\frac{\partial \ln\sigma_k}{\partial s(z)}\frac{s(y)}{\gamma(y)}.$$

Bearing in mind the fact that the vector f is smooth and the curvature of the curve S is Hölder continuous, from (19.5) we find that $g_1 \in C^{1,\alpha}(S)$

and $g_2 \in C^{0,\alpha}(S)$. These conditions guarantee the regularity of the vector function u in the domain D^+.

Let us prove that the equation (19.5) is always solvable. Suppose the contrary and denote by g_0 a nontrivial solution of the homogeneous equation corresponding to (19.5):

$$g_0(z) + \frac{1}{\pi}\int_S H(z,y)g_0(y)ds = 0, \quad z \in S. \tag{19.6}$$

Using (19.6) and the uniqueness theorem for the fourth boundary value problem in the domain D^+, we obtain

$$u(x) = 0, \quad x \in D^+. \tag{19.7}$$

Since $0(0,0) \in D^+$, we have $u(0) = 0$, and from (19.2) we obtain

$$\delta_0 = 0. \tag{19.8}$$

Using formulas (18.13) and (19.7), we get

$$Nu_0(x) = \frac{1}{m}\begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix} \frac{\partial V_0(x)}{\partial s(x)} = 0, \quad x \in D^+, \tag{19.9}$$

where

$$V_0(x) = \frac{(BC-A^2)}{\pi a_{11}}\int_S \operatorname{Im}\sum_{k=1}^{2}\left\{\begin{Vmatrix} -B_k', & A_k' \\ -D_k', & C_k' \end{Vmatrix} \frac{\partial}{\partial s(y)}\frac{\sigma_k}{(-\zeta_k)}\frac{n(y)}{\gamma(y)}g_{10}(y) + \right.$$
$$\left. + \begin{Vmatrix} A_k, B_k \\ B_k, C_k \end{Vmatrix} \ln\frac{\sigma_k}{(-\zeta_k)}\Big[\frac{\widetilde{n}(y)}{\rho(y)\gamma(y)}g_{10}(y) - \frac{a_{11}s(y)g_{20}(y)}{\omega(BC-A^2)\gamma(y)}\Big]\right\}ds, \; x \in D^+. \tag{19.10}$$

It follows from (19.9) that $V_0(x) = C_0$, $x \in D^+$, where C_0 is an arbitrary constant vector; but $V_0(0) = 0$, and therefore $C_0 = 0$. Finally we have

$$V_0(x) = 0, \quad x \in D^+. \tag{19.11}$$

Making use of the formulas (19.10) and (19.11), after elementary calculations we find

$$TV_0(x) = \frac{(BC-A^2)}{\pi a_{11}}\int_S \operatorname{Im}\sum_{k=1}^{2}\left\{\begin{Vmatrix} -q_k, & p_k \\ -s_k, & r_k \end{Vmatrix} \frac{\partial^2 \ln\sigma_k}{\partial s(x)\partial s(y)}\frac{n(y)}{\gamma(y)}g_{10}(y) + \right.$$
$$+ \begin{Vmatrix} N_k, & L_k \\ M_k, & R_k \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial s(x)}\Big[\frac{\widetilde{n}(y)}{\rho(y)\gamma(y)}g_{10}(y) -$$
$$\left. - \frac{a_{11}s(y)g_{20}(y)}{\omega(BC-A^2)\gamma(y)}\Big]\right\}dS = 0, \quad x \in D^+. \tag{19.12}$$

From (19.10) and (19.12) we have

$$\big(V_0(z)\big)^+ - \big(V_0(z)\big)^- = 2\frac{BC-A^2}{a_{11}}\frac{n(z)g_{10}(z)}{\gamma(z)},$$

$$\big(TV_0(z)\big)^+ - \big(TV_0(z)\big)^- = -2\frac{BC-A^2}{a_{11}}\Big[\frac{\widetilde{n}(z)g_{10}(z)}{\rho(z)\gamma(z)} + $$
$$+\frac{\partial}{\partial s(z)}\Big(\frac{s(z)g_{10}(z)}{\gamma(z)}\Big)\Big] + \frac{2}{\omega}\frac{s(z)g_{20}(z)}{\gamma(z)},$$

whence

$$\begin{aligned} \big(V_0(z)\big)^- &= -2\frac{BC-A^2}{a_{11}}\frac{n(z)g_{10}(z)}{\gamma(z)}, \\ \big(TV_0(z)\big)^- &= 2\frac{BC-A^2}{a_{11}}\Big[\frac{\widetilde{n}(z)g_{10}(z)}{\rho(z)\gamma(z)} + \\ &+\frac{\partial}{\partial s(z)}\Big(\frac{s(z)g_{10}(z)}{\gamma(z)}\Big)\Big] - \frac{2}{\omega}\frac{s(z)g_{20}(z)}{\gamma(z)}, \end{aligned} \tag{19.13}$$

since $(V_0(z))^+ = 0$ and $(TV_0(z))^+ = 0$. The formulas (19.13) imply

$$\big(V_0(z)TV_0(z)\big)^- = 0. \tag{19.14}$$

The vector function $V_0(x)$ is defined in the entire plane. It is unique and bounded at infinity by virtue of (19.8).

Using the uniqueness theorem for the fourth boundary value problem, we obtain $V_0(x) = 0$, $x \in D^-$, whence $TV_0(x) = 0$, $x \in D^+$, and the last formulas yield $\big(nV_0(z)\big)^- = 0$, $\big(sTV_0(z)\big)^- = 0$.

Taking now into account (19.13), we have $g_{10} = 0$, $g_{20} = 0$.

Thus we have proved that the homogeneous equation (19.6) has only the trivial solution. This means that the equation (19.5) is always solvable and the solution of the boundary value problem (the displacement vector) can be represented in the form (19.2).

Finally, consider the fourth boundary value problem in the domain D^-. To prove the existence of a solution of that problem, we write the associated to (19.5) Fredholm integral equation

$$h(z) + \frac{1}{\pi}\int_S [H(y,z)]' h(y)ds = F(z), \tag{19.15}$$

where $h = (h_1, h_2)$ is an unknown vector function, $F(F_1, F_2)$ is a vector function given on S, and

$$[H(y,z)]' = \begin{Vmatrix} H_{11}, & H_{21} \\ H_{12}, & H_{22} \end{Vmatrix},$$

$$H_{11} = \frac{(BC-A^2)}{a_{11}}\Big[\frac{n(z)}{\gamma(z)}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix} -B'_k, & -D'_k \\ A'_k, & C'_k \end{Vmatrix}\frac{\partial}{\partial s(z)}\ln\frac{\sigma_k}{z_{k0}}s(y) + $$
$$+\frac{\widetilde{n}(z)}{\rho(z)\gamma(z)}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix}\ln\frac{\sigma_k}{z_{k0}}s(y)\Big] - \frac{\widetilde{n}(z)s(y)}{\rho(z)\gamma(z)},$$

$$H_{21}=\frac{(BC-A^2)}{a_{11}}\left[\frac{n(z)}{\gamma(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}-q_k, & -s_k\\ p_k, & r_k\end{Vmatrix}\frac{\partial^2\ln\sigma_k}{\partial s(z)\partial s(y)}n(y)+\right.$$

$$\left.+\frac{\widetilde{n}(z)}{\rho(z)\gamma(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & M_k\\ L_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(y)}n(y)\right],$$

$$H_{12}=\frac{s(z)}{\omega\gamma(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{z_{k0}}s(y)-\frac{a_{11}}{\omega(BC-a^2)}\frac{s(z)s(y)}{\gamma(z)},$$

$$H_{22}=\frac{s(z)}{\omega\gamma(z)}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & M_k\\ L_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(y)}n(y).$$

The direct calculation shows that the equation (19.15) can be obtained if a solution of the boundary value problem

$$\frac{\omega(BC-A^2)}{\gamma(z)}\Big[(nLu)^-+\frac{1}{\omega a_{11}}\frac{(\widetilde{n}u)^-}{\rho(z)}\Big]=F_1(z),\quad \frac{(s(z)u(z))^-}{\omega\gamma(z)}=F_2(z) \quad (19.16)$$

in the domain D^- is sought as

$$u(x)=\frac{1}{\pi}\int_S\operatorname{Re}\sum_{k=1}^{2}\left[\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln\frac{\sigma_k}{z_k}s(y)h_1+\right.$$

$$\left.+\begin{Vmatrix}N_k, & M_k\\ L_k, & R_k\end{Vmatrix}\frac{\partial\ln\sigma_k}{\partial s(y)}n(y)h_2\right]ds-\frac{a_{11}}{\pi(BC-A^2)}\int_S s(y)h_1\,ds. \quad (19.17)$$

If $(su)^-=f_1$ and $(nTu)^-=f_2$, then taking into account the second condition in (18.8), we obtain

$$F_1(z)=\frac{\omega(BC-A^2)}{\gamma(z)}\Big[f_2-\frac{1}{\omega a_{11}}\Big(\frac{\partial f_1}{\partial s}-\frac{f_1}{\rho}\frac{\beta(z)}{\gamma(z)}\Big)\Big],\quad F_2(z)=\frac{f_1}{\omega\gamma(z)}.$$

It is obvious that (19.16) is by itself the fourth boundary value problem with modified conditions.

Consequently, the integral equation (19.15), just like the equation (19.5), is always solvable.

Thus we have proved that the fourth boundary value problem in the domain D^- has always a unique regular solution which is defined by (19.17).

A well-informed reader can easily notice that here, just as in Section 18, external boundary value problems can be studied without modification of natural boundary conditions. But then it will be necessary to construct new potentials and new integral equations.

The third and fourth boundary value problems for an anisotropic elastic body have been studied first by the author (see [7], [8]). Here we present a somewhat different method of investigation.

20. Solution of the Mixed Boundary Value Problem

In this section we consider the mixed boundary value problem, when the stress vector is given on a part S' of the boundary S, while on the remaining part S'' ($S' \cup S'' = S$) the displacement vector is prescribed. Here we present the solution of the mixed problem in a simply connected domain D^+. The case of a multiply connected domain can be considered analogously.

Let a body occupy a finite simply connected domain D^+ bounded by the contour S. The contour S is assumed to have the Hölder continuous curvature.

By D^- we denote, as ever, an infinite domain complementing $D^+ \cup S$ to the whole plane. Suppose that the arcs $S'_j = a_j b_j$, $j = 1, 2, \ldots, n$, on the curve S have no ends in common, their positive directions coincide with that of S, leaving the domain D^+ on the left, and they follow each other in that direction. The arcs $b_j a_{j+1}$, $j = 1, 2, \ldots, n$, $(a_{n+1} = a_1)$ will be denoted by S''_j, the union of arcs S'_j by S' and the union of the arcs S''_j by S''.

If u denotes the displacement vector and v is adjoint to u (see Section 8), then taking into account (8.13), we can represent the stress vector as

$$Tu = \frac{1}{m} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix} \frac{\partial v}{\partial s} + \varkappa_N \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} \frac{\partial u}{\partial s}, \tag{20.1}$$

where A, B, C are given by (4.15), the constant m by (5.17), and $\varkappa_N$ by (5.30).

In view of (20.1), the boundary conditions of the mixed problem can be written as

$$\begin{gathered} \varkappa_N \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} (u)^+ + \frac{1}{m} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix} (v)^+ = f(z) + \Gamma(z), \quad z \in S', \\ (u)^+ = f(z), \quad z \in S'', \end{gathered} \tag{20.2}$$

where f is a vector function given on S, and

$$\begin{gathered} f(z) = \int_{a_j}^{s(z)} (Tu)^+ ds, \quad \text{for} \quad z \in S'_j, \quad j = 1, 2, \ldots, n, \\ f(z) = (u)^+ \quad \text{for} \quad z \in S'', \end{gathered} \tag{20.3}$$

where Tu is the stress vector prescribed on S'; the integrals are taken along the arcs $S'_j = a_j b_j$; $S(z)$ denotes the arc coordinate of the point $z \in S'$ and, finally, Γ is a piecewise constant function on S', i.e., $\Gamma(z) = \Gamma^{(j)}$ for $z \in S'_j$, where $\Gamma^{(j)}$ are constants to be determined while solving the problem.

The function $f(z)$ will be assumed to belong to the class H_0, and $\frac{\partial f(z)}{\partial s(z)}$ to the class H^*, with the nods a_j, b_j, $j = 1, 2, \ldots, n$. Definition of the classes H_0 and H^* can be found in Muskhelishvili's monograph [39, Section 8].

The displacement vector is sought in the form

$$u(x) = \operatorname{Re}\frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_S \frac{\partial \ln(z_k - t_k)}{\partial s(t)} g(t)ds, \tag{20.4}$$

where $g = (g_1, g_2)$ is an unknown vector function, E_k, F_k, G_k, H_k are the coefficients defined by (5.37), $z_k = x_1 + \alpha_k x_2$, $t_k = y_1 + \alpha_k y_2$, (x_1, x_2) and (y_1, y_2) are the coordinates of the points x and y, respectively, and α_k are complex roots of the characteristic equation of fourth degree (see Section 3).

Obviously, the adjoint vector v has the form

$$v(x) = \operatorname{Im}\frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_S \frac{\partial \ln(z_k - t_k)}{\partial s(t)} g(t)ds. \tag{20.5}$$

Taking into account boundary properties of the potentials $u(x)$ and $v(x)$ as well as the boundary conditions (20.3), after simple transformations for determining g we obtain the system of singular integral equations with discontinuous coefficients

$$A(t_0)g(t_0) + \frac{B(t_0)}{\pi}\int_S \frac{g(t)dt}{t - t_0} + \frac{1}{\pi}\int_S K(t_0, t)g(t)ds = f(t_0) + \Gamma(t_0), \tag{20.6}$$

$t_0 \in S$, where $t_0 = z_1 + iz_2$ is the complex affix of the point z, and

$$A(t_0) = \begin{cases}\varkappa_N \begin{Vmatrix}0, & -1\\ 1, & 0\end{Vmatrix} & \text{for } t_0 \in S',\\ E, & \text{for } t_0 \in S'',\end{cases} \qquad B(t_0) = \begin{cases}-\frac{1}{m}\begin{Vmatrix}B, & -A\\ -A, & C\end{Vmatrix} & \text{for } t_0 \in S',\\ 0, & \text{for } t_0 \in S'',\end{cases}$$

$$K(t_0, t) = \frac{1}{m}\begin{Vmatrix}B, & -A\\ -A, & C\end{Vmatrix}\mathcal{P}(t_0, t) + \varkappa_N \begin{Vmatrix}0, & -1\\ 1, & 0\end{Vmatrix} Q(t_0, t) \ \text{ for } \ t_0 \in S', \ \ t \in S,$$

$$K(t_0, t) = Q(t_0, t) \quad \text{for} \quad t_0 \in S'', \quad t \in S,$$

$$\mathcal{P}(t_0, t) = i\frac{\partial\theta}{\partial s(t)}E - \operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{\partial}{\partial s(t)}\ln\Big(1 + \lambda_k \frac{\overline{t} - \overline{t}_0}{t - t_0}\Big),$$

$$Q(t_0, t) = \frac{\partial\theta}{\partial s(t)}E + \operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\frac{\partial}{\partial s(t)}\ln\Big(1 + \lambda_k \frac{\overline{t} - \overline{t}_0}{t - t_0}\Big),$$

$$E = \begin{Vmatrix}1, & 0\\ 0, & 1\end{Vmatrix}, \quad \lambda_k = \frac{1 + i\alpha_k}{1 - i\alpha_k}, \quad \theta = \arg(t_0 - t).$$

For $t_0 \in S'$, instead of (20.6) we consider the equation

$$-\varkappa_N(g_1 + Xg_2) + \frac{1}{m\pi}\int_S \frac{[(C + AX)g_2 - (A + BX)g_1]}{t - t_0}dt +$$
$$+\frac{1}{\pi}\int_S \{[K_{11}(t_0, t)X - K_{21}(t_0, t)]g_1 +$$
$$+[K_{12}(t_0, t)X - K_{22}(t_0, t)]g_2\}ds = Xf_1 - f_2 + X\Gamma_1^{(j)} - \Gamma_2^{(j)}, \ \ t_0 \in S_j', \tag{20.7}$$

where $K_{pq}(t_0,t)$, $p,q=1,2$, are the components of the matrix $K(t_0,t)$, and X is an unknown complex constant which can be defined from the equation $\frac{C+AX}{A+BX}=-X$, $BX^2+2AX+C=0$, whence $X_{1,2}=\frac{-A\pm i\sqrt{BC-A^2}}{B}$.

For $B=C$ and $A=0$, in particular, for an isotropic elastic body we have $X_{1,2}=\pm i$.

Let

$$X=\frac{-A+i\sqrt{BC-A^2}}{B}. \tag{20.8}$$

Then $BX+A=i\sqrt{BC-A^2}$, $AX+C=-iX\sqrt{BC-A^2}$, and the equation (20.7) after obvious transformations can be rewritten as

$$\begin{aligned}-\varkappa_N\omega(t_0)+\frac{\sqrt{BC-A^2}}{m\pi i}\int_S\frac{\omega(t)dt}{t-t_0}dt+\frac{1}{\pi}\int_S\bigl[K_1(t_0,t)\omega(t)+\overline{K_2(t_0,t)\omega(t)}\bigr]\,ds=\\ =Xf_1-f_2+X\Gamma_1^{(j)}-\Gamma_1^{(j)},\quad t_0\in S_j',\end{aligned} \tag{20.9}$$

where

$$\begin{aligned}\omega(t)=g_1+Xg_2,\ \ \overline{\omega}(t)=g_1+\overline{X}g_2,\ \ g_1=\frac{X\overline{\omega}-\overline{X}\omega}{X-\overline{X}},\ \ g_2=\frac{\overline{\omega}-\omega}{X-\overline{X}},\\ K_1(t_0,t)=\frac{1}{\overline{X}-X}\bigl[X\overline{X}K_{11}(t_0,t)-\overline{X}K_{21}(t_0,t)-\\ -XK_{12}(t_0,t)+K_{22}(t_0,t)\bigr],\\ \overline{K_2(t_0,t)}=\frac{1}{X-\overline{X}}\bigl[X^2K_{11}(t_0,t)-(K_{21}+\\ +K_{12})X+K_{22}(t_0,t)\bigr],\quad t_0\in S',\quad t\in S.\end{aligned} \tag{20.10}$$

For $t_0\in S''$, the equation (20.6) can now be rewritten as

$$\omega(t_0)+\frac{1}{\pi}\int_S[K_1(t_0,t)\omega(t)+\overline{K_2(t_0,t)\omega(t)}]ds=f_1+Xf_2,\ \ t_0\in S'', \tag{20.11}$$

where

$$\begin{aligned}K_1(t_0,t)=\frac{X\overline{X}K_{21}(t_0,t)+\overline{X}K_{11}(t_0,t)-XK_{12}(t_0,t)-K_{22}(t_0,t)}{\overline{X}-X},\\ \overline{K_2(t_0,t)}=\frac{X^2K_{21}(t_0,t)+(K_{11}-K_{22})X-K_{12}(t_0,t)}{X-\overline{X}},\ \text{ for } t_0\in S''.\end{aligned} \tag{20.12}$$

Obviously, (20.9) and (20.11) are singular integral equations with discontinuous coefficients, containing the unknown function $\overline{\omega(t)}$ outside the characteristic part. We rewrite the equations (20.9) and (20.11) as

$$\begin{aligned}K\omega(t_0)\equiv a(t_0)\omega(t_0)+\frac{b(t_0)}{\pi i}\int_S\frac{\omega(t)dt}{t-t_0}dt+\\ +\frac{1}{\pi}\int_S\bigl[K_1(t_0,t)\omega(t)+\overline{K_2(t_0,t)\omega(t)}\bigr]\,ds=F(t_0)+\Gamma_0(t_0),\ \ t_0\in S,\end{aligned} \tag{20.13}$$

where

$$a(t_0)=\begin{cases}-\varkappa_N, & \text{for } t_0\in S',\\ 1, & \text{for } t_0\in S'',\end{cases}\qquad b(t_0)=\begin{cases}\frac{\sqrt{BC-A^2}}{m}, & \text{for } t_0\in S',\\ 0, & \text{for } t_0\in S'',\end{cases}$$

$$F(t_0)=\begin{cases}Xf_1-f_2, & \text{for } t_0\in S',\\ f_1+Xf_2, & \text{for } t_0\in S'',\end{cases}\qquad \Gamma_0(t_0)=\begin{cases}X\Gamma_1^{(j)}-\Gamma_2^{(j)}, & \text{for } t_0\in S_j',\\ 0, & \text{for } t_0\in S'',\end{cases}$$

and the functions $K_1(t_0,t)$ and $K_2(t_0,t)$ are defined by the formulas (20.10) and (20.12).

The characteristic homogeneous equation corresponding to (20.13) is

$$a(t_0)\omega(t_0)+\frac{b(t_0)}{\pi i}\int_S\frac{\omega(t)dt}{t-t_0}=0,$$

and the corresponding to it homogeneous problem of conjugation is of the form $\Phi^+(t)=G(t)\Phi^-(t)$, where

$$G(t)=\frac{a(t)-b(t)}{a(t)+b(t)}=\begin{cases}-\frac{1+\omega\sqrt{BC-A^2}}{1-\omega\sqrt{BC-A^2}}, & t\in S',\\ 1, & t\in S''.\end{cases}$$

On the basis of the general theory (see [39]) we can easily establish that all the nodes a_j,b_j are non-singular, and the canonical solution $X(z)$ which is bounded on all the nodes, i.e., of the class h_{2n}, is given by

$$X(z)=\prod_{j=1}^n(z-a_j)^{\frac12+i\beta}(z-b_j)^{\frac12+i\beta}, \tag{20.14}$$

where

$$\beta=\frac{1}{2\pi}\ln\frac{1+\omega\sqrt{BC-A^2}}{1-\omega\sqrt{BC-A^2}}. \tag{20.15}$$

Under $X(z)$ we mean a branch which is holomorphic on the plane cut along S', and satisfies $\lim\limits_{z\to\infty}z^{-n}X(z)=1$.

It follows from (20.14) that the index of the class h_{2n} in our problem of conjugation is equal to $(-n)$ since the order of $X(z)$ at infinity equals n. Consequently, by definition, $-n$ is the index of the operator K (see (20.13)).

As is evident, the equation (20.13) does not essentially differ from the equation obtained for the mixed boundary value problem in an isotropic case.

The solution $\omega(t)$ of the equation (20.13) will be sought in the class h_{2n}. Therefore if we take into account the restrictions imposed both on the contour S and on the boundary function F, then repeating word by word the reasoning given in Manjavidze's work [34], we can show that $\omega(t)$ belongs to the Hölder class on the entire contour S, and $\omega'(t)$ belongs to the class H^*.

Since the index of the operator K in the class h_{2n} is equal to $(-n)$, according to the general theory (see [39]), we have

$$\nu - \nu' = -2n, \tag{20.16}$$

where ν is the number of independent solutions of the class h_{2n} of the homogeneous equation $K\omega = 0$, and ν' is the number of independent solutions of the class $h_0 = h'_{2n}$ of the associated homogeneous equation $K'\psi = 0$.

Let us prove that $\nu = 0$ and hence $\nu' = 2n$. Indeed, let $\omega_0(t)$ be any solution of the class h_{2n} of the equation $K\omega = 0$. We construct the potential $u_0(x) = \operatorname{Re} W_0(x)$, where

$$\begin{gathered} W_0(x) = \frac{1}{\pi i}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix}\int_S \frac{\partial \ln(z_k - t_k)}{\partial s(t)} g_0(t)ds, \\ g_0(t) = \Big(\frac{\overline{X}\omega_0 - X\overline{\omega}_0}{\overline{X} - X},\ \ \frac{\omega_0 - \overline{\omega}_0}{X - \overline{X}}\Big). \end{gathered} \tag{20.17}$$

Then $(u_0(t))^+ = 0$ for $t \in S''$ and $(Tu_0(t))^+ = 0$ for $t \in S'$, except maybe the points a_j, b_j, $j = 1, 2, \dots, n$, at which the stress vector may have only integrable singularities. Using the uniqueness theorem for the mixed boundary value problem (see Section 14), we conclude that $u_0(x) = 0$, $x \in D^+$. In this case $Nu_0(x) = 0$, $x \in D^+$, and applying the equality $\big(Nu_0(z)\big)^+ = \big(Nu_0(z)\big)^-$, $z \in S$, which takes place at all points of the contour S except possibly for the points a_j, b_j, $j = 1, 2, \dots, n$, we obtain

$$\big(Nu_0(z)\big)^- = 0. \tag{20.18}$$

Since $u_0(x)$ satisfies the decrease conditions at infinity as well as the boundary condition (20.18), using formula (2.11) for $\varkappa = \varkappa_N$, we conclude that $u_0(x) = 0$, $x \in D^-$.

The formula $(u_0(t))^+ - (u_0(t))^- = 2g_0(t)$, $t \in S$, implies $g_0(t) = 0$, and on the basis of (20.17) we obtain $\omega_0(t) = 0$ which contradicts our assumption.

Thus the homogeneous equation $K\omega = 0$ has in the class h_{2n} only the trivial solution. Hence, as above, $\nu' = 2n$, and by the general theory (see [39, Section 112]) the conditions for the solvability of (20.13) have in the class h_{2n} the form

$$\operatorname{Re}\int_S [F(t) + \Gamma_0(t)]\psi_k(t)dt = 0, \quad k = 1, 2, \dots, 2n, \tag{20.19}$$

where $\psi_k(t)$, $k = 1, 2, \dots, 2n$, is a complete system of linearly independent solutions of the class $h_0 = h'_{2n}$ of the associated homogeneous equation $K'\psi = 0$.

Taking into consideration the fact that $\Gamma_0(t) = X\Gamma_1^{(j)} - \Gamma_2^{(j)}$, $t \in S'_j$, and $\Gamma_0(t) = 0$, $t \in S''$, where $\Gamma_1^{(j)}$ and $\Gamma_2^{(j)}$ are real constants, from (20.19) we obtain the (real) system of linear equations for determining the constants $\Gamma_1^{(j)}$ and $\Gamma_2^{(j)}$, $j=1,2,\dots,n$,

$$\sum_{j=1}^{n}\left(\gamma_{kj}\Gamma_1^{(j)}+\delta_{kj}\Gamma_2^{(j)}\right)=B_k,\quad k=1,2,\dots,2n, \tag{20.20}$$

where

$$\begin{gathered}\gamma_{kj}=\operatorname{Re}\int_{S'_j}X\psi_k\,dt,\quad \delta_{kj}=-\operatorname{Re}\int_{S'_j}\Psi_k(t)dt,\\ B_k=-\int_S F(t)\psi_k(t)dt,\quad k=1,2,\dots,2n.\end{gathered} \tag{20.21}$$

Let us prove that the determinant of the system (20.20) is different from zero. Indeed, let $F(t)=0$. Then by (20.21), all B_k in (20.20) will be equal to zero and the system will turn homogeneous. If the determinant of the system under consideration is equal to zero, then it admits nonzero solutions. Let $\Gamma_{10}^{(j)}$ and $\Gamma_{20}^{(j)}$, $j=1,2,\dots,n$, be one of them. Then the equation (20.13) will be solvable (in the class h_{2n}) for $F(t)=0$, $\Gamma_0(t)=X\Gamma_{10}^{(j)}-\Gamma_{20}^{(j)}$, $t\in S'_j$, and $\Gamma_0(t)=0$, $t\in S''$. Let $\omega_0(t)$ be its solution. Composing the potential (20.17) and repeating the above arguments, we obtain $g_0(t)=0$. It is readily seen that $\omega_0(t)=0$, and hence $D_{10}^{(j)}=D_{20}^{(j)}=0$, $j=1,2,\dots,n$, which contradicts our assumption.

Thus the determinant of system (20.20) is different from zero, and hence the system is uniquely solvable with respect to the constants $\Gamma_1^{(j)}$ and $\Gamma_2^{(j)}$. Consequently, the constants $\Gamma_0^{(j)}$, $j=1,2,\dots,n$, are well-defined, the equation (20.13) for these values of constants is uniquely solvable (in the class h_{2n}), and its solution $\omega(t)$ leads to a solution of the original boundary value problem.

The above-given proof of the existence of solutions of the mixed boundary value problem is due to Basheleishvili and Zazashvili [18]. We have cited here the content of their book with some changes and additions. A version of the solution of the mixed boundary value problem in question is a natural generalization of Sherman's [45] and Manjavidze's [34] results with regard to the mixed problem for equations of statics of an isotropic elastic body.

It should be noted that the mixed boundary value problem has been solved for an anisotropic elastic body by Manjavidze [35]. The author applied the theory of linear conjugation with a shift to several unknown functions with discontinuous coefficients and to systems of singular integral equations. At the same time, Kupradze [24] and Kupradze and Burchuladze [27], using Green's matrix, the methods of the potential theory, and the theory singular integral equations, have solved the mixed boundary value problem both for isotropic and for anisotropic elastic bodies. A somewhat different method of solving the mixed boundary value problem is also given by Basheleishvili and Zazashvili [16].

CHAPTER II
EFFECTIVE SOLUTION OF TWO-DIMENSIONAL PROBLEMS

In this chapter the basic two-dimensional boundary value problems for some specific domains are solved effectively (in series or in quadratures). We will show that the potential method and the theory of integral equations we employed in Chapter I for proving the existence and uniqueness theorems can be successfully used for finding effective (explicit) solutions in a number of concrete domains. Some boundary value problems considered by us below have been already solved by other methods. In the present monograph for their solution we use the potential method for the first time. Most of the problems of this chapter are new but we have nevertheless succeeded in obtaining their solutions. It should also be noted that resources of the potential method are not confined to consideration of the problems below. Very likely, this method in combination with the theory of integral equations will largely simplify the investigation of some concrete problems of particular interest by reducing them to quasi-regular infinite systems of linear algebraic equations or by using other known methods.

We will show that the applied by us potential method allows one to solve many boundary value problems which have not been solved before. Thus we prove that the potential method is a very convenient tool for practical applications (see [20, p. 7]). Note also that our integral equations (of Fredholm type) have been successfully used by G. M. Khatiashvili (see [21], [22]) for solution of some particular problems for homogeneous and nonhomogeneous anisotropic cylindrical beams.

21. Solution of the First Boundary Value Problem for the Half-Plane

Let the axis $0x_1$ be the boundary of the half-plane. The upper half-plane will be denoted by D^+. Let the axis $0x_2$ be directed vertically upwards, the normal in the opposite direction and the tangent in the direction of $0x_1$.

A solution of the first boundary value problem will be sought in the domain D^+ in terms of the double layer potential (see (11.2)):

$$u(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_{-\infty}^{\infty} \frac{g(t)dt}{t - z_k}, \tag{21.1}$$

where g is an unknown real vector function. To determine it, we obtain the integral equation

$$g(t_0) + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_{-\infty}^{\infty} \frac{g(t)dt}{t - t_0} = f(t_0),$$

where $f(t_0)$ is a given continuous vector.

Taking into account the fact that

$$\sum_{k=1}^{2} E_k = 1, \quad \sum_{k=1}^{2} F_k = 0, \quad \sum_{k=1}^{2} G_k = 0, \quad \sum_{k=1}^{2} H_k = 1, \tag{21.2}$$

from the last equation we have $g(t_0) = f(t_0)$, and (21.1) takes the form

$$u(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_{-\infty}^{\infty} \frac{f(t)dt}{t - z_k}. \tag{21.3}$$

Thus we have obtained the Poisson formula for the solution of the first boundary value problem for the half-plane.

If $u = \operatorname{Im} W$, then $v = -\operatorname{Re} W$, where

$$W(x) = \frac{1}{\pi} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_{-\infty}^{\infty} \frac{t(t)dt}{t - z_k},$$

and using (20.1), we obtain

$$Tu = \varkappa_N \begin{Vmatrix} 0, & -1, \\ 1, & 0 \end{Vmatrix} \frac{\partial u}{\partial x_1} + \frac{1}{m} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix} \frac{\partial v}{\partial x_1}. \tag{21.4}$$

For the boundary value of the stress vector, with regard for

$$(u)^+ = f(t_0), \quad (v)^+ = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{t - t_0},$$

we have

$$(Tu)^+ = \varkappa_N \begin{Vmatrix} 0, & -1, \\ 1, & 0 \end{Vmatrix} \frac{df(t_0)}{dt_0} - \frac{1}{m} \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f'(t)dt}{t - t_0}. \tag{21.5}$$

Note that our arguments are valid if and only if $f \in C^{1,\alpha}(S)$ and satisfies the condition $f(t) = C + \frac{a}{|t|^{1+\beta}}$ at infinity, where C and a are constant vectors and $\beta > 0$. Here $S =]-\infty, \infty[$.

The above solution of the first boundary value problem can be found in author's earlier work [4].

22. Solution of the Second Boundary Value Problem for the Half-Plane

A solution of the second boundary value problem of statics of an anisotropic elastic body is sought in the domain D^+ in terms of the simple layer potential of second kind (see (11.3) for $\varkappa = 0$):

$$u(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} A'_k, & B'_k \\ C'_k, & D'_k \end{Vmatrix} \int_{-\infty}^{\infty} \ln(z_k - t) g(t)dt, \tag{22.1}$$

where g is an unknown real vector function and the coefficients are given by (6.2).

Using (6.2), we rewrite the vector $u(x)$ as

$$u(x) = \frac{a_{11}}{BC - A^2} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} \Phi(x) + \omega a_{11} \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} \Psi(x), \qquad (22.2)$$

where

$$\Phi(x) + i\Psi(x) = \frac{1}{\pi} \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \int_{-\infty}^{\infty} \ln(z_k - t) g(t) dt, \qquad (22.3)$$

and p_k, q_k, r_k, s_k are given by (4.18).

From (22.1) for the stress vector we obtain

$$Tu(x) \equiv \frac{\partial \Psi}{\partial s(x)} = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \int_{-\infty}^{\infty} \frac{g(t)dt}{z_k - t} (n_1 \alpha_k - n_2). \; (22.4)$$

For determining g, we have the integral equation

$$-g(t_0) + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \int_{-\infty}^{\infty} \frac{g(t)dt}{t - t_0} = f(t_0),$$

where $f(t_0)$ is the given value of the stress vector.

Note that

$$\operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} = 0.$$

Then $g(t_0) = -f(t_0)$, and owing to (22.4) we have

$$Tu(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \int_{-\infty}^{\infty} \frac{f(t)dt}{t - z_k} (n_1 \alpha_k - n_2). \qquad (22.5)$$

From (22.3) we readily get

$$\Big(\frac{\partial \Phi}{\partial t_0}\Big)^+ + i\Big(\frac{\partial \Psi}{\partial t_0}\Big)^+ = if(t_0) + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{t - t_0},$$

and from (22.2) we have

$$\frac{d(u)^+}{dt_0} = \omega a_{11} \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} f(t_0) + \frac{a_{11}}{BC - A^2} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(t)dt}{t - t_0}. \; (22.6)$$

Note that the formulas (21.5) and (22.6) are invertible; this can be proved by using the transposition formula (see [39])

$$\frac{1}{\pi^2} \int_{-\infty}^{\infty} \frac{dt}{t - t_0} \int_{-\infty}^{\infty} \frac{f(\tau)dt}{\tau - t} = -f(t_0). \qquad (22.7)$$

To calculate the displacement vector, we have the formula

$$\Phi(x) + i\Psi(x) = -\frac{1}{\pi}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\int_{-\infty}^{\infty}\ln(z_k - t)f(t)dt. \qquad (22.8)$$

For the regularity of the solution of the second boundary value problem it is sufficient that

$$\int_{-\infty}^{\infty} f(t)dt = 0, \quad f \in C^{0,\alpha}(-\infty,\infty), \ \alpha > 0, \quad \text{and} \quad f_k(t) = \frac{a_k}{|t|^{1+\beta}}, \ k = 1, 2,$$

for large $|t|$, where a_k are constant numbers and $\beta > 0$.

Our attention now will be given to some particular cases (see [40, p. 352]).

1. The uniformly distributed normal stresses N and tangential stresses T are applied to the boundary of the half-plane in the interval $(-a, a)$. In this case, from (22.8) we obtain

$$\Phi(x)+i\Psi(x) = \frac{1}{\pi}\sum_{k=1}^{2}\begin{pmatrix} p_kT + q_kN \\ r_kT + s_kN \end{pmatrix}\Big[(z_k + a)\ln\frac{z_k - a}{z_k + a} - 2a\ln(z_k - a)\Big]. \ (22.9)$$

2. A sloping concentrated force $\mathcal{P}$ forming the angle α with the axis $0x_1$ is applied at the origin. If the vector function f is defined on the boundary so that

$$f_1(t) = \begin{cases} \frac{\mathcal{P}}{2\varepsilon}\cos\alpha, & t \in [-\varepsilon,\varepsilon], \\ 0, & t \notin [-\varepsilon,\varepsilon], \end{cases} \quad f_2(t) = \begin{cases} \frac{\mathcal{P}}{2\varepsilon}\sin\alpha, & t \in [-\varepsilon,\varepsilon], \\ 0, & t \notin [-\varepsilon,\varepsilon], \end{cases}$$

then it is clear that the projections of the principal vector of external stresses are

$$\int_{-\infty}^{\infty} f_1(t)dt = \mathcal{P}\cos\alpha, \quad \int_{-\infty}^{\infty} f_2(t)dt = \mathcal{P}\sin\alpha.$$

Substitute the values of f_1 and f_2 into (22.8). Noticing that

$$\int_{-\infty}^{\infty}\mathcal{P}\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}\ln(z_k - t)dt =$$

$$= \frac{\mathcal{P}}{2\varepsilon}\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}\Big[(\varepsilon - z_k)\ln(z_k - \varepsilon) + (\varepsilon + z_k)\ln(z_k + \varepsilon) - 2\varepsilon\Big],$$

and passing to limit as $\varepsilon \to 0$, we obtain

$$\lim_{\varepsilon\to 0}\int_{-\varepsilon}^{\varepsilon}\mathcal{P}\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}\ln(z_k - t)dt = \mathcal{P}\begin{pmatrix}\cos\alpha \\ \sin\alpha\end{pmatrix}\ln z_k,$$

and now the formulas (22.8) and (22.5) take respectively the form

$$\Phi(x) + i\Psi(x) = -\frac{\mathcal{P}}{\pi}\sum_{k=1}^{2}\begin{pmatrix} p_k\cos\alpha + q_k\sin\alpha \\ r_k\cos\alpha + s_k\sin\alpha \end{pmatrix}\ln z_k, \qquad (22.10)$$

$$Tu(x) = -\frac{\mathcal{P}}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{pmatrix} p_k\cos\alpha + q_k\sin\alpha \\ r_k\cos\alpha + s_k\sin\alpha\end{pmatrix}\frac{(\alpha_k n_1 - n_2)}{z_k}. \qquad (22.11)$$

Taking into account (22.10), by (22.2) we can calculate the displacement vector, and for the components of the stress vector, from (22.11) we have

$$\begin{aligned}
\sigma_{x_1x_1} &= -\frac{\mathcal{P}a_{11}[(Cx_1 + Ax_2)\cos\alpha + (Ax_1 + Bx_2)\sin\alpha]x_1^2}{\pi(BC - A^2)F(x_1, x_2)}, \\
\tau_{x_1x_1} &= -\frac{\mathcal{P}a_{11}[(Cx_1 + Ax_2)\cos\alpha + (Ax_1 + Bx_2)\sin\alpha]x_1x_2}{\pi(BC - A^2)F(x_1, x_2)}, \qquad (22.12) \\
\sigma_{x_2x_2} &= -\frac{\mathcal{P}a_{11}[(Cx_1 + Ax_2)\cos\alpha + (Ax_1 + Bx_2)\sin\alpha]x_2^2}{\pi(BC - A^2)F(x_1, x_2)},
\end{aligned}$$

where

$$F(x_1, x_2) = a_{11}x_1^4 - 2a_{13}x_1^3x_2 + (2a_{12} + a_{33})x_1^2x_2^2 - 2a_{23}x_1x_2^3 + a_{22}x_2^4 \quad (22.13)$$

and A, B, C, $BC - A^2$ are given by (4.15).

3. Suppose that the concentrated normal contracting force $\mathcal{P}$ is applied at the origin. Assuming $\alpha = \frac{\pi}{2}$ in (22.12), we have

$$\sigma_{x_1x_1} = -\frac{\mathcal{P}a_{11}x_1^2(Ax_1 + Bx_2)}{\pi(BC - A^2)F(x_1, x_2)}, \quad \tau_{x_1x_1} = -\frac{\mathcal{P}a_{11}x_1x_2)(Ax_1 + Bx_2)}{\pi(BC - A^2)F(x_1, x_2)},$$
$$\sigma_{x_2x_2} = -\frac{\mathcal{P}a_{11}x_2^2(Ax_1 + Bx_2)}{\pi(BC - A^2)F(x_1, x_2)}.$$

These formulas with $\alpha_1 = ib_1$ and $\alpha_2 = ib_2$ for an orthotropic elastic body have been derived by G. N. Savin [42].

The second boundary value problem for the half-plane has been solved by a somewhat different method by S. G. Lekhnitskiĭ [30] and G. N. Savin [42]. The above-given solution is due to the author [4].

23. Solution of the Third Boundary Value Problem for the Half-Plane

The boundary conditions of the third boundary value problem for the half-plane take the form

$$(nu)^+ = -(u_2)^+ = f_1(t), \quad (sTu)^+ = -(\tau x_1x_2)^+ = f_2(t), \quad t \in]-\infty, \infty[. \quad (23.1)$$

A solution of this problem is sought in terms of (18.10). Then for the determination of g_1 and g_2 we obtain the system (18.14). Taking into account the fact that the curvature in our case is equal to zero and f_2 satisfies the condition $\int_{-\infty}^{\infty} f_2(t)dt = 0$, from (18.14) we obtain $g_1 = -f_1(t)$, $g_2 = -f_2(t)$.

Thus according (18.10) the displacement vector has the form

$$u(x) = \frac{BC - A^2}{\pi a_{11}} \int_{-\infty}^{\infty} \operatorname{Re} \sum_{k=1}^{2} \left\{ \begin{Vmatrix} B'_k, & -A'_k \\ D'_k, & -C'_k \end{Vmatrix} \frac{\partial}{\partial t} \ln\left(1 - \frac{z_k}{t}\right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{f_1(t)}{B} + \right.$$

$$\left. + \frac{a_{11}}{\omega B(BC - A^2)} \begin{Vmatrix} A_k, & B_k \\ B_k, & C_k \end{Vmatrix} \ln\left(1 - \frac{z_k}{t}\right) \begin{pmatrix} 0 \\ -1 \end{pmatrix} f_2(t) \right\} dt, \quad x \in D^+.$$

The stress vector is defined analogously.

For the displacement vector to be regular in D^+, $(x_2 > 0)$, it is sufficient that the given functions f_1 and f_2 satisfy the conditions $f_1 \in C^{1,\alpha}]-\infty, \infty[$, $f_2 \in C^{0,\alpha}] - \infty, \infty[$, $\int_{-\infty}^{\infty} f_2(t)dt = 0$, as well as the condition at infinity $f_1(t) = A_1 + \frac{B_1}{|t|^{1+\beta}}$, $f_2(t) = \frac{B_2}{|t|^{1+\beta}}$, where A_1, B_1, B_2 are constant values, and $\beta > 0$.

Quite analogously one can solve the fourth boundary value problem for the half-plane.

The above cited solution of the third boundary value problem is due to the author.

24. Solution of the First Boundary Value Problem for a Solid Ellipse

Denote the domain occupied by an ellipse by D^+. A solution of the first boundary value problem will be sought in the domain D^+ in terms of the double layer potential

$$u(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y) ds, \tag{24.1}$$

where g is an unknown real vector, $\sigma_k = x_1 - y_1 + \alpha_k(x_2 - y_2)$.

Taking into account the properties of the double layer potential, for determining g we obtain the following Fredholm integral equation of second kind:

$$g(z) + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \int_S \frac{\partial \ln \sigma_k}{\partial s(y)} g(y) ds = f(z), \tag{24.2}$$

where f is a given vector function on S, $z = (z_1, z_2) \in S$.

Let $z_1 - y_1 = a\rho \cos\theta$, $z_2 - y_2 = b\rho \sin\theta$, where a and b are the semi-axis of the ellipse, $\rho = \sqrt{(\frac{z_1 - y_1}{a})^2 + (\frac{z_2 - y_2}{b})^2}$, $\theta = \operatorname{arctg} \frac{a(z_2 - y_2)}{b(z_1 - y_1)}$ and $\frac{\partial \ln \sigma_k}{\partial s(y)} = \frac{\partial \ln \rho}{\partial s(y)} + \frac{\alpha_k b \cos\theta - a \sin\theta}{a \cos\theta + \alpha_k b \sin\theta} \frac{\partial \theta}{\partial s(y)}$.

Since

$$\operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} = \operatorname{Im} E = 0,$$

the formula (24.2) takes the form

$$g(z)+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_S \frac{\alpha_k b\cos\theta-a\sin\theta}{a\cos\theta+\alpha_k b\sin\theta}\frac{\partial\theta}{\partial s(y)}ds=f(z). \quad (24.3)$$

Find the expression of θ through φ and t, where

$$\begin{gathered} z_1=a\cos\varphi, \quad z_2=b\sin\varphi, \quad a\geq b, \quad 0\leq\varphi\leq 2\pi. \\ y_1=a\cos t, \quad y_2=b\sin t, \quad 0\leq t\leq 2\pi. \end{gathered} \quad (24.4)$$

Using (24.4), we have

$$\theta=\operatorname{arctg}\frac{\sin\varphi-\sin t}{\cos\varphi-\cos t}=\operatorname{arctg}\Big(-\operatorname{ctg}\frac{\varphi+t}{2}\Big)=\operatorname{arctg}\frac{\pi+\varphi+t}{2}.$$

Thus we have obtained

$$\theta=\frac{\pi+\varphi+t}{2}. \quad (24.5)$$

On the basis of the formulas $\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}$, we get

$$\frac{\alpha_k b\cos\theta-a\sin\theta}{a\cos\theta+\alpha_k b\sin\theta}=i\Big(1-2\frac{\lambda_k e^{-2i\theta}}{1+\lambda_k e^{-2i\theta}}\Big),$$

where

$$\lambda_k=\frac{a+i\alpha_k b}{a-i\alpha_k b}, \quad k=1,2, \quad |\lambda_k|<1. \quad (24.6)$$

Owing to (24.5), we find that

$$\frac{\alpha_k b\cos\theta-a\sin\theta}{\cos\theta+\alpha_k\sin\theta}=i\Big(1+2\sum_{n=1}^{\infty}\lambda_k^n e^{-in(\varphi+t)}\Big).$$

Substituting the last formula in (24.3), with regard for (24.5) we have

$$g(\varphi)+g_0+2\operatorname{Re}\sum_{k=1}^{2}\sum_{n=1}^{\infty}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\lambda_k^n e^{-in\varphi}g_{-n}=f(\varphi), \quad (24.7)$$

where we have used the notation

$$g_{-n}=\frac{1}{2\pi}\int_0^{2\pi}g(t)e^{-int}dt, \quad n=0,1,\dots. \quad (24.8)$$

Multiplying (24.7) successively by $\frac{1}{2\pi}e^{in\varphi}d\varphi$ $(n=0,1,\dots)$ and integrating over the interval $[0,2\pi]$, we obtain the equations

$$g_0+g_0=f_0, \quad (24.9)$$

$$\begin{cases} g_n+Q^{(n)}g_{-n}=f_n, \\ \overline{Q}^{(n)}g_n+g_{-n}=f_{-n}, \quad n=1,2\dots, \end{cases} \quad (24.10)$$

where the use is made of the notation

$$Q^{(n)}=\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\lambda_k^n,\quad \overline{Q}^{(n)}=\sum_{k=1}^{2}\begin{Vmatrix}\overline{E}_k, & \overline{F}_k\\ \overline{G}_k, & \overline{H}_k\end{Vmatrix}\bar{\lambda}_k^n,$$

$$f_n=\frac{1}{2\pi}\int_0^{2\pi}f(t)e^{int}dt,\quad f_{-n}=\frac{1}{2\pi}\int_0^{2\pi}f(t)e^{-int}dt,\quad n=1,2,\ldots. \tag{24.11}$$

The formula (24.9) yields

$$g_0=f_0/2. \tag{24.12}$$

Find from the second vector equation (24.10) g_{-n} and substitute its value into the first vector equation. We obtain

$$(E-Q^{(n)}\overline{Q}^{(n)})g_n=f_n-Q^{(n)}f_{-n}. \tag{24.13}$$

Taking into account (21.2), after obvious transformations we find

$$D_n=\det(E-Q^{(n)}\overline{Q}^{(n)})=(1-\lambda_1^n\bar{\lambda}_1^n)(1-\lambda_2^n\bar{\lambda}_2^n)+$$
$$+(E_1\overline{H}_1+H_1\overline{E}_1-F_1\overline{G}_1-G_1\overline{F}_1)(\lambda_1^n-\lambda_2^n)(\bar{\lambda}_1^n-\bar{\lambda}_2^n). \tag{24.14}$$

To write the last formula in a more convenient form, we note that

$$E_1=\frac{M_1R_2}{\Delta},\ E_2=-\frac{M_2R_1}{\Delta},\ F_1=-\frac{M_1M_2}{\Delta},\ F_2=\frac{M_1M_2}{\Delta},\ G_1=\frac{R_1R_2}{\Delta},$$
$$G_2=-\frac{R_1R_2}{\Delta},\ \ H_1=-\frac{M_2R_1}{\Delta},\ \ H_2=\frac{M_1R_2}{\Delta},\ \ \Delta=M_1R_2-M_2R_1,$$

where

$$\begin{aligned}M_1&=(1-i\omega A+i\omega C\alpha_2)r_1,\quad M_2=(1-i\omega A+i\omega C\alpha_1)r_2\\ R_1&=[-i\omega B+(1+i\omega A)\alpha_2]r_1,\quad R_2=[-i\omega B+(1+i\omega A)\alpha_1]r_2,\end{aligned} \tag{24.15}$$

by (5.5), (4.16), and (4.17).

The validity of the above formulas can be easily verified if we take into account the identity (4.17) and the equality $\Delta=\frac{m}{a_{11}}r_1$, where m is given by (5.17) and (5.37).

Thus

$$E_1\overline{H}_1+H_1\overline{E}_1-G_1\overline{F}_1-F_1\overline{G}_1=\frac{(M_1\overline{R}_1-R_1\overline{M}_1)(M_2\overline{R}_2-R_2\overline{M}_2)}{\Delta\overline{\Delta}}.$$

Using (24.15) and (4.17), we obtain

$$M_k\overline{R}_k-R_k\overline{M}_k=\Big[1+\omega^2(BC-A^2)-\frac{\omega}{b_1b_2}\Big]\frac{(\alpha_1-\overline{\alpha}_1)(\alpha_2-\overline{\alpha}_2)}{\alpha_k-\overline{\alpha}_k}r_k\overline{r}_k,\quad k=1,2,$$

and now the preceding formula can be rewritten as

$$E_1\overline{H}_1+H_1\overline{E}_1-G_1\overline{F}_1-F_1\overline{G}_1=$$
$$=\Big[\frac{1+\omega^2(BC-A^2)-\frac{\omega}{b_1b_2}}{1-\omega^2(BC-A^2)}\Big]^2\frac{(\alpha_1-\overline{\alpha}_1)(\alpha_2-\overline{\alpha}_2)}{(\alpha_1-\alpha_2)(\overline{\alpha}_1-\overline{\alpha}_2)}.$$

From (24.6) we have

$$\alpha_k = \frac{ia}{b}\frac{1-\lambda_k}{1+\lambda_k}, \quad \alpha_k - \overline{\alpha}_k = \frac{2ia}{b}\frac{1-\lambda_k\bar{\lambda}_k}{(1+\lambda_k)(1+\bar{\lambda}_k)},$$
$$\alpha_1 - \alpha_2 = \frac{2ia}{b}\frac{\lambda_2-\lambda_1}{(1+\lambda_1)(1+\lambda_2)},$$

and

$$\frac{(\alpha_1-\overline{\alpha}_1)(\alpha_2-\overline{\alpha}_2)}{(\alpha_1-\alpha_2)(\overline{\alpha}_1-\overline{\alpha}_2)} = -\frac{(1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}.$$

Thus we have obtained the formula

$$E_1\overline{H}_1 + H_1\overline{E}_1 - G_1\overline{F}_1 - F_1\overline{G}_1 =$$
$$= -\Big[\frac{1+\omega^2(BC-A^2)-\frac{\omega}{b_1b_2}}{1-\omega^2(BC-A^2)}\Big]^2\frac{(1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}. \qquad (24.16)$$

Substituting (24.16) in (24.14), we finally get

$$D_n = (1-\lambda_1^n\bar{\lambda}_1^n)(1-\lambda_2^n\bar{\lambda}_2^n) -$$
$$-\frac{(1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}(\lambda_1^n-\lambda_2^n)(\bar{\lambda}_1^n-\bar{\lambda}_2^n)\Big[\frac{1+\omega^2(BC-A^2)-\frac{\omega}{b_1b_2}}{1-\omega^2(BC-A^2)}\Big]^2. \qquad (24.17)$$

Now let us prove that $D_n > 0$ for $n = 1, 2, \ldots$. To this end, rewrite the expression for D_n as

$$D_n = \Delta_n + \frac{a_{11}^2\omega(2b_1b_2-\omega)[1-2\omega(BC-A^2)b_1b_2](1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{b_1^2b_2^2m^2(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)} \times$$
$$\times(\lambda_1^n-\lambda_2^n)(\bar{\lambda}_1^n-\bar{\lambda}_2^n), \qquad (24.18)$$

where

$$\Delta_n = (1-\lambda_1^n\bar{\lambda}_1^n)(1-\lambda_2^n\bar{\lambda}_2^n) - \frac{(1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}(\lambda_1^n-\lambda_2^n)(\bar{\lambda}_1^n-\bar{\lambda}_2^n). \qquad (24.19)$$

Because of (7.5), (7.8), and (7.13), we have

$$a_{11}^2\omega(2b_1b_2-\omega)[1-2\omega(BC-A^2)b_1b_2] = -\xi_1\xi_2(BC-A^2)b_1b_2\Big(\frac{A_{22}}{\Delta a_{11}^2} +$$
$$+\frac{a_{13}^2}{a_{11}^2} + 2\frac{\xi_3}{a_{11}} - \frac{a_{33}}{a_{11}}\Big) = -\frac{2(BC-A^2)}{a_{11}}b_1b_2\xi_1\xi_2\xi_3 = \frac{(BC-A^2)}{\Delta a_{11}}b_1b_2,$$

and for D_n we finally get

$$D_n = \Delta_n + \frac{(BC-A^2)}{\Delta a_{11}b_1b_2m^2}\frac{(1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}(\lambda_1^n-\lambda_2^n)(\bar{\lambda}_1^n-\bar{\lambda}_2^n). \qquad (24.20)$$

Obviously, the second summand is positive, and therefore if we show that Δ_n is nonnegative for all n, then we will be able to prove that $D_n > 0$ for any $n = 1, 2, \ldots$.

So we prove that Δ_n is nonnegative for any n. It is evident from (20.19) that $\Delta_1 = 0$ for $n = 1$. Let us show that $\Delta_n > 0$ for $n \geq 2$. Bearing this in mind, introduce the notation $\lambda_1 = \rho_1 e^{i\varphi_1}$, $\lambda_2 = \rho_2 e^{i\varphi_2}$. After obvious transformations we obtain

$$\frac{(\lambda_1^n - \lambda_2^n)(\bar{\lambda}_1^n - \bar{\lambda}_2^n)}{(\lambda_1 - \lambda_2)(\bar{\lambda}_1 - \bar{\lambda}_2)} \leq \Big(\frac{\rho_1^n - \rho_2^n}{\rho_1 - \rho_2}\Big)^2.$$

Taking this inequality into account, we arrive at

$$\Delta_n \geq (1-\rho_1^{2n})(1-\rho_2^{2n}) - \frac{(1-\rho_1^2)(1-\rho_2^2)}{(\rho_1-\rho_2)^2}(\rho_1^n - \rho_2^n)^2 =$$
$$= (1-\rho_1^n\rho_2^n)^2 - \frac{(1-\rho_1\rho_2)^2}{(\rho_1-\rho_2)^2}(\rho_1^n - \rho_2^n)^2 = \Delta_n' \Delta_n'',$$

where

$$\Delta_n' = 1 - \rho_1^n\rho_2^n - \frac{1-\rho_1\rho_2}{\rho_1 - \rho_2}(\rho_1^n - \rho_2^n), \quad \Delta_n'' = 1 - \rho_1^n\rho_2^n + \frac{1-\rho_1\rho_2}{\rho_1 - \rho_2}(\rho_1^n - \rho_2^n).$$

Since $\rho_k < 1$, $k = 1, 2$, with regard for the fact that $\frac{\rho_1^n - \rho_2^n}{\rho_1 - \rho_2} > 0$ for any $n = 2, 3, \ldots$, we get $\Delta_n'' > 0$, $n = 2, 3, \ldots$.

Thus it remains for us to prove that $\Delta_n' > 0$, $n = 2, 3, \ldots$.

After certain simplifications we obtain

$$\Delta_n' = \begin{cases} (1-\rho_1\rho_2)\sum_{k=0}^{p-1}\rho_1^k\rho_2^k[1-\rho_1^{2(p-k)-1}][1-\rho_2^{2(p-k)-1}] \\ \qquad \text{for } \ n = 2p, \ \ p \geq 1, \\ (1-\rho_1\rho_2)\sum_{k=0}^{p-1}\rho_1^k\rho_2^k[1-\rho_1^{2(p-k)}][1-\rho_2^{2(p-k)}] \\ \qquad \text{for } \ n = 2p+1, \ \ p \geq 1. \end{cases}$$

Obviously, $\Delta_n' > 0$ for $n = 2, 3, \ldots$. Thus we have proved that $\Delta_n > 0$ for $n = 2, 3, \ldots,$.

Taking now into account (24.20), we obtain $D_n > 0$ for any $n = 1, 2, \ldots$.

Note that for an isotropic elastic circle $D_1 = \frac{4\mu(\lambda+2\mu)}{(\lambda+3\mu)^2} > 0$, $D_n = 1$, $n = 2, 3 \ldots$.

Introduce the notation

$$\mathcal{P}^{(n)} = \sum_{k=1}^{2} \begin{Vmatrix} H_k, & -F_k \\ -G_k, & E_k \end{Vmatrix} \lambda_k^n. \tag{24.21}$$

Elementary calculations show that

$$\mathcal{P}^{(n)}Q^{(n)} = Q^{(n)}\mathcal{P}^{(n)} = \lambda_1^n\lambda_2^n E, \quad \mathcal{P}^{(n)} + Q^{(n)} = (\lambda_1^n + \lambda_2^n)E, \tag{24.22}$$

where E is the unit matrix and $Q^{(n)}$ is given by (24.11).

Using (24.11) and (24.21), after cumbersome transformations we find that

$$Q^{(n)}\overline{Q}^{(n)} + \overline{\mathcal{P}}^{(n)}\mathcal{P}^{(n)} =$$
$$= [\lambda_1^n\bar{\lambda}_1^n + \lambda_2^n\bar{\lambda}_2^n + (G_1\overline{F}_1 + F_1\overline{G}_1 - E_1\overline{H}_1 - H_1\overline{E}_1)(\lambda_1^n - \lambda_2^n)(\bar{\lambda}_1^n - \bar{\lambda}_2^n)]E.$$

The last formula and (24.16) yield

$$(E-\overline{\mathcal{P}}^{(n)}\mathcal{P}^{(n)})(E-Q^{(n)}\overline{Q}^{(n)})=ED_n. \tag{24.23}$$

Multiplying both parts of (24.13) by $(E-\overline{\mathcal{P}}^{(n)}\mathcal{P}^{(n)})$ and taking (24.23) into account, we obtain

$$g_n=\frac{(E-\overline{\mathcal{P}}^{(n)}\mathcal{P}^{(n)})}{D_n}(f_n-Q^{(n)}f_{-n}). \tag{24.24}$$

Thus g_n is expressed in terms of the Fourier coefficients of the vector function f.

Taking now (24.4) into account, after certain calculations we arrive at

$$\frac{\partial\ln\sigma_k}{\partial s(y)}ds=i\Big(1+\frac{t_{k1}}{e^{it}-t_{k1}}+\frac{t_{k2}}{e^{it}-t_{k2}}\Big)dt=i\Big[1+\sum_{n=1}^{\infty}(t_{k1}^n+t_{k2}^n)e^{-int}\Big]dt, \tag{24.25}$$

where

$$t_{k1}=\frac{z_k+\sqrt{z_k^2-(a^2+\alpha_k^2b^2)}}{a-i\alpha_kb},\quad t_{k2}=\frac{z_k-\sqrt{z_k^2-(a^2+\alpha_k^2b^2)}}{a-i\alpha_kb},$$
$$t_{k1}\cdot t_{k2}=\lambda_k,\quad k=1,2. \tag{24.26}$$

When $x\in D^+$, just as in Mikhlin's monograph [38, pp. 165–166] we prove that $|t_{k1}|<1$, $|t_{k2}|<1$, $k=1,2$.

However, if the point $x=z$ belongs to S, then $t_{k1}=e^{i\varphi}$, $t_{k2}=\lambda_ke^{-i\varphi}$.

Substituting (24.25) in (24.1) and taking into account (24.12) and (24.4), we obtain

$$u(x)=\frac{f_0}{2}+2\,\mathrm{Re}\sum_{k=1}^{2}\sum_{n=1}^{\infty}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\times$$
$$\times\frac{(E-\mathcal{P}^{(n)}\overline{\mathcal{P}}^{(n)})}{D_n}(t_{k1}^n+t_{k2}^n)(f_{-n}-\overline{Q}^{(n)}f_n), \tag{24.27}$$

which is the best possible solution of the first boundary value problem for the solid ellipse.

For the series (24.27) for the displacement vector u to converge absolutely and uniformly up to the boundary, it is sufficient that $f\in C^{0,\alpha}(S)$, $\alpha>\frac{1}{2}$. For the convergence of the series for the stress vector which is obtained from (24.27), it is sufficient that $f\in C^{1,\alpha}(S)$, $\alpha>\frac{1}{2}$. Thus we have shown that the displacement vector is regular in the domain D^+ if $f\in C^{1,\alpha}(S)$. Note also that the series (24.27) in the domain D^+ has continuous derivatives of any order, which follows from the inequalities $|t_{k1}|<1$, $|t_{k2}|<1$ for $x\in D^+$.

Assuming in the above obtained solution (24.27) $b=a$, we find the solution of the first boundary value problem for an anisotropic circle. It is interesting that in this case we have no essential simplifications, so that the solution of the first boundary value problem for a circle or an ellipse is in fact the problem of the same difficulty.

Choosing the Hooke's constants appropriately and passing to limit as $\alpha_2 \to \alpha_1 \to i$, from (24.27) we obtain the solution of the first boundary value problem both for an isotropic circle and for a solid ellipse. In the case of an isotropic circle, the series (24.27) after certain calculations take the form

$$u(x)=f_0+2\,\mathrm{Re}\left\{\frac{z}{R}f_{-1}+\sum_{n=2}^{\infty}\left(\frac{z}{R}\right)^n\left[E+\frac{R^2-z^2}{2\varkappa z^2}n\begin{Vmatrix}1, & i\\ i, & -1\end{Vmatrix}\right]f_{-n}\right\}, \tag{24.28}$$

where R is the radius, $\varkappa = \frac{\lambda+3\mu}{\lambda+\mu}$, $\rho^2 = x_1^2 + x_2^2$, $f_{-n} = \frac{1}{2\pi}\int_0^{2\pi} fe^{-int}dt$, $z = x_1 + ix_2$.

The formula (24.28) after simple transformations can be reduced to

$$u(x) = \frac{1}{\pi}\,\mathrm{Im}\int_0^{2\pi}\frac{d}{dt}\times$$
$$\times\begin{Vmatrix}\ln\sigma-\frac{1}{2}\ln\zeta-\frac{1}{2\varkappa}(\frac{\bar\sigma}{\sigma}+\frac{\bar z}{\zeta}), & -\frac{i}{2\varkappa}(\frac{\bar\sigma}{\sigma}+\frac{\bar z}{\zeta})\\ -\frac{i}{2\varkappa}(\frac{\bar\sigma}{\sigma}+\frac{\bar z}{\zeta}), & \ln\sigma-\frac{1}{2}\ln\zeta+\frac{1}{2\varkappa}(\frac{\bar\sigma}{\sigma}+\frac{\bar z}{\zeta})\end{Vmatrix}\cdot f(t)dt, \tag{24.29}$$

where $\sigma = z - \zeta$, $\zeta = y_1 + iy_2$.

We call the formula (24.29) an analogue of Poisson's formula in the theory of elasticity; this formula has been obtained by the author [2].

The above-given solution of the first boundary value problem for a solid ellipse is also due to Basheleishvili [3]. A detailed exposition of the solution can be found in V. Kupradze's monograph [24].

25. Solution of the Second Boundary Value Problem for an Ellipse

A solution of the second boundary value problem for an ellipse is sought in the form of a simple layer potential of second kind:

$$u(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix}A_k', & B_k'\\ C_k', & D_k'\end{Vmatrix}\int_s \ln\sigma_k h(y)ds, \tag{25.1}$$

where S is the contour of the ellipse and h is an unknown real vector function, $\sigma_k = x_1 - y_1 + \alpha_k(x_2 - y_2)$.

Taking into consideration (6.2), just as in Section 22, we rewrite $u(x)$ as

$$u(x) = \frac{a_{11}}{BC-A^2}\begin{Vmatrix}C, & A\\ A, & B\end{Vmatrix}\Phi(x)+\omega a_{11}\begin{Vmatrix}0, & -1\\ 1, & 0\end{Vmatrix}\Psi(x), \tag{25.2}$$

where

$$\Phi(x)+i\Psi(x) = \frac{1}{\pi}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_s \ln\sigma_k h(y)ds. \tag{25.3}$$

From (25.1), for the stress vector we obtain

$$Tu(x) \equiv \frac{\partial \Psi}{\partial s(x)} = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \left\| \begin{matrix} p_k, & q_k \\ r_k, & s_k \end{matrix} \right\| \int_s \frac{\partial \ln \sigma_k}{\partial s(x)} h(y) ds. \qquad (25.4)$$

For determining the vector function h we have the Fredholm integral equation of second kind:

$$-h(z) + \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \left\| \begin{matrix} p_k, & q_k \\ r_k, & s_k \end{matrix} \right\| \int_S \frac{\partial \ln(z_{k_0} - \zeta_k)}{\partial s(z)} h(y) ds = f(z), \quad (25.5)$$

where f is a boundary value of the stress vector, $z_{k0} = z_1 + \alpha_k z_2$, $z = (z_1, z_2) \in S$.

Using (24.4) and (24.5), after obvious transformations the equation (25.5) can be rewritten as

$$-g(\varphi) + g_0 + 2 \operatorname{Re} \sum_{k=1}^{2} \sum_{n=1}^{\infty} \left\| \begin{matrix} p_k, & q_k \\ r_k, & s_k \end{matrix} \right\| \lambda_k^n e^{-in\varphi} g_{-n} = F(\varphi), \qquad (25.6)$$

where

$$g(\varphi) = h(\varphi) \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}, \quad F(\varphi) = f(\varphi) \sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi},$$

$$\begin{gathered} g_0 = \frac{1}{2\pi} \int_0^{2\pi} g(t) dt, \quad g_{-n} = \frac{1}{2\pi} \int_0^{2\pi} g e^{-int} dt, \\ g_n = \frac{1}{2\pi} \int_0^{2\pi} g e^{int} dt, \quad n = 1, 2, \ldots \end{gathered} \qquad (25.7)$$

Introduce the notation

$$\sum_{k=1}^{2} \left\| \begin{matrix} s_k, & -q_k \\ -r_k, & p_k \end{matrix} \right\| \lambda_k^n = p^{(n)}, \quad \sum_{k=1}^{2} \left\| \begin{matrix} p_k, & q_k \\ r_k, & s_k \end{matrix} \right\| \lambda_k^n = q^{(n)}. \qquad (25.8)$$

Elementary calculations lead to

$$p^{(n)} q^{(n)} = q^{(n)} p^{(n)} = \lambda_1^n \lambda_2^n E, \quad p^{(n)} + q^{(n)} = (\lambda_1^n + \lambda_2^n) E. \qquad (25.9)$$

Taking into account (25.8), we can rewrite (25.6) as

$$-g(\varphi) + g_0 + 2 \operatorname{Re} \sum_{n=1}^{\infty} q^{(n)} e^{-in\varphi} g_{-n} = F(\varphi). \qquad (25.10)$$

For determining the unknown coefficients g_0 and g_{-n}, we obtain from (25.10) the equations

$$-g_0 + g_0 = F_0, \qquad (25.11)$$

$$-g_n + q^{(n)} g_{-n} = F_n, \quad \overline{q}^{(n)} g_n - g_{-n} = F_n, \qquad (25.12)$$

where

$$F_0=\frac{1}{2\pi}\int_0^{2\pi}F\,dt=\frac{1}{2\pi}\int_s f\,ds,\quad F_n=\frac{1}{2\pi}\int_0^{2\pi}Fe^{int}dt,\quad n=\pm1,\pm2,\ldots. \tag{25.13}$$

From (25.11) it follows the necessary condition

$$\int_s f\,ds=0. \tag{25.14}$$

Supposing that (25.14) is fulfilled, we can regard g_0 to be an arbitrary real constant. Note that the expression for the stress vector does not contain g_0. The condition (25.14) means that the principal vector of external stresses is equal to zero.

If we define g_n from the first equation (25.12) and then substitute it in the second one, then we obtain

$$(-E+\overline{q}^{(n)}q^{(n)})g_{-n}=F_{-n}+\overline{q}^{(n)}F_n. \tag{25.15}$$

Taking into account (4.18) and (25.8), after certain calculations we find

$$\Delta_n=\det(-E+\overline{q}^{(n)}q^{(n)})=(1-\lambda_1^n\bar{\lambda}_1^n)(1-\lambda_2^n\bar{\lambda}_2^n)+$$
$$+(p_1\overline{s}_1+s_1\overline{p}_1-q_1\overline{r}_1-r_1\overline{q}_1)(\lambda_1^n-\lambda_2^n)(\bar{\lambda}_1^n-\bar{\lambda}_2^n).$$

On the other hand, it is evident that

$$p_1\overline{s}_1+s_1\overline{p}_1-q_1\overline{r}_1-r_1\overline{q}_1=\frac{(\alpha_1-\overline{\alpha}_1)(\alpha_2-\overline{\alpha}_2)}{(\alpha_1-\alpha_2)(\overline{\alpha}_1-\overline{\alpha}_2)}=-\frac{(1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)},$$

so

$$\Delta_n=(1-\lambda_1^n\bar{\lambda}_1^n)(1-\lambda_2^n\bar{\lambda}_2^n)-\frac{(1-\lambda_1\bar{\lambda}_1)(1-\lambda_2\bar{\lambda}_2)}{(\lambda_1-\lambda_2)(\bar{\lambda}_1-\bar{\lambda}_2)}(\lambda_1^n-\lambda_2^n)(\bar{\lambda}_1^n-\bar{\lambda}_2^n). \tag{25.16}$$

Comparing this expression with (24.19), we can see that they coincide. The determinant appearing here has been investigated in the previous section where we have shown that $\Delta_1=0$, $\Delta_n>0$, $n=2,3,\ldots$.

For $n=1$, if the principal moment of external stresses is equal to zero, i.e., if

$$\int_s (y_1f_2(y)-y_2f_1(y))ds=\int_0^{2\pi}[a\cos tF_2(t)-b\sin tF_1(t)]dt=0,$$

then in (25.6) the expression $\int_S(y_1g_2(y)-y_2g_1(y))ds$ is an arbitrary constant which does not appear in the formula for the stress vector.

Thus we have shown that for the existence of a solution of the second boundary value problem for a solid ellipse it is necessary that the principal vector and the principal moment of external stresses be equal to zero. As is known from the general theory, these conditions are sufficient as well.

Owing to (25.8), after simple transformations we find that

$$p^{(n)}\overline{p}^{(n)}+\overline{q}^{(n)}q^{(n)}=$$

$$= \Big[\lambda_1^n\bar\lambda_1^n+\lambda_2^n\bar\lambda_2^n+\frac{(1-\lambda_1\bar\lambda_1)(1-\lambda_2\bar\lambda_2)}{(\lambda_1-\lambda_2)(\bar\lambda_1-\bar\lambda_2)}(\lambda_1^n-\lambda_2^n)(\bar\lambda_1^n-\bar\lambda_2^n)\Big]E$$

and $(-E+p^{(n)}\overline{p}^{(n)})(-E+\overline{q}^{(n)}q^{(n)})=E\Delta_n$.

Multiplying now (25.15) by $(-E+P^{(n)}\overline{P}^{(n)})$, with regard for the last formula we have

$$g_{-n}=\frac{1}{\Delta_n}(-E+p^{(n)}\overline{p}^{(n)})(F_{-n}+\overline{q}^{(n)}F_n). \tag{25.17}$$

Thus we have defined all the unknown coefficients.

If we now make use of the easily provable expansions

$$\begin{gathered}\ln\sigma_k=\ln\frac{a-i\alpha_kb}{2}+i(\pi+t)-\sum_{n=1}^{\infty}\frac{t_{k1}^n+t_{k2}^n}{n}e^{int},\\ \frac{\partial\ln\sigma_k}{\partial s(x)}=\frac{2(\alpha_kn_1-n_2)}{(a-i\alpha_kb)(t_{k1}-t_{k2})}\sum_{n=1}^{\infty}(t_{k2}^n-t_{k1}^n)e^{int},\end{gathered} \tag{25.18}$$

where t_{k1} and t_{k2} are given by (24.26), then $\Phi(x)$, $\Psi(x)$ and the stress vector will take respectively the form

$$\begin{gathered}\Phi(x)+i\Psi(x)=-2\sum_{k=1}^{2}\sum_{n=1}^{\infty}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\frac{t_{k1}^n+t_{k2}^n}{n}g_{-n},\\ Tu=\frac{\partial\Psi(x)}{\partial s(x)}=-4\,\mathrm{Im}\sum_{k=1}^{2}\sum_{n=1}^{\infty}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\frac{(\alpha_kn_1-n_2)}{(a-i\alpha_kb)(t_{k2}-t_{k1})}(t_{k2}^n-t_{k1}^n)g_{-n},\end{gathered} \tag{25.19}$$

where g_{-n} is defined by (25.17) and $\Phi(x)+i\Psi(x)$ is written to within the constant vector which does not affect the values of the stress vector.

Substituting g_{-n} appearing in (25.17) into (25.19), the expression for the stress vector after cumbersome calculations can be rewritten as

$$\begin{gathered}Tu=\frac{1}{\pi}\Big[\frac{n_1(x)}{b}\int_0^{2\pi}F\cos t\,dt+\frac{n_2(x)}{a}\int_0^{2\pi}F\sin t\,dt+\\ +\begin{Vmatrix}\gamma_1x_1+\delta_1x_2, & -(\gamma_2x_1+\gamma_1x_2)\\ -(\gamma_2x_1+\gamma_1x_2), & \delta_2x_1+\gamma_2x_2\end{Vmatrix}\begin{pmatrix}n_1(x)\\ n_2(x)\end{pmatrix}\Big]+4\,\mathrm{Im}\sum_{k=1}^{2}\sum_{n=1}^{\infty}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\times\\ \times\frac{\alpha_kn_1(x)-n_2(x)}{(\alpha-i\alpha_kb)(t_{k1}-t_{k2})}(t_{k2}^n-t_{k1}^n)\frac{(-E+p^{(n)}\overline{p}^{(n)})}{\Delta_n}(F_{-n}+\overline{q}^{(n)}F_n),\end{gathered} \tag{25.20}$$

where

$$\begin{gathered}\gamma_1=\frac{1}{\pi ab}\int_0^{2\pi}F_1\cos 2t\,dt,\quad \gamma_2=-\frac{1}{\pi ab}\int_0^{2\pi}F_2\cos 2t\,dt,\\ \delta_1=-\frac{a}{\pi b^3}\int_0^{2\pi}F_2\cos 2t\,dt+\frac{2}{\pi b^2}\int_0^{2\pi}F_1\sin 2t\,dt,\end{gathered}$$

$$\delta_2 = \frac{b}{\pi a^3}\int_0^{2\pi} F_1 \cos 2t dt + \frac{2}{\pi a^2}\int_0^{2\pi} F_2 \sin 2t dt.$$

The expression for the displacement vector can be written analogously.

For an isotropic elastic body (in the case of a circle) after long but obvious transformations we have

$$u(x) = \frac{\lambda + 2\mu}{2\mu(\lambda+\mu)}\Phi(x) + \frac{1}{2(\lambda+\mu)}\begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix}\Psi(x) - \\ -\frac{1}{4(\lambda+\mu)\pi}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\int_0^{2\pi}(f_1(t)\cos t + f_2(t)\sin t)dt, \qquad (25.21)$$

where

$$\Phi(x) + i\Psi(x) = \frac{R}{\pi}\int_0^{2\pi}\left[-E\ln\sigma + \frac{1}{2}\begin{Vmatrix} 1, & i \\ i, & -1 \end{Vmatrix}\frac{\overline{\sigma}}{\sigma}\right] f(t)dt \quad (25.22)$$

and

$$Tu = \frac{R}{\pi}\int_0^{2\pi}\frac{\partial}{\partial s(x)}\left\{ \operatorname{Im}\left[-E\ln\sigma + \frac{1}{2}\begin{Vmatrix} 1, & i \\ i, & -1 \end{Vmatrix}\left(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{z}}{\zeta}\right)\right]\right\} f(t)dt, \quad (25.23)$$

where R is the radius of the circle, $\sigma = z - \zeta$, $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$.

We call the formula (25.21) the analogue of Dini's formula in the elasticity theory. This formula has been obtained by the author [5].

The above-given solution of the second boundary value problem for a solid ellipse is due to Basheleishvili [3]. A detailed exposition of the solution can be found in V. Kupradze's monograph [24].

The second boundary value problem for a solid ellipse has been solved in a different way by Lekhnitskiĭ [30] and Kufarev [23].

26. Solution of the First Boundary Value Problem for an Infinite Anisotropic Plane with an Elliptic Hole

A solution of the first boundary value problem for an infinite anisotropic plane with a hole will be sought in the form

$$u(x) = \frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix}\int_s \frac{\partial \ln\sigma_k}{\partial s(y)} g(y)ds + \beta, \qquad (26.1)$$

where g is an unknown real vector function, β is an arbitrary real constant vector which will be defined below, and s is the contour of the ellipse.

Using the expansion (24.25) for $x \in D^-$ (D^- is the infinite plane with an elliptical hole), we obtain

$$\frac{\partial \ln\sigma_k}{\partial s(y)} ds = i\sum_{n=1}^{\infty}(\lambda_k^n e^{-int} - e^{int})\tau_{k1}^n dt, \qquad (26.2)$$

where

$$\tau_{k1} = \frac{1}{t_{k1}} = \frac{a - i\alpha_k b}{z_k + \sqrt{z_k^2 - (a^2 + b^2\alpha_k^2)}}. \tag{26.3}$$

We prove that $|\tau_{k1}| < 1$ when $x \in D^-$.

Substituting (26.2) in (26.1) and taking into account the easily verifiable identity

$$\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \lambda_k^n = \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} Q^{(n)} = Q^{(n)} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix}, \quad k = 1, 2,$$

where $Q^{(n)}$ is defined by (24.11), we find that

$$u(x) = 2\,\mathrm{Re} \sum_{k=1}^{2} \sum_{n=1}^{\infty} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} (Q^{(n)} g_{-n} - E g_n) \tau_{k1}^n + \beta.$$

Here g_{-n} and g_n are given by (24.8). Introduce the notation

$$Q^{(n)} g_{-n} - E g_n = h_n. \tag{26.4}$$

Then the preceding relation can be rewritten in the form

$$u(x) = 2\,\mathrm{Re} \sum_{k=1}^{2} \sum_{n=1}^{\infty} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \tau_{k1}^n h_n + \beta. \tag{26.5}$$

Passing to limit as $x \to z \in S$, we have

$$\beta + 2\,\mathrm{Re} \sum_{n=1}^{\infty} e^{-in\varphi} h_n = f(\varphi), \tag{26.6}$$

where f is a continuous on S vector function.

For determining β and h_n, from (26.6) we obtain

$$\beta = f_0 = \frac{1}{2\pi} \int_0^{2\pi} f\,dt, \quad h_n = f_n = \frac{1}{2\pi} \int_0^{2\pi} f e^{int}\,dt.$$

Substitution of the obtained values β and h_n in (26.5) results in

$$u(x) = f_0 + 2\,\mathrm{Re} \sum_{k=1}^{2} \sum_{n=1}^{\infty} \begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} \tau_{k1}^n f_n. \tag{26.7}$$

Thus we have obtained in the domain D^- the solution of the first boundary value problem in the form of a series. For that series to be absolutely and uniformly convergent it is sufficient that $f \in C^{0,\alpha}(s)$, $\alpha > \frac{1}{2}$. For the regularity of the displacement vector in the domain D^- it suffices that $f \in C^{1,\alpha}(s)$, $\alpha > \frac{1}{2}$.

We can easily sum the series appearing in (26.7) which then will take the integral form

$$u(x)=\frac{1}{2\pi}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_0^{2\pi}\frac{1+\tau_{k1}e^{it}}{1-\tau_{k1}e^{it}}f(t)dt. \tag{26.8}$$

This is the Poisson formula for the solution of the first boundary value problem in the domain D^-. For the regularity of that solution it suffices that $f\in C^{1,\alpha}$, $\alpha>0$.

Using now (20.1) and taking into account that

$$(V(z))^-=\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}f(t)dt,$$

for the boundary value of the stress vector we obtain

$$\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}(Tu)^-=x_N\begin{Vmatrix}0, & -1\\ 1, & 0\end{Vmatrix}\frac{df}{d\varphi}+$$
$$+\frac{\begin{Vmatrix}B, & -A\\ -A, & C\end{Vmatrix}}{m}\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\frac{df(t)}{dt}dt, \tag{26.9}$$

where $f=(u)^-$.

The solutions (26.7) and (26.8) obtained above after certain simplifications take in the case of an isotropic infinite plane with a circular hole the form

$$u(x)=f_0+2\operatorname{Re}\sum_{n=1}^{\infty}\left[E+\frac{n(\rho^2-R^2)}{2\varkappa z^2}\begin{Vmatrix}1, & i\\ i, & -1\end{Vmatrix}\right]\left(\frac{R}{z}\right)^n f_n, \tag{26.10}$$

and

$$u(x)=\frac{1}{\pi}\int_0^{2\pi}\frac{d}{dt}\operatorname{Im}\left[\left(\ln\sigma-\frac{1}{2}\ln\zeta\right)E-\frac{1}{2\varkappa}\begin{Vmatrix}1, & i\\ i, & -1\end{Vmatrix}\left(\frac{\overline{\sigma}}{\sigma}+\frac{\overline{z}}{\zeta}\right)\right]f(t)dt, \tag{26.11}$$

respectively, where R is the radius of the circle and $\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}$ is Muskhelishvili's constant.

So far we have assumed that the principal vector of external stresses, components of the stress vector and the rotation were equal to zero at infinity.

Consider now the general case. Assume that the principal vector -$\int_s F\,ds$, the stress components -$\sigma_{11}^{(\infty)}$, $\sigma_{22}^{(\infty)}$, $\tau_{12}^{(\infty)}$, and the rotation -$\varepsilon^{(\infty)}$ at infinity are constant values. In this case a solution of the first boundary value problem for an infinite anisotropic plane with an elliptical hole is sought in the form

$$u(x)=\frac{1}{2\pi}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_0^{2\pi}\frac{1+\tau_{k1}e^{it}}{1-\tau_{k1}e^{it}}g(t)dt+$$

$$+\frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln z_k\int_s F(t)ds+A^{(\infty)}x_1+B^{(\infty)}x_2, \qquad (26.12)$$

where the vectors $A^{(\infty)} = (A_1^{(\infty)}, A_2^{(\infty)})$ and $B^{(\infty)} = (B_1^{(\infty)}, B_2^{(\infty)})$ are defined as follows:

$$A_1^{(\infty)}=a_{11}\sigma_{11}^{(\infty)}+a_{12}\sigma_{22}^{(\infty)}+a_{13}\tau_{12}^{(\infty)}, \quad B_2^{(\infty)}=a_{12}\sigma_{11}^{(\infty)}+a_{22}\sigma_{22}^{(\infty)}+a_{23}\tau_{12}^{(\infty)},$$

$$A_2^{(\infty)} = \varepsilon^{(\infty)} + \frac{1}{2}(a_{13}\sigma_{11}^{(\infty)} + a_{23}\sigma_{22}^{(\infty)} + a_{33}\tau_{12}^{(\infty)}),$$

$$B_1^{(\infty)} = -\varepsilon^{(\infty)} + \frac{1}{2}(a_{13}\sigma_{11}^{(\infty)} + a_{23}\sigma_{22}^{(\infty)} + a_{33}\tau_{12}^{(\infty)}).$$

For determining g, we have the Fredholm integral equation

$$g(\varphi)+\frac{1}{2\pi}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_0^{2\pi}\frac{1+\tau_{k1}e^{it}}{1-\tau_{k1}e^{it}}g(t)dt = f_0(\varphi), \qquad (26.13)$$

where

$$f_0(\varphi)=f(\varphi)-\frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}A_k, B_k\\ B_k, C_k\end{Vmatrix}\ln z_{k0}\int_s F ds-A^{(\infty)}z_1-B^{(\infty)}z_2, \qquad (26.14)$$

$$z_{k0} = z_1 + \alpha_k z_2, \quad z_1 = a\cos\varphi, \quad z_2 = b\sin\varphi.$$

As is easily seen, the solution of the equation (26.13) has the form

$$g(\varphi) = f_0(\varphi).$$

Thus in the general case the solution of the first boundary value problem for an anisotropic plane with an elliptical hole is of the form

$$u(x) = \frac{1}{2\pi}\operatorname{Re}\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}\int_0^{2\pi}\frac{1+\tau_{k1}e^{it}}{1-\tau_{k1}e^{it}}f_0(t)dt +$$

$$+\frac{1}{2\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}A_k, & B_k\\ B_k, & C_k\end{Vmatrix}\ln z_k\int_s F(t)ds+A^{(\infty)}x_1+B^{(\infty)}x_2. \qquad (26.15)$$

The stress vector can be calculated similarly.

The above-given solution of the first boundary value problem for an infinite plane with an elliptical hole is due to Basheleishvili [3]. Detailed exposition of the solution of that problem can be found in V.Kupradze's monograph [24].

27. Solution of the Second Boundary Value Problem for an Infinite Anisotropic Plane with an Elliptical Hole

A solution of the second boundary value problem outside the elliptical domain D^- is sought in the form

$$u(x) = \frac{a_{11}}{BC - A^2} \begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix} \Phi(x) + \omega a_{11} \begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} \Psi(x), \tag{27.1}$$

where

$$\Phi(x) + i\Psi(x) = \frac{1}{\pi} \int_s \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \ln \sigma_k h(y) ds \tag{27.2}$$

and

$$Tu \equiv \frac{\partial \Psi(x)}{\partial s(x)} = \frac{1}{\pi} \int_s \operatorname{Im} \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \frac{\partial \ln \sigma_k}{\partial s(x)} h(y) ds. \tag{27.3}$$

For $x \in D^-$ there takes place the expansion

$$\ln \sigma_k = \ln\Big(\frac{a - i\alpha_k b}{2}\Big) + \ln t_{k1} - \sum_{n=1}^{\infty} \frac{e^{int} + \lambda_k^n e^{-int}}{n} \tau_{k1}^n, \tag{27.4}$$

where t_{k1} and τ_{k1} are given by (24.26) and (26.3).

We can readily establish the identity

$$\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \lambda_k^n = \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} q^{(n)}, \quad k = 1, 2, \tag{27.5}$$

where $q^{(n)}$ is given by (25.8).

Using (27.4) and (27.5) and supposing $\int_s h ds = 0$, from (27.2) we get

$$\Phi(x) + i\Psi(x) = -\sum_{k=1}^{2} \sum_{n=1}^{\infty} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \frac{\tau_{k1}^n}{n} \mu_n, \tag{27.6}$$

where we have introduced the notation

$$\begin{gathered} \mu_n = g_n + q^{(n)} g_{-n}, \quad g(\varphi) = h(\varphi)\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}, \\ g_n = \frac{1}{2\pi} \int_0^{2\pi} g e^{int} dt, \quad n = 0, \pm 1, \dots . \end{gathered} \tag{27.7}$$

The formula (27.3) yields

$$Tu = -2 \operatorname{Im} \frac{\partial}{\partial s(x)} \sum_{k=1}^{2} \sum_{n=1}^{\infty} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} \frac{\tau_{k1}^n}{n} \mu_n. \tag{27.8}$$

which in its turn for $x = z \in s$ results in

$$2\,\mathrm{Re}\sum_{n=1}^{\infty} e^{-in\varphi} g_n = F(\varphi), \tag{27.9}$$

where $F(\varphi) = f(\varphi)\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}$, $f = (Tu)^-$.

From (27.9) it follows that

$$g_n = F_n. \tag{27.10}$$

It is obvious that for the compatibility of the equation (27.9) it is necessary that the condition $\int_0^{2\pi} F\,dt = \int_s f\,ds = 0$ be satisfied.

Thus, if the principal vector of external stresses is equal to zero, then the displacement vector is defined to within the rigid displacement, while the stress vector is defined uniquely. Substituting (27.10) in (27.6) and (27.8), we obtain

$$\Phi(x) + i\Psi(x) = -2\sum_{k=1}^{2}\sum_{n=1}^{\infty}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\frac{\tau_{k1}^n}{n}F_n \tag{27.11}$$

and

$$Tu(x) = -2\frac{\partial}{\partial s(x)}\,\mathrm{Im}\sum_{k=1}^{2}\sum_{n=1}^{\infty}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\frac{\tau_{k1}^n}{n}F_n. \tag{27.12}$$

Summation of the series (27.11) and (27.12) gives

$$\Phi(x) + i\Psi(x) = \frac{1}{\pi}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\int_0^{2\pi}\ln(1 - \tau_{k1}e^{it})F(t)dt \tag{27.13}$$

and

$$Tu(x) = \frac{1}{\pi}\,\mathrm{Im}\sum_{k=1}^{2}\begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix}\int_0^{2\pi}\frac{\partial}{\partial s(x)}\ln(1 - \tau_{k1}e^{it})F(t)dt. \tag{27.14}$$

Let us now calculate the boundary value of the tangential derivative of the displacement vector. Taking into consideration the arguments of Section 25, we readily obtain

$$\begin{aligned}\frac{d(u)^-}{d\varphi} = &-\frac{a_{11}}{BC - A^2}\begin{Vmatrix} C, & A \\ A, & B \end{Vmatrix}\frac{1}{2\pi}\int_0^{2\pi}F(t)\,\mathrm{ctg}\,\frac{t-\varphi}{2}dt + \\ &+ \omega a_{11}\begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix}F(\varphi),\end{aligned} \tag{27.15}$$

where $F(\varphi) = \sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}\,(Tu)^-$.

After transformations performed for an isotropic elastic body in an infinite plane with a circular hole, the formulas (27.1), (27.13), and (27.14) take the form

$$u(x) = \frac{\lambda + 2\mu}{2\mu(\lambda + \mu)} \Phi(x) + \frac{1}{2(\lambda + \mu)} \Psi(x), \tag{27.16}$$

$$\Phi(x) + i\Psi(x) = \frac{R}{\pi} \int_0^{2\pi} \left[E \ln \sigma - \frac{1}{2} \left\| \begin{matrix} 1, i \\ i, -1 \end{matrix} \right\| \left(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{\zeta}}{z} \right) \right] f(t) dt, \tag{27.17}$$

$$Tu(x) = \frac{R}{\pi} \frac{\partial}{\partial s(x)} \int_0^{2\pi} \operatorname{Im} \left[E \ln \sigma - \frac{1}{2} \left\| \begin{matrix} 1, i \\ i, -1 \end{matrix} \right\| \left(\frac{\overline{\sigma}}{\sigma} + \frac{\overline{\zeta}}{z} \right) \right] f(t) dt, \tag{27.18}$$

where R is the radius of the circle, $\sigma = z - \zeta$, $z = x_1 + ix_2$, $\zeta = y_1 + iy_2$, $f = (Tu)^-$.

So far we have assumed that the principal vector of external stresses, the stress components and the rotation are equal to zero at infinity.

Consider now the general case. Assume that the principal vector of external stresses -$\int_s f ds$ and the stress components -$\sigma_{11}^{(\infty)}$, $\sigma_{22}^{(\infty)}$, $\tau_{12}^{(\infty)}$ at infinity are the beforehand prescribed values. In this case a solution of the second boundary value problem of statics of an anisotropic elastic body for an infinite plane with an elliptical hole is sought in the form

$$u(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \left\| \begin{matrix} A_k', & B_k' \\ C_k', & D_k' \end{matrix} \right\| \int_0^{2\pi} \ln(1 - \tau_{k1} e^{it}) g(t) dt +$$

$$+ \frac{1}{2\pi} \operatorname{Im} \sum_{k=1}^{2} \left\| \begin{matrix} A_k, & B_k \\ B_k, & C_k \end{matrix} \right\| \ln z_k \int_s f ds + A^{(\infty)} x_1 + B^{(\infty)} x_2. \tag{27.19}$$

where g is an unknown real vector function, and the vectors $A^{(\infty)} = (A_1^{\infty}, A_2^{(\infty)})$ and $B^{(\infty)} = (B_1^{(\infty)}, B_2^{(\infty)})$ are defined as

$$A_1^{(\infty)} = a_{11}\sigma_{11}^{(\infty)} + a_{12}\sigma_{22}^{(\infty)} + a_{13}\tau_{12}^{(\infty)}, \quad B_2^{(\infty)} = a_{12}\sigma_{11}^{(\infty)} + a_{22}\sigma_{22}^{(\infty)} + a_{23}\tau_{12}^{(\infty)},$$

$$A_2^{(\infty)} = B_1^{(\infty)} = \frac{1}{2}(a_{13}\sigma_{11}^{(\infty)} + a_{23}\sigma_{22}^{(\infty)} + a_{33}\tau_{12}^{(\infty)}).$$

The rotation at infinity is supposed to be $\varepsilon^{(\infty)} = 0$, since it does not affect the stress.

It follows from (27.19) that

$$Tu(x) = \frac{1}{\pi} \operatorname{Im} \sum_{k=1}^{2} \left\| \begin{matrix} p_k, & q_k \\ r_k, & s_k \end{matrix} \right\| \int_0^{2\pi} \frac{\partial}{\partial s(x)} \ln(1 - \tau_{k1} e^{it}) g(t) dt +$$

$$+ \frac{1}{2\pi} \operatorname{Im} \sum_{k=1}^{2} \left\| \begin{matrix} N_k, & L_k \\ M_k, & R_k \end{matrix} \right\| \frac{\partial \ln z_k}{\partial s(x)} \int_s f ds + \begin{pmatrix} \sigma_{11}^{(\infty)} \\ \tau_{12}^{(\infty)} \end{pmatrix} n_1 + \begin{pmatrix} \tau_{12}^{(\infty)} \\ \sigma_{22}^{(\infty)} \end{pmatrix} n_2. \tag{27.20}$$

For determining of g, we obtain the Fredholm integral equation

$$g(\varphi)+\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}\int_0^{2\pi}\frac{d}{d\varphi}\ln(1-\tau_{k1}e^{it})g(t)dt=\Phi(\varphi), \quad (27.21)$$

where

$$\Phi(\varphi)=\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}f(\varphi)-\frac{1}{\pi}\operatorname{Im}\sum_{k=1}^{2}\begin{Vmatrix}N_k, & L_k\\ M_k, & R_k\end{Vmatrix}\frac{d}{d\varphi}\ln z_{k0}\int_s f ds-$$

$$-\begin{pmatrix}\sigma_{11}^{(\infty)}\\ \tau_{12}^{(\infty)}\end{pmatrix}b\cos\varphi-\begin{pmatrix}\tau_{12}^{(\infty)}\\ \sigma_{22}^{(\infty)}\end{pmatrix}a\sin\varphi. \quad (27.22)$$

As is easily seen, the equation (27.21) can be rewritten as

$$g(\varphi)+\frac{1}{2\pi}\int_0^{2\pi}g(t)dt=\Phi(\varphi).$$

Since $\int_0^{2\pi}\Phi(t)dt=0$ and $\int_0^{2\pi}g(t)dt=0$, from the preceding equation we have $g(\varphi)=\Phi(\varphi)$.

Substituting the value found for $g(\varphi)$ into (27.19) and (27.20), we obtain the solution of the second boundary value problem in the general case. The solution obtained above is due to the author [3].

The second boundary value problem for an infinite plane with an elliptical hole has been solved in a different way by Lekhnitskiĭ [30] and Savin [42].

28. Solution of the Third and Fourth Boundary Value Problem for an Infinite Plane with an Elliptical Hole

In Chapter I (see Sections 18 and 19) we proved theorems on the existence of the solution of the third and fourth boundary value problems of anisotropic elastic body. In this section we will show that the above-mentioned problems can be proved explicitly (in quadratures) for an infinite plane with an elliptical hole. The third boundary value problem for an isotropic elastic body has been solved by Muskhelishvili [40].

It will be assumed that the stress and rotation components vanish at infinity; moreover, we suppose that the principal vector of external forces applied to the contour of the hole is equal to zero (the general case can always be reduced to that one by the techniques we have used above several times).

First let us consider the third boundary value problem. In this case the boundary conditions are of the form

$$(nu)^-=f_1, \quad (sTu)^-=f_2, \quad (28.1)$$

where f_1 and f_2 are given functions; note that $f_1\in C^{1,\alpha}(s)$, $f_2\in C^{0,\alpha}(S)$, $\alpha>0$.

By projecting (26.9) to the tangent line, after elementary calculations we obtain

$$m(a^2\sin^2\varphi+b^2\cos^2\varphi)f_2(\varphi)=\omega(BC-A^2)\begin{pmatrix}b\cos\varphi\\ a\sin\varphi\end{pmatrix}\frac{d(u)^-}{d\varphi}+$$

$$+\frac{1}{2\pi}\begin{pmatrix}-Ba\sin\varphi-Ab\cos\varphi\\ Aa\sin\varphi+Cb\cos\varphi\end{pmatrix}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\frac{d(u)^-}{dt}\,dt. \qquad (28.2)$$

Represent $(u)^-$ in the form

$$(u)^-=\begin{pmatrix}b\cos\varphi\\ a\sin\varphi\end{pmatrix}f(\varphi)+\begin{pmatrix}-a\sin\varphi\\ b\cos\varphi\end{pmatrix}g(\varphi), \qquad (28.3)$$

where

$$f(\varphi)=\frac{f_1(\varphi)}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}},\quad g(\varphi)=\frac{(su)^-}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}, \qquad (28.4)$$

$f_1(\varphi)$ is given by (28.1), and $(su)^-$ is the unknown value of the displacement vector projection to the tangent line.

From (28.3) we have

$$\frac{d(u)^-}{d\varphi}=\begin{pmatrix}b\cos\varphi\\ a\sin\varphi\end{pmatrix}\frac{df}{d\varphi}+\begin{pmatrix}-b\sin\varphi\\ a\cos\varphi\end{pmatrix}f+\begin{pmatrix}-a\sin\varphi\\ b\cos\varphi\end{pmatrix}\frac{dg}{d\varphi}-\begin{pmatrix}a\cos\varphi\\ b\sin\varphi\end{pmatrix}g.$$

Substituting this in (28.2), after obvious transformations we get

$$m(a^2\sin^2\varphi+b^2\cos^2\varphi)f_2(\varphi)=$$

$$=\omega(BC-A^2)\Big[(a^2\sin^2\varphi+b^2\cos^2\varphi)\frac{df}{d\varphi}+(a^2-b^2)\sin\varphi\cos\varphi f(\varphi)\Big]+$$

$$+\begin{pmatrix}-Ba\sin\varphi-Ab\cos\varphi\\ Aa\sin\varphi+Cb\cos\varphi\end{pmatrix}\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\left[\begin{pmatrix}b\cos t\\ a\sin t\end{pmatrix}\frac{df}{dt}+\begin{pmatrix}-b\sin t\\ a\cos t\end{pmatrix}f\right]dt-$$

$$-\omega ab(BC-A^2)g(\varphi)+$$

$$+\begin{pmatrix}-Ba\sin\varphi-Ab\cos\varphi\\ Aa\sin\varphi+Cb\cos\varphi\end{pmatrix}\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\left[\begin{pmatrix}-a\sin t\\ b\cos t\end{pmatrix}\frac{dg}{dt}-\begin{pmatrix}a\cos t\\ b\sin t\end{pmatrix}g\right]dt.$$

Introduce now the notation

$$\beta=\omega ab(BC-A^2), \qquad (28.5)$$

$$\Phi(\varphi)=-m(a^2\sin^2\varphi+b^2\cos^2\varphi)f_2(\varphi)+\omega(BC-A^2)\Big[(a^2\sin^2\varphi+$$

$$+b^2\cos^2\varphi)\frac{df}{d\varphi}+(a^2-b^2)\sin\varphi\cos\varphi f(\varphi)\Big]+\begin{pmatrix}-Ba\sin\varphi-Ab\cos\varphi\\ Aa\sin\varphi+Cb\cos\varphi\end{pmatrix}\frac{1}{2\pi}\times$$

$$\times\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\left[\begin{pmatrix}b\cos t\\ a\sin t\end{pmatrix}\frac{df}{dt}+\begin{pmatrix}-b\sin t\\ a\cos t\end{pmatrix}f\right]dt. \qquad (28.6)$$

Then the previous equation can be rewritten as

$$\beta g(\varphi) + \begin{pmatrix} -Ba\sin\varphi - Ab\cos\varphi \\ Aa\sin\varphi + Cb\cos\varphi \end{pmatrix} \frac{1}{2\pi} \times$$
$$\times \int_0^{2\pi} \operatorname{ctg}\frac{t-\varphi}{2} \left[\begin{pmatrix} a\sin t \\ -b\cos t \end{pmatrix} \frac{dg}{dt} + \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} g \right] dt = \Phi(\varphi). \qquad (28.7)$$

Thus for determining g, we have obtained a singular integral differential equation.

Bearing in mind the formulas

$$(\cos t - \cos\varphi)\operatorname{ctg}\frac{t-\varphi}{2} = -\sin t - \sin\varphi,$$
$$(\sin t - \sin\varphi)\operatorname{ctg}\frac{t-\varphi}{2} = \cos t + \cos\varphi,$$

after some calculations we can rewrite (28.7) in the form of two (equivalent) equations

$$\beta g(\varphi) + \begin{pmatrix} -Ba\sin\varphi - Ab\cos\varphi \\ Aa\sin\varphi + Cb\cos\varphi \end{pmatrix} \begin{pmatrix} a\sin\varphi \\ -b\cos\varphi \end{pmatrix} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{ctg}\frac{t-\varphi}{2} \frac{dg}{dt}\, dt +$$
$$+ \begin{pmatrix} -Ba\sin\varphi - Ab\cos\varphi \\ Aa\sin\varphi + Cb\cos\varphi \end{pmatrix} \begin{pmatrix} a\cos\varphi \\ b\sin\varphi \end{pmatrix} \frac{1}{2\pi} \int_0^{2\pi} \operatorname{ctg}\frac{t-\varphi}{2}\, g dt +$$
$$+ \begin{pmatrix} -Ba\sin\varphi - Ab\cos\varphi \\ Aa\sin\varphi + Cb\cos\varphi \end{pmatrix} \frac{1}{2\pi} \times$$
$$\times \int_0^{2\pi} \left\{ \begin{bmatrix} a(\cos t + \cos\varphi) \\ b(\sin t + \sin\varphi) \end{bmatrix} \frac{dg}{dt} + \begin{bmatrix} -a(\sin t + \sin\varphi) \\ b(\cos t + \cos\varphi) \end{bmatrix} g \right\} dt = \Phi(\varphi)$$

and

$$\beta g(\varphi) + \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} -Ba\sin t - Ab\cos t \\ Aa\sin t + Cb\cos t \end{pmatrix} \times$$
$$\times \left[\begin{pmatrix} a\sin t \\ -b\cos t \end{pmatrix} \frac{dg}{dt} + \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} g \right] \operatorname{ctg}\frac{t-\varphi}{2}\, dt +$$
$$+ \frac{1}{2\pi} \int_0^{2\pi} \begin{bmatrix} Ba(\cos t + \cos\varphi) - Ab(\sin t + \sin\varphi) \\ -Aa(\cos t + \cos\varphi) + Cb(\sin t + \sin\varphi) \end{bmatrix} \times$$
$$\times \left[\begin{pmatrix} a\sin t \\ -b\cos t \end{pmatrix} \frac{dg}{dt} + \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} g \right] dt = \Phi(\varphi).$$

Using the periodicity of g and introducing the notation

$$\Delta(\varphi) = Ba^2\sin^2\varphi + Cb^2\cos^2\varphi + 2abA\sin\varphi\cos\varphi, \qquad (28.8)$$

after obvious transformations we rewrite the above equations as

$$\beta g(\varphi) - \frac{\Delta(\varphi)}{2\pi} \int_0^{2\pi} \operatorname{ctg}\frac{t-\varphi}{2} \frac{dg}{dt}\, dt -$$

$$-\frac{\Delta'(\varphi)}{2}\,\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,g\,dt+\Delta(\varphi)g_0=\Phi(\varphi) \tag{28.9}$$

and

$$\beta g(\varphi)-\frac{1}{2\pi}\int_0^{2\pi}\Big(\Delta(t)\,\frac{dg}{dt}+\frac{\Delta'(t)}{2}\,g\Big)\operatorname{ctg}\frac{t-\varphi}{2}\,dt+C_0=\Phi(\varphi), \tag{28.10}$$

where

$$g_0=\frac{1}{2\pi}\int_0^{2\pi}g\,dt,\quad C_0=\frac{1}{2\pi}\int_0^{2\pi}\Delta(t)g\,dt. \tag{28.11}$$

Applying the formula of composition of integrals with Hilbert kernels (see [38, p. 144])

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,dt\int_0^{2\pi}u(\tau)\operatorname{ctg}\frac{\tau-t}{2}\,d\tau=-u(\varphi)+\frac{1}{2\pi}\int_0^{2\pi}u(\tau)\,d\tau,$$

from (28.10) we find

$$\begin{aligned}\Delta(\varphi)\,\frac{dg}{d\varphi}+\frac{\Delta'(\varphi)}{2}\,g+\frac{\beta}{2\pi}\int_0^{2\pi}g\operatorname{ctg}\frac{t-\varphi}{2}\,dt-g_1=\\ =\frac{1}{2\pi}\int_0^{2\pi}\Phi(t)\operatorname{ctg}\frac{t-\varphi}{2}\,dt,\end{aligned} \tag{28.12}$$

where

$$g_1=\frac{1}{2\pi}\int_0^{2\pi}\Big(\Delta\,\frac{dg}{dt}+\frac{\Delta'}{2}\,g\Big)\,dt\equiv-\frac{1}{2\pi}\int_0^{2\pi}\frac{\Delta'(t)}{2}\,g\,dt. \tag{28.13}$$

Introduce the notation

$$h(\varphi)=-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,g(t)\,dt. \tag{28.14}$$

Obviously, $\int_0^{2\pi}h dt=0$.
It follows from (28.14) that

$$\frac{dh}{d\varphi}=-\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,\frac{dg}{dt}\,dt. \tag{28.15}$$

Substituting (28.14) and (28.15) in (28.9), we obtain

$$\Delta(\varphi)\,\frac{dh}{dt}+\frac{\Delta'(\varphi)}{2}\,h+\beta g+\Delta(\varphi)g_0=\Phi(\varphi). \tag{28.16}$$

Let now

$$\mu(\varphi)=h+ig. \tag{28.17}$$

Then (28.12) and (28.16) can be rewritten in the form of one (complex) equation

$$\Delta(\varphi)\,\frac{d\mu}{d\varphi}+\frac{\Delta'(\varphi)}{2}\,\mu-i\beta\mu=\Psi(\varphi)+ig_1-\Delta(\varphi)g_0, \tag{28.18}$$

where

$$\Psi(\varphi)=\Phi(\varphi)+\frac{i}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{t-\varphi}{2}\,\Phi(t)\,dt. \tag{28.19}$$

We rewrite (28.18) as

$$\frac{d}{d\varphi}\,(\sqrt{\Delta}\,\mu)-\frac{i\beta}{\Delta(\varphi)}\,(\sqrt{\Delta}\,\mu)=i\,\frac{g_1}{\sqrt{\Delta}}-\sqrt{\Delta}\,g_0+\frac{\Psi(\varphi)}{\sqrt{\Delta}}$$

from which by integration we obtain

$$\begin{aligned}\mu(\varphi)={}&C\,\frac{e^{i\tau(\varphi)}}{\sqrt{\Delta(\varphi)}}+\\&+\frac{e^{i\tau(\varphi)}}{\sqrt{\Delta(\varphi)}}\int_0^{\varphi}\Big(i\,\frac{g_1}{\sqrt{\Delta(t)}}-g_0\sqrt{\Delta(t)}+\frac{\Psi(t)}{\sqrt{\Delta(t)}}\Big)e^{-i\tau(t)}dt,\end{aligned} \tag{28.20}$$

where C is for the time being an arbitrary constant, and

$$\tau(\varphi)=\int_0^{\varphi}\frac{\beta\,dt}{\Delta(t)}. \tag{28.21}$$

Now let us find an explicit expression $\tau(\varphi)$. Using (28.8) we have

$$\begin{aligned}\Delta(\varphi)={}&\frac{Ba^2+Cb^2}{2}+\frac{1}{2}\Big(\frac{Cb^2-Ba^2}{2}-iAab\Big)e^{2i\varphi}+\\&+\frac{1}{2}\Big(\frac{Cb^2-Ba^2}{2}+iAab\Big)e^{-2i\varphi}.\end{aligned} \tag{28.22}$$

Introduce the notation

$$\lambda=\frac{Cb^2-Ba^2+2iAab}{Cb^2+Ba^2+2ab\sqrt{BC-A^2}}. \tag{28.23}$$

Then after elementary calculations we have

$$\begin{aligned}1-\lambda\,\overline{\lambda}&=\frac{4ab\sqrt{BC-A^2}}{Cb^2+Ba^2+2ab\sqrt{BC-A^2}},\\1+\lambda\,\overline{\lambda}&=\frac{2(Cb^2+Ba^2)}{Cb^2+Ba^2+2ab\sqrt{BC-A^2}},\end{aligned} \tag{28.24}$$

whence

$$|\lambda|<1. \tag{28.25}$$

Substituting (28.24) in (28.22), we have

$$\Delta(\varphi) = \frac{Cb^2 + Ba^2 + 2ab\sqrt{BC - A^2}}{4}\,(1 + \lambda e^{-2i\varphi})(1 + \overline{\lambda} e^{2i\varphi}).$$

Taking into account (28.24), the above equation after certain transformations gives

$$\frac{1}{\Delta(\varphi)} = \frac{1}{ab\sqrt{BC - A^2}}\Big(\frac{1}{1 + \overline{\lambda} e^{2i\varphi}} - \frac{\lambda e^{-2i\varphi}}{1 + \lambda e^{-2i\varphi}}\Big),$$

where

$$\frac{\beta}{\Delta(\varphi)} = \omega\sqrt{BC - A^2}\Big[1 + 2\,\mathrm{Re}\sum_{n=1}^{\infty}(-1)^n \lambda^n e^{-2in\varphi}\Big], \qquad (28.26)$$

β is given by (28.5).

From (28.26), owing to (28.21) we obtain

$$\tau(\varphi) = \omega\sqrt{BC - A^2}\,\varphi + \omega\sqrt{BC - A^2}\;\mathrm{Im}\sum_{n=1}^{\infty}(-1)^n \lambda^n \frac{1 - e^{-2in\varphi}}{n},$$

i.e.,

$$\tau(\varphi) = \omega\sqrt{BC - A^2}\,\varphi + \omega\sqrt{BC - A^2}\;\ln\mathrm{Im}\,\frac{1 + \lambda e^{-2i\varphi}}{1 + \lambda}. \qquad (28.27)$$

The last formula yields

$$\tau(\varphi + 2\pi) - \tau(\varphi) = 2\pi\omega\sqrt{BC - A^2}. \qquad (28.28)$$

Let us now define the constant C appearing in (28.20) so that the function $\mu(\varphi)$ be periodic with the period 2π, i.e., $\mu(\varphi + 2\pi) = \mu(\varphi)$.

Using (28.27) and (28.28), from (28.20) we obtain

$$\begin{aligned}
\mu(\varphi + 2\pi) &= Ce^{2\pi i\omega\sqrt{BC-A^2}} e^{i\tau(\varphi)} + \\
&+ e^{2\pi i\omega\sqrt{BC-A^2}} e^{i\tau(\varphi)}\Big[\int_0^{2\pi}\Big(\frac{ig_1}{\sqrt{\Delta}} - \sqrt{\Delta}\,g_0 + \frac{\Psi(t)}{\sqrt{\Delta}}\Big)e^{-i\tau(t)}dt + \\
&+ e^{i\tau(\varphi)}\int_0^{\varphi}\Big(\frac{ig_1}{\sqrt{\Delta}} - \sqrt{\Delta}\,g_0 + \frac{\Psi(t)}{\sqrt{\Delta}}\Big)e^{-i\tau(t)}dt\Big] = \\
&= C\,\frac{e^{i\tau(\varphi)}}{\sqrt{\Delta}} + \frac{e^{i\tau(\varphi)}}{\sqrt{\Delta}}\int_0^{\varphi}\Big(\frac{ig_1}{\sqrt{\Delta}} - \sqrt{\Delta}\,g_0 + \frac{\Psi(t)}{\sqrt{\Delta}}\Big)e^{-i\tau(t)}dt,
\end{aligned}$$

from which for determining C we have the equation

$$Ce^{2\pi i\omega\sqrt{BC-A^2}} + e^{2\pi i\omega\sqrt{BC-A^2}}\int_0^{2\pi}\Big(\frac{ig_1}{\sqrt{\Delta}} - \sqrt{\Delta}\,g_0 + \frac{\Psi(t)}{\sqrt{\Delta}}\Big)e^{-i\tau(t)}dt = C.$$

Thus we have obtained

$$C = D_0 \int_0^{2\pi} \Big(\frac{ig_1}{\sqrt{\Delta}} - \sqrt{\Delta}\, g_0 + \frac{\Psi(t)}{\sqrt{\Delta}} \Big) e^{-i\tau(t)} dt, \tag{28.29}$$

where

$$D_0 = \frac{1}{2}\big(-1 + i \operatorname{ctg} \pi\omega \sqrt{BC - A^2}\big). \tag{28.30}$$

Substituting C from (28.29) in (28.20), we arrive at

$$\mu(\varphi) = ig_1 R(\varphi) - g_0 F(\varphi) + G(\varphi), \tag{28.31}$$

where

$$R(\varphi) = \big(D_0\, \overline{\mathcal{I}(2\pi)} + \overline{\mathcal{I}(\varphi)}\big)\mathcal{I}'(\varphi), \quad F(\varphi) = \big(D_0\, \overline{I(2\pi)} + \overline{I(\varphi)}\big)\mathcal{I}'(\varphi),$$
$$G(\varphi) = \Big(D_0 \int_0^{2\pi} \Psi(t)\, \overline{\mathcal{I}'(t)}\, dt + \int_0^{\varphi} \Psi(t) \overline{\mathcal{I}'(t)}\, dt \Big) \mathcal{I}'(\varphi) \tag{28.32}$$

and

$$\mathcal{I}(\varphi) = \int_0^{\varphi} \frac{e^{i\tau(t)}}{\sqrt{\Delta(t)}}\, dt, \quad I(\varphi) = \int_0^{\varphi} \sqrt{\Delta(t)}\, e^{i\tau(t)}\, dt. \tag{28.33}$$

Taking into account the fact that $\Delta(\varphi)\mathcal{I}''(\varphi) + (\frac{\Delta'}{2} - i\beta)\mathcal{I}'(\varphi) = 0$, we can easily show that

$$\Delta(\varphi)\frac{dR}{d\varphi} + \Big(\frac{\Delta'(\varphi)}{2} - i\beta \Big) R = 1, \quad \Delta(\varphi)\frac{dF}{d\varphi} + \Big(\frac{\Delta'(\varphi)}{2} - i\beta \Big) F = \Delta(\varphi),$$
$$\Delta(\varphi)\, \frac{dG}{d\varphi} + \Big(\frac{\Delta'(\varphi)}{2} - i\beta \Big) G = \Psi(\varphi). \tag{28.34}$$

Establish now some relations which we will need below. Using the Dirichlet formula, from (28.32) we get

$$\int_0^{2\pi} G\, d\varphi = D_0 \int_0^{2\pi} \Psi\, \overline{\mathcal{I}'(\varphi)}\, d\varphi \cdot \mathcal{I}(2\pi) + \int_0^{2\pi} \Psi(t)\, \overline{\mathcal{I}'(t)}\, dt \int_t^{2\pi} \mathcal{I}'(\varphi)\, d\varphi =$$
$$= (D_0 + 1) \int_0^{2\pi} \Psi\, \overline{\mathcal{I}'(\varphi)}\, d\varphi \cdot \mathcal{I}(2\pi) - \int_0^{2\pi} \Psi(t)\, \overline{\mathcal{I}'(\varphi)}\, \mathcal{I}(t)\, dt.$$

On the other hand,

$$\int_0^{2\pi} \Psi\, \overline{R}\, d\varphi = \overline{D}_0 \mathcal{I}(2\pi) \int_0^{2\pi} \Psi\, \overline{\mathcal{I}'(\varphi)}\, d\varphi + \int_0^{2\pi} \Psi(t)\, \overline{\mathcal{I}'(t)}\, \mathcal{I}(t)\, dt.$$

From the last two formulas we find that

$$\int_0^{2\pi} (G + \Psi\overline{R}) d\varphi = (D_0 + 1 + \overline{D}_0) \int_0^{2\pi} \Psi\, \overline{\mathcal{I}'(\varphi)}\, d\varphi \cdot \mathcal{I}(2\pi).$$

Since $D_0 + 1 + \overline{D}_0 = 0$, we have

$$\int_0^{2\pi} G\,d\varphi = -\int_0^{2\pi} \Psi\overline{R}\,d\varphi. \tag{28.35}$$

Analogously, from (28.32) we get

$$\begin{aligned}
&\int_0^{2\pi} R\,d\varphi = -\int_0^{2\pi} \overline{R}\,d\varphi, \quad \int_0^{2\pi} \Delta(\varphi)F\,d\varphi = -\int_0^{2\pi} \Delta(\varphi)\overline{F}\,d\varphi,\\
&\int_0^{2\pi} F\,d\varphi = -\int_0^{2\pi} \Delta(\varphi)\overline{R}\,d\varphi, \quad \int_0^{2\pi} \Delta(\varphi)G\,d\varphi = -\int_0^{2\pi} \Psi\overline{F}\,d\varphi.
\end{aligned} \tag{28.36}$$

Let now $R = R_1 + iR_2$, $F = F_1 + iF_2$, $G = G_1 + iG_2$. Then (28.31) can be rewritten as

$$g = g_1 R_1(\varphi) - g_0 F_2(\varphi) + G_2(\varphi), \quad h = -g_1 R_2(\varphi) - g_0 F_1(\varphi) + G_1(\varphi). \tag{28.37}$$

For determining the constants g_0 and g_1, we have two conditions

$$\frac{1}{2\pi}\int_0^{2\pi} g\,d\varphi = g_0 \quad \text{and} \quad \int_0^{2\pi} h\,d\varphi = 0. \tag{28.38}$$

It is more convenient to replace the second condition by

$$\frac{1}{2\pi}\int_0^{2\pi} \Delta(\varphi)h\,d\varphi = -g_1. \tag{28.39}$$

Let us show that this really can be done. Indeed, using the well-known formulas

$$\frac{1}{2\pi}\int_0^{2\pi} e^{ik\varphi}\operatorname{ctg}\frac{\varphi - t}{2}\,dt = ie^{ikt}, \ k = \pm 1, \pm 2, \ldots, \quad \frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\varphi - t}{2}\,d\varphi = 0,$$

and taking into account (28.14), from (28.22) we obtain

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi}\Delta(\varphi)h(\varphi)\,d\varphi &= \frac{1}{2\pi}\int_0^{2\pi} g(t)\,dt\,\frac{1}{2\pi}\int_0^{2\pi}\Delta(\varphi)\operatorname{ctg}\frac{\varphi - t}{2}\,d\varphi =\\
= \frac{1}{2\pi}\int_0^{2\pi} g(t)\Big[\frac{i}{2}\Big(\frac{Cb^2 - Ba^2}{2} &- iAab\Big)e^{2it} - \frac{i}{2}\Big(\frac{Cb^2 - Ba^2}{2} + iAab\Big)e^{-2it}\Big]\,dt =\\
&= \frac{1}{2\pi}\int_0^{2\pi} g(t)\Big(\frac{Ba^2 - Cb^2}{2}\sin 2t + Aab\cos 2t\Big)\,dt.
\end{aligned}$$

On the basis of our notation (28.8), we have

$$\frac{1}{2\pi}\int_0^{2\pi}\Delta(t)h(t)\,dt = \frac{1}{2\pi}\int_0^{2\pi} g(t)\,\frac{\Delta'(t)}{2}\,dt.$$

Taking now into consideration (28.13), we see that the condition (28.39) is valid.

Using the first formula of (28.38) and also the formula (28.39), for determining of g_0 and g_1 we obtain from (28.37) the equations

$$\begin{gathered}
g_0 = \frac{g_1}{2\pi}\int_0^{2\pi} R_1\,d\varphi - \frac{g_0}{2\pi}\int_0^{2\pi} F_2\,d\varphi + \frac{1}{2\pi}\int_0^{2\pi} G_2\,d\varphi, \\
g_1 = \frac{g_1}{2\pi}\int_0^{2\pi}\Delta(\varphi)R_2\,d\varphi| + \frac{g_0}{2\pi}\int_0^{2\pi}\Delta(\varphi)F_1\,d\varphi - \frac{1}{2\pi}\int_0^{2\pi}\Delta(\varphi)G_1\,d\varphi.
\end{gathered} \tag{28.40}$$

From (28.35) and (28.36) we have

$$\begin{aligned}
&\int_0^{2\pi} R_1\,d\varphi = 0, \quad \int_0^{2\pi}\Delta(\varphi)F_1\,d\varphi = 0, \quad \int_0^{2\pi}\Delta(\varphi)R_2\,d\varphi = \int_0^{2\pi} F_2\,d\varphi, \\
&\int_0^{2\pi} G_2\,d\varphi = \int_0^{2\pi}\Phi(\varphi)\left[R_2(\varphi) + \frac{1}{2\pi}\int_0^{2\pi} R_1(t)\operatorname{ctg}\frac{t-\varphi}{2}\,dt\right]d\varphi, \\
&\int_0^{2\pi}\Delta(\varphi)G_1\,d\varphi = \int_0^{2\pi}\Phi(\varphi)\left[F_1(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} F_2(t)\operatorname{ctg}\frac{t-\varphi}{2}\,dt\right]d\varphi
\end{aligned}$$

On the basis of the last formulas for g_0 and g_1, from (28.40) we get

$$\begin{gathered}
g_0 = \frac{1}{1+F_{20}}\,\frac{1}{2\pi}\int_0^{2\pi}\Phi(\varphi)\left[R_2(\varphi) + \frac{1}{2\pi}\int_0^{2\pi} R_1(t)\operatorname{ctg}\frac{t-\varphi}{2}\,dt\right]d\varphi, \\
g_1 = \frac{1}{-1+F_{20}}\,\frac{1}{2\pi}\int_0^{2\pi}\Phi(\varphi)\left[F_1(\varphi) - \frac{1}{2\pi}\int_0^{2\pi} F_2(t)\operatorname{ctg}\frac{t-\varphi}{2}\,dt\right]d\varphi,
\end{gathered} \tag{28.41}$$

where $F_{20} = \frac{1}{2\pi}\int_0^{2\pi} F_2(\varphi)\,d\varphi$.

Prove that $F_{20} > 1$. Substituting the above-obtained values g_0 and g_1 in (28.37), we finally find the unknown function g.

The solution of the third boundary value problem is considerably simplified if $Ba^2 = Cb^2$ and $A = 0$.

In the case under consideration, repeating the above arguments we readily obtain

$$g(\varphi) = \frac{A_0}{Cb(b+\omega aB)} + \frac{1}{Cb}\sum_{n=1}^{\infty}\frac{A_n\cos n\varphi + B_n\sin n\varphi}{nb+\omega ab},$$

where

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi}\Phi(t)\,dt, \quad A_n + iB_n = \frac{1}{\pi}\int_0^{2\pi}\Phi e^{int}dt, \quad n = 1, 2, \ldots.$$

Thus the third boundary value problem is solved. The fourth boundary value problem is solved quite analogously.

The method of effective solution of the third and fourth boundary value problems can be found in author's earlier work [10]. Note that these problems have not been solved before by other methods.

29. Effective Solution of the Mixed Boundary Value Problem for an Infinite Anisotropic Plane with an Elliptical Hole

Let an infinite anisotropic plate be weakened by an elliptical hole with the semi-axes a and b $(a > b)$. The symmetry axes of the ellipse are taken as the coordinate axes, and the major axis coincides with the real axis $0x_1$. Suppose that the ellipse contour is divided by the axis $0x_1$ into two parts. Denote by L_1 that part of the ellipse which is located in the upper $(x_2 > 0)$ half-plane and by L_2 the part located in the lower $(x_2 < 0)$ half-plane.

Consider the mixed boundary value problem: Define a stress state inside the plate if

$$\begin{aligned} (Tu)^-_{L_1} &= \frac{F_0(\varphi)}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}, \quad 0 \le \varphi \le \pi, \\ (u)^-_{L_2} &= f_0(\varphi), \quad \pi \le \varphi \le 2\pi, \end{aligned} \tag{29.1}$$

where Tu is the stress vector and u is the displacement vector; F_0 and f_0 are given vectors, where $f_0 \in C^{0,\alpha}[\pi, 2\pi]$, $\alpha > 0$, $f_0'(\varphi)$, and $F_0(\varphi)$ belong to the H^* class on $(\pi, 2\pi)$ and $(0, \pi)$, respectively (see [39]).

The components of the stress and the rotation at infinity as well as the principal vector of external forces applied to the ellipse contour will be assumed to be equal to zero.

The displacement vector for an infinite plane with an elliptical hole will be sought in the form (see Section 26)

$$u(x) = \frac{1}{2\pi}\,\mathrm{Re}\sum_{k=1}^{2}\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix}\int_0^{2\pi}\frac{1+\tau_{k1}e^{i\theta}}{1-\tau_{k1}e^{i\theta}}\,g(\theta)d\theta, \tag{29.2}$$

where $g = (g_1, g_2)$ is an unknown real vector and E_k, F_k, G_k, H_k are the given complex constants (see Section 5),

$$\tau_{k1} = \frac{a - i\alpha_k b}{z_k + \sqrt{z_k^2 - (a^2 + \alpha_k^2 b^2)}}, \qquad z_k = x_1 + \alpha_k x_2.$$

If we calculate the boundary value of the displacement vector by the formula (29.2) at the point $(a\cos\varphi, b\sin\varphi)$, $\pi \le \varphi \le 2\pi$, of the contour L_2 and take into account the fact that $\tau_{k1} = e^{-i\varphi}$, then we obtain $g(\varphi) = (u)^-_{L_2} = f_0(\varphi)$, $\pi \le \varphi \le 2\pi$.

Consequently, we have defined g on L_2. It remains to define it on L_1 so that g would belong to the Hölder class on the entire contour of the ellipse.

Note that the principal value of the stress vector $(Tu)^-$, taking into account (26.9), at the points of the ellipse contour satisfies the relation

$$\begin{aligned} \omega(BC - A^2)\begin{Vmatrix} 0, & -1 \\ 1, & 0 \end{Vmatrix} g(\varphi) + \begin{Vmatrix} B, & -A \\ -A, & C \end{Vmatrix}\frac{1}{2\pi}\int_0^{2\pi}\operatorname{ctg}\frac{\theta-\varphi}{2}\,g(\theta)\,d\theta = \\ = m\int_0^{\varphi}(Tu)^-\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}\,d\theta + C_1, \end{aligned} \tag{29.3}$$

where C_1 is an arbitrary real constant vector, $m = a_{11}[1-\omega^2(BC-A^2)] > 0$.

Since the value $g(\theta)$ for $\pi \le \theta \le 2\pi$ is known, after the transformation $t = e^{i\varphi}$, $\tau = e^{i\theta}$, in view of the obvious identity $\frac{1}{2}\operatorname{ctg}\frac{\theta-\varphi}{2}\,d\theta = \frac{d\tau}{\tau-t} - \frac{d\tau}{2\tau}$ the relation (29.3) for $0 \le \varphi \le \pi$ will take the form

$$\omega(BC-A^2)\begin{Vmatrix}0, & -1\\ 1, & 0\end{Vmatrix} g(t) + \begin{Vmatrix}B, & -A\\ -A, & C\end{Vmatrix} \frac{1}{\pi}\int_{L^+}\frac{g(\tau)\,d\tau}{\tau-t} = \Psi(t) + D, \quad (29.4)$$

$t \in L^+$, where L^+ is the semi-circumference of the unit radius located in the upper half-plane, and

$$\Psi(t) = m\int_0^{\varphi} F_0(\theta)\,d\theta - \begin{Vmatrix}B, & -A\\ -A, & C\end{Vmatrix} \frac{1}{\pi}\int_{L^-}\frac{f_0(\tau)\,d\tau}{\tau-t}, \quad 0 \le \varphi \le \pi, \quad (29.5)$$

$$D = C_1 + \begin{Vmatrix}B, & -A\\ -A, & C\end{Vmatrix} \frac{1}{2}\left[\int_{L^+}\frac{g(\tau)\,d\tau}{\tau} + \int_{L^-}\frac{f_0(\tau)\,d\tau}{\tau}\right]$$

is an arbitrary constant vector, and L^- is the semi-circumference of the unit radius located in the lower half-plane.

Since f_0 is Hölder continuous, Ψ at the ends of the line L^+ has a logarithmic singularity and belongs to the class H_ε^* (see [39, Section 8]).

Thus with respect to the vector $g(t)$ for $t \in L^+$ we have obtained a characteristic system of singular integral equations of the type (29.4).

This system is of normal type since

$$\det\left[\omega(BC-A^2)\begin{Vmatrix}0, & -1\\ 1, & 0\end{Vmatrix} \pm i\begin{Vmatrix}B, & -A\\ -A, & C\end{Vmatrix}\right] = -\frac{(BC-A^2)}{a_{11}}\,m < 0.$$

Rewrite now the system (29.4) in projections

$$-\omega(BC-A^2)g_2(t) + \frac{1}{\pi}\int_{L^+}\frac{Bg_1(\tau) - Ag_2(\tau)}{\tau-t}\,d\tau = \Psi_1(t) + D_1,$$
$$\omega(BC-A^2)g_1(t) + \frac{1}{\pi}\int_{L^+}\frac{-Ag_1(\tau) + Cg_2(\tau)}{\tau-t}\,d\tau = \Psi_2(t) + D_2.$$

If we multiply the first equation by $(-X)$ and add with the second one, we obtain

$$\omega(BC-A^2)(g_1 + Xg_2) + \frac{1}{\pi}\int_{L^+}\frac{(-A-XB)g_1 + (C+AX)g_2}{\tau-t}\,d\tau =$$
$$= \Psi_2 - X\Psi_1 + D_2 - XD_1. \quad (29.6)$$

Choosing the constant X so that the condition $C + AX = -X(A+BX)$ be satisfied, we arrive at the quadratic equation $BX^2 + 2AX + C = 0$ which has the following complex roots: $X = \frac{-A+i\sqrt{BC-A^2}}{B}$, $\overline{X} = \frac{-A-i\sqrt{BC-A^2}}{B}$, $B > 0$, $\sqrt{BC-A^2} > 0$.

Introduce now the notation

$$h = g_1 + Xg_2, \quad \Psi_0(t) = \frac{\Psi_2 - X\Psi_1}{\sqrt{BC-A^2}}, \quad D_0 = \frac{D_2 - XD_1}{\sqrt{BC-A^2}}. \quad (29.7)$$

Then after obvious transformations the equation (29.6) takes the form

$$\omega\sqrt{BC-A^2}h(t)+\frac{1}{\pi i}\int_{L^+}\frac{h(\tau)\,d\tau}{\tau-t}=\Psi_0(t)+D_0,\quad t\in L^+. \quad (29.8)$$

Thus we have reduced the system (29.4) to the characteristic equation (29.8) with respect to the scalar function h on the discontinuous contour L^+.

According to the general theory (see [39]), define the character of the end points of the line of integration. Introduce the number $\gamma = \frac{1}{2\pi i}\times \ln\frac{\omega\sqrt{BC-A^2}-1}{\omega\sqrt{BC-A^2}+1}=\frac{1}{2}+i\beta,\ \ \beta=\ln\frac{1+\omega\sqrt{BC-A^2}}{1-\omega\sqrt{BC-A^2}}>0$.

Hence the ends of the line L^+ are nonsingular. A solution of the equation (29.8) will be sought in the class of functions which are bounded at the ends. The canonical solution of the corresponding homogeneous problem of linear conjugation will have the form $X_0(z)=(z+1)^\gamma(z-1)^{1-\gamma}$, $z=x_1+ix_2$, where we mean the branch which is holomorphic on the plane cut along L^+, and satisfies $z^{-1}X_0(z)\to 1$ as $z\to\infty$.

Hence we can see that the index of that class is equal to -1, and the solution of the equation (29.8) has the form (see [39], [40])

$$h(t)=\frac{a_{11}}{m}\left[\frac{X_0^+(t)}{\pi i}\int_{L^+}\frac{(\Psi_0+D_0)}{X_0^+(\tau)(\tau-t)}\,d\tau-\omega\sqrt{BC-A^2}\,(\Psi_0(t)+D_0)\right], \quad (29.9)$$

provided the condition of solvability

$$\int_{L^+}\frac{\Psi_0(\tau)\,d\tau}{X_0^+(\tau)}+D_0\int_{L^+}\frac{d\tau}{X_0^+(\tau)}=0 \quad (29.10)$$

is satisfied, where X_0^+ is the boundary value of the function $X_0(z)$ on L^+ from the left.

Taking into account that $\int_{L^+}\frac{d\tau}{X_0^+(\tau)}=-\pi i\big(1+\omega\sqrt{BC-A^2}\big)$, from (29.10) we find that

$$D_0=\frac{1}{\pi i(1+\omega\sqrt{BC-A^2})}\int_{L^+}\frac{\Psi_0(\tau)\,d\tau}{X_0^+(\tau)}.$$

Thus we have proved the solvability of the equation (29.8) in the class of functions which are bounded at the ends.

Having found the function h, we can define the vector g on L^+ as $g=(g_1,g_2)$, where $g_1=\frac{X\overline{h}-\overline{X}h}{X-\overline{X}}$, $g_2=\frac{h-\overline{h}}{X-\overline{X}}$.

Using the results obtained by Muskhelishvili [39] and Mel'nik [37], the vector g is proved to belong to the Hölder class on the entire contour of the ellipse.

Thus the mixed boundary value problem is solved. Obviously, the general case where the ellipse is divided into several parts, can be solved in a similar way.

Note finally that an attempt of solving the problem under consideration has been made in [36] by Manjavidze. But he neither showed the normality of the obtained singular integral equation, nor established the character of the end points of L^+.

The above-cited solution is given in the work of Basheleishvili and Zazashvili [17]. We have presented it with small modifications.

30. Solution of Fundamental Boundary Value Problems for an Infinite Strip

In this section we will show that the potentials constructed in Chapter I can be successfully used for the effective solution of fundamental boundary value problems for an infinite anisotropic strip.

The fundamental boundary value problem for an infinite anisotropic strip $(0 < x_2 < h, -\infty < x_1 < \infty)$ is formulated as follows: find a regular solution of basic equations of statics of an anisotropic elastic body (see (1.9)) in the domain $0 < x_2 < h$, when on the boundary of the domain one of the following conditions is given:

I. $u|_{x_2=0} = f(x_1), \quad u|_{x_2=h} = F(x_1),$

or

II. $Tu|_{x_2=0} = f(x_1), \quad Tu|_{x_2=h} = F(x_1),$

or

III. $u|_{x_2=0} = f(x_1), \quad Tu|_{x_2=h} = F(x_1),$

where u is the displacement vector, Tu is the stress vector, f and F are the given vector functions satisfying certain smoothness conditions and also the conditions at infinity.

Solution of Problem I. A solution of the first boundary value problem is sought in the form

$$\begin{aligned} u(x) = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{k=1}^{\infty} \Bigg[\begin{Vmatrix} E_k, & F_k \\ G_k, & H_k \end{Vmatrix} e^{ip\alpha_k x_2} X(p) + \\ & + \begin{Vmatrix} \overline{E}_k, & \overline{F}_k \\ \overline{G}_k, & \overline{H}_k \end{Vmatrix} e^{-ip\overline{\alpha}_k (h-x_2)} Y(p) \Bigg] e^{ipx_1} dp, \end{aligned} \tag{30.1}$$

where $X(p)$ and $Y(p)$ are unknown vector functions, E_k, F_k, G_k, H_k are defined by (5.37) and $\alpha_k = a_k + ib_k \operatorname{sgn} p$, $\overline{\alpha}_k = a_k - ib_k \operatorname{sgn} p$, $b_k > 0$, $k = 1, 2$, are the roots of the characteristic equation of fourth degree (see (3.1)).

Since $ip\alpha_k = (ipa_k - |p|b_k)$ and $-ip\overline{\alpha}_k = -ipa_k - |p|b_k$, the functions $e^{ip\alpha_k x_2}$ and $e^{-ip\overline{\alpha}_k (h-x_2)}$ tend to zero as $p \to \pm\infty$.

Taking into account the conditions of Problem I, we have

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big[X(p) + \overline{Q} Y(p)\big] e^{ipx_1}\, dp = f(x_1), \\ & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \big[Q X(p) + Y(p)\big] e^{ipx_1}\, dp = F(x_1), \end{aligned} \tag{30.2}$$

where

$$Q=\sum_{k=1}^{2}\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}e^{ip\alpha_k h},\quad \overline{Q}=\sum_{k=1}^{2}\begin{Vmatrix}\overline{E}_k, & \overline{F}_k\\ \overline{G}_k, & \overline{H}_k\end{Vmatrix}e^{-ip\overline{\alpha}_k h}. \tag{30.3}$$

For determining the unknown vector functions $X(p)$ and $Y(p)$ from (30.2), we obtain the following equations:

$$\begin{aligned}X(p)+\overline{Q}Y(p)=\widehat{f}(p)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-pi\xi}f(\xi)\,d\xi,\\ QX(p)+Y(p)=\widehat{F}(p)\equiv\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-pi\xi}F(\xi)\,d\xi.\end{aligned}\tag{30.4}$$

Hence

$$(E-\overline{Q}Q)X(p)=\widehat{f}(p)-\overline{Q}\widehat{F}(p),\quad (E-Q\overline{Q})Y(p)=-Q\widehat{f}(p)+\widehat{F}(p), \tag{30.5}$$

where E is the unit matrix.

Taking into account the calculations performed in Section 24, we have

$$\begin{aligned}D(p)&=\det(E-\overline{Q}Q)=\det(E-Q\overline{Q})=2e^{-|p|(b_1+b_2)h}\Big\{\operatorname{ch}p(b_1+b_2)h-\\ &-\operatorname{ch}p(b_1-b_2)h-\frac{4b_1b_2}{|\alpha_1-\alpha_2|^2}\left[\frac{1+\omega^2(BC-A^2)-\frac{\omega}{b_1b_2}}{1-\omega^2(BC-A^2)}\right]^2\times\\ &\times\big[\operatorname{ch}p(b_1-b_2)h-\cos p(a_1-a_2)h\big]\Big\}.\end{aligned}\tag{30.6}$$

From (30.6), after passing to limit, for an isotropic elastic body we easily obtain $D(p)=4\varkappa^{-2}e^{-2|p|h}[\varkappa^2\operatorname{sh}^2ph-p^2h^2]$, where $\varkappa=\frac{\lambda+3\mu}{\lambda+\mu}$ is Muskhelishvili's constant. The expression in square brackets coincides with the determinant for an infinite strip in the case of isotropic elastic body (see [41]).

It follows from (30.6) that $D(p)$ at the point $p=0$ has a zero of second order, and moreover, $\lim\limits_{|p|\to\infty}D(p)=1$.

Introduce the notation

$$\mathcal{P}=\sum_{k=1}^{2}\begin{Vmatrix}H_k, & -F_k\\ -G_k, & E_k\end{Vmatrix}e^{ip\alpha_k h},\quad \overline{\mathcal{P}}=\sum_{k=1}^{2}\begin{Vmatrix}\overline{H}_k, & -\overline{F}_k\\ -\overline{G}_k, & \overline{E}_k\end{Vmatrix}e^{-ip\overline{\alpha}_k h}. \tag{30.7}$$

Taking into consideration (30.3) and (30.7), after cumbersome calculations we find

$$Q\mathcal{P}=\mathcal{P}Q=Ee^{ip(\alpha_1+\alpha_2)h},\quad \mathcal{P}+Q=E(e^{ip\alpha_1h}+e^{ip\alpha_2h}), \tag{30.8}$$

$$\begin{aligned}Q\overline{Q}+\overline{\mathcal{P}}\mathcal{P}=\overline{Q}Q+\mathcal{P}\overline{\mathcal{P}}=2Ee^{-|p|(b_1+b_2)h}\Big\{\operatorname{ch}p(b_1-b_2)h+\frac{4b_1b_2}{|\alpha_1-\alpha_2|^2}\times\\ \times\left[\frac{1+\omega^2(BC-A^2)-\frac{\omega}{b_1b_2}}{1-\omega^2(BC-A^2)}\right]^2\big[\operatorname{ch}p(b_1-b_2)h-\cos p(a_1-a_2)h\big]\Big\}.\end{aligned}\tag{30.9}$$

On the basis of (30.6), (30.8), and (30.9), we obtain

$$(E-\mathcal{P}\overline{\mathcal{P}})(E-\overline{Q}Q)=(E-\overline{\mathcal{P}}\mathcal{P})(E-Q\overline{Q})=ED(p). \qquad (30.10)$$

By the last formula, (30.5) yields

$$X(p)=\frac{1}{D(p)}\,(E-\mathcal{P}\overline{\mathcal{P}})(\widehat{f}-\overline{Q}\widehat{F}),\quad Y(p)=\frac{1}{D(p)}\,(E-\overline{\mathcal{P}}\mathcal{P})(-Q\widehat{f}+\widehat{F}).$$

If we substitute these formulas in (30.1), then after certain transformations the solution of Problem I will take the form

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Bigg\{\sum_{k=1}^{2}\left[\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}e^{-ip\alpha_k(h-x_2)}-\begin{Vmatrix}\overline{E}_k, & \overline{F}_k\\ \overline{G}_k, & \overline{H}_k\end{Vmatrix}e^{-ip\overline{\alpha}_k(h-x_2)}\right]\times$$

$$\times\big[Q-e^{ip(\alpha_1+\alpha_2)h}\overline{\mathcal{P}}\big]\,\widehat{f}(p)+\sum_{k=1}^{2}\left[\begin{Vmatrix}\overline{E}_k, & \overline{F}_k\\ \overline{G}_k, & \overline{H}_k\end{Vmatrix}e^{ip\overline{\alpha}_kx_2}-\begin{Vmatrix}E_k, & F_k\\ G_k, & H_k\end{Vmatrix}e^{ip\alpha_kx_2}\right]\times$$

$$\times\big[\overline{Q}-e^{-ip(\overline{\alpha}_1+\overline{\alpha}_2)h}\mathcal{P}\big]\,\widehat{F}(p)\Bigg\}\frac{e^{ipx_1}}{D(p)}\,dp. \qquad (30.11)$$

Using the above arguments, we can easily notice that the integrand in the last formula is bounded as $p\to 0$. Moreover, the formula (30.11) results in a regular solution of the problem, provided $\widehat{f}$, $p\cdot\widehat{f}u$ and $\widehat{F}$, $p\cdot\widehat{F}$ are absolutely integrable and vanishing at infinity vector functions.

The stress vector can be calculated from (30.11).

Solution of Problem II. A solution of the problem under consideration is sought in the domain $0<x_2<h$, $|x_1|<\infty$ in the form

$$u(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left[\begin{Vmatrix}A'_k, & B'_k\\ C'_k, & D'_k\end{Vmatrix}e^{ip\alpha_kx_2}X(p)+\right.$$

$$\left.+\begin{Vmatrix}\overline{A}'_k, & \overline{B}'_k\\ \overline{C}'_k, & \overline{D}'_k\end{Vmatrix}e^{-ip\overline{\alpha}_k(h-x_2)}Y(p)\right]\frac{e^{ipx_1}}{ip}\,dp, \qquad (30.12)$$

where $X(p)$ and $Y(p)$ are the unknown vector functions, α_k and $\overline{\alpha}_k$ are defined above, and

$$\begin{Vmatrix}A'_k, & B'_k\\ C'_k, & D'_k\end{Vmatrix}=\left[\frac{ia_{11}}{BC-A^2}\begin{Vmatrix}C, & A\\ A, & B\end{Vmatrix}\operatorname{sgn}p+\omega a_{11}\begin{Vmatrix}0, & -1\\ 1, & 0\end{Vmatrix}\right]\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}, \qquad (30.13)$$

$$p_k=-\alpha_kr_k,\ \ q_k=-\alpha_ks_k,\ \ r_k=\frac{(-1)^k}{\alpha_1-\alpha_2},\ \ s_k=\frac{(-1)^k\alpha_1\alpha_2}{\alpha_k(\alpha_1-\alpha_2)}. \qquad (30.14)$$

After elementary calculations, from (30.12) we obtain for the stress vector

$$Tu(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{k=1}^{2}\left[\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}e^{ip\alpha_kx_2}X(p)+\right.$$

$$\left.+\begin{Vmatrix}\overline{p}_k, & \overline{q}_k\\ \overline{r}_k, & \overline{s}_k\end{Vmatrix}e^{-ip\overline{\alpha}_k(h-x_2)}Y(p)\right](n_1\alpha_k-n_2)e^{ipx_1}dp. \qquad (30.15)$$

In what follows, when the point x lies on the boundary, i.e., if $x_2 = 0$ or $x_2 = h$, in the expression for the stress vector we will always suppose that $n_1 = 0$ and $n_2 = -1$, since the direction of the vector $n = (n_1, n_2)$ is opposite to that of the axis $0x_2$. Taking this remark and the boundary conditions of Problem II into account, from (30.15) we find

$$\begin{aligned} Tu|_{x_2=0} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[X(p) + \overline{Q}Y(p)\right] e^{ipx_1} dp = f(x_1), \\ Tu|_{x_2=h} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[QX(p) + Y(p)\right] e^{ipx_1} dp = F(x_1), \end{aligned} \tag{30.16}$$

where we have introduced the notation

$$Q = \sum_{k=1}^{2} \begin{Vmatrix} p_k, & q_k \\ r_k, & s_k \end{Vmatrix} e^{ip\alpha_k h}, \quad \overline{Q} = \sum_{k=1}^{2} \begin{Vmatrix} \overline{p}_k, & \overline{q}_k \\ \overline{r}_k, & \overline{s}_k \end{Vmatrix} e^{-ip\overline{\alpha}_k h}. \tag{30.17}$$

Below we will also need the matrices

$$\mathcal{P} = \sum_{k=1}^{2} \begin{Vmatrix} s_k, & -q_k \\ -r_k, & p_k \end{Vmatrix} e^{ip\alpha_k h}, \quad \overline{\mathcal{P}} = \sum_{k=1}^{2} \begin{Vmatrix} \overline{s}_k, & -\overline{q}_k \\ -\overline{r}_k, & \overline{p}_k \end{Vmatrix} e^{-ip\overline{\alpha}_k h}, \tag{30.18}$$

For determining vectors $X(p)$ and $Y(p)$, from (30.16) we obtain the equations

$$\begin{aligned} X(p) + \overline{Q}Y(p) &= \widehat{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi) e^{-ip\xi} d\xi, \\ QX(p) + Y(p) &= \widehat{F}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\xi) e^{-ip\xi} d\xi, \end{aligned}$$

whence, as above, we have

$$(E - \overline{Q}Q)X(p) = \widehat{f}(p) - \overline{Q}\widehat{F}(p), \quad (E - Q\overline{Q})Y(p) = \widehat{F}(p) - Q\widehat{f}(p). \tag{30.19}$$

Performing elementary calculations, we prove that

$$\begin{aligned} \Delta(p) = \det(E - \overline{Q}Q) = \det(E - Q\overline{Q}) = 2e^{-|p|(b_1+b_2)h} \big\{ \operatorname{ch} p(b_1 + b_2)h - \\ - \operatorname{ch} p(b_1 - b_1)h - \tfrac{4b_1b_2}{|\alpha_1 - \alpha_2|^2} \big[\operatorname{ch} p(b_1 - b_2)h - \cos p(a_1 - a_2)h \big] \big\}. \end{aligned} \tag{30.20}$$

From the latter expression, after passing to limit, for an isotropic elastic body we obtain $\Delta(p) = 4e^{-2|p|h}[\operatorname{sh}^2 ph - p^2h^2]$.

The expression in the square brackets coincides again with the determinant for an isotropic elastic body (see [41]).

Using (30.20), we can easily show that in the vicinity of the point $p = 0$ the function $\Delta(p)$ has zero of fourth order, and $\lim_{|p|\to\infty} \Delta(p) = 1$.

From (30.17), (30.18), and (30.20) we have

$$\begin{aligned} Q\mathcal{P} = \mathcal{P}Q = Ee^{ip(\alpha_1+\alpha_2)h}, \quad \mathcal{P} + Q = E\big(e^{ip\alpha_1 h} + e^{ip\alpha_2 h}\big), \\ Q\overline{Q} + \overline{\mathcal{P}}\mathcal{P} = \overline{Q}Q + \mathcal{P}\overline{\mathcal{P}} = 2Ee^{-|p|(b_1+b_2)h}\Big\{ \operatorname{ch} p(b_1 - b_2)h + \end{aligned}$$

$$+\frac{4b_1b_2}{|\alpha_1-\alpha_2|^2}\big[\operatorname{ch} p(b_1-b_2)h-\cos p(a_1-a_2)h\big]\Big\},$$

$$(E-\mathcal{P}\overline{\mathcal{P}})(E-\overline{Q}Q)=(E-\overline{\mathcal{P}}\mathcal{P})(E-Q\overline{Q})=E\Delta(p).$$

Taking into account the last formulas, (30.19) yields

$$X(p)=\tfrac{1}{\Delta(p)}\,(E-\mathcal{P}\overline{\mathcal{P}})(\widehat{f}-\overline{Q}\widehat{F}),\quad Y(p)=\tfrac{1}{\Delta(p)}\,(E-\overline{\mathcal{P}}\mathcal{P})(-Q\widehat{f}+\widehat{F}).$$

Substituting these formulas in (30.15), after certain transformations for the stress vector we obtain

$$Tu(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Big\{\sum_{k=1}^{2}\left[\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}e^{-ip\alpha_k(h-x_2)}-\begin{Vmatrix}\overline{p}_k, & \overline{q}_k\\ \overline{r}_k, & \overline{s}_k\end{Vmatrix}e^{-ip\overline{\alpha}_k(h-x_2)}\right]\times$$

$$\times\big[Q-e^{ip(\alpha_1+\alpha_2)h}\overline{\mathcal{P}}\big]\,\widehat{f}(p)+\sum_{k=1}^{2}\left[\begin{Vmatrix}\overline{p}_k, & \overline{q}_k\\ \overline{r}_k, & \overline{s}_k\end{Vmatrix}e^{ip\overline{\alpha}_kx_2}-\begin{Vmatrix}p_k, & q_k\\ r_k, & s_k\end{Vmatrix}e^{ip\alpha_kx_2}\right]\times$$

$$\times\big[\overline{Q}-e^{-ip(\overline{\alpha}_1+\overline{\alpha}_2)h}\mathcal{P}\big]\,\widehat{F}(p)\Big\}(n_1\alpha_k-n_2)\,\frac{e^{ipx_1}}{\Delta(p)}\,dp. \tag{30.21}$$

Since $\widehat{f}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ip\xi}f(\xi)\,d\xi$, $\widehat{F}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ip\xi}F(\xi)\,d\xi$, and the conditions of vanishing of the principal vector and of the principal moment of external forces $\int_{-\infty}^{\infty}f\,d\xi=\int_{-\infty}^{\infty}F\,d\xi$, $\int_{-\infty}^{\infty}\xi(F_2-f_2),d\xi=h\int_{-\infty}^{\infty}F_1\,d\xi$ will take the form

$$\widehat{f}(0)=\widehat{F}(0),\quad \widehat{f}_2'(0)-\widehat{F}_2'(0)=ih\widehat{F}_1(0). \tag{30.22}$$

Taking into account the fact that $\Delta(p)$ at the point $p=0$ is zero of the fourth order, on the basis of the equalities (30.22) we conclude that the integrand in (30.21) remains bounded at the point $p=0$.

To obtain an expression for the displacement vector we have to substitute $X(p)$ and $Y(p)$ in (30.12). But in this case the integrand possesses a singularity at the point $p=0$. To get rid of the singularity, it is necessary (besides conditions (30.22)) to assume that

$$u(0)=0,\quad \Big(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\Big)_{x=0}=0. \tag{30.23}$$

Now the integrand in (30.12) is bounded at the point $p=0$, and the displacement vector fails to be the Fourier integral.

Thus for the solvability of Problem II, it is necessary that the principal vector and the principal moment of external forces be equal to zero. Moreover, it is also necessary that the boundary values be absolutely integrable vectors. To ensure the boundedness of the displacement vector, one has to require additionally the fulfilment of condition (30.23).

Problem III can be solved analogously (see [15]).

The solution presented above is due to the author [15]. Note that this problem has not been solved by the other methods.

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