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ASYMPTOTIC REPRESENTATIONS FOR SOLUTIONS OF BISINGULAR PROBLEMS

Dedicated to the memory of my parents

Abstract. Boundary value problems for partial differential equations of elliptic, parabolic and mixed types with small parameters by higher order derivatives are considered. It is assumed that the solution of the corresponding degenerate equation has singular points and curves where this solution is non-smooth. Asymptotic representations of solutions of non-degenerate problems with respect to small parameters are constructed.

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Key words and Phrases. Partial differential equation, differential equation of elliptic type, differential equation of parabolic type, differential equation of mixed type, singular perturbation, bisingular boundary value problem, asymptotic representation.

რეზიუმე. განხილულია სასაზღვრო ამოცანები ელიფსური, პარაბოლური და შერეული ტიპის კერძოწარმოებულებიანი დიფერენციალური განტოლებისათვის მცირე პარამეტრებით მაღალი რიგის წარმოებულებთან. დაშვებულია, რომ შესაბამისი გადაგვარებული განტოლების ამონახსნს აქვს განსაკუთრებული წერტილები და წირები, სადაც ეს ამონახსნი არ არის გლუვი. აგებულია გადაუგვარებელ ამოცანათა ამონახსნების ასიმპტოტური წარმოდგენები მცირე პარამეტრების მიმართ.

Introduction

Many real processes connected with non-uniform transitions are described by means of differential equations involving large or small parameters which quantitatively characterize the influence on the process run of that factor which comes into account by the corresponding term of the differential equation. If, for example, one or another parameter is small, then we can naturally take it equal to zero and obtain thus a more simple problem. In that case solutions of the original problem with sufficiently close to zero values of the parameter can be expected to be close to a solution of the new problem, corresponding to the zero value of the parameter.

Let us consider on a set $\Omega_{\vec{t},\vec{\epsilon}} = \mathcal{T} \times \mathcal{E}$, where $\mathcal{T}$ is the domain of variation of independent variables and $\mathcal{E}$ is the set of values of one or several parameters, Problem A_ϵ, i.e., the problem of solving the differential equation

$$L_{\vec{\epsilon}} u_{\vec{\epsilon}} \equiv \left(L_{1,\vec{\epsilon}} + L_0\right) u_{\vec{\epsilon}} = h_{\vec{\epsilon}}(\vec{t}) \tag{0.1}$$

under the additional (boundary, initial, etc.) conditions

$$\mathcal{B}_{\vec{\epsilon}} u_{\vec{\epsilon}} = 0. \tag{0.2}$$

The point $\vec{\epsilon} = \vec{0}$ is assumed to be limiting for the set $\mathcal{E}$, $\vec{0} \notin \mathcal{E}$. Suppose that we have to investigate properties of the solution of Problem A_ϵ as $\vec{\epsilon} \to \vec{0}$.

Suppose that we are able to construct a formal asymptotic expansion of the solution in the form of the series

$$u(\vec{t}, \vec{\epsilon}) \sim \sum_{i=0}^{\infty} u_i(\vec{t}) \phi_i(\vec{\epsilon}), \tag{0.3}$$

where $\phi_i(\vec{\epsilon})$ are the elements of a chosen by us asymptotic sequence $\{\phi_s(\vec{\epsilon})\}$, $s = 0, 1, \ldots$, and $u_i(\vec{t})$ are the expansion coefficients; in other words, we can determine the functions $u_i(\vec{t})$ in such a way that for every partial sum $U_N(\vec{t}, \vec{\epsilon}) = \sum_{i=0}^{N} u_i(\vec{t}) \phi_i(\vec{\epsilon})$ the inequalities $\|L_{\vec{\epsilon}} U_N(\vec{t}, \vec{\epsilon}) - h_{\vec{\epsilon}}(\vec{t})\| = o(\tilde{\phi}_N(\vec{\epsilon}))$ hold, where($\{\tilde{\phi}_i(\vec{\epsilon})\}$ is an asymptotic sequence, not necessarily coinciding with the sequence $\{\phi_i(\vec{\epsilon})\}$). Suppose also that similar inequalities are fulfilled for the additional conditions.

Assume that for $\vec{\epsilon} = \vec{0}$ Problem $A_{\vec{\epsilon}}$ turns into Problem A_0, i.e., into the problem of solving the degenerate equation

$$L_0 u_0 = h_0(t) \tag{0.4}$$

under certain conditions

$$\mathcal{B}_0 u_0 = 0 \tag{0.5}$$

(which, as a rule, are a part of the conditions $\mathcal{B}_{\vec{\epsilon}} = \mathcal{B}_1 + \mathcal{B}_0$ of Problem $A_{\vec{\epsilon}}$). The problem $A_{\vec{\epsilon}}$ will be called non-degenerate and Problem A_0 will be said to be the degenerate problem corresponding to Problem $A_{\vec{\epsilon}}$.

If the series (0.1) represents an asymptotic expansion of the solution of Problem $A_{\vec{\epsilon}}$ uniformly with respect to $\vec{t} \in \mathcal{T}$, then they say that the solution depends regularly on the parameters. If, however, the asymptotic expansion (0.1) is valid not everywhere in $\overline{\mathcal{T}} = \mathcal{T} \cup \partial\mathcal{T}$, then such problems are called singularly perturbed problems. Not always one can by the type of an equation and additional conditions make a conclusion whether the perturbation is regular or singular.

Thus, the closeness of a small parameter to zero in singularly perturbed problems does not imply uniform closeness (in a norm) of solutions of the degenerate and non-degenerate problems in the whole domain of variation of independent variables. A formal criterion that the problem under consideration belongs to the class of singularly perturbed problems is, for example, the presence of small multipliers by higher derivatives of the equation; although the occurrence of such multipliers is not always an evidence of non-uniform transition from the solution of the non-degenerate problem to the solution of the degenerate problem as a small parameter tends to zero.

In constructing asymptotic expansions of solutions of singularly perturbed problems, the coefficients $u_i(\vec{t})$ of the formal asymptotic expansion (0.1) frequently happen to have themselves singularities at the points of the set $\Gamma \subset \mathcal{T} \cup \partial\mathcal{T}$ (quite often of lesser dimension than that of the set $\mathcal{T}$), and the order of singularity increases with the growth of the index i. In such singularly perturbed problems the solution, being a function of the parameter $\vec{\epsilon}$, has singularity at some point $\vec{\epsilon}_0$ of the set $\mathcal{E}$. At the same time, in the vicinity of the points of the set $\Gamma \subset \mathcal{T} \cup \partial\mathcal{T}$ the given solution, being a function of independent variables, possesses a specific behaviour which differs from the character of variation of that function at other points of the set $\mathcal{T}$. In this case the problem will be called bisingular or bisingularly perturbed problem; this term introduced by A.M. Il'in (see [31]) has proved to be highly suitable for characterization of situations arising in asymptotic analysis of solutions of differential equations.

In the sequel, partial sums $u(\vec{t}, \vec{\epsilon}) = \sum_{i=0}^{N} u_i(\vec{t})\phi_i(\vec{\epsilon})$ of asymptotic expansions will be called asymptotic representations of order N of the function $u(\vec{t}, \vec{\epsilon})$, and partial sums of formal asymptotic expansions will be called formal asymptotic representations.

In the study of properties of solutions of singularly perturbed problems the most important are the following questions: finding of conditions $\mathcal{B}_0$ for the degenerate problem; investigation of the conditions under which the solution of the non-degenerate problem tends (as the parameter $\vec{\epsilon}$ tends to zero) to the solution of the corresponding degenerate problem; possibilities of constructing an asymptotic (with respect to the parameter) expansion or asymptotic representation of a solution of the non-degenerate problem by elements of the chosen asymptotic sequence; error estimation of asymptotic representations of order N in a corresponding norm. In particular, one of the basic problems arising in the investigation of the character of variation of the solution of Problem A_ϵ (as the small parameter tends to zero) is to find conditions under which the solvability of Problem A_0 (in a space of functions) implies that of Problem A_ϵ (in a naturally corresponding space of functions; in [86], in the case of boundary value problems for ordinary differential equations such conditions are called conditions of regular degeneration of Problem A_ϵ into Problem A_0), to obtain a priori estimates for the solutions of Problems A_0 and A_ϵ and also the estimate for the closeness of solutions of the degenerate and non-degenerate problems.

The development of the asymptotic analysis of singularly perturbed differential equations and associated with them solutions of initial and boundary value problems is, of course, of great importance for more profound investigation of qualitative and quantitative characteristics of those processes and phenomena for which regularly and singularly perturbed differential equations turn out to be mathematical models, for constructing effective and stable numerical algorithms for solving the above-mentioned problem. It is, for example, known that the non-uniformities appearing in the problems of nonlinear optics [83] form a boundary layer zone. Non-uniform transitions take place in problems dealing with laminated media and composite materials (discontinuous and sharply varying coefficients, see [7], [44]), in nonlinear problems (interior transitions, see [4], [15], [84]), in problems for domains with non-smooth boundaries (see [31]), etc. The study of processes of heat transfer between a moving liquid and a heated rigid body placed in it, the description of the movement of a conductive liquid in an electromagnetic field and many other problems require the consideration of singularly perturbed equations of mixed type. Differential equations with small multipliers by derivatives, as mathematical models of the objects, processes and phenomena, arise naturally in automatic control, nonlinear oscillations, gas and magnetohydrodynamics, when describing processes taking place in physics, chemistry, biology, ecology and in some other sciences; similar equations appear in the analysis of difference schemes, upon construction of convergent numerical algorithms for solving stiff problems, and in many other problems of theoretical and applied character. Statements of various mathematical problems requiring investigation of the character of dependence on a parameter of solutions of differential equations with small maltipliers by higher derivatives can be found in [4], [6], [15], [16], [18], [19], [23]–[25], [27], [29]–[32], [35], [43], [44], [46], [51], [53], [80], [83], [84], etc. Therefore the theory of asymptotic analysis is, undoubtedly, of great significance both for the development of fundamental investigations and for the solution of concrete practical problems.

Systematic investigation of the asymptotic theory of singularly perturbed differential equations goes back to the works of A. N. Tikhonov [78]–[80]. After them we can mention the works due to M. I. Vishik and L. A. Lyusternik [86], A. B. Vasil'eva [81], A. B. Vasil'eva and V. F. Butuzov [82], S. A. Lomov [45], E. F. Mishchenko and N. Kh. Rozov [47], A. M. Il'in [31], [32] and also the works of their pupils and successors. Among the works of foreign scholars the most known are those of N. Levinson [43], P. Fife [23], [25], Fife and V. Greenly [24], S. Chang and F. Howes [16]; a more detailed bibliography can be found in [5], [7], [13], [14], [17], [21], [22], [28], [34]–[39], [43], [49], [50]–[52], [54], [55]. Different ways have been being elaborated for constructing asymptotic expansions, and in this connection there appear new terms such as "the method of boundary layer functions" (or the M. I. Vishik and L. A. Lyusternik method, or the A. B. Vasil'eva and V. F. Butuzov method), "the method of matching of asymptotic expansions" (more frequently connected with the name of A. M. Il'in when dealing with differential equations), "the regularization method" (or S. A. Lomov's method), "the method of a canonical operator" (or V. P. Maslov's method), "the averaging method", etc. However, even at present one cannot say that the general theory of constructing asymptotic expansions of singularly perturbed differential equations is completely developed. Many questions arising upon study of asymptotic behaviour of solutions of some concrete problems of applied character have neither theoretical ground nor even algorithms for investigation of properties of solutions as the parameters tend to their limiting (critical) values.

The construction of the theory of asymptotic expansions for solving singularly perturbed partial differential equations takes its origin in the works by M. I. Vishik and L.A. Lyusternik devoted to linear equations, when the coefficients of formal asymptotic expansions have no singularities (see, e.g., [86] and bibliography therein). Their method of constructing asymptotic expansions of solutions of singularly perturbed equations of boundary layer character of variation was subsequently used by many researchers and extended to nonlinear equations and to many problems for whose solutions the coefficients of formal asymptotic representations have singularities growing with the growth of the representation order. In particular, V. F. Butuzov has introduced angular boundary functions (see, e.g., [13]) and elaborated by the aid of those functions the techniques allowing one to construct asymptotic expansions for different types of singularly perturbed problems; for some bisingular problems (with angular characteristics) he suggested the method of smoothing for constructing asymptotic representations to within some order [14]. It should be noted that in many cases the error estimate of asymptotic representations of solutions of initial boundary value problems for singularly perturbed partial differential equations is performed with the help of "corrections" constructed to partial sums of formal asymptotic expansions.

The method of M. I. Vishik and L. A. Lyusternik of constructing asymptotic expansions is based on the assumption that a part of functions describing the behaviour of a solution in the neighbourhood of the set Γ and determining the character of variation of the solution in that neighbourhood, tends exponentially to zero as the small parameter tends to zero. However, this assumption in many bisingular problems is either invalid, or the behaviour of the solution in the neighbourhood of the set Γ is so complicated that it seems impossible to determine exponentially decreasing components of asymptotic expansion as solutions of sufficiently simple auxiliary problems. In such problems the most effective is the method of matching of asymptotic expansions. A valuable contribution to the development of the method has been made by A.M. Il'in [31] who, together with his pupils, justified the applicability of the method to many classes of problems connected with linear and quasi-linear ordinary and partial differential equations.

Describe briefly the sequence of operations we undertake in constructing asymptotic expansions to solve differential equations. Note that in specific problems the coefficients $u_i(\vec{t})$ may likewise depend on small parameters ϵ. To simplify the description, the problem will be assumed to involve one small positive parameter ϵ, and the solution of the problem under consideration to depend on two independent variables t_1 and t_2.

Let it be required to construct an asymptotic expansion (or an asymptotic representation) of the solution of the problem (0.1), (0.2). Along with this problem let us consider the corresponding degenerate problem (0.4), (0.5) (as is mentioned above, the correct statement of the degenerate problem requires additional, not always evident, investigations).

1. For constructing asymptotic expansions with respect to the small parameter one needs more exact a priori estimates of the solution, depending on the parameters of the equation. This need is connected firstly with the necessity to study the character of influence of each parameter involved in the initial equation and to determine the structure of the solution. Secondly, the coefficients of the resulting asymptotic expansions quite often have isolated singularities (points or lines) at which the continuity or differentiability of functions is violated, so to obtain error estimates one should know the character of violation for these functions as they approach the above-mentioned points or lines; in particular, the question how the solution behaves as the point approaches the plane of definition of initial data always arises in equations of parabolic type. Thirdly, when constructing asymptotic approximations, we always have, as a rule, to consider solutions of one or another equation in an unbounded domain. Therefore the character of variation of those estimates for infinitely increasing or decreasing arguments should be taken into account. Note that the ability of getting a priori estimates of a solution of the problem is a decisive factor in deducing error estimates of asymptotic expansions.

2. First of all we must determine an asymptotic sequence $\{\phi_i(\epsilon)\}$ which will be used in constructing the asymptotic expansion. Obviously, this sequence cannot be arbitrary. Indeed, the solution of the problem (0.1), (0.2) has a quite definite structure which depends on: (i) the kind of equation and additional (initial and boundary) conditions; (ii) the character of dependence of the operator on the small parameter; (iii) the type of the domain in which we seek for the solution; (iv) properties of the solution of the degenerate problem (0.4), (0.5). According to what has been said, partial sums of an asymptotic series must likewise possess an analogous structure. Thus, for example, if the solution of Problem A_ϵ is given in the form

$$u_\epsilon(t_1, t_2, \epsilon) = v(t_1, t_2, t_2/\epsilon, 1/\ln\epsilon),$$

where $v(t_1, t_2, \tau, \lambda)$ is an infinitely differentiable function of its arguments, then for the solution of Problem A_ϵ we can obtain the asymptotic expansion

$$u_\epsilon(t_1, t_2, \epsilon) \sim \sum_{n=0}^{\infty} u_n(t_1, t_2, t_2/\epsilon)(\ln \epsilon)^{-n} \tag{0.6}$$

as $\epsilon \to 0$.

Generally speaking, in constructing an asymptotic expansion of the function $u_\epsilon(t_1, t_2, \epsilon)$ one cannot take as asymptotic sequence the power sequence $\{\epsilon^n\}$ because the power series fails to describe the behavior of the function $1/\ln \epsilon$ as $\epsilon \to 0$.

3. Our next step is to determine the character of dependence on a small parameter and to investigate other properties of coefficients of the asymptotic expansion (for example, the small parameter involved in the coefficients of the expansion (0.6) is given by the combination t_2/ϵ). The properties of the coefficients, the character of their dependence on the small parameter as well as the type of the asymptotic sequence $\{\phi_i(\epsilon)\}$ can be "guessed" upon investigation of properties and specific features of the problem (0.1), (0.2), in deriving a priori estimates for the solution, by comparing the solutions of the problems (0.1), (0.2) and (0.4), (0.5), as well as from the well-known peculiarities of the physical process described by the mathematical model (0.1), (0.2). The necessary information can also be obtained upon considering more simple variants of the problem (0.1), (0.2).

4. Suppose that two foregoing steps are realized, i.e., the type of the asymptotic sequence $\{\phi_i(\epsilon)\}$ is determined and natural assumptions on the characteristic properties of the coefficients of the expansion are made. Substituting (0.3) in (0.1) and (0.2) and performing with regard for supposed properties of the coefficients $u_i(t_1, t_2, \epsilon)$ the needed transformations (such as the change of the coefficients, the initial and boundary functions by their Taylor expansions, introduction of new independent variables, and so on), the problem (0.1), (0.2) reduces to the form

$$\sum_{i=0}^{\infty} \left[L_i u_i(t_1, t_2, \epsilon)\right] \tilde{\phi}_i(\epsilon) \sim 0, \quad \sum_{i=0}^{\infty} \mathcal{B}_i u_i(t_1, t_2, \epsilon) \tilde{\phi}_i(\epsilon) \sim 0, \tag{0.7}$$

where $\tilde{\phi}_i(\epsilon)$ is, generally speaking, a new asymptotic sequence and L_i are some operators. As far as the initial data of the problem (the coefficients, the boundary and initial functions) may possibly be non-smooth, and the problem may have some other singularities (caused, for instance, by the nonlinearity of the problem), it is found possible to construct an asymptotic representation of some order rather than a complete asymptotic expansion of the solution; in other words, the equalities (0.7) in that case are replaced by the asymptotic equations

$$\sum_{i=0}^{N}\left[L_i u_i(t_1,t_2,\epsilon)\right]\tilde{\phi}_i(\epsilon)=o\big(\tilde{\phi}_N(\epsilon)\big),$$
$$\sum_{i=0}^{N}\left[B_i u_i(t_1,t_2,\epsilon)\right]\tilde{\phi}_i(\epsilon)=o\big(\tilde{\phi}_N(\epsilon)\big).$$

Since the relations (0.7) must be fulfilled for all sufficiently small values of the parameter, the equations (0.7) are equivalent to the family of equations

$$L_i u_i(t_1,t_2,\epsilon)=0,\quad \mathcal{B}_i u_i(t_1,t_2,\epsilon)=0,\quad i=0,1,\dots . \tag{0.8}$$

These equations represent mathematical writing of problems for determining of the asymptotic expansion coefficients $u_i(t_1,t_2,\epsilon)$.

It should be noted that in many cases the solution of the initial problem has essentially different asymptotic representations in different parts of its domain of definition, and therefore the above-mentioned procedure must necessarily be performed for each part separately.

5. The next step in constructing an asymptotic expansion of the solution of the problem (0.1), (0.2) is to solve the series of the problems (0.8).

If the sequence $\{\phi_i(\epsilon)\}$ and the basic properties of coefficients are defined correctly, then the problems (0.8), starting at least with some number, are, as a rule, of the same type and differ from each other only by the right-hand sides of the equations and by the boundary and initial functions.

Failure to carry out this criterion shows that there is an error in our previous constructions. Of course, every problem (0.8) must have more simple solution than the initial problem (0.1), (0.2), otherwise all our constructions will become senseless. Solutions of those problems must exist and be unique. Indeed, the coefficients of asymptotic expansion in the chosen asymptotic sequence $\{\phi_i(\epsilon)\}$ must, as is said above, be defined uniquely. Hence if a solution of at least one of the problems (0.8) is not unique, this means that we did not possibly take into consideration additional restrictions allowing one to distinguish a unique solution, or our hypotheses on the possible structure of the solution and based on these hypotheses constructions were erroneous from the very beginning.

Having found the solution of each problem (0.8), we must verify that the functions $u_i(t_1,t_2,\epsilon)$ constructed by us really possess the properties we have supposed at the second stage. Otherwise, substituting the approximate solution (0.3) with the coefficients found from (0.8) into the equations (0.1), (0.2), we will fail in getting the equations (0.7) (and hence the series of the problems (0.8) by means of which we have defined those coefficients), since in the absence of the above-mentioned properties of the functions $u_i(t_1,t_2,\epsilon)$ it will be impossible to carry out the needed transformations resulting in the asymptotic relations (0.7).

6. Thus as a result of our previous steps we have constructed either the formal asymptotic series (0.3) or the corresponding finite sum, the formal asymptotic representation. It is necessary now to see that this series is really an asymptotic expansion of an unknown solution of the problem (0.1), (0.2). In other words, we have to prove that everywhere in the domain of variation of the independent variables the relation

$$u_\epsilon(t_1, t_2, \epsilon) - \sum_{i=0}^{n} u_i(t_1, t_2, \epsilon)\phi_i(\epsilon) = o\big(\phi_n(\epsilon)\big) \tag{0.9}$$

is fulfilled, where $n \leq N$ if we seek for an asymptotic representation of the solution, and $n = 0, 1, \ldots$ if we seek for its full asymptotic expansion.

After proving that the above relation is valid, the process of constructing the asymptotic expansion may be considered as completed.

The emphasis should be placed on the fact that the best error estimate of an asymptotic approximation is the estimate in a norm of that functional space in which the problem under consideration is well-posed. In most of the works dealing with the construction of asymptotic expansions of solutions of the problems for singularly perturbed differential equations, the estimation of closeness of an asymptotic representation to the solution of the problem is carried out either in the norm of the space $C(\mathcal{T})$, or in integral norms. At the same time, the construction of numerical algorithms of the solution takes always into account the properties of the solution as an element of one or another functional space. Therefore the employment of asymptotic expansions in developing the above-mentioned algorithms should not take one out of the scope of the space under consideration.

In practice, however, it is too difficult to prove this, and to justify the validity of the constructed expansion, very often one takes the relations (0.8) as fulfilled to within $o\big(\phi_N(\epsilon)\big)$ when substituting that expansion into them. In this case they say that the formal expansion is a residual expansion for the unknown solution (of the equation or of boundary conditions).

One should bear in mind that the residual expansion of the solution is, in fact, far from being an asymptotic expansion. To illustrate this statement, let us consider the following example:

consider the boundary value problem

$$\epsilon y'' + yy' = 0, \quad -1 < t < 1, \qquad y(-1) = -1, \quad y(1) = 1.$$

Choose as asymptotic the sequence $\{\epsilon^i\}$. Evidently, the function $y_c(t, \epsilon) = th[(t + c)/(2\epsilon)]$ satisfies the equation for any value of the constant c. If the constant c does not vary as ϵ changes, $|c| < 1$, then the function $y_c(t, \epsilon)$ as $\epsilon \longrightarrow 0$ satisfies both boundary conditions to within $o(\epsilon^n)$, where n is a positive integer. Thus, for the chosen asymptotic sequence the function $y_c(t, \epsilon)$ for any $|c| < 1$ will be the residual expansion of the solution both of the equation and of boundary conditions. At the same time, the function

$y_0(t, \epsilon)$ is an asymptotic approximation of the solution with respect to the chosen asymptotic sequence.

Surely, if the proof of the validity of the relations (0.9) causes insuperable difficulties (and this may happen due to the objective complexity of the problem or erroneous hypotheses on the supposed structure of asymptotic expansions), then such an approach to the proof of the validity of the obtained expansion as an approximate solution will be justified. However, as the above example shows, the residual expansion of the equation and of the additional conditions is, in fact, far from being an asymptotic expansion of the solution.

In the present work we consider the methods of constructing asymptotic (as small parameters tend to zero) expansions of solutions of initial and boundary value problems for quasi-linear (Chapter I) and linear (Chapter II) singularly perturbed partial differential equations of elliptic, parabolic and mixed types, when the solutions of the corresponding degenerate problems have singularities of any kind; in other words, we consider bisingular boundary value and initial boundary value problems for equations of the above-mentioned types. Moreover, it should be noted that we consider only those problems whose asymptotic expansions of solutions possess boundary and/or interior layers of exponential type. Asymptotic expansion can, certainly, be constructed by the method of matching asymptotic expansions, however the method we present in this work for the solution of the problems is proved to be more effective, as far as it gives a more clear presentation of the solution structure.

The results stated here were obtained partially in the author's earlier works cited in references. In the present work we do not consider boundary value problems for ordinary differential equations (for the corresponding results, see [1], [59], [60], [62]–[64], [71], [73], [74], [85], etc.).

CHAPTER I
QUASI-LINEAR PARABOLIC EQUATIONS

This chapter is concerned with the study of singularly perturbed quasi-linear equations of parabolic type; special attention will be paid to the so-called model equation of gas dynamics [57]

$$L_\epsilon u \equiv \epsilon^2 \frac{\partial^2 u}{\partial x^2} - \phi'_u(u)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0.$$

A great many works have been devoted to the investigation of properties of solutions of the above equation, depending on the properties of initial and boundary functions (among recent works we may mention the works of N. S. Bakhvalov [4], T. D. Ventzel [84], A. M. Il'in [31], O. A. Ladyzhenskaya [39], O. A. Oleĭnik and T. D. Ventzel [54], O. A. Oleĭnik and S. N. Kruzhkov [53], V. I. Pryazhinskiĭ [56], V. I. Pryazhinskiĭ and V. G. Sushko [58], etc.). Asymptotic (with respect to the parameter) expansions of solutions of the above-given equation under different assumptions on properties of solutions of the corresponding degenerate problem have been constructed by the method of matching (see, for example, A. M. Il'in [32], V. I. Pryazhinskiĭ and V. G. Sushko [59]). The particular interest in the given equation is due to the fact that properties of its solutions are characteristic of properties of solutions of quasi-linear equations and their systems; in particular, singularities of analogous type can be observed in the solutions of a system of equations of gas dynamics.

In the first two sections we make a priori estimates of solutions and their derivatives for an n-dimensional quasi-linear parabolic equation (whose each second order derivative has its "own" small parameter as a multiplier) and for a system of quasi-linear equations of parabolic type. These estimates are necessary for justification of asymptotic representations of solutions and will be used in the subsequent sections.

In the third section we construct asymptotic expansions of solutions when the solution of the corresponding degenerate problem has for $t \geq 0$ a line of discontinuity (shock type wave). Boundary layer asymptotic expansion is constructed for a shock wave, and the error estimation is performed in the norm of the space C^1.

The case where the solution of the degenerate equation is continuous for $t > 0$ but has "fracture" lines on which its the derivatives are discontinuous, is considered in the fourth section. Similar situations may happen when the initial function is continuous but non-differentiable at some point (a weak discontinuity of the solution of the degenerate equation), or when the initial function is discontinuous and its limiting values from the left and from the right of the point of discontinuity are connected by certain relations (rarefaction wave). In the former case we construct a complete asymptotic expansion of the solution in powers of a small parameter, while in the latter case we give an approximation for the solution of the non-degenerate equation which is more exact than the one constructed earlier.

1.1. A Priori Estimates of Solutions of the Cauchy Problem for a Quasi-Linear Parabolic Equation

A priori estimates are of great importance for getting error estimates of formal asymptotic expansions A priori estimates for partial differential equations have been made by various authors. Complete enough results obtained in this direction can be found in the list of references (for example, [3], [16], [20], [39]–[42], [53], [57], [58], etc.).

In [4], [5], [31], [33], [52], [54], a priori estimates were obtained for solutions of the Cauchy problem for one-dimensional (in spatial variable) quasi-linear parabolic equation under various properties of the initial function.

In this section, under various assumptions on the modulus of continuity we investigate properties and deduce interior a priori estimates of solutions of a multi-dimensional singularly perturbed parabolic equation with several small parameters.

1. Consider the Cauchy problem

$$Lu \equiv \epsilon_i \frac{\partial^2 u}{\partial x_i^2} - \frac{d}{dx_i}\phi_i(t,x,u) - \psi(t,x,u) - \frac{\partial u}{\partial t} = 0, \tag{1}$$

$$u\big|_{t=0} = u_0(x). \tag{2}$$

Here $x = (x_1, x_2, \ldots, x_n)$ is a point of the space $\mathcal{R}^n$, $\prod_T = (0,T] \times \mathcal{R}^n$, the functions $\phi_i(t,x,u)$ and $\psi(t,x,u)$ are defined and continuous for all $(t,x,u) \in \overline{\prod}_T \times \mathcal{R}^1$ along with their partial derivatives in the variables x_k and u up to some order , $\epsilon_i \in (0,1]$, $u_0(x)$ is some bounded measurable function,

$$\frac{d}{dx_i}\phi_i(t,x,u) \equiv \sum_{i=1}^{n} \Big[\frac{\partial \phi_i(t,x,u)}{\partial x_i} + \frac{\partial \phi_i(t,x,u)}{\partial u}\frac{\partial u}{\partial x_i}\Big].$$

In the equation (1) and everywhere below if either term has two or more same indices, then this means summation over all these indices from 1 to n.

Introduce the following notation:

(x,t) are points of the space $\mathcal{R}^n$; $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$;

$$x_{(c,i)} = (x_1, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_n); \quad g_{(c,i)}(t,x) = g(t, x_{(c,i)});$$

$$\int_a^b g(t,x)dx = \int_{a_1}^{b_1} \int\limits_{a_2}^{b_2} \ldots \int_{a_n}^{b_n} g(t, x_1, x_2, \ldots, x_n) dx_1 dx_2 \ldots dx_n;$$

$$\int_a^b g(t,x) dx_{(c,i)} = (b_i - a_i)^{-1} \int\limits_a^b g_{(c,i)}(t,x)dx;$$

ϵ is a vector with the coordinates $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$, $\epsilon_0 = \min\limits_{1\le i\le n} \epsilon_i$; $\epsilon^\gamma = \epsilon_1^\gamma \epsilon_2^\gamma \cdots \epsilon_n^\gamma$. By M, M_k, $k = 1, 2, \ldots$ we denote independent of ϵ constants, if the value of these constants is unessential for our further reasoning.

In deducing estimates for the solution of the problem (1), (2), the solution will be assumed to be bounded everywhere in $\overline{\prod}_T$ by a constant m_0. Moreover, we will assume if needed that for $(t, x) \in \overline{\prod}_T$ and $|v(t, x)| \le m_0$ the following estimates are valid: $|\phi_i(t, x, v)| \le m_i$; $|\phi'_{iv}(t, x, v)| \le m_{i,v}$; $|\phi''_{ivv}(t, x, v)| \le m_{i,vv}$; $|\phi'_{ix_k}(t, x, v)| \le p_{i,k}$; $|\phi''_{ix_kx_s}(t, x, v)| \le p_{i,k,s}$; $|\phi''_{ivx_k}(t, x, v)| \le p_{i,k,v}$; $|\psi(t, x, v)| \le r$; $|\psi'_v(t, x, v)| \le r_v$; $|\psi'_{x_k}(t, x, v)| \le r_k$.

Let $f_a(z)$ be an infinitely differentiable function of one variable, defined for $z \in (-\infty, \infty)$ and satisfying $f_a(z) \equiv 1$ for $|z - a| \le 1$, $f_a(z) \equiv 0$ for $|z - a| \ge 2$, $0 \le f_a(z) \le 1$. Consider the function $f(x) = f_{b_1}(\epsilon_1^{-1} x_1) f_{b_2}(\epsilon_2^{-1} x_2) \cdots f_{b_n}(\epsilon_n^{-1} x_n)$, where b is a point of $\mathcal{R}^n$. The function $v(t, x) = u(t, x) f(x)$ satisfies the equation

$$L_1 v \equiv \epsilon_i \frac{\partial^2 v}{\partial x_i^2} - \frac{\partial v}{\partial t} = -2\epsilon_i \frac{\partial u}{\partial x_i}\frac{\partial f}{\partial x_i} - \epsilon_i u \frac{\partial^2 f}{\partial x_i^2} + f\frac{d}{dx_i}\phi_i(t, x, u) + f\psi(t, x, u) \quad (3)$$

and the initial condition

$$v(0, x) = v_0(x) = f(x) u_0(x). \quad (4)$$

2. Consider the modulus of continuity of $u(t, x)$ with respect to the spatial variables. To this end it suffices to estimate the difference $u(t, x) - u(t, x_{(y,j)})$ which will be done in two steps. First we will obtain a preliminary estimate (see the inequality (5)) and then, using this estimate as auxiliary, we will get the final estimate.

In obtaining the auxiliary estimate, the points x, y will be assumed to belong to the cube $b_i - \epsilon_i \le x_i$, $y_i \le b_i + \epsilon_i$, $i = 1, 2, \cdots, n$. In this case $v(t, x) = u(t, x)$, $v(t, y) = u(t, y)$, and therefore

$$u(t, x) - u(t, x_{(y,j)}) = \int\limits_{b-2\epsilon}^{b+2\epsilon} [G(t, x, z, 0) - G(t, x_{(y,j)}, z, 0)] v_0(z) dz +$$

$$+ \int\limits_0^t \int\limits_{b-2\epsilon}^{b+2\epsilon} [G(t, x, z, \tau) - G(t, x_{(y,j)}, z, \tau)] \left[\phi_i \frac{\partial f}{\partial z_i} + \epsilon_i u \frac{\partial^2 f}{\partial z_i^2} - f\psi\right] dz d\tau +$$

$$+ \int\limits_0^t d\tau \int\limits_{b-2\epsilon}^{b+2\epsilon} \left\{\frac{\partial}{\partial z_i}[G(t, x, z, \tau) - G(t, x_{(y,j)}, z, \tau)]\right\} (1 - \delta_{i,j}) \left[\phi_i f + 2u\epsilon_i \frac{\partial f}{\partial z_i}\right] dz +$$

$$+ \int\limits_0^t \int\limits_{b-2\epsilon}^{b+2\epsilon} \left\{\frac{\partial}{\partial z_j}[G(t, x, z, \tau) - G(t, x_{(y,j)}, z, \tau)]\right\} \left[\phi_i f + 2u\epsilon_i \frac{\partial f}{\partial z_i}\right] dz d\tau =$$

$$= A_1 + A_2 + A_3 + A_4,$$

where $G(t,x,z,\tau)=\epsilon^{-1/2}[4\pi(t-\tau)]^{-n/2}\exp\{-(x_i-z_i)^2/4[\epsilon_i(t-\tau)]\}$ is the fundamental solution of the heat conductivity equation, $\delta_{i,j}$ is the Kronecker symbol: $\delta_{i,j}=0$ for $i\neq j$, $\delta_{i,i}=1$.

The integrals A_1 and A_2 are estimated directly:

$$|A_1|\leq m_0\left|\int\limits_{y_j}^{x_j}d\xi\int\limits_{b-2\epsilon}^{b+2\epsilon}\left|\frac{\partial}{\partial z_j}G(t,x_{(\xi,j)},z,0)\right|dz\right|\leq M_1m_0\epsilon_j^{-1/2}t^{-1/2}|x_j-y_j|;$$

$$|A_2|\leq M_2(\epsilon_0^{-1}\overline{m}_0+\epsilon_0^{-1}m_0+r)\epsilon_j^{-1/2}t^{1/2}|x_j-y_j|,$$

where $\overline{m}_0=\max\limits_{1\leq i\leq n}m_i$. When estimating the integral A_3 we first assume that the inequality $y_j\leq x_j$ is fulfilled and then represent it as a sum $A_3=A_{3,1}+A_{3,2}+A_{3,3}+A_{3,4}+A_{3,5}+A_{3,5}$ in such a way that the interval $(b_j-2\epsilon_j,b_j+\epsilon_j)$ of integration with respect to the variable x_j be partitioned by the points $2y_j-x_j$; y_j; $2^{-1}(x_j+y_j)$; x_j; $2x_j-y_j$ into 6 segments. Since for $b_j-2\epsilon_j\leq z_j\leq 2y_j-x_j$ the inequalities $0\leq x_j-y_j\leq y_j-z_j$ are fulfilled, the estimate for the integral $A_{3,1}$ can be obtained in the form

$$|A_{3,1}|\leq M_4(m_0+\overline{m}_0)|x_j-y_j|^\gamma\int_0^t d\tau\int_{b-2\epsilon}^{b+2\epsilon}G(t,x,z,\tau)\frac{|x_i-z_i|}{\epsilon_i\epsilon_j(t-\tau)^2}dz_{(x,j)}\times$$
$$\times\int_{b_j-2\epsilon_j}^{b_j+2\epsilon_j}(y_j-z_j)^{2-\gamma}\exp\Big[-\frac{(y_j-z_j)^2}{4\epsilon_j(t-\tau)}\Big]dz_j\leq$$
$$\leq M_5(1-\gamma)^{-1}(m_0+\overline{m}_0)\epsilon_0^{-1/2}t^{(1-\gamma)/2}|x_j-y_j|^\gamma,$$

$0<\gamma<1$. The integral $A_{3,6}$ is estimated analogously.

The variable z_j in the integral $A_{3,2}$ satisfies the inequalities $2y_j-x_j\leq z_j\leq y_j$ from which we obtain $0\leq y_j-z_j\leq x_j-y_j$. Hence

$$\exp\{-(x_j-y_j)^2/[4\epsilon_j(t-\tau)]\}\leq\exp\{-(y_j-z_j)^2/[4\epsilon_j(t-\tau)]\},$$

and therefore

$$|A_{3,2}|\leq M_6(m_0+\overline{m}_0)\int_0^t d\tau\int_{b-2\epsilon}^{b+2\epsilon}G(t,x,z,\tau)dz_{(x,j)}\int_0^1|x_j-y_j|\times$$
$$\times\Big\{\int_{2y_j-x_j}^{y_j}\Big\{\frac{y_j-z_j}{2\epsilon_j(t-\tau)}\exp\Big[-\frac{(y_j-z_j)^2}{4\epsilon_j(t-\tau)}\Big]\exp\Big[-\frac{\theta^2(x_j-y_j)^2}{4\epsilon_j(t-\tau)}\Big]+$$
$$+\int_{2y_j-x_j}^{y_j}\frac{\theta(x_j-z_j)}{2\epsilon_j(t-\tau)}\exp\Big[-\frac{(y_j-z_j)^2}{4\epsilon_j(t-\tau)}\Big]\exp\Big[-\frac{\theta^2(x_j-y_j)^2}{4\epsilon_j(t-\tau)}\Big]\Big\}dz_j\Big\}d\theta.$$

From the equality $x_j-y_j=(x_j-y_j)^\gamma\theta^{\gamma-1}[\theta(x_j-y_j)]^{1-\gamma}$ we have

$$|A_{3,2}|\leq M_6(m_0+\overline{m}_0)|x_j-y_j|^\gamma\int_0^t d\tau\int_{b-2\epsilon}^{b+2\epsilon}G(t,x,z,\tau)dz_{(x,j)}\int_0^1\theta^{\gamma-1}\times$$
$$\times\Big\{\int_{2y_j-x_j}^{y_j}\frac{y_j-z_j}{2\epsilon_j(t-\tau)}[\theta(x_j-y_j)]^{1-\gamma}\exp\Big[-\frac{(y_j-z_j)^2}{4\epsilon_j(t-\tau)}-\frac{\theta^2(x_j-y_j)^2}{4\epsilon_j(t-\tau)}\Big]dz_j+$$

$$+\int_{2y_j-x_j}^{y_j}\frac{[\theta(x_j-y_j)]^{2-\gamma}}{2\epsilon_j(t-\tau)}\exp\Big[-\frac{(y_j-z_j)^2}{4\epsilon_j(t-\tau)}\Big]dz_j\exp\Big[-\frac{\theta^2(x_j-y_j)^2}{4\epsilon_j(t-\tau)}\Big]\Big\}d\theta\le$$
$$\le M_7(m_0+\overline{m}_0)[\gamma(1-\gamma)]^{-1}\epsilon_j^{-\gamma/2}\epsilon_0^{-1/2}t^{(1-\gamma)/2}|x_j-y_j|^\gamma.$$

The estimate for the integral $A_{3,5}$ is analogous to that of the integral $A_{3,2}$.

From $y_j\le z_j\le(x_j+y_j)/2$ there follow the inequalities $0\le z_j-y_j\le(x_j-y_j)/2\le x_j-z_j$ and from $(x_j+y_j)/2\le z_j\le x_j$ the inequalities $0\le x_j-z_j\le(x_j-y_j)/2\le z_j-y_j$. By virtue of these relations, the integrals $A_{3,3}$ and $A_{3,4}$ are estimated just in the same way as the integrals $A_{3,2}$ and $A_{3,5}$.

As for the integral A_4, we partition it into 6 summands $A_{4,k}$, $1\le k\le 6$ and estimate each summand by using the same techniques as for the corresponding part of the integral A_3. Thus we can consider that the intermediate estimate for the modulus of continuity of the function $u(t,x)$ is obtained:

$$|u(t,x)-u(t,x_{(y,j)})|\le$$
$$\le M[m_0\epsilon_j^{-1/2}t^{-1/2}|x_j-y_j|+(\epsilon_0^{-1}m_0+\epsilon_0^{-1}\overline{m}_0+r_0)\epsilon_j^{-1/2}t^{1/2}|x_j-y_j|+$$
$$+(1-\gamma)^{-1}\gamma^{-1}(m_0+\overline{m}_0)\epsilon_j^{-\gamma/2}\epsilon_0^{-1/2}t^{(1-\gamma)/2}|x_j-y_j|^\gamma]. \tag{5}$$

We use the estimate (5) for the determination of the estimate of the difference $u(t,x)-u(t,x_{(y,j)})$. The points x, y will now be assumed to belong to the cube $b_i-2^{-1}\epsilon_i\le x_i,\ y_i\le b_i+2^{-1}\epsilon_i$, $1\le i\le n$. It can be easily seen that the estimates for the integrals A_1, A_2 can remain unchanged. We rewrite the sum A_3+A_4 as

$$\int_0^t d\tau\int_{b-2\epsilon}^{b+2\epsilon}\Big\{\frac{\partial}{\partial z_i}[Gt,x,z,\tau)-G(t,x_{(y,j)},z,\tau)]\Big\}\Big(\phi_i f+2u\epsilon_i\frac{\partial f}{\partial z_i}\Big)dz=$$
$$=\int_0^t d\tau\int_{b-2\epsilon}^{b+2\epsilon}dz_{(z,j)}\int_{b_j-2\epsilon_j}^{b_j-\epsilon_j}(\cdots)dz_j+\int_0^t\int_{b-2\epsilon}^{b+2\epsilon}dz_{(z,j)}\int_{b_j+\epsilon_j}^{b_j+2\epsilon_j}(\cdots)dz_j+$$
$$+\int_0^t d\tau\int_{b-2\epsilon}^{b+2\epsilon}dz_{(z,j)}\int_{b_j-\epsilon_j}^{b_j+\epsilon_j}(\cdots)dz_j=B_1+B_2+B_3.$$

In the integral B_1, the inequalities

$$\sum_{i=1}^n\frac{(x_i-z_i)^2}{4\epsilon_i(t-\tau)}\ge\frac{\epsilon_j}{16(t-\tau)},\quad\frac{(y_j-z_j)^2}{4\epsilon_j(t-\tau)}+\sum_{i\ne j}\frac{(x_i-z_i)^2}{4\epsilon_i(t-\tau)}\ge\frac{\epsilon_j}{16(t-\tau)}$$

are fulfilled, and therefore

$$|B_1|\le M_9(m_0+\overline{m}_0)|x_j-y_j|\int_0^t d\tau\int_{b-2\epsilon}^{b+2\epsilon}dz_{(z,j)}\int_{b_j-2\epsilon_j}^{b_j-\epsilon_j}G(t,x,z_{(x+\epsilon/2,j)},\tau)\times$$
$$\times\Big[\frac{|x_i-z_i||x_j-z_j|}{4\epsilon_i\epsilon_j(t-\tau)^2}+\frac{\delta_{i,j}}{2\epsilon_j(t-\tau)}\Big]dz_j\le M_{10}(m_0+\overline{m}_0)|x_j-y_j|\times$$
$$\times(\epsilon_j^{-1/2}t^{-1/2}+\epsilon_j^{-3/2}t^{1/2}+\epsilon_j^{-1/2}\epsilon_0^{-1/2})\exp\{-\epsilon_j/(16t)\}.$$

It is obvious that the same estimate is valid for the integral B_2 as well.

Let us now pass to the estimation of the integral B_3. We write it as

$$B_3 = \int_0^t d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} \Big\{\frac{\partial}{\partial z_i}[Gt,x,z,\tau) - G(t,x_{(y,j)},z,\tau)]\Big\} \times$$
$$\times \Big(\phi_i f + 2u\epsilon_i \frac{\partial f}{\partial z_i}\Big) dz_j =$$
$$= \int_0^t d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} \Big\{\frac{\partial}{\partial z_i}[Gt,x,z,\tau) - G(t,x_{(y,j)},z,\tau)]\Big\}\phi_i dz_j +$$
$$+ \int_0^t d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} \Big\{\frac{\partial}{\partial z_i}[Gt,x,z,\tau) - G(t,x_{(y,j)},z,\tau)]\Big\} \times$$
$$\times 2u\epsilon_i \frac{\partial f}{\partial z_i} dz_j = B_{3,1} + B_{3,2}.$$

Since

$$\phi_i(\tau,z,u(\tau,z)) = \phi_i(\tau,z_{(\xi,j)},u(\tau,z_{(\xi,j)})) + [\phi_i(\tau,z,u(\tau,z_{(\xi,j)})) -$$
$$-\phi_i(\tau,z_{(\xi,j)},u(\tau,z_{(\xi,j)}))] + [\phi_i(\tau,z,u(\tau,z)) - \phi_i(\tau,z,u(\tau,z_{(\xi,j)}))],$$

the integral $B_{3,1}$ can be represented in the form of three summands denoted by $B_{3,1,1}$, $B_{3,1,2}$, and $B_{3,1,3}$, respectively. For the inequality $b_j - 2^{-1}\epsilon_j \le x_j$, $y_j \le b_j + 2^{-1}\epsilon_j$, the integral $B_{3,1,1}$ is estimated directly:

$$\Big| \int_0^t d\tau \int_{y_j}^{x_j} d\xi_j \int_{b-2\epsilon}^{b+2\epsilon} [\phi_i(\tau,z_{(\xi,j)},u(\tau,z_{(\xi,j)}))f(z) + 2\epsilon_j u f'_{x_i}]\, dz_{(\xi,j)} \times$$
$$\times \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} \frac{\partial^2}{\partial z_i \partial z_j} G(t,x_{(\xi,j)},z,\tau) dz_j \Big| \le$$
$$\le M_{11}\overline{m}_0 |x_j - y_j| \epsilon_0^{-1/2} \epsilon_j^{-1/2} t^{1/2} \exp\{-\epsilon_j/(16t)\}.$$

Using the inequality

$$|\phi_i(\tau,z,u(\tau,z_{(\xi,j)})) - \phi_i(\tau,z_{(\xi,j)},u(\tau,z_{(\xi,j)}))| \le p_{i,j}|\xi_j - z_j|,$$

for the summand $B_{3,1,2}$, we have

$$|B_{3,1,2}| \le M_{14} p_{0,j} \int_0^t d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} dz_j \int_{y_j}^{x_j} G(t,x_{(\xi,j)}z,\tau) \times$$
$$\times \Big[\frac{|\xi_j - z_j|}{\epsilon_j(t-\tau)}\delta_{i,j} + \frac{|\xi_j - z_j|^2 |x_i - z_i + \delta_{i,j}(\xi_j - x_j)|}{\epsilon_i \epsilon_j (t-\tau)^2}\Big] d\xi_j \le$$
$$\le M_{15} p_{0,j} \epsilon_0^{-1/2} t^{1/2} |x_j - y_j|,$$

where $p_{0,k} = \max\limits_{1 \le i \le n} p_{i,k}$. To the integral $B_{3,1,3}$ we apply the intermediate estimate (5). Since

$$|\phi_i(\tau,z,u(\tau,z)) - \phi_i(\tau,z,u(\tau,z_{(\xi,j)}))| \le$$

$$\leq M_{16} m_{i,u} \big\{ [m_0 \epsilon_j^{-1/2} t^{-1/2} + (m_0 + \overline{m}_0 + \overline{r}) \epsilon_j^{-1/2} \epsilon_0^{-1} t^{1/2}] \times$$
$$\times |\xi_j - z_j| + \gamma^{-1}(1-\gamma)^{-1}(m_0 + \overline{m}_0) \epsilon_j^{-\gamma/2} \epsilon_0^{-1/2} t^{(1-\gamma)/2} |\xi_j - z_j|^\gamma \big\},$$

the integral $B_{3,1,3}$ is estimated as

$$|B_{3,1,3}| \leq M_{17} m_{0,u} \sum_{i=1}^n \int_0^t d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b+\epsilon_j} dz_j \Big| \int_{y_j}^{x_j} \Big\{ [m_0 \epsilon_j^{-1/2} \tau^{-1/2} +$$
$$+ (\epsilon_0^{-1} m_0 + \epsilon_0^{-1} \overline{m}_0 + r) \epsilon_j^{-1/2} t^{1/2} |\xi_j - z_j|] + \gamma^{-1}(1-\gamma)^{-1} \times$$
$$\times (m_0 + \overline{m}_0) \epsilon_j^{-\gamma/2} \epsilon_0^{-1/2} \tau^{(1-\gamma)/2} |\xi_j - z_j|^\gamma \Big\} G''_{z_i z_j}(t, x_{(\xi,j)}, z, \tau) d\xi_j \Big|,$$

where $m_{0,u} = \max\limits_{1\leq i\leq n} m_{i,u}$. Evidently, without any difficulties we can obtain

$$\int_0^t \tau^\alpha d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} |\xi_j - z_j| \Big| \frac{\partial^2}{\partial z_i z_j} G(t, x_{(\xi,j)}, z, \tau) \Big| dz_j \leq$$
$$\leq \int_0^t \tau^\alpha d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} G(t, x_{(\xi,j)}, z, \tau) \times$$
$$\times \Big[\frac{|x_i - z_i||\xi_j - z_j|^2}{\epsilon_i \epsilon_j (t-\tau)^2} + \frac{\delta_{i,j} |\xi_j - z_j|}{\epsilon_j (t-\tau)} \Big] dz_j \leq M_{20} \epsilon_0^{-1/2} t^{1/2+\alpha},$$
$$\int_0^t \tau^\alpha d\tau \int_{b-2\epsilon}^{b+2\epsilon} dz_{(z,j)} \int_{b_j-\epsilon_j}^{b_j+\epsilon_j} G(t, x_{(\xi,j)}, z, \tau) \Big[\frac{|x_i - z_i||\xi_j - z_j|^{1+\gamma}}{\epsilon_i \epsilon_j (t-\tau)^2} +$$
$$+ \frac{\delta_{i,j} |\xi_j - z_j|^\gamma}{\epsilon_j (t-\tau)} \Big] dz_j \leq M_{21} \epsilon_0^{-1/2} \epsilon_j^{(\gamma-1)/2} t^{\alpha+\gamma/2}.$$

Consequently, for the integral $B_{3,1,3}$ we have the estimate

$$|B_{3,1,3}| \leq M m_{0u} \big[m_0 \epsilon_0^{-1/2} \epsilon_j^{-1/2} +$$
$$+ (m_0 + \overline{m}_0) \epsilon_0^{-1} \epsilon_j^{-1/2} t^{1/2} + (m_0 + \overline{m}_0 + \overline{r}) \epsilon_0^{-3/2} \epsilon_j^{-1/2} t \big] |x_j - y_j|.$$

The integral $B_{3,2}$ can be estimated in the same manner as the integral $B_{3,1,3}$. Combining all the obtained inequalities, we can write out the estimate for the modulus of continuity of the function $u(t,x)$ with respect to the variable x_j:

$$|u(t,x) - u(t, x_{(y,j)})| \leq M \big\{ m_0 \epsilon_j^{-1/2} t^{-1/2} + m_0 \epsilon_0^{-1/2} \epsilon_j^{-1/2} +$$
$$+ \big[m_{0u}(m_0 + \overline{m}_0) \epsilon_0^{-1} \epsilon_j^{-1/2} + (\epsilon_0^{-1} m_0 + \epsilon_0^{-1} \overline{m}_0 + r) \epsilon_j^{-1/2} +$$
$$+ p_{0,0} \epsilon_0^{-1/2} \big] t^{1/2} + m_{0u} (\epsilon_0^{-1} m_0 + \epsilon_0^{-1} \overline{m}_0 + r) \epsilon_0^{-1/2} \epsilon_j^{-1/2} t \big\} |x_j - y_j|.$$

From the last inequality, in particular, it follows that one can weaken the requirements imposed on the initial data of the problem as follows:

$$|\phi_{ix_k}(t,x,v)| \leq p_{0,k} \leq \epsilon_k^{-1} \overline{p}_{0,k}, \quad |\psi(t,x,v)| \leq r \leq \epsilon_0^{-1} \overline{r}.$$

Then the latter inequality can be written as

$$|u(t,x)-u(t,x_{(y,j)})| \le M\{m_0(\epsilon_j^{-1/2}t^{-1/2}+\epsilon_0^{-1/2}\epsilon_j^{-1/2})+$$
$$+[m_{0u}(m_0+\overline{m}_0)\epsilon_j^{-1/2}+(m_0+\overline{m}_0+\overline{r})\epsilon_j^{-1/2}+\overline{p}_{0,j}\epsilon_j^{-1/2}]\epsilon_0^{-1}t^{1/2}+$$
$$+m_{0u}(m_0+\overline{m}_0+\overline{r})\epsilon_0^{-3/2}\epsilon_j^{-1/2}t\}|x_j-y_j|. \quad (6)$$

Thus we have proved the following assertion.

Theorem 1. *Let $u_0(x)$ be a bounded measurable function. If the functions $\phi_i(t,x,u)$ are bounded for all $(t,x)\in\overline{\prod}_T$, $|u(t,x)|\le m_0$, $\epsilon_0|\psi(t,x,u)|\le\overline{r}$, $\epsilon_j|\phi'_{ix_j}(t,x,u)|\le\overline{p}_{0,j}$, $|\phi_{iu}(t,x,u)|\le m_{0u}$, where the constants $\overline{r}$, $\overline{p}_{0,j}$, m_{0u} are independent of ϵ, then for the modulus of continuity of the function $u(t,x)$ with respect to the variable x_j the estimate (6) is fulfilled. If $t\in[0,T]$, then this estimate can be written in the form*

$$|u(t,x)-u(t,x_{(y,j)}|\le M\epsilon_j^{-1/2}(t^{-1/2}+\epsilon_0^{-1/2}+\epsilon_0^{-1}t^{1/2}+\epsilon_0^{-3/2}t)|x_j-y_j|, \quad (7)$$

while if $t\in(0,\epsilon_0]$, then

$$|u(t,x)-u(t,x_{(y,j)}|\le M\epsilon_j^{-1/2}(t^{-1/2}+\epsilon_0^{-1/2})|x_j-y_j|. \quad (8)$$

The constants M depend only on the upper bounds of the functions $|u(t,x)|$, $|\phi_i(t,x,u)|$, $\epsilon_0|\psi(t,x,u)|$, $\epsilon_j|\phi'_{ix_j}(t,x,u)|$, $|\phi_{iu}(t,x,u)|$ in the cylinder $N_b=\{(t,x)|0<t\le T,\ b_i-2\epsilon_i\le x_i\le b_i+2\epsilon_i\}$.

Remark 1. Theorem 1 remains valid if instead of the differentiability of the functions $\phi_i(t,x,u)$ with respect to the variables x_j we require the fulfillment of the Lipschitz condition with respect to those variables. Moreover, our reasoning is also true if the functions $\phi_i(t,x,u)$ satisfy the Hölder condition with respect to the variables x_j, u, respectively with the exponents λ_1, λ_2, $0<\lambda_1$, $\lambda_2<1$; note that naturally the right-hand sides of the inequalities (7) and (8) will somehow change.

3. In the previous subsection we have required that the functions $\phi_i(t,x,u)$ possess the bounded first order derivatives with respect to the variables x_j, u. If, however, these functions and the function $\psi(t,x,u)$ are more smooth with respect to the above-mentioned variables, then the estimate (7) may be essentially improved.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled. Let, moreover, the functions $\phi_i(t,x,u)$ have the second derivatives with respect to the variables x_j, u, and the function $\psi(t,x,u)$ has the first derivatives with respect to the same variables. If everywhere in the strip $\{(t,x)|\epsilon_0\le t\le T,\ -\infty\le x_i\le\infty\}$ these derivatives satisfy the conditions $|\phi''_{ix_jx_k}(t,x,u)|\le\epsilon_j^{-1}\epsilon_k^{-1}p_{0,j,k}$, $|\phi''_{ix_ju}(t,x,u)|\le\epsilon_j^{-1}q_{0,j}$, $|\phi''_{iuu}(t,x,u)|\le\overline{m}_{0uu}$, $|\psi'_{x_j}(t,x,u)|\le\epsilon_j^{-1}\epsilon_0^{-1}r_0$, $|\psi'_u(t,x,u)|\le\epsilon_0^{-1}r_u$ where the constants $p_{0,j,k}$, $q_{0,j}$, $\overline{m}_{0uu}$, r_0, r_u remain unchanged when ϵ changes, then everywhere in the strip $\prod_T$ the estimate (8) is valid for the function $u(t,x)$, and the constant M does not depend* *on the values of the functions $u(t,x)$, $\phi_i(t,x,u)$, and $\psi(t,x,u)$ outside the cylinder N_b.*

Proof. Let the function $f_a(z)$, defined by us in the first section, satisfy the condition $[f_a'(z)]^2[f_a(z)]^{-1} \le M$ for all $a-2 \le z \le a+2$; obviously, such functions do exist. Consider the function $v_1(t,x) = f(x)[u^2(t,x) + \beta_j u'_{x_j}(t,x)]$, where the constant β_j is chosen from the condition $\beta_j = \min\{\sqrt{\epsilon_j \epsilon_0}; \epsilon_j \overline{m}_{0uu}^{-1}\}$. Let the function $v_1(t,x)$ at some point $P_0(t_0,x_0)$ of the strip $\epsilon_0 \le t \le T$ reach the greatest positive value. If $t_0 = \epsilon_0$, then the assertion of Theorem 2 follows from Theorem 1. Suppose $t_0 > \epsilon_0$. For $t = t_0$, $x = x_0$ the equality $f(u^2+\beta_j u'_{x_j})'_{x_k} = -f'_{x_k}(u^2+\beta_j u'_{x_j})$ is fulfilled, and therefore at the above-mentioned point we have the relation

$$f\Big[\epsilon_i \frac{\partial^2 v_1}{\partial x_i^2} - \phi'_{iu}\frac{\partial v_1}{\partial x_i} - \frac{\partial v_1}{\partial t}\Big] = 2\epsilon_i f^2\Big(\frac{\partial u}{\partial x_i}\Big)^2 + f^2\frac{\partial u}{\partial x_i}[\phi''_{iux_j}\beta_j - u^2\phi''_{iuu}] +$$
$$+v_1\Big[f\phi''_{iuu}\frac{\partial u}{\partial x_i} - \phi'_{iu}\frac{\partial f}{\partial x_i} + \epsilon_i\frac{\partial^2 f}{\partial x_i^2} + f\psi'_u + f\phi''_{iux_i} - 2\epsilon_i\frac{1}{f}\Big(\frac{\partial f}{\partial x_i}\Big)^2\Big] +$$
$$+f^2[2u\psi + 2u\phi'_{ix_i} + \beta_j\psi'_{x_j} + \beta_j\phi''_{ix_ix_j} - u^2\psi'_u - u^2\phi''_{iux_i}]. \tag{9}$$

Since the function $v_1(t,x)$ reaches at the point P_0 its greatest value, the right-hand side of the latter inequality is non-negative at that point. Hence

$$2\epsilon_j f^2\Big(\frac{\partial u}{\partial x_j}\Big)^2 + \sum_{i\ne j}\Big\{f^2\frac{\partial u}{\partial x_i}\Big[2\epsilon_i\frac{\partial u}{\partial x_i} + \phi''_{iux_j}\beta_j - u^2\phi''_{iuu}\Big]\Big\} +$$
$$+f^2\frac{\partial u}{\partial x_j}(\phi''_{jux_j}\beta_j - u^2\phi''_{juu}) + v_1\sum_{i=1}^n\Big\{f\phi''_{iuu}\frac{\partial u}{\partial x_i} + h_i\Big\} + \sum_{i=1}^n g_i \le 0, \tag{10}$$

where by h_i, g_i are denoted the corresponding summands from the right-hand side of equation (9). Consider each summand of the last inequality written in the form $f^2u'_{x_i}[2\epsilon_i u'_{x_i} + \phi''_{iux_j}\beta_j - u^2\phi''_{iuu}]$, $\quad i \ne j$. If that summand is positive, then neglecting it we will only strengthen inequality (10). If, however this summand is non-positive, then at the point P_0 one group of inequalities, either $0 \le 2\epsilon_i u'_{x_i} \le u^2\phi''_{iuu} - \phi''_{iux_j}\beta_j$ or $u^2\phi''_{iuu} - \phi''_{iux_j}\beta_j \le 2\epsilon_i u'_{x_i} \le 0$, is fulfilled. In each case the derivative u'_{x_i} for $i \ne j$ at the point P_0 satisfies the inequality $|u'_{x_i}| \le M\epsilon_i^{-1}$. Therefore inequality (10) can be written in the form

$$v_1^2 + v_1\Big\{\phi''_{juu}\beta_j(2\epsilon_j)^{-1} + (2\epsilon_j)^{-1}\beta_j^2\sum_{i=1}^n h_i\Big\} + v_1\{(2\epsilon_j)^{-1}\beta_j^2\phi''_{jux_j} -$$
$$-\beta_j^2(2\epsilon_j\epsilon_0)^{-1}M_{22} - 2fu^2 - (2\epsilon_j)^{-1}\beta_j fu^2\phi''_{juu}\} + \{(2\epsilon_j^{-1}\beta_j f^2u^4\phi''_{juu} +$$
$$+f^2u^4 - (2\epsilon_j)^{-1}\beta_j^2 f^2u^2\phi''_{jux_j} + (2\epsilon_j)^{-1}\beta_j^2\sum_{i=1}^n g_i - M_{23}(2\epsilon_j\epsilon_0)^{-1}\beta_j^2\Big\} \le 0.$$

Since $|\phi''_{iuu}|\beta_j\epsilon_j^{-1} \leq M$, the last inequality implies the upper bound of the function $v_1(t,x)$ at P_0. Hence, under our assumptions, the estimate $u'_{x_j}(t,x) \leq M\beta_j^{-1}$ is fulfilled everywhere in the strip $\epsilon_0 \leq t \leq T$. To obtain the lower bound, it suffices to consider the function $v_2(t,x) = f(t,x)[\beta_j u'_{x_j} - u^2]$. Thus, under our assumptions regarding the functions $\phi_i(t,x,u)$, $\psi(t,x,u)$, everywhere in the strip $\epsilon_0 \leq t \leq T$ the estimate (8) is valid. □

4. Pass to the estimation of the modulus of continuity of the derivatives of the function $u(t,x)$. All the assumptions on the smoothness and boundedness of the functions $u_0(x)$, $\phi(t,x,u)$, $\psi(t,x,u)$ we made in proving in Theorem 1 and Theorem 2, will be assumed to be fulfilled. With the help of the function $f_a(z)$ we can write the difference $u'_{x_j}(t,x) - u'_{x_j}(t,x_{(y,s)})$ in the form

$$u'_{x_j}(t,x)-u'_{x_j}(t,x_{(y,s)}) = -\int\limits_{b-2\epsilon}^{b+2\epsilon} v_0(z)\frac{\partial}{\partial z_j}[G(t,x,z,0)-G(t,x_{(y,s)},z,0)]dz+$$

$$+\int\limits_0^t d\tau \int\limits_{b-2\epsilon}^{b+2\epsilon}\left(f\psi + f\frac{\partial\phi_i}{\partial z_i} + \epsilon_i u\frac{\partial f}{\partial z_i}\right)\frac{\partial}{\partial z_j}[G(t,x,z,\tau) - G(t,x_{(y,s)},z,\tau)dz +$$

$$+\int\limits_0^t \int\limits_{b-2\epsilon}^{b+2\epsilon}\left(f\phi'_{iu}\frac{\partial u}{\partial z_i} + 2\epsilon_i u\frac{\partial f}{\partial z_i}\frac{\partial u}{\partial z_i}\right)\frac{\partial}{\partial z_j}[G(t,x,z,\tau) - G(t,x_{(y,s)},z,\tau)dz =$$

$$= C_1 + C_2 + C_3.$$

First we will obtain a preliminary estimate. The integral C_1 is estimated directly:

$$|C_1| \leq M\epsilon_j^{-1/2}\epsilon_s^{-1/2}t^{-1}|x_s - y_s|.$$

The integral C_2 is estimated analogously to the integrals $A_{3,1,i}$ if $j \neq s$ and A_4 if $j = s$. Therefore

$$|C_2| \leq M\gamma^{-1}(1-\gamma)^{-1}((m_0 + \overline{r} + \overline{p}_{0,0})\epsilon_j^{-1/2}\epsilon_s^{-\gamma/2}\epsilon_0^{-1}t^{(1-\gamma)/2}|x_s - y_s|^{\gamma},$$

$0 < \gamma < 1$, $\overline{p}_{0,0} = \max\limits_{1\leq i\leq n} \overline{p}_{0,i}$.

For estimation of the integral C_3 we will make use of the estimate of the first derivatives of the function $u(t,x)$ from Theorem 2. If the conditions of Theorem 2 are fulfilled, then the inequality

$$|C_3| \leq M\gamma^{-1}(1-\gamma)^{-1}(m_0 + m_{0u})\epsilon_j^{-1/2}\epsilon_s^{-\gamma/2}\epsilon_0^{-1/2}(t^{-\gamma/2} + \\ +\epsilon_0^{-1/2}t^{(1-\gamma)/2})|x_s - y_s|^{\gamma}.$$

holds. Thus the preliminary estimate for the modulus of continuity of derivatives of the function $u(t,x)$ is true. Using the auxiliary function

$w = t^{\mu} v_1(t,x)$, $\mu > 1/2$, we obtain the inequality

$$\left| \frac{\partial w(t,x)}{\partial x_j} - \frac{\partial w(t, x_{(y,s)})}{\partial x_j} \right| \le$$

$$\le \left| \int_0^t \int_{y_s}^{x_s} d\xi_s \int_{b-2\epsilon}^{b+2\epsilon} \Big(2\epsilon_i \tau^{\mu} \frac{\partial f}{\partial z_i} \frac{\partial u}{\partial z_i} + \tau^{\mu} \frac{d\phi_i}{dz_i} + \tau^{\mu} f \psi - \mu \tau^{\mu-1} u f + \right.$$

$$\left. + .\epsilon_i \tau^{\mu} u \frac{\partial^2 f}{\partial z_i^2} \Big) dz_{(\xi,s)} \int_{b_s - 2\epsilon_s}^{b_s + 2\epsilon_s} \frac{\partial^2}{\partial z_j z_s} G(t, x_{(\xi,s)}, z, \tau) dz_s \right| +$$

$$+ \left| \int_0^t d\tau \int_{b-2\epsilon}^{b+2\epsilon} \Big(2\epsilon_i \tau^{\mu} \frac{\partial f}{\partial z_i} \frac{\partial u}{\partial z_i} + \tau^{\mu} f \frac{d\phi_i}{dz_i} + \tau^{\mu} f \psi - \mu \tau^{\mu-1} u f + \right.$$

$$\left. + \epsilon_i \tau^{\mu} u \frac{\partial^2 f}{\partial z_i^2} \Big) \Big|_{z_{(\xi,s)}}^{z} \frac{\partial^2}{\partial z_i \partial z_s} G(t, x_{(\xi,s)}, z, \tau) dz \right| = D_1 + D_2,$$

where

$$\left(g((\tau, z)) \right|_{z_{(\xi,s)}}^{z} = g(\tau, z) - g\left(\tau, z_{(\xi,s)}\right).$$

Using the estimate of the modulus of continuity of the function $u(t,x)$ and the preliminary estimate of the modulus of continuity of its derivatives, we obtain

$$|D_1| \le M \epsilon_j^{-1/2} \epsilon_s^{-1/2} \epsilon_0^{-1} t^{\mu} (1 + \epsilon_0^{-1/2} t^{1/2} + \epsilon_0^{-1} t) |x_s - y_s|,$$

$$|D_2| \le M \epsilon_j^{-1/2} \epsilon_s^{-1/2} t^{\mu} (t^{-1} + \epsilon_0^{-1/2} t^{-1/2} + \epsilon_0^{-1} + \epsilon_0^{-3/2} t^{1/2} + \epsilon_0^{-2} t) |x_s - y_s|.$$

Therefore the following Theorem is true.

Theorem 3. *If the conditions of Theorem 2 are fulfilled, then everywhere in the strip* $\prod_T$ *the estimate* $|u'_{x_j}(t,x) - u'_{x_j}(t, x_{(y,s)})| \le M \epsilon_j^{-1/2} \epsilon_s^{-1/2} (t^{-1} + \epsilon_0^{-2} t) |x_s - y_s|$ *is valid, where the value of the constant* M *does not depend on the values of the functions* $u_0(x)$, $\phi_i(t,x,u)$, $\psi(t,x,u)$ *outside the cylinder* $N_b = \{(t,x) | 0 < t < T,\ |b_i - x_i| \le 2\epsilon_i\}$.

The following theorem is valid.

Theorem 4. *Let in the strip* $\prod_T$ *the conditions of Theorem 2 be fulfilled, for* $\epsilon_0 \le t \le T$ *the functions* $\phi_i(t,x,u)$ *have the third derivatives with respect to the variables* u, x_k, $k = 1, 2, \ldots, n$, *and the function* $\psi(t,x,u)$ *have the second derivatives with respect to the same variables. Supppose that every differentiation of these functions with respect to the variable* x_k *introduce the multiplier* ϵ_k^{-1} *in the estimate of their modules, while the differentiation with respect to the variable* u *leave the order of their smallness unchanged as* $\epsilon_0 \to 0$. *Then everywhere in the strip* $\prod_T$ *the estimate* $|u'_{x_j}(t,x) - u'_{x_j}(t, x_{(y,s)})| \le M \epsilon_j^{-1/2} \epsilon_s^{-1/2} (t^{-1} + \epsilon_0^{-1}) |x_s - y_s|$ *is valid, where the constant*

M remains unchanged as the functions $\phi_i(t,x,u)$, $\psi(t,x,u)$ change outside the cylinder N_b.

To prove the theorem it suffices to consider the functions

$$v_2^{\pm} = f(x)\Big\{\epsilon_0^{-1}\epsilon_s^{-1/2}\epsilon_j^{-1/2}u''_{x_j x_s} \pm \Big[\big(\epsilon_s^{-1/2}\epsilon_0^{-1/2}u'_{x_s}\big)^2 + \big(\epsilon_j^{-1/2}\epsilon_0^{-1/2}u'_{x_j}\big)^2\Big]\Big\}$$

for $\epsilon_0 \le t \le T$ and to use arguments like those done in proving Theorem 2.

Analogously to Theorems 3 and 4 we prove the following

Theorem 5. *Let $s=(s_1,s_2,\dots,s_n)$ be a multi-index, where s_i are nonnegative integers, $|s| = \sum_{i=1}^n s_i$, the functions $\phi_i(t,x,u)$ have the derivatives of order $|s|+2$ with respect to the variables u, x_k, $i,k=1,2,\dots,n$, the function $\psi(t,x,u)$ has the derivatives of order $|s|+1$ with respect to the same variables, and $|\phi_i(t,x,u)| \le m_i$, $|\psi(t,x,u)| \le \epsilon_0^{-1}\overline{r}_0$. Suppose also that every differentiation of these functions with respect to the variables x_k introduces the multiplier ϵ_k^{-1}in the estimate of their modules and the differentiation with respect to u leaves the order of smallness of these derivatives unchanged as $\epsilon \to 0$, then in the strip $\prod_T$ the estimate*

$$\begin{aligned}&\Big|\frac{\partial^{|s|}u(t,x)}{\partial x_1^{s_1}\partial x_2^{s_2}\dots\partial x_n^{s_n}} - \frac{\partial^{|s|}u(t,x_{(y,j)})}{\partial x_1^{s_1}\partial x_2^{s_2}\dots\partial x_n^{s_n}}\Big| \le \\ &\le M\epsilon^{-s/2}\epsilon_j^{-1/2}\big(t^{-(1+|s|)/2}+\epsilon_0^{-(1+|s|)/2}\big)|x_j-y_j| \end{aligned}\tag{11}$$

is valid, where $\epsilon^{-s/2} \equiv \epsilon_1^{-s_1/2}\epsilon_2^{-s_2/2}\cdots\epsilon_n^{-s_n/2}$, and the constant M does not depend on the values of the functions u, $\phi_i(t,x,u)$, $\psi(t,x,u)$ outside the cylinder N_b.

Using the estimate (11) and the equation (1), we can formulate a similar assertion for the function $\partial^{|s|+k}u(t,x)/\partial x_1^{s_1}\partial x_2^{s_2}\cdots\partial x_n^{s_n}\partial t^k$ as well.

5. Consider now the case where the initial function $u_0(x)$ is bounded, measurable and satisfying the Hölder condition with the exponent ν: $|u_0(x)-u_0(y)| \le H(|x_1-y_1|^\nu + |x_2-y_2|^\nu + \cdots + |x_n-y_n|^\nu)$. We use this inequality for the estimate of the modulus of continuity of the function $u(t,x)$. For the estimate of the integral A_1 we have

$$\begin{aligned}|A_1| \le &\Big|\int_{b-2\epsilon}^{b+2\epsilon} dz_{(\xi,j)}\int_{y_j}^{x_j} v_0(z_{(\xi,j)})\frac{\partial}{\partial z_j}G(t,x_{(\xi,j)},z,0)d\xi_j\Big| + \\ +2H&\Big|\int_{b_2\epsilon}^{b+2\epsilon} dz_{(\xi,j)}\int_{y_j}^{x_j} G(t,x_{(\xi,j)},z,0)|\xi_j-z_j|^{1+\nu}\epsilon_j^{-1}t^{-1}d\xi_j\Big| \le \\ &\le M\big(m_0 + H\epsilon_j^{(\nu-1)/2}t^{(\nu-1)/2}\big)|x_j-y_j|\end{aligned}$$

for $b_j-\epsilon_j \le x_j$ and $y_j \le b_j+\epsilon_j$. Not changing the estimates of the integrals A_2, A_3, A_4, we obtain that under the conditions of Theorem 2 for the modulus of continuity of the function $u(t,x)$ the estimate

$$|u(t,x)-u(t,x_{(y,j)})| \leq M\big(\epsilon_j^{(\nu-1)/2}t^{(\nu-1)/2}+\epsilon_j^{-1/2}\epsilon_0^{-1/2}\big)|x_j-y_j| \quad (12)$$

holds.

When we estimate the modulus of continuity of the function $u'_{x_j}(t,x)$, the estimation of the integral D_1 is made as above by using the inequality (12). For estimation of the integral D_2 let us consider the summand which contains the multiplier $\tau^{\mu-1}$. Estimating this summand by means of the inequality (12), we will obtain the following assertion.

Theorem 6. *If the function $u_0(x)$ satisfies the Hölder condition with the exponent ν and the functions $\phi_i(t,x,u)$ and $\psi(t,x,u)$ satisfy the conditions formulated in Theorem* 2, *then the estimate* (12) *is valid for the function $u(t,x)$. If, however, the functions $\phi_i(t,x,u)$ and $\psi(t,x,u)$ satisfy the conditions of Theorem* 4, *then everywhere in the strip $\prod_T$ the estimate*

$$\big|u'_{x_j}(t,x)-u'_{x_j}(t,x_{(y,s)})\big| \leq M\epsilon_j^{-1/2}\epsilon_s^{-1/2}\big(\epsilon_0^{\nu/2}t^{-1+\nu/2}+\epsilon_0^{-1}\big)|x_s-y_s| \quad (13)$$

is valid. The constants M in the inequalities (12) *and* (13) *do not depend on the values of the functions $\phi_i(t,x,u)$ and $\psi(t,x,u)$ outside the cylinder N_b.*

Analogous statements can be formulated for derivatives of any, more higher, order of the function $u(t,x)$, as it has been done in Theorem 5.

Remark 2. It should be especially noted that the estimates (12) and (13) remain valid if the initial function satisfies the Hölder condition only with respect to the variable x_j (with respect to the variables x_j, x_s, respectively). If the initial function $u_0(x)$ satisfies the Lipschitz condition, then we can put $\nu=1$ in the estimates (12) and (13). If the initial function is differentiable and its derivative satisfies the Hölder condition with the exponent λ, then under the corresponding conditions on the functions $\phi_i(t,x,u)$, $\psi(t,x,u)$, in the estimates (12) and (13) one can take $\nu=\lambda+1$.

6. All the aforementioned arguments were concerned with the case where the initial function is characterized by its belonging to one or another functional space for all $x \in \mathcal{R}^n$. Of special interest, however, may be the cases where the function $u_0(x)$ belongs to one Hölder space on some set $\Omega \subset \mathcal{R}^n$ and belongs to the other space for $x \in \mathcal{R}^n \setminus \Omega$. There naturally arises the question on the behavior of the modulus of continuity of the function $u(t,x)$ as $t \to 0$ in the neighborhood of a boundary point of the set Ω. Obviously, the corresponding estimate of the modulus of continuity of the function $u(t,x)$ must involve on the one hand the terms determined by the smoothness of the function $u_0(x)$ at the points of the set Ω, and on the other hand the terms determined by its smoothness on the set $\mathcal{R}^n \setminus \Omega$. Depending on the curve along which we reach the point, the degree of influence of the above-mentioned terms of both groups increases or decreases. The appropriate investigation has been carried out in [68]; here we restrict ourselves to the formulation of the corresponding term only.

Condition A. We will say that the point $z=(z_1,z_2,\dots,z_n)$ of the set $\overline{\Omega}$ satisfies Condition **A**, if there exists a hyperplane P containing this point and such that for some neighborhood S of the point z all the points of the set $\overline{\Omega}\bigcap S$ lie only on one side of that hyperplane. Moreover, the neighborhood S is assumed to be such that $\sup\limits_{x\in S}|x_k-z_k|\geq\beta_k\epsilon_k$, where β_k is a constant not depending on ϵ.

Theorem 7. *Let the initial function $u_0(x)$ be bounded, measurable everywhere in R^n and satisfy the Hölder condition with respect to the variable x_k with the exponent α_1 everywhere in $\mathcal{R}^n$. Let the function $u_0(x)$ satisfy the Hölder condition with the exponent $\alpha_2>\alpha_1$ with respect to the variable x_k everywhere in $\mathcal{R}^n\setminus\Omega$, where Ω is a set of points. Let the point $z=0$ belong to the set $\overline{\Omega}$ and satisfy Condition **A**, the hyperplane P coincide with the hyperplane $x_1=0$ and all the points of the set $S\bigcap\overline{\Omega}$ lie in the half-space $x_1\leq 0$. Let, finally, $\theta(t,x_1)$ be a continuous, twice continuously differentiable for $t>0$ function such that the following relations hold:*

$$\begin{array}{lll}
\theta(t,x_1)= & 0, & x_1\leq-\lambda_1t^{\alpha_1/2};\\
\theta(t,x_1)= & \epsilon_1^{-2}(x_1+\lambda_1t^{\alpha_1/2})^2, & -\lambda_1t^{\alpha_1/2}\leq x_1\leq\epsilon_1/3-\lambda_1t^{\alpha_1/2};\\
\theta(t,x_1)= & 1-\epsilon_1^{-2}(x_1-\epsilon_1+\lambda_1t^{\alpha_1/2})^2, & 2\epsilon_1/3-\lambda_1t^{\alpha_2/2}\leq x_1\leq\epsilon_1-\lambda_2t^{\alpha_2/2};\\
\theta(t,x_1)= & 1, & x_1>\epsilon_1-\lambda_2t^{\alpha_2/2};
\end{array}$$

$$0\leq\theta(t,x_1)\leq 1,\ \epsilon_1^m\,|\partial^m\theta(t,x_1)/\partial x_1^m|\leq M,\ |\partial\theta(t,x_1)/\partial t|\leq M\lambda_1\epsilon_1^{-1}t^{\alpha_1/2-1},$$

$\lambda_1=\epsilon_0\epsilon_k^{\alpha_1/2}$, $\lambda_2=\epsilon_0\epsilon_k^{\alpha_2/2}$. *Then for all $t>0$ the estimate*

$$\begin{aligned}|u(t,x)-u(t,x_{(y,k)})|\leq M\big\{&\epsilon_k^{-1/2}\epsilon_0^{-1/2}+\epsilon_k^{(\alpha_2-1)/2}t^{(\alpha_2-1)/2}\theta(t,x_1)+\\&+\epsilon_k^{(\alpha_1-1)/2}t^{(\alpha_1-1)/2}[1-\theta(t,x_1)]\big\}|x_k-y_k|\end{aligned}\tag{14}$$

is true.

The estimate (14) shows the character of the continuous passage from the estimate defined by the Hölder exponent α_1 to the estimate defined by the Hölder exponent α_2.

Corollary 1. *If the function $u_0(x)$ satisfies the condition $|u_0(x)-u_0(x_{(y,k)})|\leq H|x_k-y_k|^\alpha$ everywhere in the plane $t=0$, with the exception of the point $z=0$, and the condition x_k, y_k is fulfilled for the variables $x_ky_k>0$, then the estimate*

$$|u(t,x)-u(t,x_{(y,k)}|\leq M\epsilon_k^{-1/2}[(\lambda\epsilon_0^{-1}+\epsilon_k^{\alpha/2})t^{(\alpha-1)/2}+\epsilon_0^{-1/2}]|x_k-y_k|$$

is valid everywhere outside the set $\{(t,x)|0 \le t \le \epsilon_0, -\epsilon_s + \lambda t^{\alpha/2} \le x_s \le \epsilon_s - \lambda t^{\alpha/2}\}$. *In particular, for* $n = 1$ *the estimate*

$$|u(t,x) - u(t,y)| \le M\epsilon^{-1/2}[(\lambda\epsilon^{-1} + \epsilon^{\alpha/2})t^{(\alpha-1)/2} + \epsilon^{-1/2}]|x - y|$$

is valid for all points (t,x), (t,y) *lying outside the curvilinear trapezoid which is formed by the lines* $t = 0$; $t = \epsilon_0$; $x = -\epsilon + \lambda t^{\alpha/2}$; $x = \epsilon - \lambda t^{\alpha/2}$; $\lambda = \epsilon^{1+\alpha/2}$.

7. We can show that the estimates obtained in Theorems 1–6 are exact with respect to the order of smallness of the variables t, ϵ_k, ϵ_0 which are contained in the right-hand sides of those estimates. Indeed, let first $t < \epsilon_0$. In this case for the solution of the Cauchy problem

$$\epsilon u''_{xx} - u'_t = 0, \quad u|_{t=0} = u_0(x),$$

where $u_0(x) = 0$ for $x \le 0$ and $u_0(x) = 1$ for $x > 0$, we have $u'_x(t,0) = (4\pi\epsilon t)^{-1/2}$, and therefore for $x = 0$ the estimate of Theorem 1 is non-improvable. If as the initial we take the function $u_0(x) = 0$ for $x < 0$ and $u_0(x) = x^\alpha$ for $0 \le x < 1$, $0 < \alpha < 1$, where the function $u_0(x)$ and its derivative of the first order is bounded and continuous for all $x > 0$, then for sufficiently small values ϵ the inequality $u'_x(t,0) > \pi^{-1/2}(4t\epsilon)^{(\alpha-1)/2}e^{-1}$ will be fulfilled. Thus the estimate of Theorem 6 cannot be improved with respect to the order of smallness of the variables t, ϵ for $t < \epsilon$.

Let now $t \gg \epsilon$. Consider the problem

$$\epsilon\frac{\partial^2 u}{\partial z^2} - u\frac{\partial u}{\partial z} - \frac{\partial u}{\partial t} = 0, \tag{15}$$

$$u|_{t=0} = u_0(z) = 1 \text{ for } z \le 0, \quad u_0(z) = -1 \text{ for } z > 0. \tag{16}$$

Using Hopf's construction [30], we can write out the expression for the spatial derivative of the function $u(t,z)$. Using then an asymptotic representation for the probability integral for large values of z we have that for $t = \mathcal{O}(1)$, $z = \mathcal{O}(\epsilon)$, $\epsilon \ll 1$ the relation

$$u'_z(t,z) = t^{-1} - (2\epsilon)^{-1}\,\mathrm{ch}^{-2}\left(z/(2\epsilon)\right) + \mathcal{O}(\epsilon^{1/2}) \tag{17}$$

is valid, and therefore the above-obtained estimates are exact for $t > \epsilon$.

Consider finally the equation

$$\epsilon_1 v''_{xx} + v''_{yy} - vv'_x - vv_y - v_t = 0,$$

where ϵ_1 is the root of the equation $2\epsilon_1(1+\sqrt{\epsilon_1})^{-2} = \epsilon$, $0 < \epsilon \ll 1$. Consider the solution of that equation, satisfying for $t = 0$ the condition $v|_{t=0} = v_0(x,y) = 1$, if $x + y\sqrt{\epsilon_1} \le 0$, $\quad v_0(x,y) = -1$, if $x + y\sqrt{\epsilon_1} > 0$. Obviously, the solution of that problem is the function $v(t,x,y) = u(t,(x + y\sqrt{\epsilon_1})/(1+\sqrt{\epsilon_1}))$, where $u(t,z)$ is the solution of the problem (15), (16). Therefore it follows from (17) that the theorems proven above provide us with the estimates which are exact with respect to the order of smallness of the parameters ϵ_k, ϵ_0 for $t > \epsilon$.

Remark 3. Let us consider the spaces of functions $L_{p,x}(a,b)$ and $L_p(a,b)$, where $p \geq 1$, $a < b$,

$$||v||_{L_{p,x}(a,b)} = \sup_{0<t\leq T}\left(\int_a^b |v|^p dx\right)^{1/p}, \quad ||v||_{L_p(a,b)} = \left(\int_0^T \int_a^b |v|^p dx dt\right)^{1/p}$$

for any finite a, b, p. It follows from the relation (17) that in the spaces $L_{p,x}(a,b)$ $L_p(a,b)$, for the derivatives with respect to the spatial variables of the solution of the problem (1), (2) for the bounded initial function $u_0(x)$, the best estimates in the sense of the order of the parameter ϵ_0 take place for $p = 1$; the derivatives in a general case have in the norms $L_{1,x}$, L_1 the zero order of smallness with respect to the parameter ϵ_0, even if $b - a = \mathcal{O}(\epsilon_0)$.

8. In conclusion, we present some estimates of solutions of the Cauchy problem for the initial equation in the case of one spatial variable; those estimates make the constants appearing in the above-proven inequalities more precise and will be used later on.

For all $0 \leq t \leq T$, $-\infty < x < \infty$, $|u(t,x)| < \infty$, the functions $\phi(t,x,u)$, $\psi(t,x,u)$ will be assumed to have uniformly bounded derivatives of the first and second order, $\phi''_{uu}(t,x,u) \geq \phi_0 > 0$.

Theorem 8. *Everywhere in the strip* $\prod_T$, *the estimates*

$$\inf_{-\infty<x<\infty}[\epsilon u_0'(x) - \phi(0,x,u_0(x))] \leq \epsilon u_x'(t,x) - \phi(t,x,u) \leq$$
$$\leq \sup_{-\infty<x<\infty}[\epsilon u_0'(x) - \phi(0,x,u_0(x))]$$

hold.

Proof. Function $v_1(t,x) = [\epsilon u_x' - \phi(t,x,u)]e^{-\alpha_0 t}$ satisfies the equation

$$\epsilon 2 v''_{1xx} - \phi'_u(u)v'_{1x} - v'_{1t} - [\alpha_0 + \psi'_u]v_1 = e^{-\alpha_0 t}[\epsilon\psi'_x + \phi\psi'_u - \psi\phi'_u + \phi'_t] \equiv F_1(t,x)$$

and also the initial condition $v_1(0,x) = \epsilon u_0'(x) - \phi(0,x,u_0(x))$. Consider an auxiliary function $z(t,x) = \alpha_1(x^2+1)^{\alpha_2}$, where α_1, α_2 are some constants and $\alpha_2 < 1/2$. If we assume $\alpha_0 = 1 - \inf\limits_{\prod_T} \psi'_u + \alpha_2[2\epsilon + \sup\limits_{\prod_T} |\phi'_u|]$, $\alpha_1 = \max\{\sup\limits_{\prod_T} |F_1(t,x)|, \sup\limits_x |u_0(x)|\}$ then for any $0 < \alpha_2 < 1/2$ everywhere in the strip $\prod_T$ the inequalities $Lz < -z < 0$, $L(z \pm v_1) < 0$, $(z \pm v_1)|_{t=0} \geq 0$ are fulfilled. Choose a number N so large that the functions $z \pm v_1$ on lateral sides of the rectangle $D\{0 \leq t \leq T, |x| \leq N\}$ would take non-negative values. According to the maximum principle, these functions are non-negative everywhere in $\prod_T$, and therefore $|\epsilon u_x' - \phi(t,x,u)| \leq \alpha_1 e^{\alpha_0 t}(x^2+1)^{\alpha_2}$. Obviously, the last inequality is valid at any point of the strip $\prod_T$. Passing to the limit as $\alpha_2 \to 0$, we arrive at the assertion of the theorem. $\square$

Corollary 2. *If $\phi(t,x,u) \equiv \phi(u)$, $\psi(t,x,u) \equiv 0$, then the inequalities*

$$\inf_{-\infty<x<\infty}[\epsilon u_0'(x) - \phi(u_0(x))] \le \epsilon u_x'(t,x) - \phi'(u) \le \sup_{-\infty<x<\infty}[\epsilon u_0'(x) - \phi(u_0(x))]$$

are fulfilled.

Theorem 9. *If $\epsilon|u_0'(x)| + \epsilon^2|u_0''(x)| \le M$, then everywhere in $\prod_T$ the estimates*

$$|\epsilon^2 u_{x,x}''(t,x)| \le M, \quad |\epsilon u_t'(t,x)| \le M$$

are true.

Proof. Let $N > 0$ be any positive number and $G_N(x)$ be a twice continuously differentiable function, $G_N(x) \equiv 1$ for $|x| \le N$, $G_N(x) \equiv 0$, for $|x| \ge N+1$, $0 \le G_N(x) \le 1$, $[G_N'(x)]^2/G_N(x) \le M$. Let $v_2(t,x) = G_N(x)[\epsilon v_{1x}' + v_1^2]$, where the function $v_1(t,x)$ has been defined in Theorem 8. The function $v_2(t,x)$ on the lateral sides of the rectangle $D\{(t,x)|0 \le t \le T,\ |x| \le N+1\}$ vanishes. If the function $v_2(t,x)$ reaches its greatest (for the rectangle D) positive value for $t = 0$, then for $x \le N$ there takes place the relation

$$\epsilon^2\frac{\partial^2 u}{\partial x^2} \le \sup_D\left\{\epsilon\frac{\partial\phi(t,x,u)}{\partial x} - v_0^2(t,x) + \epsilon^2 u_0''(x) + \epsilon\frac{d\phi(0,x,u_0(x))}{dx} + v_0^2(0,x)\right\},$$

from which, with regard for Theorem 8, we obtain the upper bound for the function $\partial^2 u/\partial x^2$. However, if the largest (for the rectangle D) positive value is attained at the point $P_0(t_0,x_0)$ lying in the rectangle D or on its upper side, then at that point we have the relation

$$\begin{aligned} &G_N(x)\left[\epsilon\frac{\partial^2 v_2}{\partial x^2} - \phi_u'\frac{\partial v_2}{\partial x} - \frac{\partial v_2}{\partial t} - (\psi_u' + \phi_{ux}'')v_2\right] - \\ &-G_N^2(x)\left[2\epsilon\frac{\partial^2 v_1}{\partial x^2} + 2\epsilon\phi_{uu}''\frac{\partial u}{\partial x}\frac{\partial v_1}{\partial x} + F_2(t,x) + F_3(t,x)\right] - \\ &-v_2(t,x)[\epsilon G_N'' - \phi_u' G_N' - 2\epsilon(G_N')^2/G_N] = 0, \end{aligned} \tag{18}$$

where the functions $F_2(t,x) = \epsilon\psi_{uu}'' u_x' v_1 + \epsilon\psi_{ux}'' v_1 + \epsilon F_{1x}'$, $F_3(t,x) = 2v_0F_1(t,x) + [\psi_u' - \phi_{ux}'']v_1^2$ are bounded in the rectangle D uniformly with respect to ϵ. As it follows from equation (18), at the point P_0 the inequality

$$\begin{aligned} &G_N^2(x)\left[2\epsilon\left(\frac{\partial v_1}{\partial x}\right)^2 + \epsilon\phi_{uu}''\frac{\partial u}{\partial x}\frac{\partial v_0}{\partial x} + F_2(t,x) + F_3(t,x)\right] + \\ &+G_N(x)[\psi_u' + \phi_{ux}'']v_2 + v_2[\epsilon G_N''(x) - \phi_u' G_N'(x) - 2\epsilon(G_N'(x))^2/G_N(x)] \le 0 \end{aligned}$$

is valid, or what comes to the same thing, $v_2^2(x) + A_\epsilon(t,x)v_2 + B_\epsilon(t,x) \le 0$, where $A_\epsilon(t,x)$, $B_\epsilon(t,x)$ are some functions, bounded in $\prod_T$ uniformly with respect to ϵ. But the last inequality can be satisfied only for $v_2 \le M$, which ensures the upper bound for the function $\epsilon\partial^2 u/\partial x^2$ for all $|x| \le N$. Passing to the limit as $N \to \infty$, we obtain the required upper bound for the whole strip $\prod_T$.

To obtain the lower bound it suffices to consider the auxiliary function $v_3(t,x) = G_N(x)[\epsilon v_{1x} - v_1^2]$ and to repeat our reasoning. The estimate for the derivative with respect to the variable t follows from equation (1). □

Using the obtained estimates, we can prove the following assertions ([66]).

Theorem 10. *Let* $\|\phi''_{xx} + \psi'_x\|_{L_1(\prod_T)} \le M$. *If* $\|u'_0(x)\|_{L_1(\prod_T)} \le M$, *then* u'_t, u'_x, u''_{xx}, *are bounded uniformly with respect to* ϵ *in the norm of the space* $L_1(\prod_T)$.

Theorem 11. *If* $\|\psi + \phi'_x\|_{L_1(\prod_T)} \le M$, $\|u'_x\|_{L_1(\prod_T)} + \|\epsilon u''_{xx}\|_{L_1(\prod_T)} \le M$, *then for all* $t \in [0,T]$ *the inequality* $\int_{-\infty}^{\infty} |u(t,x) - u_0(x)|dx \le M$ *is valid. Moreover, if there exist* $\lim\limits_{x\to-\infty} u_0(x) = u^-$, $\lim\limits_{x\to\infty} u_0(x) = u^+$ *and the integrals*

$$I_1 = \int_{-\infty}^{0} |u_0(x) - u^-|dx, \qquad I_2 = \int_{0}^{\infty} |u_0(x) - u^+|dx$$

converge, then the integrals

$$\tilde{I}_1 = \int_{-\infty}^{0} |u(t,x) - u^-|dx, \qquad \tilde{I}_2 = \int_{0}^{\infty} |u(t,x) - u^+|dx$$

also converge simultaneously, and $\lim\limits_{x\to-\infty} u(t,x) = u^-$ *and* $\lim\limits_{x\to\infty} u(t,x) = u^+$.

1.2. Some a Priori Estimates for a System of Quasi-Linear Equations

In this section we deduce some integral and uniform estimates for solutions of the Cauchy problem associated with a quasi-linear system of singularly perturbed equations of parabolic type. In particular, these estimates characterize the behavior of a solution and its derivatives as a small parameter tends to zero and the time derivative increases infinitely. They are proved to be useful in investigating the properties of solutions of model problems of gas dynamics.

1. In the strip $\Pi_T = \{(t,x) | 0 < t \le T, -\infty < x < \infty\}$, let us consider the Cauchy problem

$$\epsilon \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = \frac{d}{dx}\phi(t,x,v), \tag{1}$$

$$\epsilon \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} = -\frac{\partial u}{\partial x}, \tag{2}$$

$$u|_{t=0} = u_0(x), \quad v|_{t=0} = v_0(x), \tag{3}$$

where ϵ is a non-negative constant, and $u_0(x)$ and $v_0(x)$ are continuous bounded functions possessing bounded derivatives of the first and second order. We will assume that $v_0(x) \ge m_0 > 0$, where m_0 is a constant not depending on the parameter ϵ. Suppose that the function $\phi(t,x,v)$ is continuous and bounded in each of the domains $D_a = \Pi_\infty \times \{a \le v \le \infty\}$, where a is an arbitrary positive constant, and possesses in D_a the continuous uniformly bounded derivatives up to the fourth order, inclusive, with respect to either variable, $\phi'_v(t,x,v) < 0$ and $\phi''_{vv}(t,x,v) \ge 0$, and the function $F(t,x,v) = -\int\limits_{m_0}^{v} \sqrt{-\phi'_s(t,x,s)}ds$ increases infinitely as $v \to +0$.

For $\phi(t,x,v) \equiv (\sqrt{2}v)^{-2}$, $\epsilon = 0$ the problem (1)–(3) describes motion of shallow water and isentropic gas motion in terms of the Lagrange coordinates in the case $c_p/c_v = 2$.

In [84], for the case $\phi(t,x,v) \equiv \phi(v)$, T.D. Ventzel by means of the change of variables $f^\pm = -F(t,x,v) \pm u(t,x)$ has shown that in the strip Π_T the inequalities $|u(t,x)| \le M$, $v(t,x) \ge m > 0$ hold, where the constants M, m do not depend on ϵ. In the same work it has been proved that under the above formulated conditions for the function $u_0(x)$, $v_0(x)$, $\phi(v)$ the solution of the problem exists everywhere in Π_T.

Making the change of variables $\tau = t/\epsilon$, $\xi = x/\epsilon$ and denoting again the independent variables by t, x, we obtain the problem

$$\frac{\partial^2 u_1}{\partial x^2} - \frac{\partial u_1}{\partial t} = \frac{d\phi(t,x,v_1)}{dx}, \tag{4}$$

$$\frac{\partial^2 v_1}{\partial x^2} - \frac{\partial v_1}{\partial t} = -\frac{\partial u_1}{\partial x}, \tag{5}$$

$$u_1|_{t=0} = \tilde{u}_1(x), \quad v_1|_{t=0} = \tilde{v}_1(x), \tag{6}$$

where $\tilde{u}_1(x)$, $\tilde{v}_1(x)$ are given functions. In what follows, the index in notation of the solution of the problem (4)–(6) will be omitted and for the solution of the equations (4), (5) the use will be made of the conventional notation, i.e., $u(t,x)$ and $v(t,x)$. The solution of the problem (4)–(6) will be assumed to exist everywhere in Π_∞ and the inequalities $v(t,x) \ge m > 0$, $|u(t,x)| \le M$ for that solution to be fulfilled everywhere in Π_∞.

2. If $G(t,x,\xi,\tau) = [4\pi(t-\tau)]^{-1/2} \exp\{-(x-\xi)^2/[4(t-\tau)]\}$, then the problem (4)–(6) can be written in the form

$$u(t,x) = \int_{-\infty}^{\infty} G(t,x,\xi,0)u_0(\xi)d\xi +$$

$$+\frac{1}{2}\int_0^t d\tau \int_{-\infty}^{\infty} \phi(\tau,\xi,v)(t-\tau)^{-1}(x-\xi)G(t,x,\xi,\tau)d\xi, \tag{7}$$

$$v(t,x) = \int_{-\infty}^{\infty} G(t,x,\xi,0)v_0(\xi)d\xi -$$

$$-\frac{1}{2}\int_0^t d\tau \int_{-\infty}^{\infty} (t-\tau)^{-1}(x-\xi)G(t,x,\xi,\tau)u(\tau,\xi)d\xi. \tag{8}$$

The estimate

$$v(t,x) \le M\sqrt{t+1} \tag{9}$$

can be immediately obtained from (8). Substituting (7) in (8) and changing the order of integration, after elementary transformations we obtain

$$v(t,x)=\int_{-\infty}^{\infty}G(t,x,\xi,0)v_0(\xi)d\xi-\frac{1}{2}\int_{-\infty}^{\infty}(x-\xi)G(t,x,\xi,0)u_0(\xi)d\xi+$$

$$+\frac{1}{2}\int_0^t d\tau\int_{-\infty}^{\infty}\phi(\tau,\xi,v)\left[1-\frac{(x-\xi)^2}{2(t-\tau)}\right]G(t,x,\xi,\tau)d\xi=I_1-\frac{1}{2}I_2+\frac{1}{2}I_3. \quad (10)$$

From (10) we readily obtain the estimate for the derivative of the function $v(t,x)$ with respect to the variable x:

$$|v'_x(t,x)|\le M\sqrt{t+1}. \quad (11)$$

Theorem 1. *Let the functions $u_0(x)$, $v_0(x)$ have the limits u^-, v^-, respectively, as $x\to-\infty$ and the limits u^+, v^+ as $x\to+\infty$, and the functions $\phi(t,-x,u^-)$, $\phi(t,x,u^+)$ have as $x\to\mp\infty$ the limits ϕ^-, ϕ^+. Then for every fixed value $t=t_0$ the functions $u(t,x)$, $v(t,x)$ have the same limits at infinity as they have for $t=0$.*

Proof. Consider first the function $v(t,x)$ defined by equation (10). For the definiteness, let $x\to+\infty$. Obviously,

$$I_1=\int_{-\infty}^{x/2}v_0(\xi)G(t,x,\xi,0)d\xi+\int_{x/2}^{\infty}[v_0(\xi)-v^+]G(t,x,\xi,0)d\xi+$$

$$+v^+\int_{x/2}^{\infty}G(t,x,\xi,0)d\xi=I_{1,1}+I_{1,2}+I_{1,3}.$$

For $x\to\infty$, we shall estimate each summand on the right-hand side of the last equation. For values x, such that $x/(2t)\gg 1$, we have

$$.|I_{1,1}|\le M\int_{-\infty}^{x/2}G(t,x,\xi,0)d\xi\le M\int_{x/(4\sqrt{t})}^{\infty}\exp(-z^2)dz.$$

Using Millse's relation ([48]), we obtain the inequality

$$|I_{1,1}|\le M\exp[-x^2/(4t^2)][x/(4\sqrt{t})+\sqrt{\omega+x^2/(16t)}]^{-1},$$

where $4/\pi\le\omega\le 2$. It is not difficult to see that $I_{1,3}\to v^+$ as $x\to\infty$, where

$$|I_{1,3}-v^+|=|v^+|\exp[-x^2/(4t^2)]\left\{\sqrt{\pi}[x/(4\sqrt{t})+\sqrt{\omega+x^2/(16t)}]\right\}^{-1}.$$

For the integral $I_{1,2}$ we can easily obtain the inequality $|I_{1,2}|\le\sup_{\xi\ge x/2}|v_0(\xi)-v^+|$. Further,

$$I_2=\int_{-\infty}^{x/2}G(t,x,\xi,0)(x-\xi)[u_0(\xi)-u^+]d\xi+$$

$$+\int_{x/2}^{\infty}G(t,x,\xi,0)(x-\xi)[u_0(\xi)-u^+]d\xi+$$

$$+\sqrt{t/\pi}(u^- - u^+)\exp\left[-x^2/(16t)\right] = I_{2,1} + I_{2,2} + I_{2,3}.$$

For the integral $I_{2,1}$ the estimate $|I_{2,1}| \le M\sqrt{t}\exp\left[-x^2/(16t)\right]$ is valid. Passing to the estimation of the summand $I_{2,2}$, we get

$$I_{2,2} = 2\sqrt{t/\pi}\int_{-x/(4\sqrt{t})}^{-x/(8\sqrt{t})}\left[u_0(x+2z\sqrt{t}) - u^+\right]z\exp(-z^2)dz -$$

$$-2\sqrt{t/\pi}\int_{-x/(8\sqrt{t})}^{\infty}\left[u_0(x+2z\sqrt{t}) - u^+\right]z\exp(-z^2)dz -$$

$$-\int_{-x/(8\sqrt{t})}^{\infty}\left[u_0(x+2z\sqrt{t}) - u^+\right]z\exp(-z^2)dz.$$

Obviously, for $x/(4\sqrt{t}) \gg 1$ the following inequalities hold:

$$\left|\int_{-x/(4\sqrt{t})}^{-x/(8\sqrt{t})}\left[u_0(x+2z\sqrt{t}) - u^+\right]z\exp(-z^2)dz\right| \le M\exp\Big(-\frac{x^2}{64t}\Big),$$

$$\left|\int_{-x/(8\sqrt{t})}^{\infty}\left[u_0(x+2z\sqrt{t}) - u^+\right]z\exp(-z^2)dz\right| \le M\sup_{\xi\ge 3x/4}|u_0(\xi) - u^+|.$$

Finally we pass to the limit I_3. Partition it in three summands

$$I_3 = \int_0^t d\tau\int_{-\infty}^{x-\alpha(x)}\phi(\tau,\xi,v)\Big[1 - \frac{(x-\xi)^2}{2(t-\tau)}\Big]G(t,x,\xi,\tau)d\xi +$$

$$+\int_0^t d\tau\int_{x+\alpha(x)}^{\infty}\phi(\tau,\xi,v)\Big[1 - \frac{(x-\xi)^2}{2(t-\tau)}\Big]G(t,x.\xi,\tau)d\xi +$$

$$+\int_0^t d\tau\int_{x-\alpha(x)}^{x+\alpha(x)}\phi(\tau,\xi,v)\Big[1 - \frac{(x-\xi)^2}{2(t-\tau)}\Big]G(t,x.\xi,\tau)d\xi = I_{3,1} + I_{3,2} + I_{3,3},$$

where $\alpha(x)$ a monotonically increasing function, $\alpha(x) < x/2$ for $x > 0$ and $\lim\limits_{x\to\infty}\alpha(x) = \infty$. By simple calculations, for $x/\sqrt{t} \gg 1$ we obtain the estimate

$$|I_{3,1}| \le 4M\alpha(x)\sqrt{t}\exp\Big\{-\frac{[\alpha(x)]^2}{16t}\Big\}\frac{\sqrt{16t\omega + [\alpha(x)]^2} - \alpha(x)}{\sqrt{16t\omega + [\alpha(x)]^2} + \alpha(x)},\quad \frac{4}{\pi} \le \omega \le 2.$$

Hence $|I_{3,1}| \le Mt^{3/2}[\alpha(x)]^{-1}\exp\{-[\alpha(x)]^2/(16t)\}$. An analogous estimate can be obtained for the integral $I_{3,2}$. Thus, using the above-obtained estimates, we can write equality (10) as follows:

$$z(t,x) \equiv v(t,x) - v^+ = g(t,x) +$$

$$+2^{-1}\int_0^t d\tau\int_{x-\alpha(x)}^{x+\alpha(x)}[\phi(\tau,\xi,v) - \phi(\tau,\xi,v_+)]\Big[1 - \frac{(x-\xi)^2}{2(t-\tau)}\Big]G(t,x,\xi,\tau)d\xi; \quad (12)$$

here $g(t,x)$ is the function, tending to zero as $x \to +\infty$. On considering equality (12) as the integral equation with regard to the function $z(t,x)$, we solve it by the method of successive approximations; note that as the initial approximation $z_0(t,x)$ we shall take the function $g(t,x)$.

Let inequality $x \geq 2N$ hold, where the number N is chosen in such a way that for $x \geq N$ the relation $|g(t,x)| \leq \mu$ is fulfilled. Supposing $x - 2\alpha(x) \geq N$, we shall have

$$|z_1(t,x) - z_0(t,x)| \leq 2^{-1}\int_0^t d\tau \int_{x-\alpha(x)}^{x+\alpha(x)} \int_0^1 \Big|\frac{\partial\phi(\tau,\xi,v^+ + \theta z_0(\tau,\xi))}{\partial v}\Big| d\theta \times$$

$$\times |v_0(\tau,\xi) - v^+|\Big|1 - \frac{(x-\xi)^2}{2(t-\tau)}\Big| G(t,x,\xi,\tau)d\xi e M_v t\mu/2,$$

$$M_v = \sup_{(t,x)\in\Pi_t, m\leq v\leq M\sqrt{t+1}} |\phi'_v(t,x,v)|,$$

and this inequality is valid for all $x \geq N$. The inequality $|v_2(t,x) - v_1(t,x)| \leq (M_v t/2)^2\mu$ can be obtained analogously. If $M_v t/2 < 1$, then the sequence $\{z_n(t,x)\}$ converges, and the terms of that sequence are uniformly bounded by some constant like $M\mu$. Hence, for the function $v(t,x)$ for $t < M_v^{-1}$ and $x \geq 2N$ the relation $|v(t,x) - v^+| \leq M\mu$ is fulfilled. Repeating our reasoning successively for $kM_v^{-1} \leq t \leq (k+1)M_v^{-1}$, we shall get the validity of the assertion of the theorem with respect to the function $v(t,x)$ for all $t > 0$.

Estimate now the difference $u(t,x) - u^+$. It can be easily seen that the first summand in the expression (7) is investigated exactly in the same manner as the first one in the expression (10). Let us consider the second summand for $x \geq 2N$:

$$2^{-1}\int_0^1 d\tau \int_{-\infty}^{\infty} \phi(\tau,\xi,v)(t-\tau)^{-1}(x-\xi)G(t,x,\xi,\tau)d\xi = -\tilde I_1 - \tilde I_2 =$$

$$= -\int_0^t d\tau \int_{-\infty}^{-\alpha(x)/(2\sqrt{t-\tau})} \phi(\tau, x + 2z\sqrt{t-\tau}, v)(t-\tau)^{-1/2} zG(z,4^{-1},0,0)dz -$$

$$-\int_0^t d\tau \int_{-\alpha(x)/(2\sqrt{t-\tau})}^{\infty} \phi(\tau, x + 2z\sqrt{t-\tau}, v)(t-\tau)^{-1/2} zG(z,4^{-1},0,0)dz.$$

It is not difficult to see that for for sufficiently large values of N the inequality

$$|\tilde I_1| \leq M\int_0^t (t-\tau)^{-1/2}\exp\Big[-\frac{x^2}{16(t-\tau)}\Big]d\tau \leq M\sqrt{t}\exp\left[-x^2/(16t)\right]$$

is valid. The integral $\tilde I_2$ in the following way:

$$\int_0^t (t-\tau)^{-1/2}d\tau \int_{-\alpha(x)/(2\sqrt{t})}^{\infty} [\phi(\tau, x+2z\sqrt{t-\tau}, v^+) - \phi^+] zG(z,4^{-1},0,0,)dz +$$

$$+\phi^+\int\limits_0^t G(x,t,x-\alpha(x),\tau)d\tau+\int\limits_0^t(t-\tau)^{-1/2}d\tau\int\limits_{-\alpha(x)/(2\sqrt{t})}^{\infty}[\phi(\tau,x+2z\sqrt{t-\tau},v)-$$

$$-\phi(\tau,x+2z\sqrt{t-\tau},v^+)]zG(z,4^{-1},0,0)dz=\tilde{I}_{2,1}+\tilde{I}_{2,2}+\tilde{I}_{2,3}.$$

Using the estimates for the function $v(t,x)-v^+$, we obtain for $x\geq 2N$:

$$|\tilde{I}_{2,1}|\leq \sup_{\xi\geq N, 0\leq\tau\leq t}|\phi(\tau,\xi,v^+)-\phi^+|,$$

$$|\tilde{I}_{2,2}|\leq M\sqrt{t}\exp\left\{-\frac{[\alpha(x)]^2}{4t}\right\},$$

$$|\tilde{I}_{2,3}|\leq \sup_{\xi\geq N, 0\leq\tau\leq t}|v(\tau,\xi)-v^+|.$$

From these inequalities follows the relation $\lim\limits_{x\to\infty}|u(t,x)-u^+|=0$. Obviously, the case $x\to-\infty$ is considered analogously. □

Remark 1. From the above estimates follow the estimates for the rate of convergence of the functions $u(t,x)$, $v(t,x)$ to the corresponding limiting values as $|x|\to\infty$. This rate depends on t and on the rate of convergence of the functions $u_0(x)$, $v_0(x)$, $\phi(t,x,v_-)$, $\phi(t,x,v^+)$ to their limiting values.

Remark 2. It follows from the above reasoning that the behavior of the function $v(t,x)$ as $x\to+\infty$ does not depend on the character of variation of the function $u_0(x)$ as $|x|\to\infty$ and of the function $v_0(x)$ as $x\to-\infty$.

The proof of the theorem below is the same as that of Theorem 1.

Theorem 2. *If* $\lim\limits_{|x|\to\infty}(|u_0'(x)|+|v_0'(x)|)=0$, $\lim\limits_{|x|\to\infty}(|\phi'(t,-x,v^-)|+|\phi'(t,x,v^+)|)=0$, *then* $\lim\limits_{|x|\to\infty}(|u_x'(t,x)|+|v_x'(t,x)|)=0$.

Theorem 3. *If the conditions of Theorems* 1 *and* 2 *are fulfilled and, moreover, if the integrals*

$$I_1(t,x)=\int_{-\infty}^{x}[v(t,\xi)-v^-]d\xi,\quad I_2(t,x)=\int_{x}^{\infty}[v(t,\xi)-v^+]d\xi,$$
$$I_3(t,x)=\int_{-\infty}^{x}[u(t,\xi)-u^-]d\xi,\quad I_4(t,x)=\int_{x}^{\infty}[u(t,\xi)-u^+]d\xi$$

exist for $t=0$, *then they do exist for any* $t>0$ *and the following equalities hold:*

$$\begin{cases} I_1(t,0)+I_2(t,0)=I_1(0,0)+I_2(0,0)+(u^+-u^-)t,\\ I_3(t,0)+I_4(t,0)=I_3(0,0)+I_4(0,0)+(\phi^+-\phi^-)t.\end{cases}\tag{13}$$

Theorem 4 is proved in the same way as Lemma 4 in [34].

Corollary 1. *If $v^- \neq v^+$, then*

$$I_{1,a}(t,0) + I_{2,a}(t,0) = \int_{-\infty}^{0} [v(t, x - at) - v^-]dx +$$

$$+ \int_{0}^{\infty} [v(t, x - at) - v^+]dx = I_{1,a}(0,0) + I_{2,a}(0,0),$$

where $a = (u^+ - u^-)(v^+ - v^-)^{-1}$; if $u^- \neq u^+$, then

$$I_{3,b}(t,0) + I_{4,b}(t,0) = \int_{-\infty}^{0} [u(t, x - bt) - u^-]dx +$$

$$+ \int_{0}^{\infty} [u(t, x - bt) - u^+]dx = I_{3,b}(0,0) + I_{4,b}(0,0),$$

where $b = (\phi^- - \phi^+)(u^+ - u^-)^{-1}$.

Theorem 4. *Let the function $\phi(t, x, v^-) - \phi^-$ be absolutely integrable with respect to the variable x for $x \in (-\infty, 0]$, and the function $\phi(t, x, v^+) - \phi^+$ be absolutely integrable for $x \in [0, \infty)$. Let, moreover, the integrals*

$$I_1(t) = \int_{-\infty}^{0} |v(t,x) - v^-|dx, \quad I_2(t) = \int_{0}^{\infty} |v(t,x) - v^+|dx,$$

$$_3(t) = \int_{-\infty}^{0} |u(t,x) - u^-|dx, \quad 4(t) = \int_{0}^{\infty} |u(t,x) - u^+|dx$$

converge for $t = 0$. Then these integrals converge for any $t > 0$ and the inequalities

$$I_1(t) + I_2(t) \leq Me^t(1 + t^{3/2}), \quad I_3(t) + I_4(t) \leq Me^t(1 + t^{3/2}) \qquad (14)$$

hold.

Proof. Consider first the integral $I_1(t)$. Denote the function $[1 - 2^{-1}(x - \xi)^2/(t - \tau)]G(t, x, \xi, \tau)$ by $Q(t, x, \xi, \tau)$. We have

$$v(t,x) - v^- = \int_{-\infty}^{\infty} [v_0(\xi) - v^-]G(t, x, \xi, 0)d\xi -$$

$$-2^{-1} \int_{-\infty}^{\infty} u_0(\xi)(x - \xi)G(t, x, \xi, 0)d\xi +$$

$$+2^{-1} \int_{0}^{t} d\tau \int_{-\infty}^{\infty} \phi(\tau, \xi, v)Q(t, x, \xi, \tau)d\xi = P_1 - 2^{-1}P_2 + 2^{-1}P_3.$$

Obviously,

$$P_1 = \int_{-\infty}^{0} [v_0(\xi) - v^-]Gt, x, \xi, 0)d\xi +$$

$$+ \int_{0}^{\infty} [v_0(\xi) - v^-]G(t, x, \xi, 0)d\xi = P_{1,1}(t,x) + P_{1,2}(t,x),$$

whence

$$\int_{-\infty}^{0}|P_{1,1}(t,x)|dx \le \int_{-\infty}^{0}|v_0(\xi)-v^-|\left\{\int_{-\infty}^{0}G(t,x,\xi,0)dx\right\}d\xi \le I_1(0),$$
$$\int_{-\infty}^{0}|P_{1,2}(t,x)|dx \le M\int_{0}^{\infty}\left\{\int_{-\infty}^{0}G(t,x,\xi,0)dx\right\}d\xi \le M\sqrt{t}.$$

Pass now to the estimate of the integral P_2. Let us write it in the form

$$\int_{-\infty}^{0}[u_0(\xi)-u^-](x-\xi)G(t,x,\xi,0)d\xi+\int_{0}^{\infty}[u_0(\xi)-u^+](x-\xi)G(t,x,\xi,0)d\xi+$$
$$+(u^+-u^-)\sqrt{\frac{t}{\pi}}\exp\Big[-\frac{x^2}{4t}\Big]=P_{2,1}(t,x)+P_{2,2}(t,x)+P_{2,3}(t,x),$$

and hence

$$\int_{-\infty}^{0}|P_{2,1}(t,x)|dx \le M\sqrt{t}\,I_3(0),$$
$$\int_{-\infty}^{0}|P_{2,2}(t,x)|dx \le M\sqrt{t}I_4(0),\quad \int_{-\infty}^{0}|P_{2,3}(t,x)|dx=|u^+-u^-|t.$$

Finally, let us estimate the integral $\int_{-\infty}^{0}|P_3(t,x)|dx$. We represent the function $P_3(t,x)$ as

$$\begin{aligned}P_3(t,x)&=\int_0^t d\tau\int_{-\infty}^{0}[\phi(\tau,\xi,v)-\phi(\tau,\xi,v^-)]Q(t,x,\xi,\tau)d\xi+\\&+\int_0^t d\tau\int_0^{\infty}[\phi(\tau,\xi,v)-\phi(\tau,\xi,v^+)]Q(t,x,\xi,\tau)d\xi+\\&+\Big\{\int_0^t d\tau\int_{-\infty}^{0}\phi(\tau,\xi,v^-)Q(t,x,\xi,\tau)d\xi+\\&+\int_0^t d\tau\int_0^{\infty}\phi(\tau,\xi,v^+)Q(t,x,\xi,\tau)d\xi\Big\}=\\&=P_{3,1}(t,x)+P_{3,2}(t,x)+P_{3,3}(t,x)+P_{3,4}(t,x).\end{aligned}$$

For the function $P_{3,1}(t,x)$ we shall have

$$\int_{-\infty}^{0}|P_{3,1}|dx \le$$
$$\le\int_0^t d\tau\int_{-\infty}^{0}|\phi(\tau,\xi,v)-\phi(\tau,\xi,v^-)|\int_{\xi-\sqrt{2(t-\tau)}}^{-\infty}Q(t,x,\xi,\tau)dxd\xi+$$
$$+\int_0^t d\tau\int_{-\infty}^{0}|\phi(\tau,\xi,v)-\phi(\tau,\xi,v^-)|\int_0^{\xi+\sqrt{2(t-\tau)}}Q(t,x,\xi,\tau)dxd\xi+$$

$$+\int_0^t d\tau \int_{-\infty}^0 |\phi(\tau,\xi,v)-\phi(\tau,\xi,v^-)| \int_{\xi-\sqrt{2(t-\tau)}}^{\xi+\sqrt{2(t-\tau)}} Q(t,x,\xi,\tau)dxd\xi =$$
$$= P_{3,1,1}+P_{3,1,2}+P_{3,1,3}.$$

Calculating the integral of the function $Q(t,x,\xi,\tau)$, we obtain

$$P_{3,1,1}\le M\int_0^t d\tau \int_{-\infty}^0 |v(\tau,x)–v^-|dx,\quad P_{3,1,2}\le M\int_0^t d\tau \int_{-\infty}^0 |v(\tau,\xi)–v^-|d\xi.$$

For the integral $P_{3,1,3}$ we can easily get the inequality

$$P_{3,1,3}=\left\{\frac{1}{2\sqrt{\pi}}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} e^{-s^2}ds+\frac{1}{\sqrt{2\pi}}e^{-1/2}\right\}\times$$
$$\times\int_0^t d\tau \int_{-\infty}^0 |\phi(\tau,\xi,v)-\phi(\tau,\xi,v^-)|d\xi \le$$
$$\le M\int_0^t d\tau \int_{-\infty}^0 |v(\tau,x)-v^-)|d\xi.$$

The integral of the function $P_{3,2}(t,x)$ is estimated analogously:

$$\int_{-\infty}^0 |P_{3,2}(t,x)|dx \le M\int_0^t d\tau \int_0^\infty |v(\tau,x)-v^+|dx.$$

Let us now pass to the integral $P_{3,3}$:

$$P_{3,3}=\int_0^t d\tau \int_{-\infty}^0 [\phi(\tau,\xi,v^-)-\phi^-]Q(t,x,\xi,\tau)d\xi +$$
$$+\phi^-\int_0^t d\tau \int_0^\infty t,x,\xi,\tau)d\xi = P_{3,3,1}(t,x)+P_{3,3,2}(t,x).$$

Obviously,

$$\int_{-\infty}^0 |P_{3,3,1}(t,x)|dx \le M\int_0^t d\tau \int_{-\infty}^0 |\phi(\tau,x,v^-)-\phi^-|dx,$$
$$\int_{-\infty}^0 |P_{3,3,2}(t,x)|dx = \frac{2}{3\sqrt{\pi}}t^{3/2}\phi^-.$$

We can easily see that the integral of the function $P_{3,4}$ is calculated exactly in the same manner.

Thus

$$\int_{-\infty}^0 |v(t,x)-v^-|dx \le M(1+\sqrt{t})+$$
$$+M\left\{\int_0^t d\tau \int_{-\infty}^0 |\phi(\tau,\xi,v^-)-\phi^-|d\xi + \int_0^t d\tau \int_0^\infty |\phi(\tau,\xi,v^+)-\phi^+|d\xi\right\}+$$

$$+M\int_0^t d\tau\int_{-\infty}^0 |v(t,x)-v^-|dx +^+ -u^-|t+(9\pi)^{-1/2}|\phi^+-\phi^-|t^{3/2}.$$

Similarly,

$$\int_0^\infty |v(t,x)-v^+|dx \le M(1+\sqrt{t})+$$

$$+M\left\{\int_0^t d\tau\int_{-\infty}^0 |\phi(\tau,\xi,v^-)\phi^-|d\xi+\int_0^t d\tau\int_0^\infty \phi(\tau,\xi,v^+)-\phi^+|d\xi\right\}+$$

$$+M\int_0^t d\tau\int_0^\infty |v(t,x)-v^+|dx+|u^+-u^-|t+(9\pi)^{-1/2}|\phi^+-\phi^-|t^{3/2}.$$

Adding term by term the last two inequalities and using the Gronwall-Bellman's lemma, we obtain the first inequality (14). The second one is proved in a similar manner. □

The following assertion is proved without any changes.

Theorem 5. *If* $v_0(x)-v^-$, $u_0(x)-u^-$, $\phi(t,x,v)-\phi^- \in L_{p,x}(-\infty,0)$ $v_0(x)-v^+$, $u_0(x)-u^+$, $\phi(t,x,v^+)-\phi^+ \in L_{p,x}(0,\infty)$, $p\ge 1$, *then* $v(t,x)-v^-$, $u(t,x)-u^- \in L_{p,x}(-\infty,0)$, $v(t,x)-v^+$, $u(t,x)-u^+ \in L_{p,x}(0,\infty)$.

Under appropriate assumptions on the initial data of the problem we can formulate analogous statements for the derivatives of the functions $u(t,x)$, $v(t,x)$.

Let us pass now to the estimates of the functions under consideration in the uniform norm.

Theorem 6. *Everywhere in the half-plane* $t>0$ *the estimates*

$$|u'_x(t,x)|+|u'_t(t,x)|+|v'_x(t,x)|+|v'_t(t,x)|+|u''_{xx}(t,x)|\le M\ln(e+t),$$
$$|v''_{xx}(t,x)|\le M\sqrt{\ln(e+t)}$$

are valid.

Proof. As follows from equality (8),

$$\frac{\partial^2 v}{\partial x^2}=-(2t)^{-1}\int_{-\infty}^\infty v_0(\xi)Q(t,x,\xi,0)d\xi+$$

$$+(4t)^{-1}\int_{-\infty}^\infty u_0(\xi)[3(x-\xi)-t^{-1}(x-\xi)^3]G(t,x,\xi,0)d\xi-$$

$$-4^{-1}\int_0^t d\tau\int_{-\infty}^\infty (t-\tau)^{-1}\phi(\tau,\xi,v)[3-3(t-\tau)^{-1}(x-\xi)^2+$$

$$+4^{-1}(t-\tau)^{-2}(x-\xi)^4]G(t,x,\xi,\tau)d\xi=-2^{-1}(K_1-K_2+K_3).$$

In proving the theorem we may assume $t\gg 1$, since for bounded values of the variable t the estimates of derivatives of functions can be obtained by the methods we have used in §1. Evidently, $|K_1|\le Mt^{-1}$, $|K_2|\le Mt^{-1/2}$, and hence to estimate the second derivative of the function $v(t,x)$ with respect to the variable x it sufficiently to estimate the integral K_3:

$$\frac{1}{2}\int_0^t d\tau \int_{-\infty}^{\infty} \tau^{-1}\phi(t-\tau, x+\xi, v(t-\tau, x+\xi)) \times$$
$$\times[3 - 3\tau^{-1}\xi^2 + 4^{-1}\tau^{-2}\xi^4]G(\tau,\xi,0,0)d\xi =$$
$$= 2^{-1}\int_0^\delta d\tau \int_{-\infty}^{\infty}\{\cdots\}d\xi + 2^{-1}\int_\delta^t d\tau \int_{-\infty}^{\infty}\{\cdots\}d\xi = K_{3,1} + K_{3,2},$$

where δ is some positive value which will be defined below. Since $|v'_x(t,x)| \le M\sqrt{t+1}$, we can easily obtain the estimate $|K_{3,1}| \le M\sqrt{\delta(t+1)}$. In the integral $K_{3,2}$, making the change of the variable $\xi = 2z\sqrt{\tau}$, we get $|K_{3,2}| \le M|\ln t - \ln\delta|$. Choosing $\delta = t^{-1}$, we obtain the intermediate estimate

$$|v''_{xx}(t,x)| \le M\ln(e+t).$$

The estimates

$$|u'_x(t,x)| \le M\ln(e+t), \quad |v'_x(t,x)| \le M\ln(e+t)$$

can be found analogously from formulas (7) and (8).

Estimate now the function $u''_{xx}(t,x)$. From (7) we have

$$u''_{xx}(t,x) = L_1(t,x) + L_2(t,x) = -(2t)^{-1}\int_{-\infty}^{\infty} u_0(\xi)Q(t,x,\xi)d\xi +$$
$$+4^{-1}\int_0^t d\tau \int_{-\infty}^{\infty}\phi(\tau,\xi,v)(t-\tau)^{-2}[2^{-1}(t-\tau)^{-1}(x-\xi)^3 - 3(x-\xi)]G(t,x,\xi,\tau)d\xi.$$

Obviously, $|L_1(t,x)| \le Mt^{-1}$. The integral $L_2(t,x)$ can be represented in terms of

$$L_2(t,x) = \int_0^{t-\delta} d\tau \int_{-\infty}^{\infty}\{\ldots\}d\xi + \int_{t-\delta}^t d\tau \int_{-\infty}^{\infty}\{\ldots\}d\xi = L_{2,1}(t,x) + L_{22}(t,x).$$

Integrating once by parts and using the obtained estimates for the derivatives of the function $v(t,x)$, we readily obtain the inequality $|L_{2,2}(t,x)| \le M\sqrt{\delta}\{1 + [\ln(t+1)]^2\}$. It is easily seen that the inequality $|L_{2,1}(t,x)| \le M\delta^{-1/2}$ holds. Choosing δ from the equality $\delta = [\ln(t+e)]^2$, we find the estimate $|u''_{xx}(t,x)| \le M\ln(t+e)$. Estimates for the derivatives of the functions under consideration with respect to the variable t follows from equations (4) and (5). Getting back to the estimate of the function $v''_{xx}(t,x)$, we are able, with regard for inequality (14), to find from (10) the required estimate for that derivative. $\square$

In conclusion, we can formulate an analogue of Theorem 6 in the form applicable to the problem (1)–(3).

Theorem 7. *Everywhere in the half-plane $t > 0$, for the solution of the problem* (1)–(3) *the estimates*

$$|\epsilon u'_x(t,x)|+|\epsilon u'_t(t,x)|+|\epsilon v'_x(t,x)|+|\epsilon v'_t(t,x)|+|\epsilon^2 u''_{xx}(t,x)|\leq M\ln(e+t/\epsilon),$$
$$|\epsilon^2 v''_{xx}(t,x)|\leq M\sqrt{\ln(e+t/\epsilon)}$$

are valid.

It should be noted that the results of the above theorem improve the results obtained in [84] for the cases $\epsilon\to 0$ and $t\to\infty$.

1.3. Quasi-Linear Parabolic Equation with a Discontinuous Initial Function

In this section we construct asymptotic representations for a solution of quasi-linear parabolic equation, when the solution of the corresponding degenerate problem is a discontinuous function.

1. In the strip $\prod_T\{(t,x)|0<t\leq T,-\infty<x<\infty\}$ let us consider the problem

$$L_\epsilon u\equiv\epsilon^2\frac{\partial^2 u}{\partial x^2}-\phi'_u(u)\frac{\partial u}{\partial x}-\frac{\partial u}{\partial t}=0,\ u(0,x)=f(x). \tag{1}$$

The function ϕ will be assumed to be infinitely differentiable, uniformly bounded for bounded values of the argument, $\phi(0)=\phi'(0)=0,\ \phi''(0)=1$, and $\phi''(u)\geq\phi_0>0$ for any u. Let the function $f(x)$ be continuous and uniformly bounded for $x\neq 0$ and possess uniformly bounded derivatives of any order for $x\neq 0$, which have finite limiting values as $x\to 0$ and $x\to +0$.

Along with Problems A_ϵ, we consider in the strip $\prod_T$ Problem A_0,

$$\frac{\partial u}{\partial t}+\phi'_u(u)\frac{\partial u}{\partial x}=0,\ u(0,x)=f(x). \tag{2}$$

As is known, Problem A_0 may have no solution in the class of differentiable functions even for an arbitrarily smooth initial function. In order for the problem (2) to be solvable for a continuous or piecewise continuous initial function, it is necessary to seek its solution in the class of discontinuous functions.

As it follows from O. A. Oleĭnik's work [51], for an arbitrary bounded measurable initial function $f(x)$ a solution of Problem A_ϵ is infinitely differentiable for $t>0$. At the same time, a solution of Problem A_0 may appear to be a discontinuous function even for an infinitely smooth initial function, and for a discontinuous initial function a solution of the problem may be continuous for $t>0$. In connection with these specific peculiarities, an asymptotic, as $\epsilon\to 0$, representation of a solution of Problem A_ϵ may or may not possess terms of boundary layer character; moreover, boundary layer terms of expansion can describe both the ordinary differential equations and equations of parabolic type.

Hence, the structure of an asymptotic, as $\epsilon \longrightarrow 0$, representation of a solution of Problem A_ϵ depends essentially on the properties of the solution of Problem A_0. It is not difficult to prove the following

Theorem 1. *If $f(x)$ is a function, infinitely differentiable and bounded along with its derivatives of any order, and a solution of Problem A_0 is infinitely differentiable in the strip $\prod_T$, then for the solution of Problem A_ϵ we can construct a uniform, asymptotic as $\epsilon \longrightarrow 0$ expansion of the type*

$$U(t,x,\epsilon) \sim \sum_{k=0}^{\infty} \epsilon^{2k} u_{2k}(t,x),$$

and for that expansion in the strip $\prod_T$ the estimate

$$\left\| u(t,x) - \sum_{k=0}^{N} \epsilon^{2k} u_k(t,x) \right\|_{C^2} \leq C\epsilon^{2N+2}$$

is valid.

Thus the occurrence of terms of boundary layer character in an asymptotic expansion of the solution of Problem A_ϵ is due to the occurrence of singularities in the solution of Problem A_0. In the present section we describe an asymptotic representation of the solution of Problem A_ϵ in the case, where a solution of Problem A_0 is a discontinuous function for $0 \leq t \leq T$.

Thus let the initial function satisfy the above-formulated conditions, provided $f(-0) > f(+0)$. Let the solution of the problem (3), (4) be an infinitely differentiable function everywhere in $\overline{\prod}_T$, except for an infinitely smooth line $x = x_p(t)$, $x_p(0) = 0$ at the points of which the solution of Problem A_0 and any its derivatives have finite limiting values as $x \longrightarrow x_p(t) - 0$ and $x \longrightarrow x_p(t) + 0$. As is known (see, e.g., [51]), on the line $x = x_p(t)$ the relation

$$x_p'(t) = \frac{\phi(u_0(t, x_p(t)+0)) - \phi(u_0(t, x_p(t)-0))}{u_0(t, x_p(t)+0) - u_0(t, x_p(t)-0)} \tag{3}$$

is fulfilled, where $u_0(t,x)$ is a solution of Problem A_0.

In what follows, for the sake of simplicity we will assume that the identity $x_p(t) \equiv 0$ holds; the consideration of the general case somewhat complicates technical details of the construction and is not a matter of principle.

2. An asymptotic expansion of the solution of Problem A_ϵ will be sought in the form

$$U(t,x,\epsilon) \sim \sum_{k=0}^{\infty} \epsilon^{2k} u_{2k}(t,x) + \sum_{k=0}^{\infty} \epsilon^{2k} v_k(t, x/\epsilon^2). \tag{4}$$

The function $u_0(t,x)$ is assumed to be known, and the functions $u_{2k}(t,x)$ for $k \geq 1$, $v_k(t,\xi)$ are to be defined.

The functions $u_{2k}(t,x)$ must ensure the closeness of the asymptotic expansion and the solution of Problem A_ϵ everywhere outside the neighborhood of the line of discontinuity of the solution of Problem A_0.

The functions $v_k(t,\xi)$ will be constructed under the assumption that they are of boundary layer character of variation as $|\xi| \to \infty$ and guarantee along with their derivatives of the first order the continuity of the asymptotic expansion (4).

A solution of Problem A_0 under the condition (3) is defined uniquely. Characteristics of the equation (2) are straight lines intersecting on the line $x = 0$. The function $u_0(t,x)$ is a solution of the functional equation

$$u_0 = f(x - \phi'(u_0)t), \tag{5}$$

which shows that the derivative of the function $u_0(t,x)$ with respect to the variable x for $x \neq 0$ can be found by the formula

$$u'_{0x}(t,x) = \frac{f'(x - \phi'(u_0)t)}{1 + t\phi''(u_0)f'(x - t\phi'(u_0))} = \frac{f'(x^0)}{1 + t\phi''(f(x^0))f'(x^0)}.$$

In particular, from the condition of solvability of the equation (5) it follows that in case the initial function is smooth, the smoothness of the solution of Problem A_0 cannot be violated for $t \in [0, T_0)$, where

$$T_0 = \min_{-\infty < x < \infty} \big[-\phi''(f(x))f'(x) \big]^{-1}.$$

Using the standard techniques, we can see that the functions $u_{2k}(t,x)$ for $k \geq 1$ are defined as solutions of the equations

$$L_0 u_{2k} \equiv \frac{\partial u_{2k}}{\partial t} + \frac{\partial}{\partial x}[\phi'(u_0)u_{2k}] =$$
$$= \frac{\partial^2 u_{2k-2}}{\partial x^2} - \sum_{s=0}^{2k} \frac{\phi^{(s+1)}(u_0)}{s!} \sum_{|\alpha|+\beta=2k} u_{\alpha_1}u_{\alpha_2}\ldots u_{\alpha_s}\frac{\partial u_\beta}{\partial x} \equiv F_{2k}(t,x), \tag{6}$$

satisfying the zero initial condition. Here, each of the indices β, α_1,α_2,…,α_s may be any even integer from 0 to $2k-2$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_s$. It can be easily seen that solutions of the equations (6) are, generally speaking, discontinuous for $x = 0$.

Let us now pass to the construction of the functions $v_{2k}(t,\xi)$ which are defined separately for $\xi < 0$ and $\xi > 0$. Using again the well-known procedure of constructing the asymptotic expansions, we find that the functions $v_{2k}(t,\xi)$ are obtained as solutions of the following differential equations:

$$\frac{\partial^2 v_0}{\partial \xi^2} - \phi'(u_0^\pm + v_0)\frac{\partial v_0}{\partial \xi} = 0, \tag{7}$$

$$L_1 v_{2k} \equiv \frac{\partial^2 v_{2k}}{\partial \xi^2} - \frac{\partial}{\partial \xi}[\phi'(u_0^\pm + v_0)v_{2k}] = u_{2k}^\pm \phi''(u_i^\pm + v_0)\frac{\partial v_0}{\partial \xi} + \Phi_{2k}(t,\xi); \tag{8}$$

here $u_{2k}^- = \lim_{x\to -0} u_{2k}(t,x)$, $u_{2k}^+ = \lim_{x\to +0} u_{2k}(t,x)$, $u_{2k}^\pm = u_{2k}^-$ for $\xi < 0$, $u_{2k}^\pm = u_{2k}^+$ for $\xi > 0$, and the function $\Phi_{2k}(t,\xi)$ is represented as a sum of a finite number of summands, each containing as a multiplier at least one of the functions $v_{2i}(t,\xi)$ or $v'_{2i\xi}(t,\xi)$ for $i < k$. The equations (7) and (8) are solved separately for $\xi < 0$ and $\xi > 0$; for $\xi = 0$ we impose the following conditions of continuity of:

– the asymptotic expansion

$$u_{2k}(t,-0) + v_{2k}(t,-0) = u_{2k}(t,+0) + v_{2k}(t,+0); \tag{9}$$

– the first derivatives of the asymptotic representations:

$$\begin{aligned}[v_0(t,-0)]'_\xi - [v_0(t,+0)]'_\xi = 0,\ \ [v_{2k}(t,-0)]'_\xi - [v_0(t,+0)]'_\xi = \\ = [u_{2k-2}(t,+0)]'_x - [u_{2k-2}(t,+0)]'_x, \qquad k \geq 1.\end{aligned} \tag{10}$$

Lemma 1. *The equation* (7) *possesses a solution of boundary layer type, which satisfies the conditions* (9), (10) *and tends exponentially to zero as* $|\xi| \to 0$.

Proof. In equation (7), for $\xi < 0$ and $\xi > 0$ we change the unknown functions $z^- = u_0^- + v_0$, and $z^+ = u_0^+ + v_0$, respectively. It is readily seen that both equations take the same form, and their coefficients coincide for $\xi = 0$ by virtue of the condition (9).

For $\infty < \xi < \infty$, let us consider the equation

$$z''_{\xi\xi} - \phi'(z) z'_\xi = 0 \tag{11}$$

and show that it has a solution, tending to u^- as $\xi \to -\infty$ and to u^+ as $\xi \to \infty$. Bearing this in mind, we integrate both parts of equation (11) with respect to the variable ξ and obtain $z'_\xi - \phi(z) + C(t) = 0$. Then we choose a constant $C(t)$ of integration in such a way that the derivative of the function $z(t,\xi)$ would tend to zero as $z \to u_0^-$: $C(t) = \phi(u_0^-)$. By virtue of the condition (3), $\partial z/\partial\xi \to 0$ as $z \to u_0^+$. Thus the function $z(t,\xi)$ satisfies the equation $z'_\xi = \phi(z) - \phi(u_0^-) = \phi(z) - \phi(u_0^+)$. The function $\psi(z) = \phi(z) - \phi(u_0^+)$ is convex and vanishing for $z = u_0^-$ and $z = u_0^+$. Every solution of the equation under consideration satisfies the equality

$$\xi = \int_{z(t,0)}^{z(t,\xi)} \frac{d\omega}{\psi(\omega)} = \int_{z(t,0)}^{z(t,\xi)} \frac{d\omega}{\phi(\omega) - \phi(u_0^-)}.$$

The values $z = u_0^-$ and $z = u_0^+$ are first order zeros of the function $\psi(z)$. Hence the integral on the right-hand side of the last equality diverges as $z \to u_0^-$ and $z \to u_0^+$, that is, the function $z(t,\xi)$ takes the values u_0^- and u_0^+ for none of the finite value ξ, if $z(t,0) \neq u_0^\pm$. This means that a solution of equation (11) having the above-mentioned limiting values as $\xi \to \pm\infty$ exists for $\xi \in (-\infty, \infty)$ and is defined uniquely by its valuation for $\xi = 0$. Because function $\psi(z)$ for $u_0^+ < z < u_0^-$ is of constant sign, every such solution is given in terms of the monotonically decreasing function of the variable ξ; note that $\partial z/\partial\xi$ vanishes for none of the value ξ. Since $z''_{\xi\xi} = \phi'(z)\psi(z)$, $\phi'(u_0^-) > 0$, $\phi'(u_0^+) < 0$, the function $z(t,\xi)$ has only one point of inflection $\xi = c_0(t)$, and therefore $\phi'(z(t, c_0(t))) = 0$. We fix the integral curve with inflection for $\xi = 0$, and let $z = \tilde{z}(t,\xi)$ be the equation of that curve. Then the function $z = \tilde{z}(t, \xi - C_0(t))$, where $C_0(t)$ is an arbitrary smooth function, defined for all $t \in [0,T]$, satisfies equation (11) and has an inflection for $\xi = C_0(t)$. Therefore $C_0(t)$ can be interpreted as the constant of integration distinguishing a unique solution of equation (7):

$$v_0(t,\xi) = \begin{cases} \tilde{z}(t, \xi - C_0(t)) - u_0^-, & \xi < 0, \\ \tilde{z}(t, \xi - C_0(t)) - u_0^+, & \xi > 0. \end{cases} \tag{12}$$

Obviously, the condition (10) is fulfilled for any choice of the function $C_0(t)$; thus, in constructing a zero approximate asymptotic expansion $u^{(0)} = u_0(t,x) + v_0(t, x/\epsilon)$ one constant of integration (being a function of the parameter t) remains still undetermined. The proof presents no difficulties when solutions of equation (7), satisfying the conditions $0 < v_0(t,0) < u_0^- - u_0^+$, tend exponentially to zero. $\square$

Lemma 2. *Solutions of the boundary layer type equation* (8) *exponentially tending to zero as* $|\xi| \to \infty$ *and satisfying the conditions* (9), (10) *exist. For every fixed k the set of functions $\{v_s(t,\xi)\}$, $s = 0, 1, \ldots, k$ is defined to within one integration constant $C_k(t)$ which is a function of the parameter t; moreover, each of the integration constants $C_s(t)$, $s = 0, 1, \ldots, k$ is defined as a solution of some linear, for $s \geq 1$, first order ordinary differential equation. The initial value $C_s(0)$ for the solution of each of these equations is arbitrary.*

Proof. The existence and exponential tending to zero as $|\xi| \to \infty$ of solutions of equations (7) is obvious. By virtue of the above-described properties of the function $v_0(t,\xi) = \tilde{v}_0(t, \xi - C_0(t))$, the solution of either equation (8) which tends to zero as $|\xi| \to \infty$, exists and depends on the two constants of integration being the functions of the variable t. Let us consider equation (8) for $k = 1$. Obviously, we shall have

$$\frac{\partial v_2}{\partial \xi} - \phi'(u_0^\pm + v_0)v_2 = \Phi_2^\pm(t,\xi), \tag{13}$$

$$\Phi_2^\pm(t,\xi) = \int_{\pm\infty}^{\xi} \Big\{ \frac{\partial v_0}{\partial t} + [\phi'(u_0^\pm + v_0) - \phi'(u_0^\pm)]\Big(\frac{\partial u_0}{\partial x}\Big)^\pm + \\ + \phi''(u_0^\pm + v_0)\Big[u_2^\pm + \eta\Big(\frac{\partial u_0}{\partial x}\Big)^\pm\Big]\frac{\partial v_0}{\partial \eta}\Big\} d\eta,$$

where the indices "$-$" and "$+$" are chosen for $\xi < 0$ and $\xi > 0$, respectively. A solution of equation (13) is defined by means of two constants of integration. Condition (9) can be satisfied by choosing only one of the constants. As is follows from (13), no choice of the second constant of integration can satisfy that condition for the function $v_2(t,\xi)$; only the remaining, for the time being unknown constant of integration $C_0(t)$ can satisfy the condition (9). From equation (13) and condition (10) we have

$$\frac{\partial u_0(t,+0)}{\partial x} - \frac{\partial u_0(t,-0)}{\partial x} = \int_{-\infty}^{0} \frac{\partial v_0}{\partial t} d\xi + \int_{0}^{\infty} \frac{\partial v_0}{\partial t} d\xi + \tilde{\Phi}_2(t), \qquad (14)$$

where the function $\tilde{\Phi}_2(t)$ is defined by means of limiting, as $x \to \pm 0$, values of the function $u_0(t,x)$ and its derivative with respect to the variable x. Using the representation (12) for the function $v_0(t,\xi)$, form the last equation it is not difficult to get for the function $C_0(t)$ the differential equation $C_0'(t) = q_0(t, C_0)$, whose right-hand side is the continuous function and satisfies the Lipschitz condition in the second argument for all $t \in [0,T]$. It follows from the relations (8) and (13) that the conditions (9) and (10) will be fulfilled for any choice of the initial condition for $\xi = 0$ for the solution of equation (13). The lemma is proved for the case $k = 1$.

To prove the lemma for the general case $k \geq 2$, we write the relation for the jump of first derivatives of the function $v_{2k}(t,\xi)$ with respect to the variable ξ for $\xi = 0$ in the form

$$\frac{\partial v_{2k}(t,-0)}{\partial \xi} - \frac{\partial v_{2k}(t,+0)}{\partial \xi} = \int_{-\infty}^{\infty} \frac{\partial v_{2k-2}}{\partial t} d\xi + g_k(t), \qquad (15)$$

where $g_k(t)$ is the given function. Using the general type of the solution of equation (8) for $k \geq 2$ and performing integration with respect to the variable ξ, we can get a differential equation for the integration constant $C_{2k-2}(t)$:

$$a(t) C_{2k-2}' = p_k(t) C_{2k-2} + q_k(t), \qquad (16)$$

where

$$a(t) = \int_{-\infty}^{\infty} \exp\left[\int_0^{\xi} \phi'(u_0^{\pm} + v_0) d\eta\right] d\xi \geq a_0 > 0,$$

$p_k(t)$, $q_k(t)$ are the known smooth functions.

Note that the conditions (9) and (10) are fulfilled for any choice of the initial value for the solution of equation (16). Moreover, in the approximation $u^{(2k)}(t,x,\epsilon)$, the integration constant $C_{2k}(t)$ and all the initial values $C_{2s}(0)$, $s \leq k-1$, remain still undetermined. □

Remark 1. Note that we have completely constructed a formal asymptotic expansion of the solution which, in fact, is the asymptotic residual expansion of the equation. By no choice of values of the constants $C_{2s}(0)$ can one achieve that the given expansion would satisfy the initial condition uniformly for all $|x| < \infty$. At the same time, it is obvious that in order that the constructed by us formal expansion approximate the exact solution for $t > 0$, it is necessary to determine these constants uniquely, as long as the choice of constants enables us to determine the structure of the solution in the neighborhood of the line of discontinuity of the solution of the degenerate problem.

The following assertion is a consequence of the theorems proved in the first section.

Lemma 3. *The values of $C_{2s}(0)$ can be chosen in such a way that for $t = 0$ the equality*

$$\int_{-\infty}^{\infty} v_{2s}(0,\xi)d\xi = 0$$

would be fulfilled.

Proof. Here we shall prove only the case $s = 0$, since the validity of assertions of the lemma for $s > 0$ can be easily proved by means of the explicit representation of the solution. We have

$$\int_{-\infty}^{\infty} v_0(0,\xi)d\xi = \int_{-\infty}^{0} \left[\tilde{z}(0,\xi - c_0) - u_0^-\right]d\xi + \int_{0}^{\infty} \left[\tilde{z}(0,\xi - c_0) - u_0^+)\right]d\xi = J_0(c_0),$$

where $c_0 = C_0(0)$. Since the function $\tilde{z}(0,\xi - c_0)$ is continuously differentiable for $\xi \in (-\infty,\infty)$ and the integrals of that derivative converge uniformly as c_0 varies on any finite interval, we can write the equality

$$\frac{dJ_0(c_0)}{dc_0} = \int_{-\infty}^{\infty} \frac{\partial\tilde{z}(0,\xi - c_0)}{\partial c_0}d\xi = -\int_{-\infty}^{\infty} \frac{\partial v_0(0,\xi)}{\partial\xi}d\xi = u_0^- - u_0^+ = \text{ const } > 0.$$

Thus the value $J_0(c_0)$ is the linear function of the variable c_0. Hence there exists a finite value $\overline{c}_0$ such that the equality $J_0(\overline{c}_0) = 0$ holds. $\square$

3. Let us pass to the error estimate of the constructed asymptotic expansion. We will use some a priori estimates of the solution of the problem (1), cited in the foregoing sections.

Under our assumptions, the initial function $f(x)$ satisfies the inequality $\sup\limits_{x_1 \neq x_2} [f(x_1) - f(x_2)][x_1 - x_2]^{-1} \le K_1$, and for the solution of Problem A_ϵ the estimate $u'_x(t,x) \le K_1$ is valid. As it follows from the construction of asymptotic representation, the similar estimate is also valid for the derivative with respect to the spatial variable of the function $u^{(N)}(t,x)$. Moreover, from the results of the first section the lower estimate $u'_x(t,x) \ge -M\epsilon^{-1}(t^{-1/2} + \epsilon^{-1})$ follows . We will also use the known estimates

$$\int_a^b \left|\frac{\partial u(t,x)}{\partial x}\right| dx \le M, \quad \int_a^b |u(t,x) - u_0(x)|dx \le M\epsilon,$$

which in the case under consideration are valid for all $t \in [0,T]$ and any finite a, b, $a < b$.

Let the constant α be such that $1/3 < \alpha \le 1/2$, $\delta = M\epsilon^{1-2\alpha}$. First we obtain the error estimate outside the δ – neighborhood of the straight line $x = 0$ which is the line of discontinuity of the solution of the degenerate equation. Using the auxiliary functions

$$z_{1,2}(t,x,\epsilon) = 2M_0 \exp\left[\epsilon^{-1}(M_1 t - |x| + M_1 T + \delta)\right] +$$

$$+\epsilon^{2n+2}M_2(t+1) \pm e^{-\alpha_1 t}\big[u(t,x) - u^{(N)}(t,x)\big],$$

where the constants M_0, M_1, M_2, α_1 not depending on ϵ are such that for $|x| \geq M_1 T + \delta$ the inequalities

$$a_1(t,x,\epsilon) = \alpha_1 + \frac{\partial u^{(N)}(t,x)}{\partial x}\int_0^1 \phi''(u(t,x)\theta + u^{(N)}(t,x)(1-\theta))d\theta \geq 1,$$
$$\big|L_\epsilon u^{(N)}(t,x)\big| \leq M\epsilon^{2N+2}$$

are fulfilled.

By means of the maximum principle we can easily prove that for $|x| \geq 2M_1T + 1 + \delta$ the estimate $|u(t,x\epsilon) - u^{(N)}(t,x,\epsilon)| \leq M\epsilon^{2N+2}$ is fulfilled.

Lemma 4. *Let $\alpha = const$, $0 < \alpha < 1/2$. For $|x| \geq \delta_0 = \epsilon^{1-2\alpha}$ the inequality $|u(t,x) - u^{(N)}(t,x)| \leq M\epsilon^{\alpha}$ holds.*

Proof. By virtue of the properties of the function $u^{(N)}(t,x)$ for $|x| \geq \delta_0$ there takes place the inequality $\big[u(t,x) - u^{(N)}(t,x)\big]'_x \leq K_2$, where K_2 is some constant, not depending on ϵ. Moreover, for any a, b, $a \leq b$ the relation $\int_a^b |u(t,x) - u^{(N)}(t,x)|dx \leq K_3\epsilon$ is valid, and therefore on the segment $[\delta_0/2, \delta_0]$ of the straight line $t = t_1$, $t_1 \in [0,T]$ there is at least one point $x = x_1$ such that the function $z(t,x,\epsilon) = u(t,x) - u^{(N)}(t,x)$ satisfies at the point (t_1,x_1) the inequality $\delta_0 z(t_1,x_1,\epsilon)/2 = \delta_0 \min\limits_{\delta_0/2 \leq x \leq \delta_0} z(t_1,x,\epsilon)/2 \leq \int_{\delta_0/2}^{\delta_0} z(t,x,\epsilon)dx \leq K_3\epsilon$. Suppose that at the point (t_1,x_2), $\delta_0 < x_2 < 2M_1T + 2$ the inequality $z(t,x,\epsilon) > C_0\epsilon^{\alpha}$ is fulfilled, where $C_0 = 2\sqrt{K_1K_2}$. Let (t_1,x_3) be a point of the segment $[x_1,x_2]$ of the straight line $t = t_1$ at which $z(t,x,\epsilon) = 2^{-1}C_0\epsilon^{\alpha}$, and for $x \in (x_3,x_2]$ the inequality $z(t_1,x,\epsilon) > 2^{-1}C_0\epsilon^{\alpha}$ is fulfilled; such a point exists because the function $z(t_1,x,\epsilon)$ is continuous for $x > \delta_0/2$. Hence, $C_0\epsilon^{\alpha}/2 < z(t_1,x_2,\epsilon) - z(t_1,x_3,\epsilon) = z'_x(t_1, x_3\theta + x_2(1-\theta), \epsilon)(x_2 - x_3) \leq K_2(x_2 - x_3)$, $0 < \theta < 1$, whence $x_2 - x_3 \geq (2K_2)^{-1}C_0\epsilon^{\alpha}$. Using this inequality, we get $K_3\epsilon \geq \int_{x_3}^{x_2} z(t_1,x,\epsilon)dx > (4K_2)^{-1}C_0^2\epsilon^{2\alpha} = K_3\epsilon^{2\alpha}$, which contradicts the condition $\alpha < 1/2$.

Analogously, the upper bounds for the integral of the function $z(t,x,\epsilon)$ can be obtained in a half-strip $x \leq -\delta_0$, $0 \leq t \leq T$ and the lower bounds in half-strips $|x| \geq \delta_0$, $0 \leq t \leq T$. □

Lemma 5. *For $|x| \geq M\delta_0$ the estimate $|u(t,x) - u^{(N)}(t,x)| \leq M\epsilon^{2N+2}$ is valid.*

Proof. Let $\rho > 0$ be some number. Let us construct a function $f_1(t,x)$ possessing the following properties: $f_1(t,-\rho) = f_1(t,\rho) = 0$; $f_1(t,x) \geq 1$ for $|x| \geq m_1\rho$, $m_1 = \text{const}$,

$$\left|\rho^s \frac{\partial^l f_1}{\partial x^s \partial t^{l-s}}\right| \leq M; \quad \frac{\partial f_1}{\partial t} + \phi'(u^{(N)})\frac{\partial f_1}{\partial x} = g_1(t,x) \leq 0 \ \text{ for } \ |x| > \rho.$$

It is not difficult to see that if $\rho \gg -\epsilon \ln \epsilon$, then the function exists. Consider an auxiliary function $z_1(t,x,\epsilon) = e^{-\alpha_2 t} f_1(t,x) z(t,x,\epsilon)$ which for $x > \rho$ satisfies the equation

$$L_1 z_1 \equiv \epsilon^2 \frac{\partial^2 z_1}{\partial x^2} - \phi'(u)\frac{\partial z_1}{\partial x} - \frac{\partial z_1}{\partial t} - a_2(t,x,\epsilon) z_1 = \Phi_1(t,x,\epsilon),$$

where $a_2(t,x,\epsilon) \geq 1$ for $|x| \geq \rho$, and the function $\Phi_1(t,x,\epsilon)$ is written in an obvious manner. Note that by virtue of the properties of coefficients of the asymptotic expansion the constant α_2 can be chosen independent of ϵ. For $t = 0$, $|x| \geq \rho$ the condition $|z_1(0,x,\epsilon)| \leq M\epsilon^{2N+2}$ is fulfilled for the function $z_1(t,x,\epsilon)$.

Suppose that the function $z_1(t,x,\epsilon)$ at some point $P_1(t_1,x_1)$ of the half-strip $x \geq \rho$, $0 < t \leq T$ reaches its largest (for that half-strip) positive value. Since the equality $z f'_{1x} + f_1 z'_x = 0$ is fulfilled at the point P_1, that is $f'_{1x} z'_x = -z\left(f'_{1x}\right)^2 / f_1$, the function $\tilde{z}_1 = z_1 - \beta_0$, where β_0 is some constant, satisfies at that point P_1 the relation

$$\begin{aligned} L_1 \tilde{z}_1 \equiv \epsilon^2 \frac{\partial^2 \tilde{z}_1}{\partial x^2} - \phi'(u)\frac{\partial \tilde{z}_1}{\partial x} - a_2 \tilde{z}_1 - \frac{\partial \tilde{z}_1}{\partial t} = \\ = \beta_0 a_2(t,x,\epsilon) + e^{-\alpha_2 t} z \left[\epsilon_{1x}^2 - g_1(t,x,\epsilon)\right] - 2\epsilon^2 e^{-\alpha_2 t} \left(f'_{1x}\right)^2 / f_1 - \\ e^{-\alpha_2 t} f_1 F_N(t,x,\epsilon) - e^{-\alpha_2 t} z_{1x}^2 \int_0^1 \phi''(u\theta + u^{(N)}(1-\theta))\theta. \end{aligned} \tag{17}$$

Consider the value of the right-hand side of (17) at the point P_1. Denote

$$8e^{-\alpha_2 t_1}\left[\rho f'_{1x}\right] = d_1, \quad 4e^{-\alpha_2 t_1} \rho f'_{1x} \int_0^1 \phi''(u\theta + u^{(N)}(1-\theta)) d\theta = d_2,$$
$$4e^{-\alpha_2 t_1} \rho^2 f''_{1xx} = d_3, \quad 4e^{-\alpha_2 t_1} \epsilon^{-(2N+2)} f_1 F_N = d_4$$

and choose a constant β_0 as follows:

$$\beta_0 = \max\left\{\epsilon^2 \rho^{-2} d_1 f_1^{-1} z;\ \rho^{-1} d_2 z^2;\ \epsilon^2 \rho^{-2} d_3 z;\ \epsilon^{2N+2} d_4\right\}.$$

For such a choice of the constant β_0, the expression $L_1 \tilde{z}_1$ is nonnegative at the point P_1, since $a_2(P_1) \geq 1$, $g_1(P_1,\epsilon) \leq 0$. As it follows from equation (17), the largest value the function $z_1(t,x,\epsilon)$ reaches at the point P_1 is nonnegative, i.e.,

$$f_1(P_1) z(P_1,\epsilon) \leq e^{-\alpha_2 t_1} \beta_0. \tag{18}$$

Suppose first that $\beta_0 = \epsilon^2 \rho^{-2} d_1 f_1^{-1}(P_1) z(P_1,\epsilon)$. Then inequality (18) implies $f_1^2(P_1) \leq \epsilon^2 \rho^{-2} d_1 e^{-\alpha_2 t_1}$. Therefore, by virtue of Lemma 4, $f_1(P_1) z \times (P_1,\epsilon) \leq M\epsilon^{1+\alpha}\rho^{-1}$.

If $\beta_0 = \rho^{-1} d_2 z^2(P_1,\epsilon)$, then $f_1(P_1) z(P_1,\epsilon) \leq M\epsilon^{2\alpha}\rho^{-1}$. However, if $\beta_0 = \epsilon^2 \rho^{-2} d_3 z(P_1,\epsilon)$, then $f_1(P_1) z(P_1,\epsilon) \leq M\epsilon^{2+\alpha}\rho^{-2}$.

Finally, for $\beta_0 = \epsilon^{2N+2} d_4$ we obtain the inequality $f_1(P_1) z(P_1,\epsilon) \leq M\epsilon^{2N+2}$. Comparing these estimates and supposing $\alpha = (m_0+3)/(3m_0)$,

$m_0 \geq 6$, $\rho = \delta_0$, we can state that the estimate $z(t,x,\epsilon) \leq M\epsilon^{4\alpha-1} = M\epsilon^{\alpha+3/m_0}$ holds for all $x \geq m_1\rho$, $0 \leq t \leq T$.

Let now $z_2(t,x,\epsilon) = f_2(t,x)z(t,x,\epsilon)$, where the function $f_2(t,x)$ is constructed in the half-strip $x \geq m_1\rho$, $0 \leq t \leq T$ just in the same way as the function $f_1(t,x)$ in the half-strip $x \geq \rho$, $0 \leq t \leq T$. Repeating for the function $z_2(t,x,\epsilon)$ the same arguments as for the function $z_1(t,x,\epsilon)$, we shall arrive at the conclusion that for $x \geq 2m_1\rho$, $0 \leq t \leq T$, when the function $z_2(t,x,\epsilon)$ reaches at the the point P_2 its positive maximum, we have the estimate

$$u(P_2) - u^{(N)}(P_2) \leq M\epsilon^{\alpha+9/m_0}.$$

Repeating our reasoning successively in the half-strips $x \geq 3m_1\rho$, $x \geq 4m_1\rho$ and so on, we shall step by step raise the exactness of the estimate until we obtain the inequality $z(P_r,\epsilon) \leq M\epsilon^{2N+2}$. It is easily seen that a number of steps r, which we have to do for getting a final estimate, does not exceed the value $[(2N+2)m_1]/3+1$.

The lower bound for the asymptotic expansion error can be found analogously. $\square$

Lemma 6. *For $|x| \geq M\delta_0$, $\delta_0 = \epsilon^{1/3-2/m_0}$, $m_0 \geq 6$ the estimate*

$$|z_x(t,x,\epsilon)| \equiv \left|[u(t,x) - u^{(N)}(t,x)]_x\right| \leq M\epsilon^{2N+2} \tag{19}$$

is valid.

Proof. First let us prove the unique boundedness of the function $\partial u/\partial x$ in the domain under consideration. Towards this end we consider in the half-strip $x \geq M\delta_0 = M\epsilon^{1/3-2/m_0}$, $0 \leq t \leq T$ the function $\overline{z}(t,x,\epsilon) = [\epsilon^2 u'_x - \phi(u) + \phi(u^{(N)})]$. As is easily seen, the function $\overline{z}(t,x,\epsilon)$ in that strip is uniformly bounded with respect to ϵ and satisfies both the equation

$$L_1\overline{z} = \epsilon^2\phi''(u^{(N)})\Big[\frac{\partial u^{(N)}}{\partial x}\Big]^2 - (u-u^{(N)})\frac{\partial\phi(u^{(N)})}{\partial x}\int_0^1 \phi''(u\theta+u^{(N)}(1-\theta))d\theta +$$
$$+\phi'(u^{(N)})F_N(t,x,\epsilon) = \epsilon^2\Phi_1(t,x,\epsilon), \quad |\Phi_1(t,x,\epsilon)| \leq M,$$

and the initial condition $\overline{z}(0,x,\epsilon) = \epsilon^2 f'(x) - [\phi(u) - \phi(u^{(N)})]\big|_{t=0}$. In the half-strip $x \geq sM_1\rho$, let us consider now the function $f_s(t,x)$ which has been used in proving the above lemma. Let $P_1(t_1,x_1)$ be the point of the above-mentioned half-strip in which the function $z_2(t,x,\epsilon) = e^{-t}f_s(t,x)\overline{z}(t,x,\epsilon)$ reaches its smallest negative value. If the constant β_0 is chosen from the condition

$$\beta_0 = 4e^{-t_1}\max\big\{ -2\epsilon^2 f_s(P_1)\overline{z}(P_1,\epsilon)[f'_{sx}(P_1)]^2, \quad \epsilon^2\overline{z}(P_1,\epsilon)f''_{sxx}(P_1),$$
$$\epsilon^2 f_s(P_s,\epsilon)\Phi_1(P_1,\epsilon), \quad \overline{z}(P_1,\epsilon)[\phi'(u(P_1)) - \phi'(u^{(N)}(P_1))]f'_{sx}(P_1)\big\},$$

then at the point P_1 for the function $z_3(t,x,\epsilon) = z_2(t,x,\epsilon) + \beta_0$ the relation

$$\epsilon^2\frac{\partial^2 z_3}{\partial x^2} - \phi'(u)\frac{\partial z_3}{\partial x} - z_3 - \frac{\partial z_3}{\partial t} = -\beta_0 + e^{-t}\overline{z}\Big[\epsilon^2\frac{\partial^2 f_s}{\partial x^2} - g_s(t,x,\epsilon)\Big] -$$

$$-2\epsilon^2 e^{-t} f_s^{-1}\overline{z}\Big[\frac{\partial f_s}{\partial x}\Big]^2 - e^{-t}\overline{z}[\phi'(u)-\phi'(u^{(N)})]\frac{\partial f_s}{\partial x} + \epsilon^2 e^{-t} f_s \Phi_1 \leq 0$$

is fulfilled. It follows from this inequality that the minimum, the function $z_3(t,x,\epsilon)$ reaches at the point P_1, is nonnegative, i.e., at that point there takes place the inequality $e^{-t}f_s(t,x)\overline{z}(t,x,\epsilon)\geq -\beta_0$. Consequently, at the point P_1 the relations $f_s(t,x)\overline{z}(t,x,\epsilon)\geq -\epsilon\rho^{-1}\overline{z}(t,x,\epsilon)\geq -K\epsilon^{2/3+2/m_0}$ are valid, and therefore the inequality $\overline{z}(t,x,\epsilon)\geq -M\epsilon^{2/3+2/m_0}$ is fulfilled for $x\geq m_2\rho$. Just as in the previous lemma, through a finite number of steps we shall obtain that for $x\geq M\rho$ the estimate $\overline{z}(t,x,\epsilon)\geq -M\epsilon^2$ is fulfilled, whence for $x\geq M\rho$, $0\leq t\leq T$ there follows the estimate $|u_x(t,x)|\leq M$. To complete the proof of the lemma, it suffices to consider the function $z_4(t,x,\epsilon)=z_x(t,x,\epsilon)$ which satisfies for $x\geq M\rho$ the condition

$$L_2 z_4 \equiv \epsilon^2\frac{\partial^2 z_4}{\partial x^2} - \phi' u\frac{\partial z_4}{\partial x} - \phi''(u)\big[u(t,x)-u^{(N)}(t,x)\big]z_4 - \frac{\partial z_4}{\partial t} =$$
$$= \big[\phi'(u)-\phi'(u^{(N)})\big]\frac{\partial^2 u^{(N)}}{\partial x^2} - \frac{\partial F_N(t,x,\epsilon)}{\partial x} + \big[\phi''(u)-\phi''(u^{(N)})\big]\Big[\frac{\partial u^{(N)}(t,x)}{\partial x}\Big]^2.$$

Taking into account the uniform (with respect to the parameter ϵ) boundedness of the function $(u-u^{(N)})'_x$ for $x\geq M\rho$ and repeating the proof of the previous lemma, we can prove the assertions of Lemma 6. $\square$

Analogously is proved the following

Lemma 7. *For $|x|\geq M\epsilon^{1/3+2/m_0}$, the inequality $|z_t(t,x,\epsilon)|\leq M\epsilon^{2N+2}$ is valid.*

Proof. Let R be an arbitrary number, not depending on ϵ. Denote by D a set of points of the rectangle $\{(t,x)|,\ 0\leq t\leq T,\ |x|<R\}$ and consider in the rectangle D the potential $w_N(t,x,\epsilon)$ of the difference $-z=u^{(N)}(t,x)-u(t,x)$:

$$w_N(t,x,\epsilon) = -\int_\Gamma z\,dx + \big[z_x - \phi'_u\cdot z - h^{(N)}\big]\,dt,$$

where $\Gamma=\Gamma(t,x)$ is a piecewise smooth curve connecting the point $(0,-R)$ with an arbitrary point (t,x) of the rectangle D,

$$\phi'_u(u) = \int_0^1 \phi'(u\theta + u^{(N)}(1-\theta))d\theta, \quad h^{(N)} = \int_{-R}^x F_N(t,y,\epsilon)dy.$$

Obviously, the function $w_N(t,x,\epsilon)$ for $x\neq 0$ satisfies both the equation

$$L_2 w_N \equiv \epsilon^2\frac{\partial^2 w_N}{\partial x^2} - \phi'_u\frac{\partial w_N}{\partial x} - \frac{\partial w_N}{\partial t} = h^{(N)}(t,x,\epsilon) \qquad (20)$$

and the conditions on the boundary of the rectangle

$$w_N(0,x,\epsilon) = \int_{-R}^x \sum_{i=0}^N \epsilon^{2i} v_{2i}\left(0,y/\epsilon^2\right)dy,$$

$$w_N(t,-R,\epsilon) = -\int_0^t \left[\, \epsilon^2 z_x - \phi'_u \cdot z + h^{(N)} \right]\Big|_{x=-R} d\tau + \mathcal{O}(\epsilon^{2N+2}),$$
$$w_N(t,R,\epsilon) = -\int_0^t \left[\, \epsilon^2 z_x - \phi'_u \cdot z + h^{(N)} \right]\Big|_{x=R} d\tau + \mathcal{O}(\epsilon^{2N+2}).$$

It is readily seen that if for the boundary layer functions $v_{2s}(t,\xi)$, $0 \le s \le N$ the estimate $|v_{2s}(t,\xi)| \le Me^{-m|\xi|}$ holds, then for $t = 0$ the function $w_N(t,x,\epsilon)$ satisfies the condition

$$|w_N(0,x,\epsilon)| \le \epsilon^2 M \exp\left[-m|x|/\epsilon^2\right] + M\epsilon^{2N+2}.$$

The function $w_N(t,x,\epsilon)$ is continuous together with its derivatives everywhere in the rectangle D for $x \ne 0$. For $x = 0$, $t > 0$ the function $w_N(t,x,\epsilon)$ is discontinuous, and

$$w_N(t,+0,\epsilon) - w_N(t,-0,\epsilon) = \epsilon^2 \int_0^t \Big[\frac{\partial u^{(N)}(s,+0)}{\partial x} - \frac{\partial u^{(N)}(s,-0)}{\partial x}\Big] ds =$$
$$= g_N(t,\epsilon), \quad \frac{\partial w_N(t,+0,\epsilon)}{\partial t} - \frac{\partial w_N(t,-0,\epsilon)}{\partial t} = g'_N(t,\epsilon),$$
$$\frac{\partial w_N(t,+0,\epsilon)}{\partial x} - \frac{\partial w_N(t,-0,\epsilon)}{\partial x} = 0.$$

Moreover, the relations $|g_N(t,\epsilon)| \le M\epsilon^{2N+2}$, $|g'_N(t)| \le M\epsilon^{2N+2}$ are valid. Therefore, applying the generalization of the maximum principle to piecewise continuous functions (see, e.g., [67]) it is not difficult to get the estimate $|w_N(t,x,\epsilon)| \le M\epsilon^2$. From this inequality it, in particular, follows that for any finite a, b, $a < b$, $t_1 \in [0,T]$ there holds the relation

$$\Big| \int_a^b \left[u(t_1,x) - u^{(N)}(t_1,x)\right] dx \Big| \le M\epsilon^2.$$

Using this relation and arguing simply, we can show that the coefficient ϕ'_u in equation (20) satisfies the inequalities $\phi'_u > \phi_0 > 0$ for $-R \le x \le -m_1\epsilon^2$, $\phi'_u < -\phi_0 < 0$ for $m_1\epsilon^2 \le x \le R$, $\phi'_u(u^{(N)}) \ge -m_2\epsilon^2$ for $-m_1\epsilon^2 \le x \le -\epsilon^2 C_0(t)$, and $\phi'_u(u^{(N)}) \le m_2\epsilon^2$ for $\epsilon^2 C_0(t) \le x \le m_1\epsilon^2$, where $C_0(t)$ is the function defined by us upon constructing the function $v_0(t,\xi)$. □

Lemma 8. *In the rectangle D, the estimate*

$$|w_N(t,x,\epsilon)| \le M\epsilon^{2N+2} + M\epsilon^2 \exp\left\{-\gamma_0 t/[\epsilon^2 \ln \epsilon^{-1}] - \gamma_0 x^2/[\epsilon^4 \ln \epsilon^{-1}]\right\} \quad (21)$$

is valid, where γ_0 is a constant not depending on ϵ.

Proof. Consider auxiliary functions

$$z_{1,2}(t,x,\epsilon) = \epsilon^{2N+2} K_1(t+1) +$$
$$+\epsilon^2 K_2 \exp\left\{-\gamma\epsilon^{-4}[x^2 - \epsilon^2 C_0(t)]^2 - \lambda\epsilon^{-2} t\right\} + \psi_{1,2}(t,x,\epsilon) \mp w_N(t,x,\epsilon),$$

where K_1, K_2 are some positive constants, not depending on ϵ, and $\psi_{1,2}(t,x,\epsilon) = \delta_1 e^{-|x|} \pm 2^{-1} g_N(t,\epsilon) \operatorname{sign} x$, γ, λ, δ_1 are positive numbers,

$\delta_1 \le \epsilon^{2N+2}$. The functions $z_{1,2}(t,x,\epsilon)$ for a sufficiently large value of K_1 are positive on the lateral faces of the rectangle D, since $|\psi_{1,2}(t,x,\epsilon)| \le M\epsilon^{2N+2}$. For $t=0$ there holds the inequality $|w_N(0,x,\epsilon)| \le \epsilon^{2N+2}M + \epsilon^2 M\exp\left(-m|x|/\epsilon^2\right)$, and hence

$$z_{1,2}(0,x,\epsilon) \ge K_1\epsilon^{2N+2} - |\psi_{1,2}(t,x,\epsilon)| - M\epsilon^{2N+2} +$$
$$+K_2\epsilon^2\exp[-\gamma\epsilon^{-4}(x-\epsilon^2c_0)^2]\{1-m_3K_2^{-1}\exp[\gamma\epsilon^{-4}(x-\epsilon^2c_0)^2-m|x|\epsilon^{-2}]\},$$

$c_0 = C_0(0)$. Choose a constant γ in such a way that the expression in braces would be positive for $|x| \le [2(N+1)\epsilon^2\ln\epsilon^{-1}]/m$. To this end, it is sufficient that for $K_2 \ge m_3$ the index of the exponent would take negative values at the ends of the segment $[-2(N+1)\epsilon^2\ln(\epsilon^{-1})/m, 2(N+1)\epsilon^2\ln(\epsilon^{-1})/m]$. Obviously, for this in its turn it suffices to satisfy the inequality $\gamma \le m^2/\left[2(N+1)\ln\epsilon^{-1}\right]$. The fact that the function $z_{1,2}(t,x,\epsilon)$ is positive for $|x| > [2(N+1)\epsilon^2\ln\epsilon^{-1}]/m$ is obvious.

Let the functions $z_{1,2}(t,x,\epsilon)$ take in the rectangle D negative values. Since the functions under consideration are continuous, there exist points at which either of the functions reaches its negative minimum. For example, let $P_1(t_1,x_1)$ be a point of negative minimum of the function $z_1(t,x,\epsilon)$. Since $z'_{1x}(t,+0,\epsilon)-z'_{1x}(t,-0,\epsilon) = -2\delta_1 < 0$, $x_1 \ne 0$. Hence the function $z_1(t,x,\epsilon)$ at the point P_1 has continuous derivatives up to the second order inclusive. With regards to the estimates for the functions $g_N(t,\epsilon)$, $h^{(N)}(t,x,\epsilon)$ and value δ we shall have

$$L_2z_1 = -\epsilon^{2N+2}K_1 + K_2\exp\left[-\gamma\epsilon^{-4}(x_1-\epsilon^2C_0(t_1))^2 - \epsilon^{-2}\lambda t_1\right] \times$$
$$\times\left[-2\gamma + 4\gamma^2\epsilon^{-4}(x_1-\epsilon^2C_0(t_1))^2 + 2\gamma\epsilon^{-2}(x_1-\epsilon^{-2}C_0(t_1))\phi'_u +\right.$$
$$\left.+\lambda - 2\gamma(x_1-\epsilon^2C_0(t_1))C'_0(t_1)\right] + \delta_1e^{-|x_1|}(x_1+\phi'_u\,\mathrm{sign}\,x_1) - 2^{-1}g'_N(t_1,\epsilon)\times$$
$$\times\mathrm{sign}\,x_1 - h^{(N)}(t_1,x_1,\epsilon) \le -\frac{1}{2}\epsilon^{2N+2}K_1 + K_2\exp\left[-\gamma\epsilon^{-4}(x_1-\epsilon^2C_0(t_1))^2-\right.$$
$$\left.-\epsilon^{-2}\lambda t_1\right]\left[-2\gamma + 4\gamma^2\epsilon^{-4}(x_1-\epsilon^2C_0(t_1))^2 +\right.$$
$$\left.+2\gamma\epsilon^{-2}(x_1-\epsilon^2C_0(t_1))\phi'_u + \lambda - 2\gamma(x_1-\epsilon^2C_0(t_1))C'_0(t_1)\right].$$

First, let $|x_1-\epsilon^2C_0(t_1)| \le m_1\epsilon^2$. In that case $4\epsilon^{-4}\gamma^2[x_1-\epsilon^2C_0(t_1)]^2 \le M[\ln\epsilon^{-1}]^{-2}$, $2\gamma\epsilon^{-2}[x_1-\epsilon^2C_0(t_1)]\phi'_u \le M\epsilon^2[\ln\epsilon^{-1}]^{-1}$, $-2\gamma[x_1-\epsilon^2C_0(t_1)]\times C'_0(t_1) \le M\epsilon^2[\ln\epsilon^{-1}]^{-1}$, and therefore $L_2z_1 < 0$ if we choose $\lambda = \gamma$. If $m_1\epsilon^2 < |x_1-\epsilon^2C_0(t_1)| \le m_4\epsilon^2\ln\epsilon^{-1}$, then we obtain $4\gamma^2\epsilon^{-4}[x_1-\epsilon^2C_0(t_1)]^2 \le 4m_4^2[\ln\epsilon^{-1}]^2\gamma^2$, $2\gamma\epsilon^{-2}[x_1-\epsilon^2C_0(t_1)]\phi'_u \le -2m_4\gamma\phi_0\ln\epsilon^{-1}$, $-2\gamma[x_1-\epsilon^2C_0(t_1)]C'_0(t_1) \le M\epsilon^2$. Let it be required that in the case under consideration the inequality $4\gamma^2\epsilon^{-4}[x_1-\epsilon^2C_0(t_1)]^2-\gamma\epsilon^{-2}|x_1-\epsilon^2C_0(t_1)|\phi_0 \le 0$ be fulfilled. For this to be so it is sufficient that the constant γ to satisfy the inequality $\gamma \le \phi_0[4m_4\ln\epsilon^{-1}]^{-1}$. Obviously, under such a condition the inequality $L_2z_1(t_1,x_1,\epsilon) < 0$ will be fulfilled on the interval $m\epsilon^2 \le |x_1-\epsilon^2C_0(t_1)| \le m_4\epsilon^2\ln\epsilon^{-1}$ at the point P_1.

Let, finally, $|x_1 - \epsilon^2 C_0(t_1)| \geq m_4 \epsilon^2 \ln \epsilon^{-1}$. If the constant m_4 is sufficiently large, then the validity of the inequality $L_2 z_1(t_1, x_1, \epsilon) < 0$ is evident. Thus at the extremum point of the function $z_1(t_1, x_1, \epsilon)$ the inequality $L_2 z_1 < 0$ is fulfilled, and therefore this point cannot be the point of negative minimum. Consequently, the function $z_1(t, x, \epsilon)$ everywhere in the rectangle D is nonnegative. Carrying out analogous investigation with respect to the function $z_2(t, x, \epsilon)$, we finally have

$$|w_N(t, x, \epsilon)| \leq M\epsilon^{2N+2} + M\epsilon^2 \exp\left\{-\gamma[x - \epsilon^2 C_0(t)]^2/\epsilon^4 - \gamma t/\epsilon^2\right\},$$

where $\gamma = \min\left\{\phi_0/[4m_4 \ln \epsilon^{-1}]^{-1},\ 2(N+1)[m \ln \epsilon^{-1}]^{-1}\right\}$. From the above inequality we immediately obtain the assertion of the lemma. $\square$

Theorem 2. *Outside of the neighborhood* $\Omega = \{(t, x) \mid 0 \leq t \leq M\epsilon^2 \ln \epsilon^{-1}$, $|x| \leq M\epsilon^2 \sqrt{\ln \epsilon^{-1}}\}$ *of the point where the line of discontinuity of the solution of the degenerate equation and the initial straight line meet, the inequality* $|u(t, x, \epsilon) - u^{(N)}(t, x, \epsilon)| \leq M\epsilon^{2N+2}$ *is valid.*

Proof. Let $t_1 \geq M\epsilon^2 \ln \epsilon^{-1}$. It follows from (21) that for any a, b, $a < b$ the inequality

$$\left|\int_a^b [u(t_1, x) - u^{(2N+3)}(t_1, x)]dx\right| \leq M\epsilon^{4N+6}$$

is fulfilled. Let x_1 be a point of the line $t = t_1$ at which the difference $u(t_1, x) - u^{(2N+3)}(t_1, x)$ reaches its largest positive value m_0. Let a, b be chosen in such a way that $b = x_1$, and on the interval $[a, b]$ the function $z(t_1, x, \epsilon) = u(t_1, x) - u^{(2N+3)}(t_1, x)$ is nonnegative. Since $\partial z/\partial x \leq m_1 \epsilon^{-2}$ in the strip $\prod_T$, the length of the interval $[a, b]$ is not less than $m_0 \epsilon^2/m_1$. Therefore

$$\frac{m_0^2 \epsilon^2}{2m_1} \leq \int_a^b z(t_1, x, \epsilon)dx \leq M\epsilon^{4N+6},$$

whence $m_0 \leq M\epsilon^{2N+2}$. Analogously one can prove the assertions of the lemma in the case of negative minimum of the function $z(t, x, \epsilon)$ for an arbitrary point lying outside the neighborhood of Ω. $\square$

Theorem 3. *Everywhere in the strip* $\prod_T$, *with the exception of the neighborhood* $\Omega\{(t, x) \mid t \leq M\epsilon^2 \ln \epsilon^{-1}$, $|x| \leq \epsilon^2 \ln \epsilon^{-1}\}$ *of the origin, the estimate* $|z'_x(t, x, \epsilon)| \leq M\epsilon^{2N+2}$ *is valid.*

Proof. As is shown in the first paragraph, in the case under consideration there takes place the inequality $|u''_{xt}(t, x)| \leq M\epsilon^{-1}(\epsilon^{-3} + t^{-3/2})$. Moreover, from the construction of the asymptotic expansion it follows that for an approximate solution $u^{(N)}(t, x)$ the estimate $\left|[u^{(N)}(t, x)]''_{xt}\right| \leq M\epsilon^{-4}$, $x \neq 0$ is valid. Thus for $t > M\epsilon^2 \ln \epsilon^{-1}$, $x \neq 0$, we have the estimate for the function $\tilde{z}_N(t, x, \epsilon) = u(t, x) - u^{(N)}(t, x)$:

$$\left|\frac{\partial^2 \tilde{z}_N(t, x, \epsilon)}{\partial x \partial t}\right| e M\epsilon^{-4}. \qquad (22)$$

It follows from Lemma 8 and estimate (21) that everywhere in the strip $\prod_T$ for any finite a, b, $a < b$, the inequality

$$\left| \int_a^b \epsilon^2 \frac{\partial \tilde{z}_N(\tau, x, \epsilon)}{\partial x} d\tau \right| \leq M\epsilon^{2N+2}$$

is valid. Using the estimate (22) for the function $\tilde{z}_{2N+5}(t, x, \epsilon)$, we can to receive, just as in Theorem 2, the validity of the assertion of Theorem 3.

Let $V(t, x, \epsilon) = \{\epsilon^2 u'_x - \phi(u)\} - \{\epsilon^2 [u^{(N)}]'_x - \phi(u^{(N)})\}$. On writing for that function the corresponding potential we can prove the assertions similar to those of Lemmas 6 and 7 and then to prove for the function $z(t, x, \epsilon)$ the analog of Theorem 3, i.e., to get an estimate for the function $\partial^2/\partial x^2 [u(t, x) - u^{(N)}(t, x)]$. Thus we can obtain an estimate for the derivative with respect to the variable t. Hence we have the following assertion. $\square$

Theorem 4. *If the solution of the problem* (3), (4) *is a function which is bounded, infinitely differentiable in the closure of the half-strips* $\{0 \leq t \leq T,\ -\infty < x \leq -0\}$ *and* $\{0 \leq t \leq T,\ 0 \leq x < \infty\}$, *then for the asymptotic expansion of the solution of the problem* (1), (2) *everywhere in the strip* $\prod_T$ *with the exception of the neighborhood of the origin of radius* $M\epsilon^2 \ln \epsilon^{-1}$ *the estimate*

$$\left\| u(t, x, \epsilon) - \sum_{k=0}^{N} \epsilon^{2k} \left[u_k(t, x) + v_k(t, x/\epsilon^2) \right] \right\|_{C^1} \leq M\epsilon^{2N+2}$$

is valid.

Remark 2. Obviously, the asymptotic expansion of a solution of the initial problem has similar structure even in the cases, where Problem A_0 has several isolated lines of discontinuity defined for $0 \leq t \leq T$. Moreover, the obtained estimates enable one to consider "composite" approximate solutions which are represented on one part of the strip by means of the asymptotic expansion and by some other means on the other part of the strip (for example, by numerical methods).

1.4. Expansion in the Case of Weak Discontinuity of the Solution of the Degenerate Problem

In this section we consider a quasi-linear parabolic equation under the assumption that a solution of the corresponding degenerate problem has for $t > 0$ one or several lines of discontinuity of derivatives. In case of one line of discontinuity of derivatives generated by a "breaking" of thecontinuous initial function (weak discontinuity of a solution), we construct a complete asymptotic expansion of the solution of the nondegenerate problem. In case of two intersecting lines of discontinuity of derivatives generated by a discontinuous initial function (rarefaction wave), we construct two terms of asymptotic expansion of a solution which specify earlier known representations.

1. In the strip $\prod_T = \{(t,x)|0 < t \leq T, \ -\infty < x < \infty\}$, let us consider the Cauchy problem

$$L_\epsilon u \equiv \epsilon^2 \frac{\partial^2 u}{\partial x^2} - \phi'(u)\frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0, \quad u\mid_{t=0} = f(x). \tag{1}$$

The function $f(x)$ will be assumed to be continuous and bounded for $x \in (-\infty, \infty)$, possessing for $x \neq 0$ bounded continuous derivatives of any order and having finite limiting values as $x \to -0$ and $x \to +0$.

Along with the problem (1) we consider Problem A_0:

$$\frac{\partial u}{\partial t} + \phi'(u)\frac{\partial u}{\partial x} = 0, \quad u\mid_{t=0} = f(x). \tag{2}$$

A solution $u_0(t,x)$ of Problem A_0 will be supposed to be a continuous function everywhere in the strip $\prod_T$, possessing continuous bounded derivatives everywhere in $\prod_T$, with the exception of the points of the characteristic which passes through the origin; the derivatives of the function $u_0(t,x)$ on that characteristic, generally speaking, are not continuous, although they have limiting values when their arguments tend to the points of the characteristic from the left and from the right. In this case we will say that the function $u_0(t,x)$ has a weak discontinuity at the points of the above-mentioned characteristic. Note once more that the characteristics of the problem (3), (4) are straight lines.

Our aim is to construct a uniform asymptotic expansion of the solution of the problem (1), (2). It would be more natural to suppose that the asymptotic expansion of the solution of Problem A_ϵ in the case under consideration could be obtained from the asymptotic expansion constructed by us in the previous section for the discontinuous initial function by the passage to limit as the jumping values of the initial function tend at the point $x = 0$ to zero. However, the constructions of the present section show that the boundary layer terms of the asymptotic expansion in the case under consideration are determined by means of parabolic equations, and in this connection their character of variation differs in principle from the above-considered case where the boundary layer terms were described in terms of ordinary differential equations.

For the sake of simplicity it will be assumed that a weak discontinuity of the solution of the degenerate problem takes place along the straight line $x = 0$; bearing this in mind, under the above assumptions on the properties of the function $\phi(u)$, it is sufficient that the equality $f(0) = 0$ be fulfilled.

2. An asymptotic expansion of the solution of Problem A_ϵ will be sought in the form

$$u(t,x) \sim \sum_{k=0}^{\infty} \epsilon^{2k} u_{2k}(t,x) + \sum_{k=1}^{\infty} \epsilon^k v_k\left(t, x/\epsilon\right). \tag{3}$$

Here the functions $u_{2k}(t,x)$ are defined just as in the preceding section. It can be easily seen that the function $u_0(t,x)$ is continuous everywhere in the strip $\prod_T$, and its derivatives with respect to the variable x have the jumps as their arguments pass through the straight line $x = 0$; note that the jump of the derivative tends to a different from zero constant as $t \to 0$. The functions $u_{2k}(t,x)$, $k \geq 1$, being solutions of the equation (6) in the previous section under zero initial conditions, are, generally speaking, discontinuous for $x = 0$ functions, bounded along with their derivatives of any order for $x \neq 0$. Moreover, jumps of the functions $u_{2k}(t,x)$, $k \geq 1$, as well as of their derivatives with respect to the variable x have as $t \to 0$ the order $\mathcal{O}(t)$.

As usual, it will be assumed that the functions $v_k(t,\xi)$ as functions of the variable ξ are of boundary layer character as $|\xi| \to \infty$. Taking into account the expression written out in the previous section for the derivative of the function $u_0(t,x)$, we can write a recursion system of equations

$$L_1 v_1 \equiv \frac{\partial^2 v_1}{\partial \xi^2} - \frac{\partial v_1}{\partial t} - \frac{\partial}{\partial \xi}\left[\frac{1}{2}v_1^2 + \frac{a^{\pm}\xi}{1+a^{\pm}t}v_1\right] = 0, \tag{4}$$

$$\frac{\partial^2 v_k}{\partial \xi^2} - \frac{\partial v_k}{\partial t} - \frac{\partial}{\partial \xi}\left[\left(v_1 + \frac{a^{\pm}\xi}{1+a^{\pm}t}\right)v_k\right] = \frac{\partial}{\partial \xi}\Phi_k(t,\xi), \tag{5}$$

$k \geq 2$. Here $a^{\pm} = \lim\limits_{x \to \pm 0} f'(x)$, and the functions $\Phi_k(t,\xi)$ can be easily defined successively for $k = 2, 3, \ldots$ by using the standard algorithms; the functions $\Phi_k(t,\xi)$ are represented by a sum whose each summand is a product of a polynomial $P_s(t,\xi)$ of degree s with functions of the variable t as coefficients by one or several functions $v_i(t,\xi)$, $s \leq k$, $i = 1, 2, \ldots, k-1$. Moreover, by virtue of our assumptions on the properties of the solution of the problem (2), the functions $1 + a^{+}t$, $1 + a^{-}t$ do not vanish for $t \in [0, T]$.

The equations (4), (5) are solved separately for $\xi < 0$ and $\xi > 0$. We will seek for such solutions of the equation (4), (5) which satisfy the conditions

$$[v_{2i+1}(t,\xi)] = 0, \quad \left[(v_{2i+1}(t,\xi))'_{\xi}\right] = -\left[(u_{2i}(t,x))'_x\right], \tag{6}$$

$$[v_{2i}(t,\xi)] = -[u_{2i}(t,x)], \quad \left[(v_{2i}(t,\xi))'_{\xi}\right] = 0, \tag{7}$$

$$v_k(0,\xi) = 0, \tag{8}$$

$[z(t,y)] \equiv z(t,+0) - z(t,-0)$, $i = 1, 2, \ldots$. The fulfilment of the conditions (6), (7) implies the continuity (along with the first order derivatives) of the formal asymptotic expansion (3) of the solution of the problem (1).

3. Consider the problem (4), (6), (8) for $i = 0$. The change of the unknown function $w_1(t,\xi) = v_1(t,\xi) + a^{\pm}\xi/[1+a^{\pm}t]$ leads us to the equation

$$w''_{1\xi\xi} - w_1 w'_{1\xi} - w'_{1t} = 0, \tag{9}$$

whose solution must satisfy the additional conditions $w_1(0,\xi) = a^{\pm}\xi$, $w_1(t,+0) = w_1(t,-0)$, $\left[w'_{1\xi}(t,\xi)\right] = 0$. Thus the solution of the equation (9) must be continuous in the domain $t > 0$ and possess in that domain the continuous derivative with respect to the variable ξ. Note that the equation

(9), Hopf's equation [30], and the solution of that equation can be written out explicitly:

$$
\begin{aligned}
w_1(t,\xi) = \Big\{ & \frac{(a^+ - a^-)\sqrt{t}}{\sqrt{(1+a^-t)(1+a^+t)}} \exp\Big(-\frac{\xi^2}{4t}\Big) + \\
+\xi & \frac{a^-\sqrt{1+a^+t}}{1+a^-t} \exp\Big[-\frac{a^-\xi^2}{4(1+a^-t)}\Big] \int_{\sigma(t,\xi)}^{\infty} e^{-\omega^2} d\omega + \\
+\xi & \frac{a^+\sqrt{1+a^-t}}{1+a^+t} \exp\Big[-\frac{a^+\xi^2}{4(1+a^+t)}\Big] \int_{-\sigma^+(t,\xi)}^{\infty} e^{-\omega^2} d\omega \Big\} \times \\
\times \Big\{ & \sqrt{1+a^+t} \exp\Big[-\frac{a^-\xi^2}{4(1+a^-t)}\Big] \int_{\sigma(t,\xi)}^{\infty} e^{-\omega^2} d\omega + \\
+ & \sqrt{1+a^-t} \exp\Big[-\frac{a^+\xi^2}{4(1+a^+t)}\Big] \int_{-\sigma^+(t,\xi)}^{\infty} e^{-\omega^2} d\omega \Big\}^{-1},
\end{aligned} \tag{10}
$$

where $\sigma^+(t,\xi) = \xi/\big(2\sqrt{1+a^+t}\big)$, $\sigma(t,\xi) = \xi/\big(2\sqrt{1+a^-t}\big)$. The expression (10) implies that the equality $v_1(t,0) = \mathcal{O}(\sqrt{t})$ is fulfilled for $\xi = 0$.

To investigate the behavior of the function $w_1(t,\xi)$ as $|\xi| \to \infty$, we will use the well-known asymptotic formulas for the integrals appearing in (10).

Applying these formulas, for $|\xi| \gg 1$ we can get

$$
\begin{aligned}
w_1(t,\xi) = {} & a^{\pm}\xi/[1+a^{\pm}t] + 2t^{3/2}(a^+ - a^-)/\big(\sqrt{\pi}\xi^2\big) \times \\
\times & \sqrt{(1+a^-t)(1+a^+t)} \exp\big\{-\xi^2/[4t(1+a^{\pm}t)]\big\}\big(1+o(1)\big),
\end{aligned}
$$

where the symbol "$\pm$" takes the values "$-$" for $\xi \ll -1$ and "$+$" for $\xi \gg 1$. On the basis of the above-obtained asymptotic representations we can formulate the following

Lemma 1. *A solution of the problem* (4), (6), (8) *exists and exponentially tends to zero as* $|\xi| \to \infty$*; moreover, for that solution there holds the estimate*

$$
|v_1(t,\xi)| \le M\sqrt{t}\Big\{\exp\Big[-\frac{\xi^2}{4t(1+a^{\pm}t)}\Big] + \exp\Big[-\frac{\xi^2}{4t}\Big]\Big\}.
$$

For our further investigation we have to study the behavior of the derivatives of the function $v_1(t,\xi)$ as $|\xi| \to \infty$ and $t \to 0$.

Lemma 2. *For the derivatives of the solution of the problem* (4), (6), (8) *with respect to the variable* ξ *the estimates*

$$
\Big|\frac{\partial v_1(t,\xi)}{\partial \xi}\Big| \le M\Big\{\exp\Big[-\frac{\xi^2}{4t(1+a^{\pm}t)}\Big] + \exp\Big[-\frac{\xi^2}{4t}\Big]\Big\},
$$

$$
\Big|\frac{\partial^2 v_1(t,\xi)}{\partial \xi^2}\Big| \le Mt^{-1/2}(1+\xi^2/t)\Big\{\exp\Big[-\frac{\xi^2}{4t(1+a^{\pm}t)}\Big] + \exp\Big[-\frac{\xi^2}{4t}\Big]\Big\}
$$

are valid.

We can prove thet lemma by means of an explicit expression for the function $v_1(t,\xi)$. From Lemmas 1 and 2 and the equation (4) it follows the estimate for the function $\partial v_1/\partial t$.

Let us pass to the consideration of the functions $v_k(t,\xi)$, $k \ge 2$. Suppose that the estimates

$$\Big|\frac{\partial \Phi_k}{\partial \xi}\Big| \le M(1+|\xi|^{n_k})\Big\{\exp\Big[-\frac{\xi^2}{4t(1+a^{\pm}t)}\Big] + \exp\Big[-\frac{\xi^2}{4t}\Big]\Big\},$$
$$\Big|\frac{\partial^2 \Phi_k}{\partial \xi^2}\Big| \le M\sqrt{t}(1+|\xi|^{n_k+1})\Big\{\exp\Big[-\frac{\xi^2}{4t(1+a^{\pm}t)}\Big] + \exp\Big[-\frac{\xi^2}{4t}\Big]\Big\}$$

hold for the right-hand side of the equation (5). Note that by the change of variables

$$y = \xi\,/[1+a^{\pm}t]\,, \quad \tau = t/\left[1+a^{\pm}t\right], \quad \tilde{v}_k = (1+a^{\pm}t)v_k \tag{11}$$

the equations (4), (5) are reduced to those with bounded coefficients which, generally speaking, are discontinuous for $y=0$:

$$\tilde{v}''_{1yy} - \tilde{v}_1\tilde{v}'_{1y} - \tilde{v}'_{1\tau} = 0,$$
$$\tilde{v}''_{kyy} - (\tilde{v}_1\tilde{v}_k)'_y - \tilde{v}'_{k\tau} = \Big\{\tilde{\Phi}_k(\tau,y)/[1-a^{\pm}\tau]\Big\}'_y\,.$$

The existence of bounded solutions of either equation can be substantiated, for example, by the methods developed in [40]. To do this, it suffices to write the solution separately for $y<0$ and $y>0$ and then, using Green's function, to write out an integral equation with respect to the function $\tilde{v}_k(\tau,0)$. From the known estimates for Green's function (see, e.g., [26]) it follows that the obtained integral equation of Abelian type is uniquely solvable. Thus the consideration of the problem (5), (7), (8) is reduced to two problems in half-strips $\xi<0$ and $\xi>0$. As is known, owing to the continuity of the boundary (for $y=0$) and initial functions, as well as by virtue of the properties of the coefficients and right-hand sides of the equation (5) for different values of the index k, each problem has a unique bounded solution.

Let us show that the functions $v_k(t,\xi)$ and their derivatives are of boundary layer character as $|\xi|\to\infty$.

Lemma 3. *Every solution of the problem* (5), (7), (8) *satisfies the estimates*

$$|v_k(t,\xi)| + \sqrt{t}|v'_{k\xi}(t,\xi)| + t|v'_{kt}(t,\xi)| + t|v''_{k\xi\xi}(t,\xi)| \le$$
$$\le Mt(1+|\xi|^{n_k+2})\Big\{\exp\Big[-\frac{\xi^2}{4t(1+a^{\pm}t)}\Big] + \exp\Big[-\frac{\xi^2}{4t}\Big]\Big\}.$$

Proof. First we estimate the function $v_k(t,\xi)$. The function $w_k(t,\xi) = v_k(t,\xi)e^{-m_1t}$ satisfies the equation

$$L_3w_k \equiv \frac{\partial^2 w_k}{\partial \xi^2} - \Big[v_1 + \frac{a^{\pm}\xi}{1+a^{\pm}t}\Big]\frac{\partial w_k}{\partial \xi} - \frac{\partial w_k}{\partial t} -$$

$$-[v'_{1\xi} + m_1 + a^{\pm}/(1 + a^{\pm}t)]w_k = e^{-m_1 t}\Phi_k(t,\xi), \qquad (12)$$

and also the conditions $w_k(0,\xi) = 0$, $w_k(t,0) = p_k(t)$ or $w_k(0,\xi) = 0$, $w'_{k\xi}(t,0) = q_k(t)$. Moreover, $p_k(t) = \mathcal{O}(t)$, $q_k(t) = \mathcal{O}(t)$ as $t \to 0$, $k \geq 2$, owing to the above-mentioned character of jumps (as $t \to 0$) of the functions $u_{2s}(t,x)$ and their first order derivatives on the line $x = 0$. The solution of equation (12) will be considered in the half-strip $\{0 < t \leq T,\ 0 < \xi < \infty\}$.

Let first $a^+ < 0$. Consider auxiliary functions $z_{1,2}(t,\xi) = m_2(1 + \xi^{n_k})t \exp\left[-\xi^2/(4t)\right] \pm w_k(t,\xi)$, where m_2 is a positive constant and n_k is a positive nonnegative integer. Obviously,

$$\begin{aligned}L_3 z_{1,2} = (1+\xi^{n_k})m_2 \exp\left[-\xi^2/(4t)\right]\Big\{ &-\frac{3}{2(1+\xi^{n_k})} + \\ +t\frac{n_k(n_k-1)}{1+\xi^{n_k}} - \frac{n_k\xi^{n_k}}{(1+\xi^{n_k})} &- \frac{a^+ n_k \xi^{n_k}}{(1+a^+t)(1+\xi^{n_k})} - \\ -t\frac{n_k\xi^{n_k-1}v_1}{1+\xi^{n_k}} - \frac{a^+\xi^2}{2(1+a^+t)} &+ \frac{v_1(t,\xi)\xi}{2} - \frac{3\xi^{n_k}}{2(1+\xi^{n_k})} - \\ -\frac{a^+t}{1+a^+t} - t\frac{\partial v_1(t,\xi)}{\partial \xi} &- m_1 t\Big\} \pm e^{-m_1 t}\frac{\partial}{\partial \xi}\Phi_k(t,\xi)). \qquad (13)\end{aligned}$$

Taking into account the estimates obtained for the function $v_1(t,\xi)$, we can choose a constant m_1 so large that for $\xi \geq 0$ the inequality

$$\begin{aligned}m_1 + \frac{\partial v_1(t,\xi)}{\partial \xi} + \frac{a^+}{1+a^+t} + \frac{a^+ n_k \xi^{n_k}}{(1+a^+t)(1+\xi^{n_k})} + \\ +\frac{n_k \xi v_1}{1+\xi^{n_k}} - \frac{n_k(n_k-1)\xi^{n_k-2}}{1+\xi^{n_k}} > 0\end{aligned}$$

be fulfilled. It can be easily shown that if $\xi \geq 0$ and $a+ < 0$ the inequality

$$\xi|v_1(t,\xi)| \leq \overline{m}_0 t \exp[-\xi^2/(4t) - \tilde{m}_0\xi^2/(1+a^+t)]$$

where $\overline{m}_0$ and $\tilde{m}_0$ are positive constants, is valid for the function $v_1(t,\xi)$. Therefore there exists a constant $m_1 > 0$, such that the expression appearing in the braces in (13) is less than -1. Taking into consideration the inequalities to which satisfy the function $F_k(t,\xi,v_1,\ldots,v_{k-1})$ and its derivative with respect to the variable ξ, and choosing a constant m_2, we find that the inequality $L_3 z_{1,2} < 0$ is fulfilled for all $0 < t \leq T$, $\xi \geq 0$. Using the maximum principle, for the function $w_k(t,\xi)$ we obtain for $a^+ < 0$ the estimate $|w_k(t,\xi)| \leq Mt(1+\xi^{n_k})\exp\left[-\xi^2/(4t)\right]$.

If $a^+ > 0$, then on considering auxiliary functions

$$z_{3,4}(t,\xi) = m_2 t(1+\xi^{n_k})\exp\left\{-\xi^2/[4t(1+a^+t)]\right\} \pm w_k(t,\xi),$$

and arguing as above, we can prove that the function $z_{3,4}(t,\xi)$ for $0 \leq t \leq T$, $\xi \geq 0$, are positive. Hence, the second assertion of the lemma is valid for $\xi \geq 0$, $k \geq 2$. Analogously we can prove the same estimate for the half-strip $0 \leq t \leq T$, $-\infty < \xi \leq 0$.

To estimate a derivative of the function $v_k(t,\xi)$ with respect to the variable ξ, it is sufficient to make in equation (12) the change of variables (11) and then to consider the function $\overline{v}_k(\tau,y) \equiv \tau\tilde{v}_k(\tau,y)$ which can be written in the form

$$\overline{v}_k(\tau,y) = \frac{2}{\sqrt{\pi}}\int_{\gamma(\tau,y)}^{\infty} e^{-z^2}\Big(\tau - \frac{y^2}{z^2}\Big)\tilde{p}_k\Big(\tau - \frac{y^2}{4z^2}\Big)\,dz -$$
$$-\int_0^{\tau}\frac{d\theta}{2\sqrt{\pi(\tau-\theta)}}\int_0^{\infty}\exp\Big[-\frac{(y-\eta)^2}{4(\tau-\eta)}\Big]\times$$
$$\times\Big\{\frac{\partial}{\partial\eta}\Big[\frac{\theta}{1-a^+\theta}\tilde{v}_1(\theta,\eta)\tilde{v}_k(\theta,\eta) + \frac{\theta}{(1-a^+\theta)^2}\tilde{\Phi}_k(\theta,\eta)\Big] +$$
$$+\frac{\tilde{v}_k(\theta,\eta)}{1-a^+\theta}\Big\}d\eta + \int_0^{\tau}\frac{d\theta}{2\sqrt{\pi(\tau-\theta)}}\int_0^{\infty}\exp\Big[-\frac{(y+\eta)^2}{4(\tau-\theta)}\Big]\times$$
$$\times\Big\{\frac{\partial}{\partial\eta}\Big[\frac{\theta}{1-a^+\theta}\tilde{v}_1(\theta,\eta)\tilde{v}_k(\theta,\eta) + \frac{\theta}{(1-a^+\theta)^2}\tilde{\Phi}_k(\theta,\eta)\Big] + \frac{\tilde{v}_k(\theta,\eta)}{1-a^+\theta}\Big\}d\eta =$$
$$= J_1 - J_2 + J_3.$$

Here $\gamma(\tau,y) = y/(2\sqrt{\tau})$, and the functions $\tilde{v}_1(\tau,y)$, $\tilde{\Phi}_k(\tau,y)$, $\tilde{p}_k(\tau)$ are obtained by the above-mentioned change of variables of already known functions $v_1(t,\xi)$, $\Phi_k(t,\xi)$, $p_k(t)$.

Let us estimate now the derivatives of the function $J_1(\tau,y)$. Differentiating this function with respect to the variable y and integrating by parts, we readily get the estimate

$$|J'_{1y}| \le M\big(\sqrt{\tau} + y + y^2\sqrt{\tau} + y^3/\tau\big)\exp[-y^2/(4\tau)].$$

In a similar manner we obtain the second derivative of the function $J_1(\tau,y)$ with respect to the variable y:

$$|J''_{1yy}| \le M(1 + y\sqrt{\tau} + y^2/\tau)\exp[-y^2/(4\tau)].$$

Estimates of derivatives of the functions J_2 and J_3 can be obtained exactly in the same way is done in the first paragraph, provided the initial function is smooth. $\square$

Remark. Thus we have constructed the formal asymptotic expansion of the solution of the problem under consideration. Its partial sums are asymptotic residual representations of the equation and boundary conditions. To prove the asymptotic character of the constructed by us formal expansion, it is necessary to obtain the corresponding error estimates.

Theorem 1. *For the solution of Problem A_ϵ under the above-mentioned conditions the asymptotic expansion* (3) *is valid. Moreover, the estimate*

$$\big\|u(t,x,\epsilon) - u^{(N)}(t,x,\epsilon)\big\|_{C^1} \equiv$$

$$\equiv \left\| u(t,x,\epsilon) - \sum_{k=0}^{N} \epsilon^{2k} u_{2k}(t,x) - \sum_{k=1}^{2N+1} \epsilon^k v_k(t,x,\epsilon) \right\|_{C^1} \leq M\epsilon^{2N}$$

holds.

Proof. Consider the difference $z_N(t,x,\epsilon) = u(t,x,\epsilon) - u^{(N)}(t,x,\epsilon)$. The function $z_N(t,x,\epsilon)$ satisfies the zero initial condition for $t = 0$ and is twice continuously differentiable for $t > 0$, $x \neq 0$. The function $z_N(t,x,\epsilon)$ for $x = 0$ is continuous and has continuous for $t > 0$ derivatives of the first order. Everywhere in the strip $\prod_T$, with the exception of the points of the axis $x = 0$, the function $z_N(t,x,\epsilon)$ satisfies the equation

$$\epsilon^2 \frac{\partial^2 z_N}{\partial x^2} - \phi'(u)\frac{\partial z_N}{\partial x} - \phi_u'' \frac{\partial u^{(N)}}{\partial x} z_N - \frac{\partial z_N}{\partial t} = -\Psi_N(t,x,\epsilon), \qquad (14)$$

where

$$\phi_u'' = \int_0^1 \phi''(u^{(N)}(1-\theta) + u\theta)d\theta, \quad |\Psi_N(t,x,\epsilon)| \leq M\epsilon^{2N+2}.$$

From the results of the previous paragraphs it follows that the coefficients of equation (14) are continuous for $t > 0$ bounded functions, and the coefficient $\phi'(u)$ is differentiable for $t > 0$, while the derivative of the function $u^{(N)}(t,x,\epsilon)$ with respect to the variable x is uniformly (with respect to t) bounded by a constant of the type $M\epsilon^{-1}$. According to the maximum principle, everywhere in the strip $\prod_T$ the estimate $z_N(t,x,\epsilon)$ for the function $|z_N(t,x,\epsilon)| \leq M\epsilon^{2N+2}$ is valid.

To estimate the first derivative of the function $z_N(t,x,\epsilon)$ with respect to the variable x we consider the function $\tilde{z}_N(t,x,\epsilon) = \epsilon^2 z'_{Nx} - [\phi(u) - \phi(u^{(N)})]$. For $t = 0$ this function satisfies the zero initial condition. For $t > 0$, $x = 0$, the function $\tilde{z}_N(t,x,\epsilon)$ is continuous and has, generally speaking, a jump of derivatives with respect to x. It is easily seen that standard reasoning allows us to obtain the estimate $|\tilde{z}_N(t,x,\epsilon)| \leq M\epsilon^{2N+2}$ which leads to the estimate of the derivative of the function $z_N(t,x,\epsilon)$ with respect to the variable x.

To estimate the value of the first derivative of the function $z_N(t,x,\epsilon)$ with respect to the variable t, we again consider equation (14) and write it in the form

$$\epsilon^2 \frac{\partial^2 z_N}{\partial x^2} - \frac{\partial z_N}{\partial t} = \frac{\partial}{\partial x}[\phi(u) - \phi(u^{(N)})] - \Psi_N(t,x,\epsilon).$$

As is mentioned above, the function $z_N(t,x,\epsilon)$ is continuous everywhere in the strip $\prod_T$ and has the continuous first derivative with respect to the variable x. Hence the derivative of the function $z_N(t,x,\epsilon)$ with respect to the variable t is also continuous in the strip $\prod_T$. Therefore the jump of the second derivative of the function $z_N(t,x,\epsilon)$ with respect to the variable x for $x = 0$ is equal to within the multiplier ϵ^2 to that of the right-hand side of that equation. By an algorithm for constructing coefficients of asymptotic expansion a jump of the function $\Psi_N(t,x,\epsilon)$ can be written as $\Psi_N(t,+0,\epsilon)-\Psi_N(t,-0,\epsilon)=g_N(t,\epsilon)$, where $g_N(t,\epsilon)$ is the continuous function, such that the inequality $|g_N(t,\epsilon)|\leq M\epsilon^{2N+2}$ holds. Moreover, the function $\tilde\Psi_N(t,x,\epsilon)=\Psi_N(t,x,\epsilon)-2^{-1}g_N(t,\epsilon)\theta(x)$, $\theta(x)=-1$ for $x<0$, $\theta(x)=1$ for $x\geq 0$, is continuous, and for the modulus of continuity of that function we have the estimate $|\tilde\Psi_N(t,x,\epsilon)-\tilde\Psi_N(t,y,\epsilon)|\leq M\epsilon^{2N+1}t^{-1/2}|x-y|$. Using this estimate, we can, as when proving Theorems 3 and 4 of the first paragraph, to get the inequality

$$\left|[z_N(t,x,\epsilon)]'_{xx}\right|\leq M\epsilon^{2N}\left(\sqrt{t}+\epsilon^{-2}\right),$$

which provides us with the estimate for the function $\partial z_N(t,x,\epsilon)/\partial t$. $\square$

4. Let now the initial function $f(x)$ be discontinuous for $x=0$, $f(x)=0$ for $x<0$, $f(x)=b>0$ for $x>0$. Then the solution of Problem A_0 is continuous and many times differentiable everywhere in the strip $\prod_T$, with the exception of the points of the straight lines $x=0$, $x=\phi'(b)t$. As is known, the solution of Problem A_0 under the above-mentioned assumptions on the function $f(x)$ is termed a rarefaction wave.

The properties of the solution of Problem A_ϵ under the above-indicated assumptions on the function $f(x)$ have been considered in [5], [51], [52], [54], etc. In particular, the estimate

$$\int_a^b |u(t,x,\epsilon)-u_0(t,x)|dx\leq M\epsilon^2,$$

where a, b are arbitrary finite numbers, $a\leq b$, has been obtained in [5].

In this section we will give an estimate for the above-mentioned difference in the uniform norm

Introduce the new variables $\tau=t/\epsilon^2$, $y=x/\epsilon^2$ and consider in the half-plane $\{0<\tau<\infty,\ |y|<\infty\}$ the problem

$$\frac{\partial^2 u}{\partial y^2}-\phi'(u)\frac{\partial u}{\partial y}-\frac{\partial u}{\partial \tau}=0,\quad u\mid_{\tau=0}=f(x). \tag{15}$$

The solution of the corresponding degenerate problem can be written in the form

$$u_0(\tau,y)=\begin{cases}0 & \text{for } y\leq 0,\\ b & \text{for } y\geq\phi'(b)\tau,\\ q(y/\tau) & \text{for } 0<y\leq\phi'(b)\tau,\end{cases}$$

where the function $q(y/\tau)$ is defined as the solution of equation $\phi'(q(r))=r$.

Lemma 4. *For $y\leq 0$ and $\phi'(b)\tau\leq y$, for the difference $z_0(\tau,y)=u(\tau,y,\epsilon)-u_0(\tau,y)$ the inequality*

$$|z_0(\tau,y)|\leq M\{\exp\left[-y^2/(4\tau)\right]+\exp\left[-(y-\phi'(b)\tau)^2/(4\tau)\right]\}$$

is fulfilled.

Proof. Let us consider an auxiliary function $\psi(\tau, y) = \exp\left[-y^2/(4\tau)\right]$ for $\tau > 0$, $\psi(\tau, y) = 0$ for $\tau = 0$. Obviously,

$$L_1\psi \equiv \frac{\partial^2\psi}{\partial y^2} - \phi'(u)\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial\tau} = \frac{\psi(\tau, y)}{2\tau}[-1 + y\phi'(u)],$$

and hence for $y<0$ the relation $L_1(M\psi \pm z_0) = M\psi[-(2\tau)^{-1} + \phi'(u)y/(2\tau)] < 0$ holds, since the function $\phi'(u)$ for the initial function is nonnegative. Moreover, the function $M\psi \pm z_0$ is nonnegative for $\tau = 0$ and for large values of y, if the constant M is sufficiently large. According to the maximum principle, the function $M\psi \pm z_0$ is positive for $y \leq 0$, which implies the validity of the assertion of the lemma for $y \leq 0$.

Using the function $\psi(\tau, y) = \exp\left[-\left(y - \phi'(b)\tau\right)^2/(4\tau)\right]$, we can analogously prove the assertion of the lemma for the case $y \geq \phi'(b)\tau$. $\square$

Lemma 5. *The function $z_0(\tau, y)$ tends to zero as $\tau \to \infty$; note that $|z_0(\tau, y)| \leq M\tau^{-\alpha}$, where $0 < \alpha < 1/2$ is an arbitrary constant.*

Proof. We introduce into consideration an auxiliary function

$$\psi(\tau, y) = \begin{cases} \exp\left[-m(y - \beta\sqrt{\tau})^2\tau^{-1}\right]/\tau^{\alpha}, & -\infty < y \leq 0, \\ \exp\left(-m\beta^2\right)/\tau^{\alpha}, & 0 < y \leq \phi'(b)\tau, \\ \exp\left[-m(y - \phi'(b)\tau + \beta\sqrt{\tau})^2\tau^{-1}\right]/\tau^{\alpha}, & \phi'(b)\tau < y < \infty, \end{cases}$$

where α, m, β are some nonnegative constants. For $y \leq 0$ we shall have

$$\begin{aligned} L_2\psi &\equiv \frac{\partial^2\psi}{\partial y^2} - \phi'(u)\frac{\partial\psi}{\partial y} - \int_0^1 \phi''(u_0 + \theta z_0)d\theta\frac{\partial u_0}{\partial y}\psi - \frac{\partial\psi}{\partial\tau} = \\ &= \psi(\tau, y)\left\{(\alpha - 2m)/\tau + m(4m-1)(y - \beta\sqrt{\tau})^2/\tau^2 + \right. \\ &\qquad \left. + m(y - \beta\sqrt{\tau})\left[2\phi'(u) - \beta/\sqrt{\tau}/\tau\right]\right\}. \end{aligned}$$

Choose a positive constant m so small that the inequality $4m - 1 < 0$ be fulfilled and then define a constant α from the condition $\alpha - 2m < 0$. If $\phi'(u)\sqrt{\tau} - \beta > 0$, then for any choice of the constant $\beta > 0$ the whole right-hand side of the latter equation will be negative for $y \leq 0$. If $\phi'(u)\sqrt{\tau} - \beta < 0$ for sufficiently large values of the variable τ, then the assertion of the lemma in the case under consideration is obvious. For the rest values of τ the negativeness of the expression in braces is equivalent to the fulfilment of the inequality

$$z^2 - 2z\frac{\left(\phi'(u)\sqrt{\tau} - \beta\right)}{\left[\sqrt{\tau}(1 - 4m)\right]} - \frac{(\alpha - 2m)}{\tau m(1 - 4m)} > 0.$$

This inequality will be fulfilled in the case if we choose the constant β such that the inequality $\beta < 2\sqrt{(1 - 4m)(2m - \alpha)/m}$ be fulfilled. Hence there exists a constant $\delta > 0$ such that for $y \leq 0$ the inequality $L_2\psi \leq -\delta\psi\tau^{-1}$ is fulfilled.

Similarly, choosing for $y \geq \phi'(b)\tau$ successively the constants m, α, β, we can achieve the validity of the inequality $L_2\psi \leq -\delta\psi\tau^{-1}$ for all $y \geq \phi'(b)\tau$.

For $0 \le y \le \phi'(b)\tau$, the function $z_0(\tau, y)$ satisfies the equation $L_2 z_0 = -q''(y/\tau)/\tau^2$. Obviously, the equality $L_2\psi = -\tau^{-1}\psi[\phi''(u_0+\theta z_0)q'(y/\tau) - \alpha]$ is fulfilled for $0 \le y \le \phi'(b)\tau$. Let $\alpha < \min\big\{ \min\limits_{0\le u, u_0 \le b} \phi''(u_0+\theta z_0)/\phi''(u_0)$, $2^{-1}\big\}$. As is easily seen, $\alpha \ne 0$. Since $q'(y/\tau) = [\phi''(q(y/\tau))]^{-1}$, the inequality $L_2\psi \le -\delta\tau^{-1}\psi$ is fulfilled for $0 \le y \le \phi'(b)\tau$.

Consider now auxiliary functions $R_{1,2}(\tau, y) = M\psi \pm z_0(\tau, y)$. For these functions, for $y \ne 0$, $y \ne \phi'(b)\tau$ the relations $L_2 R_{1,2} \le -\delta\tau^{-1}\psi \pm L_2 u_0(\tau, y) = -M\delta\tau^{-1}\psi \mp q''(y/\tau)/\tau^2$ are fulfilled. Since the functions $R_{1,2}(\tau, y)$ are continuous for all $\tau > 0$, $-\infty < y < \infty$ and, moreover, the relations

$$[R_{1,2}]'_y\big|_{y=0, y=\phi'(b)\tau} = -2Mm\beta\tau^{-\alpha-1/2}\exp(-m\beta^2) + \mathcal{O}(\tau^{-1})$$

are fulfilled, the negative minimum of these functions for sufficiently large values of the variable τ cannot be achieved for $y = 0$ or $y = \phi'(b)\tau$. It is easily seen that if the constant T is sufficiently large, then all the above relations will be fulfilled for $t > T$. We now choose the constant M so large that the functions $R_{1,2}(\tau, y)$ for $\tau = T$ be positive. Owing to the maximum principle, these functions will be positive for all $\tau \ge T$, which proves the assertion of the lemma for the above-mentioned choice of the constant α.

Using the obtained estimate for the chosen value of α and repeating, (if necessary) all the above arguments, it is not difficult to see that the constant α can take any positive value, less than $1/2$. $\square$

We introduce into consideration auxiliary functions $w((\phi'(b)\tau - y)/\sqrt{\tau})$, $w(y/\sqrt{\tau})$ which can be defined by the equality

$$w(s) = \exp\Big[-\frac{s^2}{4}\Big]\Big[\int_{s/2}^{\infty} e^{-\theta^2} d\theta\Big]^{-1}. \tag{16}$$

First we consider an auxiliary solution of the problem (15) written in the form

$$u_1(\tau, y) = \begin{cases} w(y/\sqrt{\tau})/\sqrt{\tau} & \text{if } \ y \le 0, \\ b - w((\phi'(b)\tau - y)/\sqrt{\tau})/\sqrt{\tau} & \text{if } \ 0 < y \le \phi'(b)\tau, \\ q\,(y/\tau) + \rho_1\,(y/\tau)\,[w(y/\sqrt{\tau})/\sqrt{\tau} - y/\tau] - & \\ -1/\phi''(b)\rho_2\,((\phi'(b)\tau - y)/\tau)\,[w((\phi'(b)\tau - y)/\tau)/\sqrt{\tau} + & \\ \qquad +(y - \phi'(b))\tau)/\tau] & \text{if } \ y \ge \phi'(b)\tau, \end{cases} \tag{17}$$

where $\rho_1(s)$ and $\rho_2(s)$ are sufficiently smooth shearing functions, $0 \le \rho_i(s) \le 1$, $\rho_i(s) \equiv 1$ for $s \le m_1\phi'(b)$, $\rho_i(s) \equiv 0$ for $s \ge m_2\phi'(b)$, $i = 1, 2$, and m_1, m_2 are some positive, sufficiently small constants whose values will be specified below.

It is readily seen that the function $u_1(\tau, y) - u_0(\tau, y)$ is by itself the “correction” to the solution of the degenerate problem (2) which we have to estimate.

The difference $z_1(\tau, y) = u(\tau, y) - u_1(\tau, y)$ for $\tau > 0,\ y \neq 0,\ y \neq \phi'(b)\tau$ satisfies the equation

$$L_3 z_1 \equiv \frac{\partial^2 z_1}{\partial y^2} - \phi'(u)\frac{\partial z_1}{\partial y} - \phi''(u_1 + \theta z_1)\frac{\partial u_1}{\partial y} z_1 - \frac{\partial z}{\partial \tau} = -L u_1,$$

where $L u_1 \equiv u''_{1yy} - \phi'(u_1)u'_{1y} - u'_{1\tau}$. It is evident that the relation $L u_1 = -\tau^{-1} w'(s)[\phi'(u_1) - u_1],\quad s = y/\sqrt{\tau}$ is fulfilled for $y < 0$. Let us consider the auxiliary function $\psi(\tau, y) = \tau^{-\gamma} w'(s) e^{ms^2}$ for $y \leq 0$, where $\gamma > 0,\ m > 0$ are some constants. We have

$$L_3 \psi = \tau^{-\gamma-1} e^{ms^2} w' \Big\{ \gamma - 1 + \frac{w w''}{w'} + w'[1 - \phi''(u_1 + \theta z_1)] +$$
$$+ 2m + 2ms[2w'' + s(2m + 2^{-1})w'] - \sqrt{\tau}\phi'(u)[w'' + 2msw'] \Big\}. \quad (18)$$

It offers no difficulty to prove the following

Lemma 6. *For $y \leq 0$ and sufficiently small values of the constant m the relations $2w'' + s(2m + 2^{-1})w' > 0,\ w'' + 2msw' > 0,\ w''w/w' < 2^{-1}$ are fulfilled.*

The proof can be performed by direct calculations by using the representation (17).

Taking into account the assertion of Lemma 6, from the equation (18) we can get the estimate

$$L_1 \psi < \tau^{-\gamma-1} e^{my^2/\tau} w' \Big\{ \gamma + 2m - 1 + \frac{w'' w}{w'} + w'[1 - \phi''((u_1 + \theta z_1)] \Big\}.$$

For $y \leq 0$ and sufficiently large values of τ, by Lemma 5 we can make the summand $w'[1 - \phi''(u_1 + \theta z_1)]$ arbitrarily small in modulo. Hence for $y \leq 0$ and sufficiently large values of τ the relation

$$L_1 \psi \leq -\delta \tau^{-\gamma-1} w' \left(y/\sqrt{\tau}\right) \exp\left(my^2/\tau\right) = -\delta \tau^{-1} \psi(\tau, y)$$

is fulfilled, where $\delta > 0,\ \gamma > 0$ are some constants; note that the constant γ can be chosen so that the inequality $\gamma > 1/2$ be fulfilled.

Consider now the domain $0 \leq y \leq \phi'(b)\tau$ and the auxiliary function $\psi(\tau, y) = \tau^{-\gamma} w'(0)$. Obviously, for $0 \leq y/\tau \leq m_2\phi'(b)$ there takes place the equality

$$L_1 \psi = \gamma \tau^{-\gamma-1} w'(0) - \tau^{-\gamma-1} w'(0)\phi''(u_1 + \theta z_1)[q'(y/\tau) +$$
$$+ \rho_1'(y/\tau)(w/\sqrt{\tau} - y/\tau) + \rho_1(y/\tau)(-y/(2\sqrt{\tau})w - 1 + w^2/2)].$$

By using for the function $w(s)$ Millse's relation [48], we can get the equality

$$L_1 \psi = \tau^{-\gamma-1} w'(0) \Big\{ \gamma - \phi''(u_1 + \theta z_1) \Big[q'(y/\tau) +$$
$$+ \rho_1' \Big(\frac{y}{\tau}\Big) \frac{\omega}{\tau[\sqrt{\omega/\tau + y^2/(4\tau^2)} + y/(2\tau)]} + \rho_1 \Big(\frac{y}{\tau}\Big) \Big(\frac{\omega}{2} - 1\Big) \Big] \Big\}.$$

If $0 \leq y/\sqrt{\tau}$, then by decreasing (if necessary) the constant m_2, we get $L_1\psi \leq 2\tau^{-\gamma-1}\left\{\gamma - 2/\pi + \mathcal{O}\left(m_2 + \tau^{-\alpha}\right)\right\}/\pi$, where the constant α is chosen in the same way as in Lemma 5. Choosing now $\gamma < 2/\pi$, we obtain for $0 \leq y/\tau \leq m_2\phi'(b)$ the inequality $L_1\psi \leq -\delta\tau^{-1}\psi(\tau, y)\left[1 + \mathcal{O}\left(m_2 + \tau^{-\alpha}\right)\right]$. If, however, $m_2\phi'(b) < y/\tau \leq (1 - m_2)\phi'(b)$, then we obtain $\rho_i(s) = \rho_i'(s) = 0$, $i = 1, 2$, and hence $L_1\psi \leq \frac{2}{\pi}\tau^{-\gamma-1}\left[\gamma - 1 + \mathcal{O}\left(\tau^{-\alpha}\right)\right]$. A similar inequality can be obtained for $(1 - m_2)\phi'(b) \leq y/\tau \leq \phi'(b)$. Consequently, in the case under consideration, for sufficiently large values of τ we will have $L_1\psi \leq -\delta\tau^{-1}\psi$, if and only if $\gamma < 1$, $0 \leq y/\tau \leq \phi'(b)$.

The construction of the barrier function for $y/\tau > \phi'(b)$ is carried out by means of the function $w((\phi'(b)\tau - y)/\sqrt{\tau})$ exactly in the same manner as we did it for $y < 0$ by means of the function $w(y/\sqrt{\tau})$. The function $\psi(\tau, y)$ is continuous in the half-plane $\tau > 0$, and if the variable τ is sufficiently large, for $y = 0$ and $y = \phi'(b)\tau$ there takes place a jump of derivatives with respect to the variable y of order $\mathcal{O}(\tau^{-\gamma-1})$. By simple but cumbersome calculations we can show that the relation $L_1u_1 = \mathcal{O}(\tau^{-2})$ is fulfilled for sufficiently large values of τ and for all $-\infty < y < \infty$, $y \neq 0$, $y \neq \phi'(b)\tau$. Consider the auxiliary functions $R_{1,2}(\tau, y) = M\psi(\tau, y) \pm z_1(\tau, y)$, where M is a large enough constant. It is easily seen that those functions are continuous for all $-\infty < y < \infty$, and there take place the inequalities $[R_{1,2}]_y'\big|_{y=0} < 0$, $[R_{1,2}]\big|_{y=\phi'(b)\tau} < 0$. Therefore the functions $R_{1,2}(\tau, y)$ fail to achieve their minimal values on the lines $y = 0$, $y = \phi'(b)\tau$. Choosing the constants M and T sufficiently large and using the maximum principle, we can conclude that the inequality $|z_1(\tau, y)| \leq M\psi(\tau, y)$ is fulfilled for all $\tau \geq T$.

Lemma 7. *For the function* (17), *the relation* $|u(\tau, y) - u_1(\tau, y)| \leq M\tau^{-\gamma}$, *where* $1/2 < \gamma < 1$ *is a positive constant, is fulfilled for all sufficiently large values of* τ.

Using the obtained relations, we can get the main result on the problem (15).

Theorem 2. *The function* $u_1(\tau, y)$ *defined by the formula* (17) *satisfies for* $\tau \geq T$ *the inequalities*

$$|u(\tau, y) - u_1(\tau, y)| \leq M\tau^{-\gamma}\exp\left(-m_3y^2/\tau\right) \quad \text{if} \quad y \leq 0,$$
$$|u(\tau, y) - u_1(\tau, y)| \leq M\tau^{-\gamma} \quad \text{if} \quad 0 \leq y \leq \phi'(b)\tau,$$
$$|u(\tau, y) - u_1(\tau, y)| \leq M\tau^{-\gamma}\exp\left\{-m_3[y - \phi'(b)\tau]^2/\tau\right\} \ \text{if} \ y \geq \phi'(b)\tau.$$

Here M *and* T *are sufficiently large constants,* γ *is an arbitrary positive constant such that* $1/2 < \gamma < 1$, *and* m_3 *is a constant,* $0 < m_3 < 1/4$.

The proof of Theorem 2 is analogous to that of Lemma 7.

Getting back to the variables t and x, we can rephrase the obtained results in terms of the following

Theorem 3. *Everywhere outside some neighborhood of the origin the estimate*

$$|u(t,x) - u_1\left(t/\epsilon^2, y/\epsilon^2\right)| \leq$$
$$\leq \begin{cases} M\epsilon^{2\gamma}t^{-\gamma}\exp\left(-m_3x^2/\epsilon^2 t\right) & \text{if } x \leq 0, \\ M\epsilon^{2\gamma}t^{-\gamma} & \text{if } 0 \leq x \leq \phi'(b)t, \\ M\epsilon^{2\gamma}t^{-\gamma}\exp\{-m_3[x-\phi'(b)t]^2/\epsilon^2 t\} & \text{if } \phi'(b)t \leq x \end{cases}$$

is valid (m_3 is a positive constant), note that the radius of that neighborhood is of order $\mathcal{O}(\epsilon^2)$.

Obviously, Theorem 3 may be considered to be valid for all $t > 0$ if the constant M is sufficiently large; note once more that one can obtain the corresponding estimates for finite values of t by using the same techniques we have used in the first section.

Theorem 3 gives us an idea of the character of variation of the solution of the problem under consideration as $t \to \infty$.

CHAPTER II
LINEAR EQUATIONS OF ELLIPTIC AND PARABOLIC TYPES

In this chapter we consider linear, singularly perturbed problems connected with equations of elliptic, parabolic and mixed types.

Many problems of physics, chemistry and engineering lead to mathematical models representing singularly perturbed problems for elliptic and parabolic equations in different parts of a domain. For example, when modeling the descending straight motion of liquids and gases in plates of rectification columns, there arises a boundary value problem for a parabolic equation with discontinuous coefficients [12]. In mathematical description of electromagnetic fields appearing in moving of trains on a magnetic cushion there arises the necessity in studying solutions of singularly perturbed boundary value problems for differential equations which are elliptic in one part D^+ of the domain D and parabolic in the other part $D^- = D \setminus \overline{D}^+$; note that solutions of an elliptic equation depend in D^- on the time variable t as on a parameter [18], [19]. Many problems dealing with foliated and periodic media lead to problems for mixed type differential equations.

When we solve practical problems connected with singularly perturbed differential equations of mixed type or with discontinuous coefficients (the so-called stiff problems), quite often we encounter the problem of choosing a numerical method which would be stable with respect to simple parameter variations. To justify the choice of one or another numerical algorithm, it is necessary to divide the initial problem into several consecutive problems to which standard methods of numerical calculation could be applied. Such a “deparallelization” of the initial problem becomes possible owing to preliminary investigation of the asymptotic character of a solution.

In the first two sections of Chapter II we deal with the boundary value problems for elliptic equations, when coefficients of the equation are either discontinuous or a higher coefficient of the degenerate equation vanishes. If a higher coefficient of the degenerate equation vanishes on some line, then in constructing an asymptotic expansion of a solution there arise problems for parabolic equations with the so-called varying time direction, when the coefficient of the derivative with respect to the “time” variable vanishes. In the second section, for problems of similar type, theorems on the solvability of corresponding problems in unbounded domains are proved; estimates of the behavior of solutions for unboundedly increasing arguments are obtained; asymptotic (in powers of the small parameter) expansion of a solution is constructed.

In the third section we consider an equation of elliptico-parabolic type. Under different additional assumptions, asymptotic expansions of solutions of the problems under consideration are constructed; estimates of closeness of partial sums of asymptotic expansions to exact solutions of the problems are presented.

2.1. Elliptic Equations with Strongly Varying Coefficients

Not infrequently, numerical modeling of processes in science and engineering lead us to the necessity of investigating solutions of elliptic equations with strongly varying coefficients. Problems of similar type are most frequently encountered when studying processes which take place in foliated media and also in media with small-grained or periodic structure. Application of standard numerical algorithms of solvong such equations is ineffective by the following reasons. As a rule, in discretization of the initial problem there appears a system of linear algebraic equations with a symmetric matrix A whose eigenvalues λ have great scattering, for example, $\max \lambda_A [\min \lambda_A]^{-1} \sim \max k(\overline{x})[\min k(\overline{x})]^{-1} = \Lambda \gg 1$ ($k(\overline{x})$ is the diffusion coefficient). Application of traditional numerical methods for solving a system of linear algebraic equations with matrices with $\Lambda \gg 1$ requires a significant waste of the processor time (a number of iterations necessary to obtain the solution of a system of linear algebraic equations is at best proportional to $\sqrt{\Lambda}$)).

In their work [7], N.S. Bakhvalov and G.P. Panasenko suggested an iterative method of solving the Dirichlet problem for the elliptic equation with strongly varying diffusion coefficient

$$-\operatorname{div}(k(\overline{x})\nabla u) = f(\overline{x}), \quad \overline{x} \in \Omega, \quad u\,|_{\partial\Omega} = 0.$$

The rate of convergence of the suggested iterative process does not depend on the value of Λ; in particular, the case is quite possible where the coefficient Ω in the domain $k(\overline{x})$ is equal to infinity. Moreover, it was assumed that $\overline{\Omega} = \bigcup\limits_{i=1}^{N} \overline{\Omega}_i$,, where Ω_i are nonintersecting subdomains with piecewise smooth boundaries, and the coefficient $k(\overline{x})$ satisfies the conditions:

1^0. $k(\overline{x}) = \alpha_i k_i(\overline{x}), \quad \overline{x} \in \Omega_i,\ \alpha_i = const \geq 1, \quad 0 < K_0 \leq k_i(\overline{x}) \leq K_i;$

2^0. $\Omega_1, \Omega_2, \dots, \Omega_N$ are topologically separable.

For such a problem we have managed to construct an iteration process with the rate of convergence not depending on the value $\Lambda = \max\limits_i \alpha_i$ for the case where every subdomain Ω_i, $i = 1, 2, \dots, N$ with a large value of α_i is in the vicinity of the subdomain Ω_j with values α_j of order $\mathcal{O}(1)$ only.

As it turns out, without knowledge of the character of variation of a solution in different parts of the domain α_i, α_j, α_s and without corresponding a priori estimates we are unable to extend that method to problems in which the orders of Ω_i, Ω_j, Ω_s in the three neighbouring subdomains Ω are different. In this connection, it is of great importance to study asymptotic behavior of a solution for large values of the diffusion coefficient in different parts of the domain Ω.

1. Let us study asymptotic properties of solutions of the model equation

$$-\operatorname{div}(k_\epsilon(x, y)\nabla u_\epsilon) = f(x, y), \tag{1}$$

whose solution is sought in the rectangle $\overline{\Omega} = \bigcup\limits_{i=1}^{5} \overline{\Omega}_i$, $\Omega_i = [i-1, i] \times [0, b]$, $1 \le i \le 5$; as for the diffusion coefficient, we assume that it is of the form

$$k_\epsilon(x,y) = \begin{cases} k_1(x,y) & \text{if } (x,y) \in \Omega_1 = \{(x,y) \mid 0 < x < 1, 0 < y < 1\}, \\ \epsilon^{-1} k_2(x,y) & \text{if } (x,y) \in \Omega_2 = \{(x,y) \mid 1 < x < 2, 0 < y < 1\}, \\ \epsilon^{-2} k_3(x,y) & \text{if } (x,y) \in \Omega_3 = \{(x,y) \mid 2 < x < 3, 0 < y < 1\}, \\ \epsilon^{-1} k_4(x,y) & \text{if } (x,y) \in \Omega_4 = \{(x,y) \mid 3 < x < 4, 0 < y < 1\}, \\ k_5(x,y) & \text{if } (x,y) \in \Omega_5 = \{(x,y) \mid 4 < x < 5, 0 < y < 1\}, \end{cases}$$

$0 \le \epsilon \ll 1$ is a small parameter.

Let a solution $u_\epsilon(x,y)$ of the equation (1) satisfy the boundary conditions

$$u_\epsilon(x,y)\big|_{x=0} = u_\epsilon(x,y)\big|_{x=5} = 0, \quad u'_{\epsilon y}(x,y)\big|_{y=0} = u'_{\epsilon y}(x,y)\big|_{y=1} = 0. \quad (2)$$

We impose the following additional conditions on the lines of discontinuity of the coefficients and on the right-hand side of the equation:

$$\begin{gathered} u_\epsilon(i-0,y) = u_\epsilon(i+0,y), \\ k_\epsilon(i-0,y)\partial u_\epsilon(i-0,y)/\partial\vec{n} = k_\epsilon(i+0,y)\partial u_\epsilon(i+0,y)/\partial\vec{n}, \end{gathered} \quad (3)$$

where $\vec{n}$ is a normal to the interface line, $1 \le i \le 4$. Suppose $f(x,y)$, $k_i(x,y) \ge k_0 > 0$ are piecewise continuous functions which are uniformly bounded along with all their derivatives in the closure of each subdomain. The solvability of the problem (1)–(3) follows from the work [40].

Let us formulate the problem of constructing an asymptotic, as $\epsilon \to 0$, representation of the solution of the problem under consideration. We will naturally assume that the solution of the equation (1) in domains Ω_1 and Ω_5 under the conditions (2) and boundary condition of the first or second kind for $x = 1$ and $x = 4$ can be found to any degree of precision by means of some algorithm (e.g., a numerical one). As for the subsequent application of the obtained results, it suffices to prove that the asymptotic representation is close to the exact solution in the integral norm, although by simple calculations we can obtain corresponding estimates in a uniform norm.

Asymptotic representation $U^{(N)}(x,y,\epsilon)$ of a solution of the problem (1)–(3) for an arbitrary N is sought in the form

$$U_N(x,y,\epsilon) \sim \sum_{m=0}^{N} \epsilon^m u_m(x,y). \quad (4)$$

The coefficients of the asymptotic representation (4) will be defined for each subdomain of the domain Ω_i, $1 \le i \le 5$. We begin with the domain Ω_3.

To define the functions $u_m(x,y)$ in the domain Ω_3, we have the problem

$$L_3 u_m \equiv -\operatorname{div}\big(k_3(x,y)\operatorname{grad} u_m(x,y)\big) = \delta_{2,m} f(x,y), \quad (5)$$

$$-k_3(x,y)u'_{mx}(x,y)\big|_{x=2+0} = k_2(x,y)u'_{m-1x}(x,y)\big|_{x=2-0}, \quad (6)$$

$$k_3(x,y)u'_{mx}(x,y)\big|_{x=3-0} = -k_4(x,y)u'_{m-1x}(x,y)\big|_{x=3+0}, \quad (7)$$

$$u'_{my}(x,y)\big|_{y=0} = u'_{my}(x,y)\big|_{y=1} = 0. \tag{8}$$

In particular, for the function $u_0(x,y)$ we have the homogeneous equation (5) with homogeneous boundary conditions. Consequently, $u_0(x,y) \equiv C_0$, where C_0 is a constant to be defined below.

In the domain Ω_2, the functions $u_m(x,y)$ are defined in terms of solutions of the problems

$$L_2 u_m \equiv -\operatorname{div}(k_2(x,y)\operatorname{grad} u_m(x,y)) = \delta_{2,m} f(x,y), \tag{9}$$

$$-k_2(x,y)u'_{mx}(x,y)\big|_{x=1+0} = k_1(x,y)u'_{m-1x}(x,y)\big|_{x=1-0}, \tag{10}$$

$$u_m(x,y)\mid_{x=2-0} = u_m(x,y)\mid_{x=2+0}, \tag{11}$$

$$u'_{my}(x,y)\big|_{y=0} = u'_{my}(x,y)\big|_{y=1} = 0. \tag{12}$$

It follows from the relations (9)–(12) that for $(x,y) \in \Omega_2$ the identity $u_0(x,y) \equiv C_0$ should be fulfilled. Analogously one can determine that the relation $u_0(x,y) \equiv C_0$ holds for $(x,y) \in \Omega_4$. In the domain Ω_1, for the function $u_m(x,y)$ we obtain the problem

$$L_1 u_m \equiv -\operatorname{div}\left(k_1(x,y)\operatorname{grad} u_m(x,y)\right) = \delta_{0,m} f(x,y), \tag{13}$$

$$\begin{aligned} &u'_{my}(x,y)\big|_{y=0} = u'_{my}(x,y)\big|_{y=1} = 0, \\ &u_m(x,y)\big|_{x=0} = 0, \quad u_m(x,y)\big|_{x=1-0} = u_m(x,y)\big|_{x=1+0}. \end{aligned} \tag{14}$$

Similar problems for the function $u_m(x,y)$ can be written out in the domain Ω_5. The solvability of elliptic equations in a rectangle with mixed boundary conditions can be found in [40]. In particular, for the function $u_0(x,y)$ the corresponding boundary value problem for $0 < x < 1$ has the form

$$\begin{aligned} &L_1 u_0(x,y) = f(x,y), \quad (x,y) \in \Omega_1, \\ &u'_{0y}(x,y)\mid_{y=0} = u'_{0y}(x,y)\mid_{y=1} = 0, \\ &u_0(x,y)\mid_{x=0} = 0, \quad u_0(x,y)\mid_{x=1-0} = C_0. \end{aligned} \tag{15}$$

Leaving the constant C_0 for the time being undefined, we pass to constructing the function $u_1(x,y)$. In the domain Ω_3 we have for this purpose the problem (5)–(8). Hence $u_1(x,y) \equiv C_1$, $(x,y) \in \Omega_3$, where C_1 is a constant which, like a constant C_0, will be defined below.

For the function $u_1(x,y)$ in the domain Ω_2 we have the problem

$$L_2 u_1(x,y) = f(x,y), \tag{16}$$

$$\begin{aligned} &\partial u_1(x,y)/\partial y\big|_{y=0,y=1} = 0, \quad u_1(x,y)\mid_{x=2-0} = C_1, \\ &-k_2(x,y)u'_{1x}(x,y)\big|_{x=1+0} = k_1(x,y)u'_{1x}(x,y)\big|_{x=1-0}. \end{aligned} \tag{17}$$

The problem for the function $u_1(x,y)$ in the domain Ω_4 is formulated analogously.

Finally, we define the function $u_2(x,y)$ in the domain Ω_3 in terms of the solution of the problem

$$L_3 u_2(x,y) = f(x,y), \tag{18}$$

with the boundary conditions

$$\begin{array}{l} u'_{2y}(x,y)\big|_{y=0,y=1} = 0, \\ -k_3(x,y)u'_{2x}(x,y)|_{x=2+0} = k_2(x,y)u'_{1x}(x,y)|_{x=2-0}\,, \\ k_3(x,y)u'_{2x}(x,y)|_{x=3-0} = -k_4(x,y)u'_{1x}(x,y)|_{x=3+0}\,, \end{array} \tag{19}$$

Choose now a constant C_0 so as to ensure the solvability of the problem (18)–(19). Using Green's formula, we write the condition for its solvability as follows:

$$I_{2,3} = \int_0^1\int_2^3 f(x,y)dxdy = \int_{\partial\Omega_3} k_3(x,y)\frac{\partial u_2(x,y)}{\partial\vec{n}}ds. \tag{20}$$

Taking into account the condition (19), we can write the relation (20) in the form of the following equality:

$$I_{2,3} = \int_1^0\left[k_2(x,y)\frac{\partial u_2(x,y)}{\partial x}\right]\Big|_{x=2-0}dy - \int_0^1\left[k_4(x,y)\frac{\partial u_1(x,y)}{\partial x}\right]\Big|_{x=3+0}dy. \tag{21}$$

For the problem (16)–(17) we write out the relation similar to (19) and define from it the first summand in the right-hand side of (21):

$$I_{1,2} = \int_1^0\left[-k_2(x,y)\frac{\partial u_1(x,y)}{\partial x}\right]\Bigg|_{x=1+0}dy + \int_{y=0}^{y=1}\left[k_2(x,y)\frac{\partial u_1(x,y)}{\partial x}\right]\Bigg|_{x=2}dy,$$

$$\int_0^1\left[k_2(x,y)\ \frac{\partial u_1(x,y)}{\partial x}\right]\Bigg|_{x=2}dy = \int_1^0\left[k_2(x,y)\frac{\partial u_1(x,y)}{\partial x}\right]\Bigg|_{x=1-0}dy +$$

$$+I_{1,2} = \int_{y=0}^{y=1}\left[k_1(x,y)\frac{\partial u_0(x,y)}{\partial x}\right]\Bigg|_{x=1-0}dy + I_{1,2}.$$

Reasoning analogously, we obtain the expression for the second summand appearing in the right-hand side of (21). As a result, this equality will take the form

$$I_{1,4} = \int_1^0\left[k_1(x,y)\frac{\partial u_0(x,y)}{\partial x}\right]\Big|_{x=1-0}dy - \int_0^1\left[k_5(x,y)\frac{\partial u_0(x,y)}{\partial x}\right]\Big|_{x=4-0}dy. \tag{22}$$

Having written out for problems of the type (15) the relations analogous to the relation (20), we can express the summands in the right-hand side of the equation (22) which will take the form

$$I_{0,5} = \int_1^0\left[k_1(x,y)\frac{\partial u_0(x,y)}{\partial x}\right]\Bigg|_{x=0}dy - \int_0^1\left[k_5(x,y)\frac{\partial u_0(x,y)}{\partial x}\right]\Bigg|_{x=5}dy. \tag{23}$$

Get back to the solution of the problem (15). It can be represented as follows: $u_0(x,y) = z_1(x,y) + C_0 z_2(x,y)$, where C_0 is a constant to be defined, and $z_j(x,y)$, $j = 1, 2$, are solutions of the problems

$$\begin{aligned} L_1 z_1 = f(x,y), \ \ (x,y) \in \Omega_1, \quad z'_{1y}|_{y=0,\, y=1} = 0, \ \ z_1|_{x=0,\, x=1} = 0, \\ L_1 z_2 = 0, \ \ (x,y) \in \Omega_1, \quad z'_{2y}|_{y=0,\, y=1} = 0, \ \ z_2|_{x=0} = 0, \ \ z_2|_{x=1} = 1. \end{aligned} \tag{24}$$

In a similar form we can represent a solution of the problem (8): $u_0(x,y) = \overline{z}_1(x,y) + C_0 \overline{z}_2(x,y)$, $(x,y) \in \Omega_5$. The equation (23) will take the form

$$\begin{aligned} I_{0,5} = \int_1^0 k_1(x,y) \left.\frac{\partial \overline{z}_1}{\partial x}\right|_{x=0} dy + \int_1^0 k_5(x,y) \left.\frac{\partial \overline{z}_1}{\partial x}\right|_{x=5} dy + \\ + C_0 \int_1^0 k_1(x,y) \left.\frac{\partial \overline{z}_2}{\partial x}\right|_{x=0} dy + C_0 \int_1^0 k_5(x,y) \left.\frac{\partial \overline{z}_2}{\partial x}\right|_{x=5} dy. \end{aligned} \tag{25}$$

Let us show that the equation (25) is solvable with respect to the constant C_0. Since the problems for the functions $z_2(x,y)$, $\overline{z}_2(x,y)$ are of the same type, it is sufficient to show that the value, for example $K_{0,1} = \int_{y=1}^{y=0} k_1(x,y) \frac{\partial z_2(x,y)}{\partial x}\big|_{x=0} dy$, has the property of having a fixed sign. According to the corollary from Hopf's theorem (see, e.g., [42]), the solution of the problem (24) can have negative (positive) relative minimum at none of the interior points of the domain Ω_1. Suppose that the relation $\min\limits_{y=0,\, y=1} z_2(x,y) = z_2(x_0,y_0) \leq 0$ holds. Then it follows from the Zaremba-Giraud principle that the inequality $z'_{2y}(x,y)\big|_{(x_0,y_0)} > 0$ is fulfilled at the point (x_0,y_0). However, by the hypothesis from (24) it follows that the derivative with respect to the variable y of the function $z_2(x,y)$ is equal on the horizontal sides of the rectangle Ω to zero, and therefore the function $z_2(x,y)$ reaches its minimum for $x = 0$. Note that there takes place the inequality $\partial z_1(0,\overline{y})/\partial x > 0$ which, as it follows from the relations (24), is fulfilled for an arbitrary point $(0,\overline{y})$, $0 < \overline{y} < 1$. This implies the relation $K_{0,1} > 0$.

Consequently, the equation (23) is uniquely solvable with respect to the constant C_0, and hence the function $u_0(x,y)$ is defined in the domain Ω.

Just in the same way we can define the constants $C_1, C_2, \ldots$. Note that to construct the function $U^{(N)}(x,y,\epsilon)$ uniquely, it is necessary to consider the function $U^{(N+2)}(x,y,\epsilon)$ and then, with the help of the solvability conditions for the coefficients of the expansion $u_{N+1}(x,y)$, $u_{N+2}(x,y)$, to define the constants C_{N-1}, C_N. which arise in the process of constructing.

Theorem 1. *There exists a value $\epsilon_0 > 0$ such that for all positive $\epsilon \leq \epsilon_0$ the estimate*

$$||u_\epsilon(x,y) - U^{(N)}(x,y,\epsilon)||_{L_2} \leq M\epsilon^{N-1}$$

is valid.

Proof. Obviously, the function $w_N(x,y,\epsilon)=u_\epsilon(x,y)-U^{(N)}(x,y,\epsilon)$ is the solution of the problem

$$
\begin{aligned}
&Lw_N \equiv \operatorname{div}\Big(k_\epsilon(x,y)\operatorname{grad} w_N(x,y,\epsilon)\Big)=0, \quad (x,y)\in \bigcup_{i=1}^{5}\Omega_i,\\
&w'_{Ny}(x,0,\epsilon)=w'_{Ny}(x,1,\epsilon)=0,\ w_N(0,y,\epsilon)=w_N(5,y,\epsilon)=0,\\
&w_N(i+0,y,\epsilon)-w_N(i-0,y,\epsilon)=0, \quad i=1,2,3,4,\\
&k_\epsilon(i+0,y)\partial w_N(i+0,y,\epsilon)/\partial\vec{n}-k_\epsilon(i-0,y)\partial w_N(i-0,y,\epsilon)/\partial\vec{n}=\\
&\quad=\epsilon^{N+1}[k_\epsilon(x,y)\partial u_{N+1}(x,y,\epsilon)/\partial\vec{n}]\Big|_{x=i+0}^{x=i-0}, \quad 1\le i\le 4.
\end{aligned}
\tag{26}
$$

It is readily seen that the right-hand side of the last of equations (26) can be estimated as follows:

$$
\Big|\Big[k_\epsilon(i+0,y)\frac{\partial u_{N+1}(i+0,y)}{\partial\vec{n}}-k_\epsilon(i-0,y)\frac{\partial u_{N+1}(i-0,y)}{\partial\vec{n}}\Big]\Big|\le M.
$$

Formally integrating once by parts the identity

$$
\int_0^1\int_0^5 \eta(x,y)Lw_N(x,y,\epsilon)dxdy=0,
$$

where $\eta(x,y)$ is an arbitrary function from the space $W^1_{2,0}$ – the subspace of the space $W^1_2(\Omega)$, in which a set of all functions from $C^1(\overline{\Omega})$ is dense and equal in the vicinity of boundaries $x=0$, $x=5$ of the rectangle $\eta(x,y)$ to zero. Taking $\eta(x,y)$ as the function $w_N(x,y,\epsilon)$, we can easily obtain the equality

$$
\begin{aligned}
&\sum_{i=1}^{4}\int_{y=0}^{y=1}\Big[k_\epsilon(x,y)\frac{\partial w_N(x,y,\epsilon)}{\partial\vec{n}}w_N(x,y,\epsilon)\Big]\Big|_{x=i}dy=\\
&\quad=\int_0^1\int_0^5 k_\epsilon(x,y)\Big\{\Big(\frac{\partial w_N}{\partial x}\Big)^2+\Big(\frac{\partial w_N}{\partial y}\Big)^2\Big\}dxdy.
\end{aligned}
$$

With regard for the relation (16), the left-hand side of the latter equation can be estimated from over:

$$
\Big|\sum_{i=1}^{4}\int_{y=0}^{y=1}\Big[k_\epsilon(x,y)\frac{\partial w_N(x,y,\epsilon)}{\partial\vec{n}}w_N(x,y,\epsilon)\Big]\Big|_{x=i}dy\Big|\le M\epsilon^{N-1} \tag{27}
$$

and the right-hand side estimated from upper:

$$
\int_0^1\int_0^5 k_\epsilon(x,y)\Big\{\Big(\frac{\partial w_N}{\partial x}\Big)^2+\Big(\frac{\partial w_N}{\partial y}\Big)^2\Big\}dxdy\ge k_0\|\operatorname{grad} w_N\|^2_{L_2}. \tag{28}
$$

As far as the function $w_N(x,y,\epsilon)$ is an element of the space $W^1_{2,0}$, for it the inequality

$$
\int_0^1\int_0^5 w_N^2(x,y,\epsilon)dxdy\le C\int_0^1\int_0^5\Big(\operatorname{grad} w_N\Big)^2dxdy \tag{29}
$$

with the constant C, depending only on the domain Ω, is valid (see [42]). Taking into account relations (27)–(29), we arrive at the inequality $||w_N(x,y,\epsilon)||_{L_2} \leq M\epsilon^{N+1}$ for any value N, which proves the theorem. □

As is indicated in [40], in the considered by us case of high smoothness in each of subdomains of the functions $k_i(x,y)$, $f(x,y)$ the solution of the problem (1)–(3) satisfies (in the classical sense) the equation (1) and the conditions (2)–(3) at every interior point of the set $\Omega = \bigcup_{i=1}^{5} \Omega_i$ and of each of the segments composing the boundary of domains Ω_i, $1 \leq i \leq 5$, respectively. Using this fact and the estimate of Theorem 1, we can prove the assertion of the following.

Theorem 2. *For the partial sum $U^{(N)}(x,y,\epsilon)$ of an asymptotic expansion of a solution of the problem* (1)–(3) *the estimate*

$$||u_\epsilon(x,y) - U^{(N)}(x,y,\epsilon)||_{C(\Omega)} \leq M\epsilon^{N-1}$$

is valid.

It is not likewise difficult to estimate the first derivatives of the difference $u_\epsilon(x,y) - U^{(N)}(x,y,\epsilon)$ and to obtain an error estimate in the norm of the space C^1 for the asymptotic representation.

Remark. Consideration of the problem on constructing an asymptotic expansion of a solution of the problem (1)–(2) under the assumption that the coefficient $k_\epsilon(x,y)$ may be of order ϵ in some strips $x_1 < x < x_2$ of the original rectangle is not a matter of much difficulty; moreover, some parts of the rectangle, being the strips of the type $x_3 < x < x_4$, may have the width μ, where the value μ is a small parameter not depending on the parameter ϵ. Finally, instead of the equation (1) one can consider a more general equation which contains terms with first derivatives.

2.2. Elliptic Equation with Degeneration in Lowest Terms

In this section we construct asymptotic expansions of solutions for solving some problems which are connected with a singularly perturbed equation of elliptic type, for which the coefficient at a higher derivative of the corresponding degenerate equation vanishes on some smooth line. In constructing a formal asymptotic expansion there arises the need to study properties of solutions of parabolic equations with varying time direction.

1. In the rectangle $D\{(x,y) \mid 0 < x < 1,\ |y| < 1\}$ let us consider the boundary value problem

$$L_\epsilon u \equiv \epsilon^2 \Delta u - a(x,y)u'_y - k(x,y)u = -f(x,y), \quad u\,|_{\partial D} = 0. \qquad (1)$$

Suppose the functions $a(x,y)$, $k(x,y)$ are defined in the closure of the rectangle D and are infinitely differentiable and uniformly bounded along with the derivatives of any order of functions, $k(x,y) \geq k_0 > 0$.

Asymptotic expansions of various types for solving the problem (1) under different assumptions on initial data have been previously constructed by many authors (see, e.g., [13], [32], etc.). As a rule, one of the basic assumptions we use in considering such kind of problems is the assumption about regular solvability of the appropriate degenerate equation; in the problem under consideration this assumption leads, in particular, to the requirement that everywhere in the rectangle $\overline{D}$ the inequality $|a(x,y)| > 0$ should be fulfilled.

We will assume that $a(x,\phi(x)) \equiv 0$, where $y = \phi(x)$ is some infinitely differentiable for $x \in [0,1]$ function such that $|\phi(x)| < 1$. Under such an assumption, the type of an asymptotic representation of a solution of the problem (1) depends substantially on the signs of the function $a(x,y)$ for $y < \phi(x)$ and $y > \phi(x)$. If we assume that the functions involved in the equation are very smooth, then it becomes essential only the sign of the function $a(x,y)$ as the variable y increases. Therefore the description of the techniques of constructing asymptotic expansions of solutions will cover the cases: 1) $a(x,y) \equiv yb(x)$; 2) $a(x,y) \equiv -yb(x)$; 3) $a(x,y) \equiv y^2 b(x)$; 4) $a(x,y) \equiv -y^2 b(x)$. Here $b(x)$ is an infinitely differentiable for $x \in [0,1]$ function, $b(x) \geq b > 0$. The construction of asymptotic expansions for some other types of dependence of the coefficient $a(x,y)$ on its arguments differs by insignificant technical details only.

It is not difficult to see that it is sufficient to describe the asymptotic expansion for the above-mentioned cases only in the neighborhood of the straight lines $x = 0$, $y = 0$ since the types of the expansion in the two neighboring portions of the boundary can be obtained by a simple change of variables.

In accordance with the results obtained in [1], we will additionally assume in case (2) that there exists a positive integer N such that for all $x \in [0,1]$ the inequality $Nb(x) + k(x,\phi(x)) > 0$ holds.

Asymptotic expansion of a solution of the problem (1) in the neighborhood of the left and lower portions of the boundary of the rectangle D will be sought in the form

$$U^{(N)}(x,y,\epsilon) = \sum_{m=0}^{N} \epsilon^{2m} \Big[u_{2m}(x,y) + v_{2m}\Big(x, \frac{1+y}{\epsilon^2}\Big) \Big] +$$

$$+ \sum_{m=0}^{2N} \epsilon^{m} \Big[P_m\Big(\frac{x}{\epsilon}, y\Big) + w_m\Big(\frac{x}{\epsilon}, \frac{1+y}{\epsilon^2}\Big) + Q_m\Big(\frac{x}{\epsilon^2}, \frac{y}{\epsilon^2}\Big) \Big]. \qquad (2)$$

When deducing the equations which determine the functions in the representation (2), the functions will be assumed to tend uniformly to zero as at least one of the arguments increases unboundedly, that is, these functions are of boundary layer character of variation with respect to their arguments.

2. Let $a(x,y) \equiv yb(x)$. In this case, the coefficients of the representation (2) can be constructed for any value of the number N. The coefficients $u_{2m}(x,y)$ of the so-called smooth parts of the asymptotic expansion, approximating the solution of the initial problem outside a fixed neighborhood of the set ∂D, can be found in terms of solutions of the equations

$$L_0 u_{2m} \equiv a(x,y)\frac{\partial u_{2m}}{\partial y} + k(x,y)u_{2m} = \delta_{0,m} f(x,y) + \Delta u_{2m-2} \equiv f_{2m}(x,y); \quad (3)$$

here and everywhere in what follows, the functions with negative indices are assumed to be identically equal to zero. The solution of each of the equations (3) exists for $|y| \leq 1$, is unique and infinitely differentiable in $\overline{D}$ (see [1]). In the neighborhood of the boundary $x = 0$ of the rectangle D, the boundary layer terms of the representation (2) are found for $0 < \xi < \infty$, $|y| < 1$ in terms of the solutions of the problems

$$L_\xi P_m \equiv \frac{\partial^2 P_m}{\partial \xi^2} - b_0 y \frac{\partial P_m}{\partial y} - k_0(y) P_m =$$

$$= -\frac{\partial^2 P_{m-2}}{\partial y^2} + \sum_{j=1}^{m} k_j(y)\xi^j P_{m-j} + y\sum_{j=1}^{m} b_j \xi^j \frac{\partial P_{m-j}}{\partial y} = \tilde{P}_m(\xi, y), \quad (4)$$

$$P_m(0,y) = -u_m(0,y) \equiv p_m(y), \quad (5)$$

where b_j, $k_j(y)$ are the coefficients of the expansions of the functions $b(x)$, $k(x,y)$ in powers of the variable x. Without restriction of generality we may assume $b_0 = 1$. Note that the equation (4) is parabolic for $y \neq 0$, with varying time direction; boundary value problems for equations of similar type in bounded domains under zero boundary conditions and zero right-hand side have been considered in [77].

Now we proceed to deducing the equations which determine the functions $v_{2m}(x,\eta)$, $w_m(\xi,\eta)$ guaranteeing the fulfilment of the boundary condition for $y = -1$. By means of a standard procedure we readily obtain that these functions are the solutions of the problems

$$L_{x,\eta} v_{2m} \equiv \frac{\partial^2 v_{2m}}{\partial \eta^2} + b(x)\frac{\partial v_{2m}}{\partial \eta} =$$

$$= \sum_{j=1}^{m} \tilde{k}_j(x)\eta^j v_{2m-2j} - \frac{\partial^2 v_{2m-2}}{\partial x^2} + \eta b(x)\frac{\partial v_{2m-2}}{\partial \eta}, \quad (6)$$

$$L_{\xi,\eta} w_m \equiv \frac{\partial^2 w_m}{\partial \eta^2} + \frac{\partial w_m}{\partial \eta} = \sum_{2l+j}^{m-2} k_{j,l}\xi^j \eta^l w_{m-j-2l-2} -$$

$$- \sum_{j=1}^{m} b_j \xi^j \frac{pa w_{m-j}}{\partial \eta} - \frac{\partial^2 w_{m-2}}{\partial \xi^2} + \eta \sum_{j=0}^{m-2} b_j \xi^j \frac{\partial w_{m-j-2}}{\partial \eta}, \quad (7)$$

where $\tilde{k}_j(x)$, $k_{j,l}$ are the coefficients of the expansions of the function $k(x,y)$ by the Taylor formula,

$$v_{2m}(x,0) = -u_{2m}(x,-1), \quad w_m(\xi,0) = -P_m(\xi,-1). \tag{8}$$

The functions $v_{2m}(x,\eta)$, $w_m(\xi,\eta)$, which ensure the fulfilment of the boundary condition for $y=-1$, cause discrepancy in the boundary conditions for $x=0$. To remove that discrepancy, we use the functions $Q_m(\zeta,\eta)$ which are found in terms of the solutions of the problems

$$L_{\zeta,\eta}Q_m \equiv \Delta Q_m + Q'_{m\eta} = \sum_{j+l=1}^{[\frac{m}{2}]-2} k_{j,l}\zeta^j\eta^l Q_{m-2j-2l-2} -$$
$$-\sum_{j=1}^{[\frac{m}{2}]-1} b_j\zeta^j \frac{\partial Q_{m-2j}}{\partial\eta} + \eta\sum_{j=0}^{[\frac{m}{2}]-2} b_j\zeta^j\frac{\partial Q_{m-2j-2}}{pa\eta} \equiv \tilde{Q}_m(\zeta,\eta), \tag{9}$$

$$Q_m(\zeta,0)=0, \quad Q_m(0,\eta) = -v_m(0,\eta) - w_m(0,\eta) \equiv q_m(\eta), \tag{10}$$

$0<\zeta$, $\eta<\infty$, tending to zero as $\zeta+\eta\to\infty$.

3. Now we pass to the problems of existence and uniqueness of solutions of the problems written out by us and to investigation of properties of these problems. We begin with the functions $P_m(\xi,y)$ which guarantee the fulfilment of the boundary condition for $x=0$.

To solve the problem (4), (5) for $m=0$ under the assumption that the function $P_0(\xi,y)$ vanishes as $\xi\to 0$, we will use the Fourier sine-transformation; supposing the application of that transformation is legitimate, we will find an explicit form of the function $P_0(\xi,y)$, investigate its properties and prove the validity of all transformations used.

Applying to both parts of the equation (4) the Fourier sine-transformation, we obtain for the Fourier image $\tilde{P}_0(s,y)$ the equation

$$y\tilde{P}'_{0y} + \left[s^2 + k_0(y)\right]\tilde{P}_0 = sp_0(y)\sqrt{2/\pi}. \tag{11}$$

Obviously, the equation (11) is analogous to the equation (3). As it follows from the results of [1], the equation (11) has for $|y|\le 1$ a unique solution which is infinitely differentiable for all $|y|\le 1$. It is easily seen that the Fourier preimage in that case can be written out explicitly:

$$|P_0(\xi,y)| = \frac{2}{\sqrt{\pi}}\int_0^\infty p_0(y\omega)\exp\left[-r^2 + \int_y^{y\omega} t^{-1}k_0(t)dt\right]dr, \tag{12}$$

$\omega = \exp\left[-\xi^2/(4t^2)\right]$.

Lemma 1. *The function in* (12) *is well defined; it is continuous in the half-strip* $0\le\xi<\infty$, $|y|\le 1$. *This function is continuously differentiable* *in the domain of its definition, and there exists a constant $\gamma > 0$ such that the estimate*

$$|P_0(\xi,y)| + \left|P'_{0\xi}(\xi,y)\right| + \left|P'_{0y}(\xi,y)\right| +$$
$$+ \left|P''_{0yy}(\xi,y)\right| + \left|P''_{0y\xi}(\xi,y)\right| + \left|P'''_{0yy\xi}(\xi,y)\right| \le M\exp\left(-\gamma\xi\right) \qquad (13)$$

holds.

Proof. First we show that the function $P_0(\xi,y)$ is defined for $y=0$. Obviously,

$$P_0(\xi,y) = \frac{2}{\sqrt{\pi}}\int_0^\infty p_0(y\omega)\exp\Big[-r^2 - \frac{k_0(\tilde t)\xi^2}{4r^2}\Big]dr.$$

This implies that the limit of the function $y \to 0$ as $P_0(\xi,y)$ exists and can be written in the form

$$\lim_{y\to 0} P_0(\xi,y) = \lim_{y\to 0}\frac{2}{\sqrt{\pi}}\int_0^\infty p_0(y\omega)\exp\Big[-r^2 - \frac{k_0(\tilde t)\xi^2}{4r^2}\Big]dr =$$
$$= \frac{2}{\sqrt{\pi}}\left[\int_0^{\rho_0}\exp\left[-r^2 - \frac{k_{0,0}\xi^2}{4r^2}\right]dr + \int_{\rho_0}^\infty \exp\left[-r^2 - \frac{k_{0,0}\xi^2}{4r^2}\right]dr\right] =$$
$$= \frac{2}{\sqrt{\pi}}\int_{\rho_0}^\infty\Big(1 + \frac{\sqrt{k_{0,0}\xi}}{2r^2}\Big)\exp\Big[-r^2 - \frac{k_{0,0}\xi^2}{4r^2}\Big]dr,$$

$\rho_0 = [k_{0,0}\xi/4]^{1/4}$. It is clear that the integral, appearing in the right-hand side of the latter equality, exists for all $0 \le \xi < \infty$, being a function, differentiable for $\xi > 0$.

To obtain the estimate (13), let us consider formula (12):

$$|P_0(\xi,y)| \le M\int_{\rho_1}^\infty \exp\left(-r^2\right)dr \le M(1+\sqrt{\xi})^{-1}\exp\left(-\overline{k}\xi/2\right),$$

$\rho_1 = (\overline{k}\xi^2/4)^{1/4}$. To estimate the derivative of the function $P_0(\xi,y)$ with respect to the variable ξ, we consider the integral obtained from (2) by means of a formal differentiation with respect to the variable ξ under the integral sign. Obviously, $y\omega p'_0(y\omega) + k_0(y\omega)p_0(y\omega) = -f(0,y\omega)$. We have

$$\int_0^\infty \frac{\xi}{2r^2}\exp\Big[-r^2 + \frac{\overline{k}\xi^2}{4r^2}\Big]dr \le M\int_{\rho_1}^\infty \exp\left(-r^2\right)dr,$$

and therefore the integral $I(\xi,y)$ converges uniformly with respect to variables ξ, y. Consequently, the function $P_0(\xi,y)$ is differentiable with respect to the variable ξ everywhere in the domain of definition, and there takes place the estimate

$$\Big|\frac{\partial P_0}{\partial\xi}\Big| \le M\int_0^\infty \frac{\xi}{4r^2}\exp\Big(-r^2 - \frac{\overline{k}\xi^2}{4r^2}\Big)dr \le M\frac{1}{1+\sqrt{\xi}}\exp\Big(-\frac{\overline{k}\xi}{2}\Big).$$

Let us pass to the estimation of the first derivative of the function $P_0(\xi, y)$ with respect to the variable y. With this end in view, we consider the integral

$$J(\xi, y) = \frac{2}{\sqrt{\pi}} \int_0^\infty \left[\omega p_0'(y\omega) + (\omega - 1) p_0(y\omega) \times \right.$$
$$\left. \times \int_0^1 k_0'(y + \theta y(\omega - 1)) d\theta \right] \exp\left[- r^2 + \int_y^{y\omega} t^{-1} k_0(t) dt \right] dr,$$

which is obtained by a formal differentiation with respect to the variable y of the integrand in (12). As in the previous case, we shall have the inequality $|J(\xi, y)| \le M(1 + \sqrt{\xi})^{-1} \exp\left(-\overline{k}\xi/2\right)$, from which follow the existence and the estimate for the function $P_{0y}'(\xi, y)$.

It is readily seen that in a similar manner one can estimate the function $\partial^2 P_0/\partial\xi\partial y$, derivatives of the function $P_0(\xi, y)$ with respect to the variable y of any order and also the function $\partial^3 P_0/\partial\xi\partial y^2$ with the same majorizing function in the right-hand side of the inequality. □

Lemma 2. *A solution of the problem* (4), (5) *exists for* $m > 0$ *and satisfies the estimates of the type* (13).

Proof. Evidently, to deduce the estimates for the solution of problem (4), (5) for $m > 0$ it is sufficient to estimate the influence exerted by the right-hand side of the equation. Using the Fourier sine-transformation, a corresponding component of a solution can be written as

$$P_{m,1}(\xi, y) = -\frac{1}{2\sqrt{\pi}} \int_0^\infty e^{-\overline{k}z} \frac{dz}{\sqrt{z}} \exp\left[\int_y^{\sigma_1} \frac{k_0(t)}{t} dt \right] \int_0^\infty \tilde{P}_m(\eta, ye^{-z}) \times$$
$$\times \left\{ \exp\left[- \frac{(\eta - \xi)^2}{4\ln(y/r)} \right] - \exp\left[- \frac{(\eta + \xi)^2}{4\ln(y/r)} \right] \right\} d\eta, \tag{14}$$

$\sigma_1 = ye^{-z}$. Supposing that the estimate (13) is valid for the function $\tilde{P}_m(\xi, y)$ and for its derivatives to within the factor $1 + \xi^{m-1}$, we can write the inequality

$$|P_{m,1}(\xi, y)| \le$$
$$\le M\left\{ e^{-\sqrt{\overline{k}}\xi/2} \int_0^\infty e^{-3\overline{k}z/4} dz \int_{\sigma_2}^\infty (1 + \xi - \sqrt{\overline{k}}z + 2\sqrt{z}\omega)^{m-1} e^{-\omega^2} d\omega - \right.$$
$$\left. - e^{\sqrt{\overline{k}}\xi/2} \int_0^\infty e^{-3\overline{k}z/4} dz \int_{\sigma_3}^\infty (1 - \xi - \sqrt{\overline{k}}z + 2\sqrt{z}\omega)^{m-1} e^{-\omega^2} d\omega \right\},$$

$\sigma_2 = [\sqrt{\overline{k}}z - \xi]/(2\sqrt{z})$, $\sigma_3 = [\sqrt{\overline{k}}z + \xi]/(2\sqrt{z})$. Let us estimate the first integral in braces. Obviously,

$$(1 + \xi - \sqrt{\overline{k}}z + 2\sqrt{z}\omega)^{m-1} \le M \sum_{i+j+l=0}^{m-1} \xi^i z^{j+l/2} \omega^l, \tag{15}$$

where i, j, l are nonnegative integers. Therefore

$$\int_0^\infty e^{-3\overline{k}z/4}dz\int_{\sigma_2}^\infty (1+\xi-\sqrt{\overline{k}}z+2\sqrt{z}\omega)^{m-1}e^{-\omega^2}d\omega\le M(1+\xi^{m-1}).$$

Consider the second integral

$$\int_0^\infty e^{-3\overline{k}z/4}dz\int_{\sigma_3}^\infty (1-\xi-\sqrt{\overline{k}}z+2\sqrt{z}\omega)^{m-1}e^{-\omega^2}d\omega\le$$

$$\le M\sum_{i+j+l=0}^{m-1}\xi^i\int_0^\infty e^{-3\overline{k}z/4}z^{j+l/2}\omega^l dz\int_{\sigma_3}^\infty e^{-\omega^2}d\omega\le$$

$$\le M\sum_{i+j+l=0}^{m-1}\sum_{s=1}^{l-1}\xi^s\int_0^\infty z^{j+l-s-1/2}\exp\Big[-\frac{(\sqrt{\overline{k}}z+\xi)^2}{4z}-\frac{3}{4}\overline{k}z\Big]dz.$$

By virtue of the validity of the sequence of the inequalities we have

$$\int_0^\infty z^{j+l-s-1/2}\exp\Big(-\frac{\xi^2}{4z}-\overline{k}z\Big)dz\le$$

$$\le M\int_{\sigma_4}^\infty \xi^{2j+2l-2s-2}\omega^{-2j-2l+2s-2}e^{-\omega^2}d\omega+M\int_{\sigma_4}^\infty \omega^{2j+2l-2s}e^{-\omega^2}d\omega\le$$

$$\le M\left(1+\xi^{j+l}\right)\int_{\sigma_4}^\infty e^{-\omega^2}d\omega+M\left(1+\xi^{j+l-s-1/2}\right)\int_{\sigma_4}^\infty e^{-\omega^2}d\omega,$$

$\sigma_4=(\overline{k}\xi^2/4)^{1/4}$. Therefore

$$\int_0^\infty z^{j+l-s-1/2}\exp\Big(-\frac{\xi^2}{4z}-\overline{k}z\Big)dz\le M\left(1+\xi^{j+l-s}\right)e^{-\sqrt{\overline{k}}\xi/2},$$

whence

$$e^{\sqrt{\overline{k}}\xi/2}\int_0^\infty e^{-3\overline{k}z/4}\int_{\sigma_3}^\infty \left(1-\xi-\sqrt{\overline{k}}z+2\sqrt{z}\right)^{m-1}e^{-\omega^2}d\omega\le$$

$$\le M\left(1+\xi^{m-1}\right)e^{-\sqrt{\overline{k}}\xi/2}.$$

Thus the estimation of the module of the function $P_{m,1}(\xi,y)$ is complete.

We shall now proceed to the estimation of the first derivative of the function $P_{m,1}(\xi,y)$ with respect to the variable ξ. To this end we consider the integral

$$I(\xi,y)=-\frac{1}{2\sqrt{\pi}}\int_0^\infty\frac{dz}{\sqrt{z}}\exp\Big[\int_y^{\sigma_1}\frac{k_0(t)}{t}dt\Big]\int_0^\infty\tilde{P}_m(\eta,ye^{-z})\times$$

$$\times\Big\{\frac{\eta-\xi}{2z}\exp\Big[-\frac{(\eta-z)^2}{4z}\Big]-\frac{\eta+\xi}{2z}\exp\Big[-\frac{(\eta+z)^2}{4z}\Big]\Big\}d\eta=I_1(\xi,\eta)+I_2(\xi,\eta),$$

which is obtained from equation (18) by means of a formal differentiation under the integral sign. Evidently,

$$|I_1(\xi,y)| \le Me^{-\sqrt{k}\xi/2} \times$$
$$\times \int_0^\infty e^{-3\overline{k}z/4}dz \int_{\sigma_2}^\infty (1+\xi-\sqrt{\overline{k}}z+2\sqrt{z}\omega)^{m-1}\left|\sqrt{\overline{k}}z-2\omega\right|e^{-\omega^2}d\omega.$$

Using inequality (15), we obtain

$$|I_1(\xi,y)| \le Me^{-\sqrt{k}\xi/2}\sum_{i+j+l=0}^{m-1}\xi^i\int_0^\infty z^{j+l/2}e^{-3\overline{k}z/4}dz\int_{-\infty}^\infty \omega^l(\sqrt{z}+\omega)e^{-\omega^2}d\omega.$$

As is easily seen, the integrals in the right-hand side of the latter inequality converge, and therefore $|I_1(\xi,y)| \le M(1+\xi^{m-1})e^{-\sqrt{k}\xi/2}$.

Let us pass to the estimation of the integral $I_2(\xi,y)$:

$$|I_2(\xi,y)| \le Me^{\sqrt{k}\xi/2}\sum_{i+j+l=0}^{m-1}\xi^i\int_0^\infty z^{j+l/2}e^{-3\overline{k}z/4}dz\int_{\sigma_3}^\infty \omega^l(\sqrt{z}+\omega)e^{-\omega^2}d\omega.$$

Acting similarly to that we have done when estimating the function $P_{m,1}(\xi,y)$, we can get the estimate $|I_2(\xi,y)| \le M(1+\xi^{m-1})e^{-\sqrt{k}\xi/2}$. This implies that the function $P_{m,1}(\xi,y)$ is differentiable with respect to the variable ξ, and for the derivative of that function the estimate $|(P_{m,1}((\xi,y))'_\xi| \le M(1+\xi^{m-1})e^{-\sqrt{k}\xi/2}$ is valid.

To obtain the estimates for the derivatives of the function $P_{m,1}(\xi,y)$ appearing in the left-hand side of inequality (13), it is sufficient to note that, as is shown in Lemma 1, the estimate (13) is valid for the derivatives of the function $P_0(\xi,y)$ with respect to the variable y of any order. Supposing that analogous estimates are valid for the corresponding derivatives of the function $P_{m-1,1}$, without any difficulty we obtain the desired estimates for the derivatives of the function $P_m(\xi,y)$ with respect to the above-mentioned variables. □

4. The existence of solutions of the problems (6), (8) and (7), (8) and the boundary layer character of variation of the functions $v_{2m}(x,\eta)$, $w_m(\xi,\eta)$ are obvious. Therefore the following assertion is valid.

Lemma 3. *For the functions $v_{2m}(x,\eta)$ and $w_m(\xi,\eta)$ the estimates*

$$|v_{2m}(x,\eta)|+|v'_{2mx}(x,\eta)|+\left|v'_{2m\eta}(x,\eta)\right|+|v''_{2mxx}(x,\eta)| \le M(1+\eta^{2m})e^{-b(x)\eta},$$
$$|w_m(\xi,\eta)|+\left|w'_{m\xi}(\xi,\eta)\right|+\left|w'_{m\eta}(\xi,\eta)\right|+\left|w''_{m\xi\xi}(\xi,\eta)\right| \le$$
$$\le M(1+\xi^{m+1}+\eta^{m+1})e^{-\eta-\sqrt{k}\xi/2}$$

are valid.

In proving the lemma it suffices only to take into account that by virtue of Hölder's inequality, $\xi^j\eta^l \le \left[j\xi^{j+l} + l\eta^{j+l}\right](j+l)^{-1}$.

Consider now the problem (9), (10). Passing to the function $R_m(\zeta,\eta) = Q_m(\zeta,\eta)e^{-\eta/2}$, we obtain in the quarter of the plane $0<\zeta<\infty$, $0<\eta<\infty$ the problem for the elliptic equation with fixed coefficients:

$$\Delta R_m - 4^{-1}R_m = \tilde{Q}_m(\zeta,\eta)e^{\eta/2} \equiv \tilde{R}_m(\zeta,\eta), \tag{16}$$

$$R_m(\zeta,0) = 0,\ R_m(0,\eta) = q_m(\eta)e^{\eta/2}. \tag{17}$$

Note that as it directly follows from the relations (5), (8), (10), the boundary values of the functions $Q_m(\zeta,\eta)$ for any $m \ge 0$ are consistent with continuity. Moreover, direct calculations show that $q_0(\eta) \equiv q_1(\eta) \equiv 0$, $q_2(\eta) \not\equiv 0$; in this connection the identities $Q_0(\zeta,\eta) \equiv 0$, $Q_1(\zeta,\eta) \equiv 0$ are fulfilled.

We can write out the solution of the problem (16), (17) explicitly by using McDonald's function $K_0(r)$ of the zero order. Thus the fact that a solution of the problem (9), (10) exists for $m = 2$, becomes obvious.

To estimate the rate by which the function $Q_2(\zeta,\eta)$ tends to zero as $\eta+\zeta \to \infty$, we will use the maximum principle.

Lemma 4. *For the functions $Q_m(\zeta,\eta)$ for $0<\zeta<\infty$, $0<\eta<\infty$ the estimates*

$$|Q_m(\zeta,\eta)| + \left|\frac{\partial Q_m(\zeta,\eta)}{\partial\zeta}\right| + \left|\frac{\partial Q_m(\zeta,\eta)}{\partial\eta}\right| \le M\exp\left(-\alpha_m\zeta - \beta_m\eta\right)$$

are valid, where $\alpha_m = 2^{-3/2} - \lambda_m$, $\beta_m = 2^{-1} + 2^{-3/2} - \lambda_m$, $0 < \lambda_m < 2^{-3/2}$ are some constants.

Proof. Obviously, there takes place the inequality

$$|q_2(\eta)| \le M(1+\eta^2)e^{-\eta}. \tag{18}$$

Consider the function $\Phi_2(\zeta,\eta) = M_2\exp\left(-\alpha_2\zeta - \beta_2\eta\right)$, where M_2, α_2, β_2 are some positive constants, and estimate the function $Q_2(\zeta,\eta)$ and its derivatives. It is readily seen that $L_{\zeta,\eta}\Phi_2 = \left(\alpha_2^2 + \beta_2^2 - \beta_2\right)\Phi_2$. Let us choose constants α_2, β_2 such that the inequality $\alpha_2^2 + \beta_2^2 - \beta_2 < 0$ be fulfilled. We choose the constant M_2 is such a way that for $\beta_2 < 1$ there would take place the relation $\Phi_2(0,\eta) \ge |q_2(\eta)|$. Under these conditions the functions $\Phi_2(\zeta,\eta) \pm Q_2(\zeta,\eta)$ admit nonnegative values on the boundary of the domain $D_{\zeta,\eta}\{(\zeta,\eta) \mid 0<\zeta<\infty,\ 0<\eta<\infty\}$, and at the points of the domain $D_{\zeta,\eta}$ itself the inequalities $L_{\zeta,\eta}\left(\Phi_2 \pm Q_2\right) \le 0$ are valid. Owing to the maximum principle, everywhere in the closure of the domain $D_{\zeta,\eta}$ there takes place the estimate $|Q_2(\zeta,\eta)| \le A_2\exp\left(-\alpha_2\zeta - \beta_2\eta\right)$.

Let us now estimate derivatives of the function $Q_2(\zeta,\eta)$ (see [33]). Consider an auxiliary function $\Psi_3(\zeta,\eta) = N\exp\left(-N_1\eta - \alpha_2\zeta\right)$, where the positive constants N, N_1 are to be defined. Obviously, $L_{\zeta,\eta}\Psi_3 = (N_1^2 - N_1 + \alpha_2^2)\Psi_3(\zeta,\eta)$. For $\eta = 0$, the function $\Psi_3(\zeta,\eta) + Q_2(\zeta,\eta)$ takes a positive value $Ne^{-\alpha_2\zeta}$. Let the constant N be so large that for $\eta = N_1^{-1}$ the inequality $0 < N\exp(-1-\alpha_2\zeta) + Q_2(\zeta, N_1^{-1}) < N$ is fulfilled; for that inequality to be valid, it suffices to take $N \geq A_2/[e^{\alpha_2\zeta} - e^{-1}] \geq A_2 e/(e-1)$. Now we choose a constant $N_1 > 1$ so large that for $\zeta = 0$ the function $\Psi_2(\zeta,\eta) + Q_2(\zeta,\eta)$ decreases on the segment $\eta \in [0, N_1^{-1}]$; we can see that such a choice of the constant N_1 is quite possible; under such a choice the constants N and N_1 the function $\Psi_2(\zeta,\eta) + Q_2(\zeta,\eta)$ on the boundary of the half-strip $0 \leq \eta \leq N_1^{-1}$, $0 \leq \zeta < \infty$ takes its largest value for $\eta = 0$. Moreover, increasing (if necessary) the constant N_1, we can assume that the inequality $L_{\zeta,\eta}(\Psi_2 + Q_2) = (N_1^2 - N_1 + \alpha_2^2)\Psi_2 > 0$ holds at the interior points of that half-strip. According to the maximum principle, the function $\Psi_2(\zeta,\eta) + Q_2(\zeta,\eta)$ cannot take its largest value at the above-mentioned points. Consequently, the function reaches its largest positive value for $\eta = 0$, and therefore $(\Psi'_{2\eta} + Q'_{2\eta})\big|_{\eta=0} \leq 0$, whence $Q'_{2\eta}\big|_{\eta=0} \leq NN_1 e^{-\alpha_2\zeta}$.

In a similar way one can obtain lower bound to the function $Q'_{2\eta}$ for $\eta = 0$. Evidently, exponential estimate for that derivative is valid for $\zeta = 0$ as well. Reasoning as above, we can estimate on the boundary of the domain $D_{\zeta,\eta}$ the derivative of the function $Q_2(\zeta,\eta)$ with respect to the variable ζ. Differentiating now both parts of equation (9) for $m = 2$ with respect to the appropriate variable and using the maximum principle, we obtain exponential estimates for the first derivatives of the function $Q_2(\zeta,\eta)$. Thus, everywhere in $D_{\zeta,\eta}$ the estimates

$$|Q_2(\zeta,\eta)| + \left|Q'_{2\zeta}(\zeta,\eta)\right| + \left|Q'_{2\eta}(\zeta,\eta)\right| \leq M\exp(-\alpha_2\zeta - \beta_2\eta)$$

hold.

When we estimate functions $Q_m(\zeta,\eta)$ for $m > 2$, it is more convenient to consider problems (16), (17). Suppose the inequalities

$$|\tilde{Q}_m(\zeta,\eta)| \leq M(1 + \zeta^{m+1} + \eta^{m+1})\exp(-\alpha_m\zeta - \beta_m\eta)$$

hold, where α_m and β_m are some constants. Under such a choice of constants α_m and β_m, the right-hand side of equation (16) will be a function, decreasing exponentially; moreover, in this case there takes place the inequality $|\tilde{R}_m(\zeta,\eta)| \leq M\exp[-\alpha_m(\zeta+\eta)]$. Represent a solution of problem (16), (17) in terms of the sum $R_m(\zeta,\eta) = R_{m,1}(\zeta,\eta) + R_{m,2}(\zeta,\eta)$, where the function $R_{m,1}(\zeta,\eta)$ satisfies equation (16) and the homogeneous boundary conditions (17), while the function $R_{m,2}(\zeta,\eta)$ satisfies the homogeneous equation (16) and the boundary conditions (17). Make a change of variables $Z_{m,2}(\zeta,\eta) = R_{m,2}(\zeta,\eta) - e^{-\alpha_m\zeta}R_m(0,\eta)$. The function $Z_{m,2}(\zeta,\eta)$ satisfies the homogeneous boundary conditions for $\zeta = 0$, $\eta = 0$. For $\zeta > 0$, $\eta > 0$, this function satisfies the equation $\Delta Z_{m,2} - 4^{-1}Z_{m,2} = \tilde{Z}_m(\zeta,\eta)$, where $\tilde{Z}_m(\zeta,\eta)$ decreases exponentially as $\zeta + \eta \to \infty$ and is uniformly bounded for $(\zeta,\eta) \in \overline{D}_{\zeta,\eta}$. Hence, to estimate derivatives of the function $R_m(\zeta,\eta)$ in the domain $D_{\zeta,\eta}$, it is sufficient to estimate derivatives of solutions of the equation

$$\Delta Z - 4^{-1}Z = \tilde{Z}(\zeta,\eta), \quad (\zeta,\eta) \in D_{\zeta,\eta}, \tag{19}$$

whose right-hand side is uniformly bounded, decreases exponentially as $\zeta + \eta \to \infty$ and has uniformly bounded derivatives of the first order which vanish exponentially as $\zeta + \eta \to \infty$. Moreover, the function $Z(\zeta, \eta)$ on the boundary of the domain $D_{\zeta,\eta}$ satisfies zero boundary conditions of the first kind.

Let us write a solution of problem (16), (17) in the form

$$Z(\zeta, \eta) =$$

$$= \frac{1}{2\pi} \int_0^\infty \int_0^\infty \tilde{Z}(s,t) \Big[K_0\Big(\frac{r_1}{2}\Big) - K_0\Big(\frac{r_2}{2}\Big) - K_0\Big(\frac{r_3}{2}\Big) + K_0\left(\frac{r_4}{2}\right) \Big] ds dt, \quad (20)$$

$$r_1 = \sqrt{(\zeta - s)^2 + (\eta - t)^2}, \quad r_2 = \sqrt{(\zeta - s)^2 + (\eta + t)^2},$$

$$r_3 = \sqrt{(\zeta + s)^2 + (\eta - t)^2}, \quad r_4 = \sqrt{(\zeta + s)^2 + (\eta + t)^2},$$

$K_0(r)$ is the McDonald's function of zero order.

Using the representation (20) and taking into account asymptotic representations of the McDonald's functions and also the properties of the functions $\tilde{R}_m(\zeta, \eta)$, we can conclude by induction that the functions $Z_2(\zeta, \eta)$ exist, are differentiable in the domain $D_{\zeta,\eta}$ and for them the estimates

$$|Z_2| + \left|Z'_\zeta(\zeta, \eta)\right| + \left|Z'_\eta(\zeta, \eta)\right| \le M \exp\left[-\alpha_m(\zeta + \eta)\right]$$

are valid, where α_m are constants chosen by the above-mentioned manner. The above inequality implies that the assertions of the lemma are valid. $\square$

Thus the formal construction of the asymptotic representation (2) can be considered to be fulfilled.

We will now proceed to the error estimation of the above-constructed asymptotic representation.

Theorem 1. *If $a(x, y) \equiv yb(x)$, where $b(x)$ is a function which is infinitely differentiable and positive for $x \in [0, 1]$, then the solution of the problem* (1) *has the asymptotic representation* (2) *for which the estimate*

$$||u(x, y) - u^{(N)}(x, y)||_{C^1} \le M\epsilon^{2N}$$

is valid.

The proof of the theorem is carried out by means of the maximum principle.

5. Let us consider the case $a(x, y) \equiv -yb(x)$. In that case, in the representation (2) we suppose that $v_{2m}(x, \eta) \equiv w_m(\xi, \eta) \equiv Q_m(\zeta, \eta) \equiv 0$ and the condition $\overline{k} > N \max\limits_{x \in [0,1]} b(x)$ is fulfilled. In this connection the coefficients of the asymptotic representation (2) will be considered only for those values of m for which the inequality $2m \le N - 3$ is fulfilled.

The asymptotic behavior of the solution of the problem under consideration is well-known; for the coefficients of the expansion to remain bounded as they reach an angular point of the domain, we require that the function

$f(x, y)$ as well as all its derivatives up to order N be vanishing at the angular points of the domain D.

Just as in the previous case, the functions $u_{2m}(x, y)$ are found in terms of the solutions of the equations (3). However, solutions of these equations in the case under consideration will be sought under the following additional conditions:

$$u_{2m}(x, -1) = 0, \quad u_{2m}(x, 1) = 0. \tag{21}$$

As follows from the results of [1], solutions of the problems (3), (21) exist and are $N - 2m$ times differentiable everywhere in the rectangle D. To eliminate residual exerted in the boundary condition by the functions $u_{2m}(x, y)$ for $x = 0$, the functions $P_m(\xi, y)$ will be constructed in terms of solutions of the problems

$$\tilde{L}_{\xi,y} P_m \equiv \frac{\partial^2 P_m}{\partial \xi^2} + y \frac{\partial P_m}{\partial y} - k_0(y) P_m =$$

$$= -\frac{\partial^2 P_{m-2}}{\partial y^2} + \sum_{j=1}^{m} k_j(y) \xi^j P_{m-j} - y \sum_{j=1}^{m} b_j \xi^j \frac{\partial P_{m-j}}{\partial y} = \tilde{P}_m(\xi, y), \tag{22}$$

$$P_m(\xi, -1) = P_m(\xi, 1) = 0, \quad P_m(0, y) = -u_m(0, y) \equiv p_m(y). \tag{23}$$

Note that the equality $p_m(-1) = p_m(1) = 0$ holds by virtue of the conditions (23). Moreover, the derivatives up to the order $N - m$ of the function $p_m(y)$ vanish for $y = \pm 1$.

Let us consider the problem (22), (23) for $m = 0$. To solve that problem, we again make use of the Fourier sine-transformation. For the Fourier image $\tilde{P}_0(s, y)$ we obtain the problem

$$y \tilde{P}'_{0y} - [s^2 + k_0(y)] \tilde{P}_0 = s p_0(y) \sqrt{2/\pi}, \tag{24}$$

$$\tilde{P}_0(s, -1) = 0, \quad \tilde{P}_0(s, 1) = 0. \tag{25}$$

Writing out the solution of the problem (24), (25) and using the inverse Fourier sine-transformation, we obtain the solution of the problem (22), (23) for $m = 0$:

$$P_0(\xi, y) = \sqrt{\frac{2}{\pi}} \int_{\sigma_6}^{\infty} p_0(y\omega) \exp\left[\int_{y\omega}^{y} \frac{k_0(t)}{t} dt - r^2\right] dr, \tag{26}$$

$\sigma_6 = \xi / 2\sqrt{-\ln |y|}$, $\omega = \exp\left(\xi^2 / 4r^2\right)$. As before, it is not difficult to get the estimate

$$|P_0(\xi, y)| \le M \exp\left(-\sqrt{\overline{k}}\xi/2\right). \tag{27}$$

Let us pass now to the estimation of the derivative of the function $P_0(\xi, y)$ with respect to the variable y. Obviously,

$$\left|\frac{\partial P_0(\xi, y)}{\partial y}\right| \le M \int_{\sigma_6}^{\infty} \exp\left\{\frac{\xi^2}{4r^2} - \frac{\overline{k}\xi^2}{4r^2} - r^2\right\} dr \le M \exp\left[-\sqrt{\overline{k}_1}\xi/2\right],$$

$\overline{k}_1 = \overline{k} - 1$. Analogously, we can obtain the estimate for the second and third derivatives of the function $P_0(\xi, y)$ with respect to the variable y. We have

$$\frac{\partial^2 P_0(\xi,y)}{\partial y^2} = \frac{2}{\sqrt{\pi}} \int_{\sigma_6}^{\infty} \left\{ p_0''(y\omega)\omega^2 + p_0'(y\omega)\omega \left[\frac{k_0(y)}{y} - \frac{k_0(y\omega)}{y}\right] + \right.$$
$$+ p_0(y\omega)\Big[\frac{k_0'(y) - \omega k_0'(y\omega)}{y} - \frac{k_0(y) - k_0(y\omega)}{y^2}\Big] + \frac{k_0(y) - k_0(y\omega)}{y} \times$$
$$\left. \times \left[p_0'(y\omega)0\omega + p_0(y\omega)\frac{k_0(y) - k_0(y\omega)}{y}\right]\right\} \exp\left[\int_{y\omega}^{y} \frac{k_0(t)}{t} dt - r^2\right] dr.$$

Taking into account the relation $p_0(y\omega) = y\omega \int_0^1 p_0'(\theta y\omega) d\theta$, it is not difficult to get the inequalities

$$\Big|\frac{\partial^2 P_)(\xi,y)}{\partial y^2}\Big| \le M \int_{\sigma_6}^{\infty} \exp\Big[-\frac{\overline{k}_2\xi^2}{4r^2} - r^2\Big] dr \le M \exp\Big(-\frac{\sqrt{\overline{k}_2}\xi}{2}\Big), \quad (28)$$
$$\Big|\frac{\partial^3 P_0(\xi,y)}{\partial y^3}\Big| \le M \int_{\sigma_6}^{\infty} \exp\Big[-\frac{\overline{k}_3\xi^2}{4r^2} - r^2\Big] dr \le M \exp\Big(-\frac{\sqrt{\overline{k}_3}\xi}{2}\Big), \quad (29)$$

$\overline{k}_2 = \overline{k} - 2$, $\overline{k}_3 \le \overline{k}_2$. Finally, we obtain the estimate for the derivatives of the functions $P_0(\xi, y)$, $P_{0y}'(\xi, y)$, $P_{0yy}''(\xi, y)$ with respect to the variable ξ. If $\xi \ge -2\sqrt{\overline{k}_1}\ln|y|$, then from the latter relation we immediately obtain the estimate $\Big|P_{0\xi}'(\xi, y)\Big| \le M \exp\Big(-\sqrt{\overline{k}_1}\xi/2\Big)$. If, however, $\xi < -2\sqrt{\overline{k}_1}\ln|y|$, then in that case we can write a chain of inequalities

$$\big|P_{0\xi}'\big| \le M \exp\Big(-\frac{\sqrt{\overline{k}_1}\xi}{2}\Big) + M \int_{\sigma_6}^{\sigma_7} \frac{\xi}{r^2} \exp\Big[-\frac{\overline{k}_1\xi^2}{4r^2}\Big] dr \le$$
$$\le M \exp\Big(-\frac{\sqrt{\overline{k}_1}\xi}{2}\Big). \quad (30)$$

For the derivative of the function $\partial P_0(\xi, y)/\partial\xi$ with respect to the variable y we will have

$$\frac{\partial^2 P_0(\xi,y)}{\partial\xi\partial y} = \frac{2}{\sqrt{\pi}} \int_{\sigma_6}^{\infty} \Big\{ p_0''(y\omega)\omega + p_0'(y\omega)[k_0(y) - k_0(y\omega)] +$$
$$+ p_0'(y\omega) - p_0(y\omega)k_0'(y\omega) - \left[p_0(y\omega)\omega + p_0(y\omega)\frac{k_0(y) - k_0(y\omega)}{y}\right] \times$$
$$\times \frac{k_0(y\omega)}{\omega}\Big\} \frac{\xi}{2r^2} \exp\left[\frac{\xi^2}{4r^2} + \int_{y\omega}^{y} \frac{k_0(t)}{t} dt - r^2\right] dr.$$

Evidently, just in the same way as in estimating the function $\partial P_0(\xi, y)/\partial\xi$, we can obtain the inequality

$$\big|P_{0\xi y}''(\xi, y)\big| \le M \exp\big(-\sqrt{\overline{k}_1}\xi/2\big) \quad (31)$$

and exactly the same estimate for the function $\partial^3 P_0(\xi, y)/\partial\xi\partial y^2$.

Before passing to the functions $P_m(\xi, y)$, first of all it should be noted that it suffices for us to estimate only those components of the functions which are generated by the right-hand sides of the equations (22). Using the Fourier sine-transformation, these components can be written as follows:

$$P_{m,1}(\xi, y) = \frac{1}{2}\int_{\operatorname{sign} y}^{y} \frac{dr}{r\sqrt{\pi \ln|y/r|}} \int_0^\infty \tilde{P}_m(s, r) \exp\left[\int_r^y \frac{k_0(z)}{z} dz\right] \times$$
$$\times\Big\{\exp\Big[-\frac{(s-\xi)^2}{4\ln|r/y|}\Big] - \exp\Big[-\frac{(s+\xi)^2}{4\ln|r/y|}\Big]\Big\}ds =$$
$$= -\frac{1}{2\sqrt{\pi}}\int_0^{-\ln|y|} t^{-1/2}\exp\left[\int_\rho^y \frac{k_0(z)}{z}dz\right] \times$$
$$\times\int_0^\infty \tilde{P}_m(\eta, \rho)\left\{\exp\left[-\frac{(\eta-\xi)^2}{4t}\right] - \exp\left[-\frac{(\eta+\xi)^2}{4t}\right]\right\} d\eta dt, \tag{32}$$

$\rho = ye^t$. Taking into account the type of the right-hand side of the equation (22) and also the estimates (27)–(32), we can assume that the functions $\tilde{P}_m(\xi, y)$ satisfy the inequality $|\tilde{P}_m(\xi, y)| \le M\exp\left(-\sqrt{\overline{k}_m}\xi/2\right)$. Consequently, we can obtain for the function $P_{m,1}(\xi, y)$ the following relations:

$$|P_{m,1}(\xi, y)| \le M\int_0^{-\ln|y|} t^{-1/2}e^{-\overline{k}t}\int_0^\infty \exp\Big[-\frac{\sqrt{\overline{k}}\eta}{2} - \frac{(\xi-\eta)^2}{4t}\Big]d\eta dt +$$
$$+M\int_0^{-\ln|y|} t^{-1/2}e^{-\overline{k}t}\int_0^\infty \exp\Big[-\frac{\sqrt{\overline{k}}\eta}{2} - \frac{(\xi+\eta)^2}{4t}\Big]d\eta dt = A + B.$$

Simple transformations result in

$$A \le Me^{-\sqrt{\overline{k}}\xi/2}\int_0^{-\ln|y|} t^{-1/2}e^{-3\overline{k}t/4}dt\int_{\sigma_8}^\infty e^{-z^2}dz \le Me^{-\sqrt{\overline{k}}\xi/2},$$
$$B \le M\int_0^{-\ln|y|}\exp\left[-\frac{\xi^2}{4t} - \overline{k}t\right]dt \le$$
$$\le M\int_0^{\sigma_9}\exp\left(-\frac{\xi^2}{4t}\right)dt + M\int_{\sigma_9}^\infty \exp(-\overline{k}t)dt \le$$
$$\le M\int_{\sqrt{\overline{k}}\xi/2}^\infty \frac{\xi^2}{4z^2}e^{-z}dz + Me^{-\sqrt{\overline{k}}\xi/2} \le Me^{-\sqrt{\overline{k}}\xi/2},$$

$\sigma_8 = (\sqrt{\overline{k}}t - \xi)/(2\sqrt{t})$, $\sigma_9 = \xi/(2\sqrt{\overline{k}})$. Thus, $|P_m(\xi, y)| \le Me^{-\sqrt{\overline{k}}\xi/2}$. If we take into consideration the circumstance that for $2m \le N-3$ the functions $P_{m,1}(\xi, y)$ and their derivatives up to the third order vanish for $y = \pm 1$, then we will see that estimates of the derivatives with respect to the variable y are of the form (28), and therefore for these derivatives the estimates of the same type are valid, if only we replace the constant $\overline{k}$ by the constants $\overline{k}_1$, $\overline{k}_2$, $\overline{k}_3$ when differentiating with respect to y one, two or three times, respectively.

The functions $\partial P_{m,1}(\xi,y)/\partial\xi$, $\partial^2 P_{m,1}(\xi,y)/\partial\xi\partial y$, $\partial^3 P_{m,1}(\xi,y)/\partial\xi\partial^2 y$. can be estimated analogously.

Error estimation for the above-constructed asymptotic representation can be performed exactly in the same manner as it was done in the aforegoing case. Therefore the following theorem is valid.

Theorem 2. *Let $a(x,y) \equiv -yb(x)$, where $b(x)$ is an infinitely differentiable positive for $x \in [0,1]$ function, $\overline{k} > N \max\limits_{x\in[0,1]} b(x)$, where N is a natural number, and the function $f(x,y)$ and its derivatives up to order N vanish at the angular points of the domain D. Then for the asymptotic representation of the type (4) the estimate*

$$\|u(x,y) - u^{(N)}(x,y)\|_{C^1} \le M\epsilon^{2N} \tag{33}$$

for $v_{2m} \equiv w_m \equiv Q_m \equiv 0$, $2m \le N-3$ is valid.

We can make similar constructions for the cases $a(x,y) \equiv y^2 b(x)$, $a(x,y) \equiv -y^2 b(x)$,, where the function $b(x)$ possesses the above-mentioned properties. Moreover, we can construct asymptotic representations for an arbitrary number N, if the function $f(x,y)$ and its derivatives up to order $N+3$ vanish at the points $(0,-1)$, $(1,-1)$, when $a(x,y) \equiv y^2 b(x)$, and at the points $(0,1)$, $(1,1)$, when $a(x,y) \equiv -y^2 b(x)$. One can also write out a kind of asymptotic representation in the neighborhood of either of the portions of the boundary of the rectangle D, taking into account the sign of the function $a(x,y)$ by analogy with the above considered cases. Estimate of the closeness of the asymptotic representation to the exact solution under the above-mentioned assumptions has the type (33).

6. The problem (1) can be considered analogously in the case where the function $a(x,y)$ vainshes on the lines $y_i = \phi_i(x)$, $i = 1,2,\ldots,m$, $0 < |\phi_i(x)| < 1$ for $x \in [0,1]$. The lines $y_i = \phi_i(x)$ may intersect at the points of the interval $(0,1)$. Note that all the constructions in that case maintain the same singularities as in the above-considered cases; a type of asymptotic representation in the neighborhood of either lines $y=-1$, $y=1$ depends on the sign of the coefficient $a(x,y)$ in the neighborhood of the corresponding line. Just in the same way one can consider the case where the function $a(x,y)$ vanishes, for example, for $y=1$. Note that the construction of an asymptotic representation of order N does not differ from that in the cases where the function $f(x,y)$ and all its derivatives up to order $N+3$ vanish for $y=1$.

2.3. Boundary Value Problems for Equations of Elliptic-Parabolic Type

In this section we construct asymptotic expansions of solutions of boundary value problems which are connected with elliptic-parabolic equations. Problems of similar type can arise in the cases where stationary processes of heat conductivity are described mathematically.

1. Let D be a rectangle $\{(x,z) \mid 0 < x < 1,\ Z_1 < z < Z_2\}$, $D_1 = D \cap \{z < 0\}$, $D_2 = D \cap \{z > 0\}$. Consider the boundary value problem

$$\epsilon^2 \Delta u - k^2(x,z)u = -f_1(x,z), \quad (x,z) \in D_1, \tag{1}$$

$$\epsilon^2 \frac{\partial^2 u}{\partial x^2} - \epsilon^\alpha \frac{\partial u}{\partial z} - b^2(v,z)u = -f_2(x,z), \quad (x,z) \in D_2, \tag{2}$$

$$u(0,z) = \phi_0(z), \quad u(1,z) = \phi_1(z), \quad z \in [Z_1, Z_2], \tag{3}$$

$$u(x, Z_1) = \psi(x), \quad x \in [0,1]. \tag{4}$$

Here α is a nonnegative integer, the coefficients and right-hand sides of the equations (1), (2) as well as boundary functions in the relations (1), (2) are infinitely differentiable functions which are bounded along with their derivatives everywhere in the closure of the corresponding domains of definition, $k(x,z) \geq \overline{k} > 0$, $b(x,z) \geq \overline{b} > 0$, $\phi_1(0) = \psi(0)$, $\phi_2(0) = \psi(1)$.

By a solution of the problem (1)–(2) will be meant a function $u(x,z)$ such that

$$u(x,z) \in C(\overline{D}) \cap C^1(D) \cap C^2(D_1 \cup D_2),$$

satisfies in the domains D_1 and D_2 the equations (1), (2), respectively, while on the lateral and lower faces of the rectangle D it satisfies the conditions (3)–(4). The existence of a solution of the problem (1)–(4) follows from the results presented in the monograph [22].

2. Let $\alpha = 2$. An asymptotic representation of the solution of the problem (1)–(4) will be sought in the form

$$U^{(N)}(x,z,\epsilon) = \sum_{m=0}^{N} \epsilon^m \Big[u_m(x,z) + v_m\Big(x, \frac{z}{\epsilon}\Big) + r_m\Big(\frac{x}{\epsilon}, z\Big) +$$
$$+ w_m\Big(x, \frac{Z_1 - z}{\epsilon}\Big) + q_m\Big(\frac{1-x}{\epsilon}, z\Big) + P_m\Big(\frac{x}{\epsilon}, \frac{z}{\epsilon}\Big) + R_m\Big(\frac{x}{\epsilon}, \frac{Z_1 - z}{\epsilon}\Big) +$$
$$+ Q_m\Big(\frac{1-x}{\epsilon}, \frac{z}{\epsilon}\Big) + S_m\Big(\frac{1-x}{\epsilon}, \frac{Z_1 - z}{\epsilon}\Big)\Big], \quad (x,z) \in D_1, \tag{5}$$

$$U^{(N)}(x,z,\epsilon) = \sum_{m=0}^{N} \Big[\tilde{u}_m(x,z) + \tilde{v}_m\Big(x, \frac{z}{\epsilon^2}\Big) + \tilde{r}_m\Big(\frac{x}{\epsilon}, z\Big) +$$
$$+ \tilde{q}_m\Big(\frac{1-x}{\epsilon}, z\Big) + \tilde{P}_m\Big(\frac{1-x}{\epsilon}, \frac{z}{\epsilon^2}\Big) + \tilde{Q}_m\Big(\frac{1-x}{\epsilon}, \frac{z}{\epsilon^2}\Big)\Big], \quad (x,z) \in D_2. \tag{6}$$

As throughout before, it will be assumed that if at least one of the arguments of an arbitrary function takes unbounded values as $\epsilon \to 0$, then that function is of a boundary layer character as a small parameter tends to zero.

Using a customary procedure, we can write out problems which later will be applied for determining successively the coefficients of the representations (5) and (6); note that the functions with negative indices will be assumed to be identically equal to zero. By $\Delta_{r,s}$ we denote the Laplace operator with independent variables r, s.

$$u_m(x,z) = \delta_{0,m} f_1(x,z)/k^2(x,z) + [k(x,z)]^{-2}\Delta_{x,z} u_{m-2}, \tag{7}$$

$$\tilde{u}_m(x,z) = \delta_{0,m}\frac{f_2(x,z)}{b^2(x,z)} + \frac{1}{b^2(x,z)}\left[\frac{\partial^2 \tilde{u}_{m-2}(x,z)}{\partial x^2} - \frac{\partial \tilde{u}_{m-2}(x,z)}{\partial z}\right];$$

$$\frac{\partial^2 r_m(\xi,z)}{\partial \xi^2} - k^2(0,z) r_m(\xi,z) = \sum_{i=1}^{m} k_{i,0}\xi^i r_{m-i}(\xi,z) - \frac{\partial^2 r_{m-2}(\xi,z)}{\partial z^2}, \tag{8}$$

$$r_m(0,z) = \delta_{0,m}\phi_0(z) - u_m(0,z), \tag{9}$$

$$\frac{\partial^2 \tilde{r}_m(\xi,z)}{\partial \xi^2} - b^2(0,z)\tilde{r}_m(\xi,z) = \sum_{i=1}^{m} b_{i,0}\xi^i \tilde{r}_{m-i}(\xi,z) + \frac{\partial \tilde{r}_{m-2}(\xi,z)}{\partial z}, \tag{10}$$

$$\tilde{r}_m(0,z) = \delta_{0,m}\phi_0(z) - \tilde{u}_m(0,z). \tag{11}$$

Here $k_{i,0}$, $b_{i,0}$ are the coefficients of the expansions of the functions $k^2(x,z)$, $b^2(x,z)$ in powers of the variable x in the neighborhood of the point $x = 0$;

$$v''_{m\tau\tau}(x,\tau) - k^2(x,z)v_m(x,\tau) = \sum_{i=1}^{m} k_{0,i}(x)\tau^i v_{m-i}(x,\tau) - \frac{\partial^2 v_{m-2}(x,\tau)}{\partial x^2}, \tag{12}$$

$$\tilde{v}'_{m\rho}(x,\rho) + b^2(x,0)\tilde{v}_m(x,\rho) = -\sum_{i=1}^{[m/2]} b_{0,i}(x)\rho^i \tilde{v}_{m-2i}(x,\rho) + \frac{\partial^2 \tilde{v}_{m-2}(x,\rho)}{\partial x^2},$$

$$\frac{\partial \tilde{v}_0(x,0)}{\partial \rho} = 0, \quad \frac{\partial \tilde{v}_1(x,0)}{\partial \rho} = \frac{\partial v_0(x,0)}{\partial \tau},$$

$$\frac{\partial \tilde{v}_m(x,0)}{\partial \rho} = \frac{\partial v_{m-1}(x,0)}{\partial \tau} + \frac{\partial u_{m-2}(x,0)}{\partial z} - \frac{\partial \tilde{u}_{m-2}(x,0)}{\partial x}, \; m \geq 2, \tag{13}$$

$$v_m(x,0) = \tilde{v}_m(x,0) + \tilde{u}_m(x,0) - u_m(x,0), \tag{14}$$

where $k_{0,i}(x)$, $b_{0,i}(x)$ are the coefficients of the expansions of the functions $k^2(x,z)$, $b^2(x,z)$ in powers of the variable z in the neighborhood of the point $z = 0$;

$$\Delta_{\xi\tau} P_m - k^2(0,0)P_m(\xi,\tau) = \sum_{i+j=1}^{m} k_{i,j}\xi^i\tau^j P_{m-i-j}(\xi,\tau), \tag{15}$$

$$\frac{\partial^2 \tilde{P}_m}{\partial \xi^2} - \frac{\partial \tilde{P}_m}{\partial \rho} - b^2(0,0)\tilde{P}_m = \sum_{i+2j=1}^{m} b_{i,j}\xi^i\rho^j \tilde{P}_{m-i-2j}(\xi,\rho), \tag{16}$$

$$P_m(0,\tau) = -v_m(0,\tau), \quad \tilde{P}_m(0,\rho) = -\tilde{v}_m(0,\rho),$$

$$P_m(\xi,0) - \tilde{P}_m(\xi,0) = \tilde{r}_m(\xi,0) - r_m(\xi,0),$$

$$\frac{\partial \tilde{P}_0(\xi,0)}{\partial \rho} = 0, \quad \frac{\partial \tilde{P}_1(\xi,0)}{\partial \rho} = \frac{\partial P_0(\xi,0)}{\partial \tau},$$

$$\frac{\partial \tilde{P}_m(\xi,0)}{\partial \rho} = \frac{\partial P_{m-1}(\xi,0)}{\partial \tau} + \frac{\partial r_{m-2}(\xi,0)}{\partial z} - \frac{\partial \tilde{r}_{m-2}(\xi,0)}{\partial z};$$

here $k_{i,j}$, $b_{i,j}$ are the coefficients of the expansions of the functions $k^2(x,z)$, $b^2(x,z)$ in powers of the variables x, z;

$$\frac{\partial^2 w_m}{\partial \omega^2} - k^2(x,Z_1)w_m = \sum_{i=1}^{m} k_i(x)\omega^i w_{m-i}(x,\omega) - \frac{\partial^2 w_{m-2}(x,\omega)}{\partial x^2},$$
$$w_m(x,0) = -u_m(x,Z_1) + \delta_{0,m}\psi(x),$$
$$\Delta_{\xi\omega} R_m - k^2(0,Z_1)R_m = \sum_{i+j=1}^{m} k^0_{i,j}\omega^i \xi^j R_{m-i-j}(\xi,\omega),$$
$$R_m(0,\omega) = -w_m(0,\omega), \quad R_m(\xi,0) = -r_m(\xi,Z_1);$$

in these equalities by $k_i(x)$, $k^0_{i,j}$ we have denoted the coefficients of the expansions of the functions $k^2(x,z)$ in the neighborhood of the straight line $z = Z_1$ and of the point $(0,Z_1)$, respectively.

It is evident that the problems for determining the remaining coefficients of the expansions (5) and (6) are similar to those written out by us above.

3. As is easily seen, the functions $r_m(\xi,z)$, $\tilde{r}_m(\xi,z)$ exist, are infinitely differentiable with respect to either variable and tend exponentially to zero, as the corresponding independent variable tends in its absolute value to infinity. To clarify the properties of the function $R_m(\xi,\omega)$, it should be noted that the conditions for continuity of boundary and initial values of these functions are fulfilled at the angular point of the boundary of the domain of definition. Therefore the estimates obtained in the second section of Chapter II for the function $R_m(\xi,\omega)$ are valid:

$$|R_m(\xi,\omega)| + \left|\frac{\partial R_m(\xi,\omega)}{\partial \xi}\right| + \left|\frac{\partial R_m(\xi,\omega)}{\partial \omega}\right| \le M \exp[-\alpha_m(\xi+\omega)],$$

where α_m is a positive constant.

Let us pass to the problems which determine the functions $P_m(\xi,\tau)$, $\tilde{P}_m(\xi,\rho)$. Obviously, $\tilde{P}_0(\xi,\rho) \equiv 0$. The function $P_0(\xi,\tau)$ is a solution of the problem

$$\Delta_{\xi\tau} P_0 - k_0^2 P_0 = 0,$$
$$P_0(0,\tau) = -v_0(0,\tau) \equiv [u_0(0,0) - \tilde{u}_0(0,0)]e^{-k_0\xi},$$
$$P_0(\xi,0) = [\phi_0(0) - \tilde{u}_0(0,0)]e^{-b_0\xi} - [\phi_0(0) - u_0(0,0)]e^{-k_0\xi},$$

$k_0 = k(0,0)$, $b_0 = b(0,0)$. As is known, a solution of that problem exists, and since the values of the boundary functions at the angular points of the boundary are compatible, there take place the following estimates:

$$\|P_0(\xi,\tau)\|_{C^1} \le M \exp[-\alpha(\xi+\tau)],$$

($\alpha_0 \le k_0/\sqrt{2}$ is a positive constant). Without restriction of generality we may assume that $\alpha_0 \le b_0$. Consequently, the function $\tilde{P}_1(\xi,\rho)$ is found in terms of the solution of the problem

$$\frac{\partial^2 \tilde{P}_1(\xi,\rho)}{\partial \xi^2} - \frac{\partial \tilde{P}_1(\xi,\rho)}{\partial \rho} - b_0 \tilde{P}_1(\xi,\rho) = 0,$$

$$\tilde{P}_1(0,\rho) = \frac{k_0}{b_0^2}[u_0(0,0) - \tilde{u}(0,0)]e^{-b_0^2\rho}, \quad \frac{\partial \tilde{P}_1(\xi,0)}{\partial \rho} = \frac{\partial P_0(\xi,0)}{\partial \tau}.$$

To solve the problem, we introduce into consideration the function $\overline{P}_1 = \tilde{P}'_{1\rho}e^{b_0^2\rho}$. The function $\overline{P}_1(\xi,\rho)$ satisfies the equation

$$\frac{\partial^2 \overline{P}_1(\xi,\rho)}{\partial \xi^2} - \frac{\partial \overline{P}_1(\xi,\rho)}{\partial \rho} = 0$$

as well as the additional conditions

$$\overline{P}_1(0,\rho) = k_0[\tilde{u}_0(0,0) - u_0(0,0)], \quad \overline{P}_1(\xi,0) = \frac{\partial P_0(\xi,0)}{\partial \tau}.$$

Hence the function $\overline{P}_1(\xi,\rho)$ can be written out as follows:

$$\overline{P}_1(\xi,\rho) = \frac{1}{2\sqrt{\pi}}\int_0^\rho k_0[\tilde{u}_0(0,0) - u_0(0,0)]\frac{\xi}{(\rho-\sigma)^{3/2}}\exp\Big[-\frac{\xi^2}{4(\rho-\sigma)}\Big]d\sigma +$$
$$+\frac{1}{2\sqrt{\pi\rho}}\int_0^\infty \frac{\partial P_0(\eta,0)}{\partial \tau}\exp\Big[-\frac{(\xi-\eta)^2}{4\rho}\Big]d\eta -$$
$$-\frac{1}{2\sqrt{\pi\rho}}\int_0^\infty \frac{\partial P_0(\eta,0)}{\partial \tau}\exp\Big[-\frac{(\xi+\eta)^2}{4\rho}\Big]d\eta = I_1 + I_2 - I_3.$$

First let us estimate each of summands on the right-hand side of the last equality. We have

$$|I_1| = 2\frac{k_0|\tilde{u}_0(0,0) - u_0(0,0)|}{\sqrt{\pi}}\int_{\zeta_1}^\infty \frac{\xi}{2\sqrt{\rho}}e^{-\sigma^2}d\sigma \le M\exp\Big[-\frac{\xi^2}{4\rho}\Big],$$

where $\zeta_1 = \xi/(2\sqrt{\rho})$. Taking into account the estimates for the functions $P_0(\xi,\tau)$, we obtain

$$|I_2(\xi,\rho)| \le M\exp[-\alpha_0\xi + \alpha_0^2\rho]\int_{\gamma_1}^\infty e^{-\eta^2}d\eta,$$

$\gamma_1 = -\xi/(2\sqrt{\rho}) + \alpha_0\sqrt{\rho}$. If $\xi \le 2\alpha_0\rho$, then $|I_2(\xi,\rho)| \le M\exp[-\xi^2/(4\rho)]$. If, however, $\xi > 2\alpha_0\rho$, then $|I_2(\xi,\rho)| \le M\exp(-\alpha_0\xi + \alpha_0^2\rho)$. The integral $I_3(\xi,\rho)$ can be estimated analogously:

$$|I_3(\xi,\rho)| \le M\exp()\alpha_0\xi + \alpha_0^2\rho)\int_{\gamma_2}^\infty \exp(-\sigma^2)d\sigma \le M\exp[-\xi^2/(4\rho)],$$

where $\gamma_2 = \xi/(2\sqrt{\rho}) + \alpha_0\sqrt{\rho}$, Thus we are able to get for the function $\tilde{P}'_{1\rho}(\xi,\rho)$ the following estimate:

$$|\tilde{P}'_{1\rho}(\xi,\rho)| \le M\exp[-b_0\rho - \xi^2/(4\rho)] + M\exp[-\alpha_0\xi - (b_0^2 - \alpha_0^2)\rho].$$

Since $\xi \ge 0$, $\rho \ge 0$ $b_0^2\rho + \xi^2/(4\rho) \ge \alpha_0\xi + (b_0^2 - \alpha_0^2)\rho$, the estimate

$$|\tilde{P}'_{1\rho}(\xi,\rho)| \le M\exp[-\alpha_0\xi - (b_0^2 - \alpha_0^2)\rho]$$

holds, and thus there exists a constant $\alpha_1 > 0$ such that the inequality $|\tilde{P}'_{1\rho}(\xi,\rho)| \le M\exp[-\alpha_1(\xi+\rho)]$ is valid. By virtue of that inequality, $|\tilde{P}_1(\xi,\rho)| \le M\exp[-\alpha_1(\xi+\rho)]$. Using the equation (16), we can get for the derivative of the function $\tilde{P}_1(\xi,\rho)$ with respect to the variable ξ the inequality

$$\Big|\frac{\partial\tilde{P}_1(\eta,\rho)}{\partial\xi}\Big| \le \int_\xi^\infty \Big|\frac{\partial^2\tilde{P}_1(\eta,\rho)}{\partial\eta^2}\Big| d\eta \le M\exp[-\alpha_1(\xi+\rho)].$$

It is readily seen that for $m > 1$ the estimates of similar type can be obtained for the functions $P_m(\xi,\tau)$, $\tilde{P}_m(\xi,\rho)$ under a suitable choice of the constants α_m.

Theorem 1. *For the asymptotic representation* (5), (6) *of a solution of the problem* (1)–(4) *in case* $\alpha = 2$ *the estimate*

$$\|u(x,z,\epsilon) - U^{(N)}(x,z,\epsilon)\|_{C^1} \le M\epsilon^{N-1}$$

is valid, where the constant M *does not depend on* ϵ.

Proof. The function $U^{(N)}(x,z,\epsilon)$ is, generally speaking, not continuous at the points of the straight line $z = 0$. It is not difficult to see that the relations

$$\overline{u}_N(x,\epsilon) \equiv U^{(N)}(x,+0,\epsilon) - U^{(N)}(x,-0,\epsilon) = \mathcal{O}(\epsilon^{N+1}),$$
$$\overline{u}_{N,z}(x,\epsilon) \equiv \frac{\partial U^{(N)}(x,+0,\epsilon)}{\partial z} - \frac{\partial U^{(N)}(x,-0,\epsilon)}{\partial z} = \mathcal{O}(\epsilon^{N-1})$$

hold. Introduce the function

$$\omega_N(x,z,\epsilon) = U^{(N)}(x,z,\epsilon) - \theta(z)[\overline{u}_N(x,\epsilon) + z\overline{u}_{N,z}(x,\epsilon)],$$

where $\theta(z)$ is the Heaviside's function, equal to zero for $z < 0$ and to 1 for $z \ge 0$. Suppose $G_N(x,z,\epsilon) = u(x,z) - \omega_N(x,z,\epsilon)$. Obviously, the function $G_N(x,z,\epsilon)$ is continuous and continuously differentiable everywhere in the rectangle D. Making use of the maximum principle, without any difficulty we can prove the assertion of the theorem. □

4. Consider now the problem (1)–(4) for $\alpha = 0$. In this case the functions $\phi_0(z)$, $\phi_1(z)$ will be assumed to be identically equal to zero, and the right-hand sides of the equations (1), (2) to vanish along with their derivatives of high enough order at the points $(0,0)$ and $(1,0)$.

As an asymptotic representation, $u^{(N)}(x,z,\epsilon)$ will be sought in the form (5) for the rectangle D_1 and in the form (6) for the rectangle D_2; however, the functions $\tilde{v}_m(x,\rho)$, $\tilde{P}_m(x,\rho)$, $\tilde{Q}_m(x,\rho)$ in the rectangle D_2 will be assumed to be identically equal to zero. The remaining functions are defined and estimated similarly to what we have done above. Therefore there takes place the following assertion.

Theorem 2. *For the asymptotic representation* (5), (6) *of the solution of the problem* (1)–(4) *the function* $\tilde{v}_m(x,\rho)$, $\tilde{P}_m(\xi,\rho)$, $\tilde{Q}_m(\eta,\rho)$ *for* $\alpha = 0$ *is identically equal to zero. Under these conditions, for that asymptotic representation the inequality*

$$\|u(x,z,\epsilon) - u^{(N)}(x,z,\epsilon)\|_{C^1} \leq M\epsilon^{N-1}$$

is fulfilled, where the constant M *does not depend on* ϵ.

The proof of the theorem is analogous to that of Theorem 1.

Note that if the additional conditions imposed on the boundary functions and on the right-hand sides of the equations (1), (2) are not fulfilled, then the boundary layer components of the asymptotic representation of the solution of the problem under consideration do not possess the character we have supposed by the construction (5), (6). In this case it is necessary to complement the asymptotic representation with functions describing the behavior of the solution in the neighborhood of the points $(0,0)$, $(1,0)$ for $z < 0$. It should be noted that to construct an asymptotic expansion of a solution of the problem it is necessary to apply the method of matching asymptotic expansions [31].

5. Consider the problem (1)–(4) for $\alpha = 1$. An asymptotic representation of the solution of the problem under consideration will be sought in the form (5) for the domain D_1 and in the form

$$u^{(N)}(x,z,\epsilon) = \sum_{m=0}^{N} \Big[u_m(x,z) + \tilde{v}_m\Big(x, \frac{z}{\epsilon}\Big) + \tilde{r}_m\Big(\frac{x}{\epsilon}\Big) + \\ + \tilde{q}_m\Big(\frac{1-x}{\epsilon}, z\Big) + \tilde{P}_m\Big(\frac{x}{\epsilon}, \frac{z}{\epsilon}\Big) + \tilde{Q}_m\Big(\frac{1-x}{\epsilon}, \frac{z}{\epsilon}\Big)\Big] \qquad (17)$$

for the domain D_2.

Not dwelling on the description of the problems which determine the functions $u_m(x,z)$, $\tilde{u}_m(x,z)$, $r_m(\xi,z)$, $v_m(x,\tau)$, $\tilde{v}_m(x,\tau)$, $w_m(x,\nu)$, $q_m(\eta,z)$, $\tilde{q}_m(\eta,z)$, $R_m(\xi,\nu)$, $Q_m(\eta,\tau)$, $\tilde{Q}_m(\eta,\tau)$, $S_m(\eta,\nu)$, we pass to the problems specifying the functions $P_m(\xi,\tau)$, $\tilde{P}_m(\xi,\tau)$. These functions satisfy the equations

$$\Delta_{\xi\tau} P_m - k_0^2 P_m = \sum_{i+j=1}^{m} k_{i,j}\xi^i\tau^j P_{m-i-j}(\xi,\tau), \tag{18}$$

$$\frac{\partial^2 \tilde{P}_m}{\partial\xi^2} - \frac{\tilde{P}_m}{\partial\tau} - b_0^2\tilde{P}_m = \sum_{i+j=1}^{m} b_{i,j}\xi^i\tau^j\tilde{P}_{m-i-j}(\xi,\tau) \tag{19}$$

and also the additional conditions

$$\begin{aligned} &P_m(0,\tau) = -v_m(0,\tau), \quad \tilde{P}_m(0,\tau) = -\tilde{v}_m(0,\tau), \\ &P_m(\xi,0) - \tilde{P}_m(\xi,0) = \tilde{r}_m(\xi,0) - r_m(\xi,0), \end{aligned} \tag{20}$$

$$\frac{\partial P_m(\xi,0)}{\partial\tau} - \frac{\partial\tilde{P}_m(\xi,0)}{\partial\tau} = \frac{\partial\tilde{r}_m(\xi,0)}{\partial z} - \frac{\partial r_m(\xi,0)}{\partial z}. \tag{21}$$

Let us show that the functions $P_m(\xi,\tau)$, $\tilde{P}_m(\xi,\tau)$ exist and are the boundary layer functions with respect to a collection of variables. For $m=0$ we have

$$\frac{\partial^2 P_0}{\partial\xi^2} + \frac{\partial^2 P_0}{\partial\tau^2} - k_0^2 P_0 = 0,\ \tau<0,\ \xi>0, \tag{22}$$

$$\frac{\partial^2\tilde{P}_0}{\partial\xi^2} - \frac{\partial\tilde{P}_0}{\partial\tau} - b_0^2\tilde{P}_0 = 0, \quad \tau>0, \quad \xi>0,$$

$$P_0(\xi,0) - \tilde{P}_0(\xi,0) = \tilde{r}_0(\xi,0) - r_0(\xi,0), \quad \frac{\partial P_0(\xi,0)}{\partial\tau} = \frac{\partial\tilde{P}_0}{\partial\tau}, \tag{23}$$

$$P_0(0,\tau) = -v_0(0,\tau), \quad \tilde{P}_0(0,\tau) = -\tilde{v}_0(0,\tau). \tag{24}$$

Note that the equality $\tilde{v}_0(0,0) - v_0(0,0) = \tilde{r}_0(0,0) - r_0(0,0)$ is fulfilled at the origin of coordinates of the plane of the variables ξ,τ. To solve the obtained problem, we will apply the Fourier sine-transformation of the functions $P_0(\xi,\tau)$, $\tilde{P}_0(\xi,\tau)$ with respect to the variable ξ. For the images $P_{0,F}(t,\tau)$, $\tilde{P}_{0,F}(t,\tau)$ we obtain the following system of equations:

$$\frac{\partial^2 P_{0,F}(t,\tau)}{\partial\tau^2} - (k_0^2+t^2)P_{0,F}(t,\tau) = \sqrt{\frac{2}{\pi}}tb_0^2\frac{u_0(0,0)-\tilde{u}_0(0,0)}{k_0+b_0^2}e^{k_0\tau}, \tag{25}$$

$$\frac{\partial\tilde{P}_{0,F}(t,\tau)}{\partial\tau} + (b_0^2+t^2)\tilde{P}_{0,F}(t,\tau) = \sqrt{\frac{2}{\pi}}tk_0\frac{u_0(0,0)-\tilde{u}_0(0,0)}{k_0+b_0^2}e^{-b_0^2\tau}, \tag{26}$$

and also the additional conditions

$$\begin{aligned} P_{0,F}(t,0) - \tilde{P}_{0,F}(t,0) &= [\phi_0(0) - \tilde{u}_0(0,0)]\sqrt{\frac{2}{\pi}}\frac{t}{b_0^2+t^2} - \\ &-[\phi_0(0) - u_0(0,0)]\sqrt{\frac{2}{\pi}}\frac{t}{k_0^2+t^2}, \end{aligned} \tag{27}$$

$$\frac{\partial P_{0,F}(t,0)}{\partial\tau} = \frac{\partial\tilde{P}_{0,F}(t,0)}{\partial\tau}. \tag{28}$$

As is easily seen,the problem (25)–(28) has a unique solution vanishing as $|\tau| \to \infty$. Moreover, one can apply to that solution, as a function of the parameter t, the Fourier sine-transformation, and its image is a continuous function which tends to zero as $\xi \to \infty$. Consequently, we can write out a solution of either of the equations (22), and for these solutions the estimates

$$\|P_0(\xi,\tau)\|_{C^1} + \|\tilde{P}_0(\xi,\tau)\|_{C^1} \le M \exp\left[-\alpha_0(\xi + |\tau|)\right]$$

will be valid; here α_0 is a positive constant which can be defined with the help of the constants k_0 and b_0.

The problems (18)–(21) for $m \ge 1$ can be considered analogously. Moreover, the same kind estimates as for $m = 0$ will be valid for solutions of these problems.

Theorem 3. *For the asymptotic representation* (5), (7) *of the solution of the problem* (1)–(4) *for* $\alpha = 1$ *the estimate*

$$\|u(x,z,\epsilon) - u^{(N)}(x,z,\epsilon)\| \le M\epsilon^{N-1},$$

where the constant M *does not depend on* ϵ*, is valid.*

Remark 1. Having introduced into consideration the function $\mu(\xi) = \partial P_0(\xi,0)/\partial\tau$, we can write the solution of the problem (22)–(24) for the domains $\{\xi > 0,\ \tau > 0\}$ and $\{\xi > 0,\ \tau < 0\}$ separately by means of the corresponding Green's functions and then obtain a singular integral equation with respect to the function $\mu(\xi)$:

$$\int_0^\infty \mu(\eta)\left\{\left[K_0(k_0|\xi-\eta|) - K_0(k_0|\xi+\eta|)\right] - \pi\left[\exp\left(-b_0|\xi-\eta|\right) - \right.\right.$$
$$\left.\left. - \exp\left(-b_0|\xi+\eta|\right)\right]\right\} d\eta = A\exp\left(-b_0\xi\right) + B\exp\left(-k_0\xi\right) -$$
$$-2\int_{-\infty}^0 v_0(\lambda) K_1\left(k_0\sqrt{\xi^2+\lambda^2}\right)\frac{\lambda}{\sqrt{\xi^2+\lambda^2}}\,d\lambda,$$

where $K_0(t)$, $K_1(t)$ are the McDonald's functions, and A, B are some constants. Following V. A. Fok (see [2]), the above-given integral equation can be solved by the Fourier transformation. Integral equations connected with solutions of the problems (18)–(21) for $m > 0$ are solved in a similar manner.

6. Consider in the rectangle D the boundary value problem

$$\epsilon^2\Delta u + a(x,z)\frac{\partial u}{\partial z} - k^2(x,z)u = -f_1(x,z), \quad (x,z) \in D_1, \tag{29}$$

$$\epsilon^2\frac{\partial^2 u}{\partial x^2} - c(x,z)\frac{\partial u}{\partial z} - b^2(x,z)u = -f_2(x,z),\ (x,z) \in D_2, \tag{30}$$

with homogeneous boundary conditions (3) and (4). It will be assumed that the coefficients $k(x,z)$, $b(x,z)$ and the functions $f_1(x,z)$, $f_2(x,z)$ satisfy the above-formulated requirements, and also that the functions $a(x,z)$, $c(x,z)$ are infinitely differentiable and positive in the closure of the corresponding domains of definition. Moreover, it will be supposed that the functions $f_1(x,z)$, $f_2(x,z)$ along with their partial derivatives of high enough order vanish at the points $(0,0)$ and $(1,0)$; as it will follow from our further considerations, these conditions are necessary for the boundedness of the coefficients of asymptotic (for the chosen collection of asymptotic sequences) representation of the solution of the problem under consideration.

As asymptotic representation of a solution of the problem (29), (30) will be sought in the form

$$u^{(N)}(x,z,\epsilon)=\sum_{m=0}^{N}\epsilon^m\Big[u_m(x,z)+r_m\Big(\frac{x}{\epsilon},z\Big)+s_m\Big(\frac{1-x}{\epsilon},z\Big)+$$

$$+v_m\Big(x,\frac{Z_1-z}{\epsilon^2}\Big)+w_m\Big(\frac{x}{\epsilon},\frac{Z_1-z}{\epsilon^2}\Big)+R_m\Big(\frac{x}{\epsilon^2},\frac{Z_1-z}{\epsilon^2}\Big)+$$

$$+Q_m\Big(\frac{1-x}{\epsilon},\frac{Z_1-z}{\epsilon^2}\Big)+S_m\Big(\frac{1-x}{\epsilon^2},\frac{Z_1-z}{\epsilon^2}\Big)\Big],\quad (x,z)\in D_1, \tag{31}$$

$$u^{(N)}(x,z,\epsilon)=\sum_{m=0}^{N}\Big[\tilde u_m(x,z)+\tilde r_m\Big(\frac{x}{\epsilon},z\Big)+\tilde s_m\Big(\frac{1-x}{\epsilon},z\Big)\Big],\ (x,z)\in D_2. \tag{32}$$

The functions $u_m(x,z)$ and $\tilde u_m(x,z)$ can be found in a standard way. It can be easily seen that problems for determining these functions are solvable in a class of sufficiently smooth functions; note that under the above-formulated requirements imposed on the functions $f_1(x,z)$, $f_2(x,z)$, the functions $u_m(x,z)$, $\tilde u_m(x,z)$ and their partial derivatives of high enough order vanish at the points $(0,0)$, $(0,1)$.

The functions $r_m(\xi,z)$ and $\tilde r_m(\xi,z)$ are needed for the boundary condition to be fulfilled at $x=0$, and also for preserving the continuity of the asymptotic representation and its derivative with respect to the variable z at $z=0$; these functions must satisfy the following relations:

$$\frac{\partial^2 r_m}{\partial\xi^2}+a(0,z)\frac{\partial r_m}{\partial z}-k^2(0,z)r_m=\frac{\partial^2 r_{m-2}}{\partial z^2}-$$

$$-\sum_{i=1}^{m}a_{i,0}(z)\xi^i\frac{\partial r_{m-i}(\xi,z)}{\partial z}+\sum_{i=1}^{m}k_{i,0}(z)\xi^i r_{m-i}(\xi,z),\ z<0, \tag{33}$$

$$\frac{\partial^2 \tilde r_m}{\partial\xi^2}-c(0,z)\frac{\partial \tilde r_m}{\partial z}-b^2(0,z)\tilde r_m=$$

$$=\sum_{i=1}^{m}c_{i,0}(z)\xi^i\frac{\partial \tilde r_{m-i}(\xi,z)}{\partial z}+\sum_{i=1}^{m}b_{i,0}(z)\xi^i\tilde r_{m-i}(\xi,z), \tag{34}$$

$$r_m(\xi,0)=\tilde r_m(\xi,0),\quad \frac{\partial r_m(\xi,0)}{\partial z}=\frac{\partial \tilde r_m(\xi,0)}{\partial z}, \tag{35}$$

$$r_m(0,z)=-u_m(0,z),\quad \tilde r_m(0,z)=-\tilde u_m(0,z). \tag{36}$$

Let us introduce the notation $r_m(\xi,0)=r_m^0(\xi)$. As it follows from the relations (33)–(36), the function $r_m^0(\xi)$ satisfies the relation

$$\Big[\frac{1}{a(0,0)} + \frac{1}{c(0,0)}\Big](r_m^0)'' - \Big[\frac{k^2(0,0)}{a(0,0)} + \frac{b^2(0,0)}{c(0,0)}\Big] r_m^0 = \theta_m(\xi) \qquad (37)$$

for some function $\theta_m(\xi)$ defined both by the right-hand sides of the equations (33), (34) for $z = 0$ and by the initial condition $r_m^0(0) = -u_m(0,0)$. Obviously, the equation (37) under that condition has a unique solution which vanishes as $\xi \to \infty$. Having found the function $r_m^0(\xi)$, we can write out the corresponding solutions of the equations (33), (34).

We will now proceed to investigating properties of the problems under consideration. It can be easily seen that solutions of the problems (33)– (36) exist, tend exponentially to zero as $\xi \to \infty$ and possess bounded derivatives of the first order which tend exponentially to zero as $\xi \to \infty$. Moreover, by virtue of restrictions imposed on the functions $f_1(x,z)$, $f_2(x,z)$ and owing to the properties of the functions $u_m(x,z)$ and $\tilde{u}_m(x,z)$, the function $r_m(\xi,z)$ possesses bounded derivatives up to some order which tend to zero as $\xi \to 0$. The conclusions made earlier for the functions $\tilde{r}_m(\xi,z)$ hold true for the functions $m \geq 0$ as well.

Thus we can reckon that for the functions $r_m(\xi,z)$, $\tilde{r}_m(\xi,z)$ in the closure of the corresponding domains of definition the estimates

$$||r_m(\xi,z)||_{C^1} + ||\tilde{r}_m(\xi,z)||_{C^1} \leq M \exp\left(-\alpha_m \xi\right),$$

are fulfilled, where α_m, $m = 0, 1, \ldots$ are some positive constants.

Remark 2. It is easy to see that to construct the coefficients $r_m(\xi,z)$, $\tilde{r}_m(\xi,z)$ of the asymptotic representation (31), (32),with bounded first and second order derivatives, it is sufficient that the functions $f_1(x,z)$, $f_2(x,z)$ and their derivatives up to the order $N+2$ inclusive vanish at the points $(0,0)$, $(0,1)$.

The definition and investigation of properties of the functions $v_m(x,\nu)$, $w_m(\xi,\nu)$ and $R_m(\zeta,\nu)$ do not cause difficulties, and therefore without any trouble we can prove the following assertion.

Theorem 4. *For the asymptotic representation* (31), (32) *of the solution of the problem* (29), (30), (3), (4) *there takes place the estimate*

$$||u(x,z,\epsilon) - u^{(N)}(x,z,\epsilon)||_{C^1} \leq M\epsilon^{N-1},$$

where the constant M does not depend on ϵ.

Solution of the equations (29), (30) for the case $a(x,z) < 0$ can be studied in a similar way. It should be noted that the functions $u_m(x,z)$ satisfy the initial conditions for $z = Z_1$, and therefore $v_m(x,z) \equiv 0$. An algorithm for constructing an asymptotic representation of the solution of the problem under investigation consists of successive fulfilment of the following steps: first we introduce into consideration the functions $h_m(x, z/\epsilon^2)$ which would guarantee the fulfilment of the condition (4); then, to fulfil the condition (3), we define the functions $\tilde{u}(x,z)$ by means of the initial conditions prescribed for $z=0$. Since all the estimates taking place in the case $a(x,z)>0$ hold true, there also takes place the assertion similar to that of Theorem 4.

7. Consider, finally, the problem

$$\epsilon^2\Delta u - \mu a(x,z)\frac{\partial u}{\partial z} - k^2(x,z)u = -f_1(x,z), \quad (x,z)\in D_1, \tag{38}$$

$$\epsilon^2\frac{\partial^2 u}{\partial x^2} - \epsilon^\alpha c(x,z)\frac{\partial u}{\partial z} - b^2(x,z)u = -f_2(x,z), \quad (x,z)\in D_2, \tag{39}$$

under the boundary conditions (3), (4). Here $\mu>0$ is an independent small parameter, and $a(x,z)>0$, $c(x,z)>0$, α is a positive constant. The type of an asymptotic representation of the solution of the given problem depends essentially on the ratio of small parameters μ and ϵ. According to what has been said as well as depending on the value of the parameter α, the questions of solvability of the problems which specify the coefficients of asymptotic representations are reduced either to the question of solvability of a certain singular integral equation, or to a series of breaking up problems for one of the three types of differential equations – ordinary, elliptic or parabolic. Because of a large number of possible cases we will not describe in detail the process of constructing an asymptotic approximation of a solution for each case separately, but only indicate typical peculiarities which arise in the course of constructing.

When constructing asymptotic expansions it is convenient to introduce one more parameter $\delta=\delta(\epsilon,\mu)$ and then construct the expansion with respect to three small parameters.

Let first $0<\mu\ll\epsilon\ll 1$. An asymptotic representation of the solution for $(x,z)\in D_1$ will be sought in the form

$$\begin{aligned} u^{(N)}(x,z,\mu,\epsilon,\delta) = \sum_{i,j,m=0}^{N} \left(\frac{\mu}{\epsilon}\right)^i \epsilon^j\delta^m \Big[u_{i,j,m}(x,z) + r_{i,j,m}\Big(\frac{x}{\epsilon},z\Big) + \\ + v_{i,j,m}\Big(x,\frac{z}{\epsilon}\Big) + w_{i,j,m}\Big(x,\frac{Z_1-z}{\epsilon}\Big) + q_{i,j,m}\Big(\frac{1-x}{\epsilon},z\Big) + \\ + P_{i,j,m,}\Big(\frac{x}{\epsilon},\frac{z}{\epsilon}\Big) + Q_{i,j,m}\Big(\frac{1-x}{\epsilon},\frac{z}{\epsilon}\Big) + R_{i,j,m}\Big(\frac{x}{\epsilon},\frac{Z_1-z}{\epsilon}\Big) + \\ + S_{i,j,m}\Big(\frac{1-x}{\epsilon},\frac{Z_1-z}{\epsilon}\Big)\Big], \quad (x,z)\in D_1, \end{aligned} \tag{40}$$

$$\begin{aligned} u^{(N)}(x,z,\mu,\epsilon,\delta) = \sum_{i,j,m=0}^{N} \left(\frac{\mu}{\epsilon}\right)^i \epsilon^j\delta^m \Big[\tilde{u}_{i,j,m}(x,z) + \tilde{r}_{i,j,m}\Big(\frac{x}{\epsilon},z\Big) + \\ + \tilde{v}_{i,j,m}\Big(x,\frac{z}{\epsilon^\alpha}\Big) + \tilde{q}_{i,j,m}\Big(\frac{1-x}{\epsilon},z\Big) + \tilde{P}_{i,j,m}\Big(\frac{x}{\epsilon},\frac{z}{\epsilon^\alpha}\Big) + \\ + \tilde{Q}_{i,j,m}\Big(\frac{1-x}{\epsilon},\frac{z}{\epsilon^\alpha}\Big)\Big], \quad (x,z)\in D_2. \end{aligned} \tag{41}$$

If $\alpha = 1$, then $\delta = \epsilon$. For all functions appearing in the representation (40), (41), except the functions $P_{i,j,m}(\xi,\tau)$, $\tilde{P}_{i,j,m}(\eta,\tau)$ and $Q_{i,j,m}(\eta,\tau)$, $\tilde{Q}_{i,j,m}(\eta,\tau)$, we obtain a series of breaking up problems (i.e., a family of problems whose solutions at every step of constructing depend only on those coefficients of the asymtotic representation which by that moment have already been defined). Problems of each series are similar to the above-considered ones, and therefore it is not difficult to determine conditions for their solvability and to investigate properties of the corresponding coefficients of the asymptotic representation.

In constructing the functions $P_{i,j,m}(\xi,\tau)$ and $\tilde{P}_{i,j,m}(\xi,\tau)$ either the use can be made of the Fourier sine-transformation, or one can obtain for the values of the derivative of one of these functions with respect to the variable τ for $\tau = 0$ a singular integral equation of the type indicated in Remark 1. Consequently, as is quoted in subsection 5 of the present section, using the Fourier transformation, we can find the functions $P_{i,j,m}(\xi,\tau)$, $\tilde{P}_{i,j,m}(\xi,\tau)$ for any values of the indices . The functions $Q_{i,j,m}(\eta,\tau)$, $\tilde{Q}_{i,j,m}(\eta,\tau)$ can be determined in a similar manner.

If $\alpha = 0$, then $\delta = \epsilon$. In the representation (40), (41) we have $\tilde{v}_{i,j,m}(x,\tau) \equiv 0$, and the continuity of the asymptotic representation for $z = 0$ is achieved by the choice of initial conditions for first order ordinary differential equations determining the functions $\tilde{u}_{i,j,m}(x,z)$. The functions $P_{i,j,m}(\xi,\tau)$ and $Q_{i,j,m}(\eta,\tau)$ are constructed in terms of solutions of elliptic equations with constant coefficients and given boundary conditions of the second kind on the straight lines $\xi = 0$, $\tau = 0$. Finally, the functions $\tilde{P}_{i,j,m}(\xi,\tau)$ and $\tilde{Q}_{i,j,m}(\eta,\tau)$ in the case under consideration are identically equal to zero for any set of indices i, j, m.

If $0 < \alpha < 1$, then $\delta = \epsilon^{\alpha}$, and for $P_{i,j,m}(\xi,\tau)$, $\tilde{P}_{i,j,m}(\xi,\rho)$, $\rho = z/\epsilon^{\alpha}$ we easily obtain the problems where we first solve an elliptic equation with prescribed boundary condition of second kind for $\tau = 0$ and then we seek for the solution of a parabolic equation with the given initial condition for $\rho = 0$. The problems determining the functions $Q_{i,j,m}(\xi,\tau)$, $\tilde{Q}_{i,j,m}(\xi,\rho)$ are considered analogously.

If $\alpha > 1$, then in the representation (41) the functions $\tilde{v}_{i,j,m}$ depend on the arguments $(x, z/\epsilon)$, the functions $\tilde{P}_{i,j,m}$ on the arguments $(x/\epsilon, z/\epsilon)$ and the functions $\tilde{Q}_{i,j,m}$ on the arguments $((1-x)/\epsilon, z/\epsilon)$, $\delta = \epsilon^{\alpha-1}$. Moreover, to determine the functions $P_{i,j,m}(\xi,\tau)$ and $\tilde{P}_{i,j,m}(\xi,\tau)$, we obtain the problems analogous to those which were applied for determination of similar functions in case $\alpha = 1$.

Let now $0 < \epsilon \ll \mu \ll 1$. The asymptotic representation is in this case of more complicated type which depends not only on the ratio of coefficients ϵ and μ, but also on the ratio of the parameters ϵ^{α} and μ. The asymptotic representation of the solution of the problem (38), (39), (3), (4) will be sought in the form

$$u^{(N)}(x,z,\mu,\epsilon,\delta)=\sum_{i,j,m=1}^{N}\Big(\frac{\epsilon}{\mu}\Big)^{i}\mu^{j}\delta^{m}\Big[u_{i,j,m}(x,z)+r_{i,j,m}\Big(\frac{x}{\epsilon},z\Big)+$$
$$+v_{i,j,m}\Big(x,\frac{\mu z}{\epsilon^{2}}\Big)+w_{i,j,m}\Big(x,\frac{Z_1-z}{\mu}\Big)+q_{i,j,m}\Big(\frac{1-x}{\epsilon},z\Big)+$$
$$+P_{i,j,m}\Big(\frac{x\mu}{\epsilon^{2}},\frac{z\mu}{\epsilon^{2}}\Big)+Q_{i,j,m}\Big(\mu\frac{1-x}{\epsilon^{2}},\frac{z\mu}{\epsilon^{2}}\Big)+R_{i,j,m}\Big(\frac{x}{\epsilon},\frac{Z_1-z}{\mu}\Big)+$$
$$+S_{i,j,m}\Big(\frac{1-x}{\epsilon},\frac{Z_1-z}{\mu}\Big)\Big],\quad (x,z)\in D_1, \tag{42}$$
$$u^{(N)}(x,z,\mu,\epsilon,\delta)=\sum_{i,j,m=1}^{N}\Big(\frac{\epsilon}{\mu}\Big)^{i}\mu^{j}\delta^{m}\Big[\tilde{u}_{i,j,m}(x,z)+\tilde{r}_{i,j,m}\Big(\frac{x}{\epsilon},z\Big)+$$
$$+\tilde{v}_{i,j,m}\Big(x,\frac{z}{\epsilon^{\alpha}}\Big)+\tilde{q}_{i,j,m}\Big(\frac{x}{\epsilon},z\Big)+\tilde{P}_{i,j,m}\Big(\frac{x}{\epsilon},\frac{z}{\epsilon^{\alpha}}\Big)+\tilde{H}_{i,j,m}\Big(\frac{x\mu}{\epsilon^{2}},\frac{z\mu^{2}}{\epsilon^{\alpha+2}}\Big)+$$
$$+\tilde{Q}_{i,j,m}\Big(\frac{1-x}{\epsilon},\frac{z}{\epsilon^{\alpha}}\Big)+\tilde{S}_{i,j,m}\Big(\frac{x\mu}{\epsilon^{2}},\frac{z\mu^{2}}{ep^{\alpha+2}}\Big)\Big],\quad (x,z)\in D_2. \tag{43}$$

If $\alpha=1$, then $\delta=\epsilon$. For all the functions appearing in the representation (42), (43) we obtain a breaking up system of analogous problems which enable one to indicate without any additional complications the conditions for solvability of these problems and to investigate properties of the corresponding coefficients of the asymptotic representation. Moreover, it should be noted that for determination of the functions $R_{i,j,m}$ and $S_{i,j,m}$ we obtain parabolic equations whose boundedness and smoothness of solutions depend on the boundedness and smoothness of derivatives with respect to the time variable of the same functions with smaller indices. This means that the initial data of the problem which affect the boundedness and smoothness of the functions $R_{i,j,m}$ and $S_{i,j,m}$ must possess high enough degree of smoothness and compatibility (of initial and boundary functions, for example), and on that degree of smoothness and compatbility of the corresponding initial data depends the number of the terms of the asymptotic representation (42), (43) we are able to construct. It is not difficult to formulate the conditions imposed on the initial data of the problem, which enable one to construct the required number of coefficients of the asymptotic representation.

For the other values of the parameter α the situation is analogous.

If $\alpha=0$, then $\delta=\epsilon$, $\tilde{v}_{i,j,m}\equiv\tilde{P}_{i,j,m}\equiv\tilde{Q}_{i,j,m}\equiv 0$. For each unknown function we get the initial and boundary value problem with given initial and boundary conditions.

If $0<\alpha<1$, then $\delta=\epsilon^{\alpha}$, and the type of the asymptotic representation depends on the ratio of the parameters μ and ϵ^{α}. If $\mu=\mathcal{O}(\epsilon^{\alpha})$, then the functions $P_{i,j,m}$ and $\tilde{H}_{i,j,m}$ are defined respectively from elliptic and parabolic equations, and they are continuous along with the first derivative with respect to the time variable on the common boundary of the two subdomains. If, however, the orders of smallness of μ and ϵ^α differ, then for the determination of the coefficients of the asymptotic representation we obtain a breaking up system of problems.

Finally, if $\alpha > 1$, then $\delta = \epsilon^{\alpha-1}$, and for the determination of the coefficients of the asymptotic representation we also obtain a breaking up system of problems.

Using the methods described in the present and the previous sections, we can prove the following assertion.

Theorem 5. *To a solution of the problem* (38), (39), (3), (4) *one can construct an asymptotic representation in powers of the small parameters in the form* (40), (41) *or* (42), (43)*; moreover, when additional conditions imposed on the initial data and guaranteeing the boundedness of the coefficients of the representation are fulfilled, then the estimate*

$$\|u(x,z,\mu,\epsilon) - u^{(N)}(x,z,\mu,\epsilon,\delta)\|_{C^1} \leq M\left(\epsilon^{\beta(N)} + \mu^{\gamma(N)}\right)$$

hold, where $\beta(N)$, $\gamma(N)$ *are some constants increasing unboundedly as the number* N *increases.*

The problems (38), (39), (3), (4) can be considered analogously for the case where the coefficient $a(x,z)$ is negative for $z \in [Z_1, 0]$.

Remark 3. For certain relations between the small parameters included in the problem for elliptico-parabolic equations as multipliers of derivatives, the functions $a(x,z)$, $c(x,z)$ may be assumed to vanish on some lines none of which intersect the straight lines $z = 0$, $z = Z_1$ (although the case of the coincidence of a line, at the points of which the coefficient vanishes, with the straight lines $z = 0$ or $Z = Z_1$ is quite possible). In such cases, for the coefficients of the asymptotic representation there arise, as a rule, problems analogous to those we have considered in the present section and in Section 2.

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