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ON NEW NECESSARY CONDITION OF OPTIMALITY OF THE INITIAL MOMENT IN CONTROL PROBLEMS WITH DELAY

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Necessary conditions of optimality are obtained for optimal control problems with non fixed initial moment. The condition at the optimal initial moment, unlike the early known condition [1], contains a new term.

Let $O \subset R^n$, $G \subset R^r$ be open sets, $J = [a, b]$ be a finite interval and let the function $f : J \times O^2 \times G^2 \to R^n$ satisfies the following conditions:

1) for a fixed $t \in J$ the function $f(t, x_1, x_2, u_1, u_2)$ is continuous with respect to $(x_1, x_2, u_1, u_2) \in O^2 \times G^2$ and continuously differentiable with respect to $(x_1, x_2) \in O^2$;

2) for a fixed $(x_1, x_2, u_1, u_2) \in O^2 \times G^2$ the functions f, f_{x_i}, $i = 1, 2$ are measurable with respect to t; for arbitrary compacts $K \subset O$, $V \subset G$ there exists a function $m_{K,V}(\cdot) \in L_1(J, R_0^+)$, $R_0^+ = [0, \infty)$, such that

$$|f(t, x_1, x_2, u_1, u_2)| + \sum_{i=1}^{2} |f_{x_i}(\cdot)| \le m_{K,V}(t), \quad \forall (t, x_1, x_2, u_1, u_2) \in J \times K^2 \times V^2.$$

The following asumptions will be made: $\tau : R^1 \to R^1$ and $\theta : R^1 \to R^1$ are absolutely continuous functions satisfying $\tau(t) \le t$, $\dot\tau(t) > 0$, $\theta(t) < t$, $\dot\theta(t) > 0$ for $t \in R^1$; Δ is the set of piecewise continuous functions $\varphi : [\tau(a), b] \to N$ with a finite number of discontinuity points with $N \subset O$ convex bounded set, $\|\varphi\| = \sup\{|\varphi| : t \in J_1\}$; Ω is the set of measurable functions, $u : [\theta(a), b] \to U$ is such that $cl\{u(t) : t \in [\theta(a), b]\}$ is a compact lying in G, $U \subset G$ is an arbitrary set; $q^i : J^2 \times O^2 \to R^n$, $i = 0, \dots, l$, are continuously differentiable functions; $\gamma(t)$ is the function inverse to $\tau(t)$.

To every element $\sigma = (t_0, t_1, x_0, \varphi, u) \in A = J \times O \times \Delta \times \Omega$, $t_0 < t_1$ there corresponds the differential equation

$$\dot x(t) = f(t, x(t), x(\tau(t)), u(t), u(\theta(t))), \quad t \in [t_0, t_1], \tag{1}$$

with the initial condition

$$x(t) = \varphi(t), \quad t \in [\tau(t_0), t_0), \qquad x(t_0) = x_0. \tag{2}$$

Definition 1. The function $x(t) = x(t, \sigma) \in O$, $t \in [\tau(t_0), t_1]$, $t_0 \in [a, t_1)$, is said to be a solution corresponding to the element $\sigma \in A$ if on $[\tau(t_0), t_0]$ it satisfies the condition (2), while on the interval $[t_0, t_1]$ is absolutely continuous and almost everywhere satisfies the equation (1).

Definition 2. The element $\sigma \in A$ is said to be admissible if the corresponding solution $x(t)$ satisfies $q^i(t_0, t_1, x_0, x(t_1)) = 0$, $i = 1, \dots, l2$.

The set of admissible elements will denoted by A_0.

Definition 3. The element $\sigma = (\tilde t_0, \tilde t_1, \tilde x_0, \tilde\varphi, \tilde u) \in A_0$ is said to be locally optimal if there exist a number $\delta > 0$ and a compact set $K \subset O$ such that for an arbitrary element

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$\sigma \in A_0$ satisfying $|\tilde t_0 - t_0| + |\tilde t_1 - t_1| + |\tilde x_0 - x_0| + \|\tilde\varphi - \varphi\| + \|\tilde f - f\|_K \le \delta$ the inequality $q^0(\tilde t_0, \tilde t_1, \tilde x_0, \tilde x(\tilde t_1)) \le q^0(t_0, t_1, x_0, x(t_1))$ holds.

Here

$$\|\tilde f - f\|_K = \int_J H(t; f, K)dt,$$

$$H(t; f, K) = \sup\Big\{|\tilde f(t, x_1, x_2) - f(t, x_1, x_2)| + \sum_{i=1}^{2} |\tilde f_{x_i}(\cdot) - f_{x_i}(\cdot)| : (x_1, x_2) \in K^2\Big\};$$

$\tilde f(t, x_1, x_2) = f(t, x_1, x_2, \tilde u(t), \tilde u(\theta(t)))$, $f(t, x_1, x_2) = f(t, x_1, x_2, u(t), u(\theta(t)))$, $\tilde x(t) = x(t, \tilde\sigma)$.

The problem of optimal control consists in finding a locally optimal element.

Theorem 1. *Let $\tilde\sigma \in A_0$, $\tilde t_0 \in (a, b)$, $\tilde t_1 \in (a, b]$, $\gamma_0 = \gamma(\tilde t_0) \in [\tilde t_0, \tilde t_1)$ be a locally optimal element and there exist the finite limits*

$$\lim_{\omega\to\omega_0^-} \tilde f(\omega) = f_0^-,\ \omega \in R_{\tilde t_0}^- \times O^2,\ R_{\tilde t_0}^- = (-\infty, \tilde t_0],\ \omega_0^- = (\tilde t_0, \tilde x_0, \tilde\varphi(\tau(\tilde t_0^-))),$$

$$\lim_{(\omega_1,\omega_2)\to(\omega_1^0,\omega_2^-)} [\tilde f(\omega_1) - \tilde f(\omega_2)] = f_1^-,\ \omega_i \in R_{\tilde\gamma_0}^- \times O^2,\ i = 1, 2,\ \omega_1^0 = (\gamma_0, \tilde x(\gamma_0), \tilde x_0),$$

$$\omega_2^- = (\gamma_0, \tilde x(\gamma_0), \tilde\varphi(\tilde t_0^-)),\quad \lim_{t\to\tilde t_0^-} \dot\gamma(t) = \dot\gamma^-,$$

$$\lim_{\omega\to\omega_3^-} \tilde f(\omega) = f_2^-,\ \omega \in R_{\tilde t_1}^- \times O^2,\ \omega_3^- = (\tilde t_1, \tilde x(\tilde t_1), \tilde x(\tau(\tilde t_1^-))). \tag{3}$$

Then there exist a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$ and a solution $\psi(t)$, $t \in [\tilde t_0, \gamma(\tilde t_1)]$ of the adjoint equation

$$\dot\psi(t) = -\psi(t)\tilde f_{x_1}[t] - \psi(\gamma(t))\tilde f_{x_2}[\gamma(t)]\dot\gamma(t),\ \ t \in [\tilde t_0, \tilde t_1],\quad \psi(t) = 0,\ \ t \in (\tilde t_1, \gamma(\tilde t_1)], \tag{4}$$

such that the following conditions are fulfilled:

$$\int_{\tau(\tilde t_0)}^{\tilde t_0} \psi(\gamma(t))\tilde f_{x_2}[\gamma(t)]\dot\gamma(t)\tilde\varphi(t)dt \ge \int_{\tau(\tilde t_0)}^{\tilde t_0} \psi(\gamma(t))\tilde f_{x_2}[\gamma(t)]\dot\gamma(t)\varphi(t)dt,\quad \forall\varphi \in \Delta, \tag{5}$$

$$\int_{\tilde t_0}^{\tilde t_1} \psi(t)\tilde f[t]dt \ge \int_{\tilde t_0}^{\tilde t_1} \psi(t) f(t, \tilde x(t), \tilde x(\tau(t)), u(t), u(\theta(t)))dt,\quad \forall u \in \Omega; \tag{6}$$

$$\pi\tilde Q_{x_0} = -\psi(\tilde t_0),\quad \pi\tilde Q_{x_1} = \psi(\tilde t_1), \tag{7}$$

$$\pi\tilde Q_{t_0} \ge \psi(\tilde t_0) f_0^- + \psi(\gamma_0) f_1^- \gamma_0^-,\quad \pi\tilde Q_{t_1} \ge -\psi(\tilde t_1) f_2^-. \tag{8}$$

Here $Q = (q^0, \dots, q^l)$, the tilde over Q means that there corresponding gradient is calculated at the point $(\tilde t_0, \tilde t_1, \tilde x_0, \tilde x(\tilde t_1))$; $\tilde f_{x_i}[t] = \tilde f_{x_i}(t, \tilde x(t), \tilde x(\tau(t)))$, $\tilde f[t] = \tilde f(t, \tilde x(t), \tilde x(\tau(t)))$.

If rank $\big(\tilde Q_{x_0}, \tilde Q_{x_1}\big) = 1 + l$, then in the theorem 1 $\psi(t) \not\equiv 0$.

Theorem 2. *Let $\tilde\sigma \in A_0$, $\tilde t_0 \in [a, b)$, $\tilde t_1 \in (a, b)$, $\gamma_0 \in [\tilde t_0, \tilde t_1)$ be a locally optimal element and there exist the finite limits*

$$\lim_{\omega\to\omega_0^+} \tilde f(\omega) = f_0^+,\ \omega \in R_{\tilde t_0}^+ \times O^2,\ \omega_0^+ = (\tilde t_0, \tilde x_0, \tilde\varphi(\tau(\tilde t_0^+))),$$

$$\lim_{(\omega_1,\omega_2)\to(\omega_1^0,\omega_2^+)} [\tilde f(\omega_1) - \tilde f(\omega_2)] = f_1^+,\ \omega_i \in R_{\tilde\gamma_0}^+ \times O^2,\ i = 1, 2,\ \omega_2^+ = (\gamma_0, \tilde x(\gamma_0), \tilde\varphi(\tilde t_0^+)),$$

$$\lim_{t\to\tilde t_0^+} \dot\gamma(t) = \dot\gamma^+,$$

$$\lim_{\omega\to\omega_3^+} \tilde f(\omega) = f_2^+,\ \omega \in R_{\tilde t_1}^+ \times O^2,\ \omega_3^+ = (\tilde t_1, \tilde x(\tilde t_1), \tilde x(\tau(\tilde t_1^+))). \tag{9}$$

Then there exist a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$ and a solution $\psi(t)$, $t \in [\tilde t_0, \gamma(\tilde t_1)]$ of the equation (4) such that the conditions (5)–(7) *are fulfilled. Moreover,*

$$\pi\tilde Q_{t_0} \le \psi(\tilde t_0) f_0^+ + \psi(\gamma_0) f_1^+ \gamma_0^+,\quad \pi\tilde Q_{t_1} \le -\psi(\tilde t_1) f_2^+. \tag{10}$$

Theorem 3. *Let $\tilde{\sigma} \in A_0$, $\tilde{t}_i \in (a,b)$, $i = 0,1$, $\gamma_0 \in (\tilde{t}_0, \tilde{t}_1)$ be a locally optimal element and the assumptions of Theorems* 1, 2 *are fulfilled. Let, besides,*

$$f_0^+ = f_0^- = f_0, \quad f_1^+\dot{\gamma}^+ = f_1^-\dot{\gamma}^- = f_1, \quad f_2^+ = f_2^- = f_2. \tag{11}$$

Then there exist a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$ and a solution $\psi(t)$, $t \in [\tilde{t}_0, \gamma(\tilde{t}_1)]$ of the equation (4) *such that the conditions* (5)–(7) *hold. Moreover,*

$$\pi \tilde{Q}_{t_0} = \psi(\tilde{t}_0) f_0 + \psi(\gamma_0) f_1, \quad \pi \tilde{Q}_{t_1} = -\psi(\tilde{t}_1) f_2. \tag{12}$$

If rank $\left(\tilde{Q}_{t_0}, \tilde{Q}_{t_1}, \tilde{Q}_{x_0}, \tilde{Q}_{x_1}\right) = 1 + l$, *then in Theorem* 3 $\psi(t) \not\equiv 0$.

Remark 1. Assume that the function $(\dot{\gamma}(t), \tilde{\varphi}(t), \tilde{\varphi}(\tau(t)))$ is continuous at point $\tilde{t}_0$ and the function $\tilde{f}(t, x_1, x_2)$ is continuous at the points $(\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0)))$, $(\gamma_0, \tilde{x}(\gamma_0), \tilde{x}_0)$, $(\gamma_0, \tilde{x}(\gamma_0), \tilde{\varphi}(t_0))$, $(\tilde{t}_1, \tilde{x}(\tilde{t}_1), \tilde{x}(\tau(\tilde{t}_1)))$, then, it is clear that in Theorem 3

$$f_0 = \tilde{f}(\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tau(\tilde{t}_0))), \; f_1 = [\tilde{f}(\gamma_0, \tilde{x}(\gamma_0), \tilde{x}_0) - \tilde{f}(\gamma_0, \tilde{x}(\gamma_0), \tilde{\varphi}(t_0))]\dot{\gamma}(\tilde{t}_0),$$
$$f_2 = \tilde{f}(\tilde{t}_1, \tilde{x}(\tilde{t}_1), \tilde{x}(\tau(\tilde{t}_1))).$$

Theorem 4. *Let $\tilde{\sigma} \in A_0$, $\tilde{t}_i \in (a,b)$, $i = 0,1$, $\gamma_0 = \tilde{t}_0$ be a locally optimal element and the conditions of theorems* 1, 2 *and* (11) *are fulfilled. Moreover $f_0^- + f_1^-\dot{\gamma}^- = f_0^+ + f_1^+\dot{\gamma}^+ = \hat{f}_1$. Then there exist a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$ and a solution $\psi(t)$, $t \in [\tilde{t}_0, \gamma(\tilde{t}_1)]$ of the equation* (4) *such that the conditions* (5), (7), (12) *hold. Let, besides,*

$$\pi \tilde{Q}_{t_0} = \psi(\tilde{t}_0) \hat{f}_1.$$

Theorem 5. *Let $\tilde{\sigma} \in A_0$, $\tilde{t}_i \in (a,b)$, $i = 0,1$, $\gamma_0 = \tilde{t}_0$ be a locally optimal element, the condition* (11) *hold, the function $(\dot{\gamma}(t), \tilde{\varphi}(t))$ be continuous at the point $\tilde{t}_0$ and*

$$f_3^- = f_3^+ = f_3, \tag{13}$$

where

$$\lim_{\omega \to \omega_4} \tilde{f}(\omega) = f_3^-, \; \omega \in R_{\tilde{t}_0}^- \times O^2, \; \omega_4 = (\tilde{t}_0, \tilde{x}_0, \tilde{x}_0), \; \lim_{\omega \to \omega_4} \tilde{f}(\omega) = f_3^+, \; \omega \in R_{\tilde{t}_0}^+ \times O^2.$$

Let, besides, there exist a neighborhood $V(\omega_5)$ of the point $\omega_5 = (\tilde{t}_0, \tilde{x}_0, \tilde{\varphi}(\tilde{t}_0)$ such that the function $\tilde{f}(\omega)$, $\omega \in V(\omega_5)$ is bounded. Then there exist a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$ and a solution $\psi(t)$, $t \in [\tilde{t}_0, \gamma(\tilde{t}_1)]$ of the equation (4) *such that the conditions* (6), (7), (12) *hold. Moreover,*

$$\pi \tilde{Q}_{t_0} = \psi(\tilde{t}_0) f_3. \tag{14}$$

Theorem 6. *Let $\tilde{\sigma} \in A_0$, $\tilde{t}_i \in (a,b)$, $i = 0,1$, be a locally optimal element. The conditions* (11), (13) *hold. Let, besides, in some neighborhood of $\tilde{t}_0$ the function $\tau(t) = t$. Then there exist a non-zero vector $\pi = (\pi_0, \dots, \pi_l)$, $\pi_0 \le 0$ and a solution $\tilde{\psi}(t)$, $t \in [\tilde{t}_0, \gamma(\tilde{t}_1)]$ of the equation* (4) *such that the conditions* (6), (7), (12), (14) *hold.*

References

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