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ESTIMATES OF HIGHER DERIVATIVES OF THE CAUCHY PROBLEM SOLUTION WITH RESPECT TO INITIAL VALUES

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For ordinary differential equations the question of estimation of higher derivatives of the Cauchy problem solution with respect to initial values is considered, when the field of the equation is continuous with respect to time and has higher derivatives with respect to phase variable. The used technique is close to [1] and is developed in [2] and especially in [3]. We will formulate the basic result (similar, but more general results are proved in [3]) in the near-ring of special type, and we will obtain the estimates from it.

Let us determine $C^n_{Lip}(X)$, where X is a Banach space, $n \in \mathbb{Z}_+$ ($\mathbb{Z}_+$ is the set of non-negative integer numbers). $g \in C^n_{Lip}(X)$, if $g \in C^n(X)$ ($g : X \mapsto X$ is n-times continuously differentiable function) and, for each $i \in \{0, 1, \dots, n\}, f^{(i)} : X \mapsto \mathcal{L}_i(X, X)$ is a Lipschitz mapping, where $\mathcal{L}_i(X, Y)$ denotes the Banach space consisting of the continuous multilinear forms $A : \underbrace{X \times \cdots \times X}_{i-times} \mapsto Y$ with the norm (see [4]):

$$\|A\|_{\mathcal{L}_i} = \sup |A(x_1, \dots, x_i)| \quad \text{when} \quad \|x_1\| \le 1, \dots, \|x_i\| \le 1.$$

As usual, we assume $\mathcal{L}_0(X, Y) = Y$. In the sequel, when $A \in \mathcal{L}_i(X, Y)$ and $x_1, \dots, x_i \in X$, we will use the notation $(\dots(Ax_1)x_2 \dots x_i)$ insted of its equivalent one $A(x_1, \dots, x_i)$; thus it is clear that $Ax_1 \in \mathcal{L}_{i-1}(X, Y)$.

Let us determine the convergence on $C^n_{Lip}(X)$. For each $m \in \mathbb{N}$, it is correctly defined the bounded deviation $d_m : C^n_{Lip}(X) \times C^n_{Lip}(X) \mapsto R_+$:

$$d_m(g_1, g_2) = \sup_{|x| \le m} \|g_1(x) - g_2(x)\|,$$

and for each $i \in \{1, \dots, n+1\}$

$$\delta_i(g) = \sup_{\substack{x_1 \ne x_2 \\ x_1, x_2 \in X}} \left\{ \frac{\|g^{(i-1)}(x_1) - g^{(i-1)}(x_2)\|_{\mathcal{L}_{i-1}}}{\|x_1 - x_2\|} \right\}, \quad \forall g \in C^n_{Lip}(X),$$

$\delta_i : C^n_{Lip}(X) \mapsto \mathbb{R}_+$ will be called by the restriction.

It is easy to verify that

$$\delta_i(g) = \sup_{x \in X} \|g^{(i)}(x)\|_{\mathcal{L}_i}, \forall g \in C^n_{Lip}(X), \quad \forall i \in \{1, \dots, n\}, \tag{1}$$

when $n \ge 1$.

Let $\{g_i\}_{i=0}^\infty \subset C^n_{Lip}(X)$. We will say that $\{g_i\}_{i=1}^\infty$ converges to g_0, if:

a) $\lim\limits_{i \to \infty} d_m(g_i, g_0) = 0, \forall m \in \mathbb{N}$;

b) $\sup\{\delta_m(g_i)\}_{i=1}^\infty < \infty, \forall m \in \{1, \dots, n+1\}$.

We say that $\{g_i\}_{i=1}^\infty$ is fundamental in $C^n_{Lip}(X)$, if:

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a) $\forall \varepsilon > 0$ and $\forall m \in N$ $\exists i_0 \in N$ such that $d_m(g_{i_1}, g_{i_2}) < \varepsilon$ when $i_1, i_2 \geq i_0$;
b) $\sup\{\delta_m(g_i)\}_{i=1}^{\infty} < \infty$, $\forall m \in \{1, \ldots, n+1\}$.

The convergent and fundamental directednesses can be defined analogously, but we do not need such definitions, because the convergent sequences and the convergent directednesses determine the same limit structure in $C^n_{Lip}(X)$ (as in metric space).

$C^n_{Lip}(X)$ is a complete space and each δ_m is lower semi-continuous, i.e., from $\lim\limits_{i\to\infty} g_i = g_0$ it follows

$$\delta_m(g_0) \leq \sup\{\delta_m(g_i)\}_{i=1}^{\infty} < \infty, \forall m \in \{1, \ldots, n+1\}.$$

The operations $+$ and 0 (null function) introduce a structure of additive group, $\circ$ (composition) and $I_X (I_X : X \mapsto X$ is the identical mapping) introduce a structure of monoid, and

$$(g_1 + g_2) \circ g = g_1 \circ g + g_2 \circ g, \ \ \forall g, g_1, g_2 \in C^n_{Lip}(X),$$

so $C^n_{Lip}(X)$ is a near-ring.

The restrictions have very interesting properties:

$$\delta_m(0) = 0, \ \ \delta_m(g_1 + g_2) \leq \delta_m(g_1) + \delta_m(g_2),$$

$$\delta_1(I_X) = 1, \delta_i(I_X) = 0 \quad \text{if} \quad i \geq 1,$$

$$\delta_m(g_1 \circ g_2) \leq \sum \frac{m!}{i_1! \cdots i_k!} \left[\frac{\delta_1(g_2)}{1!}\right]^{i_1} \cdots \left[\frac{\delta_k(g_2)}{k!}\right]^{i_k} \delta_{i_1+\cdots+i_k}(g_1),$$

where the sum $\sum$ is spread over all collections $(i_1, \ldots, i_k)$ such that $i_j \in \mathbb{Z}_+$ and $1i_1 + \cdots + ki_k = n$.

Let us determine a special sequence of polynorms $\{P_m\}_{m=1}^{\infty}$ as follows: $P_1 \equiv 1$, and for $m \geq 1$ and each $\{\eta_1, \ldots, \eta_m\} \subset [0, \infty)$

$$P_m(\eta_1, \ldots, \eta_m) = \eta_1 \widehat{\sum} \frac{m!}{i_1! \cdots i_k!} \left[\frac{P_1}{1!}\right]^{i_1} \cdots \left[\frac{P_k(\eta_1, \ldots, \eta_k)}{k!}\right]^{i_k} \eta_{i_1+\cdots+i_k}, \qquad (2)$$

where the sum $\widehat{\sum}$ is spread over all the solutions of equation $1i_1 + \cdots + ki_k = m$ ($i_j \in \mathbb{Z}_+$) except $i_m = 1$ (i.e., $k = m$ and $i_1 = \cdots = i_{m-1} = 0$, $i_m = 1$); therefore P_m do not take part in the right-hand side of (2). Thus, (2) is the recurrent formula.

Let us formulate the basic result in the form which is sufficient for our purposes.*

Theorem 1. *Let $\{f(t, \cdot)\}_{t\in[a,b]}$ be a continuous family in some $C^n_{Lip}(X)$, $-\infty < a < b < +\infty$. Then there exists a two-parameter continuous family $\{\varphi_{t_0,t}\}_{t_0,t\in[a,b]}$ in $C^n_{Lip}(X)$, such that for each $(t_0, x_0) \in [a, b] \times X$ the equalities*

$$\frac{d}{dt}\varphi_{t_0,t}(x_0) = f(t, \varphi_{t_0,t}(x_0)), \ \ \forall t \in (a, b),$$
$$\varphi_{t_0,t}(x_0) = x_0$$

are valid in X and for each $m \in \{1, \ldots, n+1\}$

$$d_m(\varphi_{t_0,t}) \leq P_m(|t - t_0|, \gamma_2, \ldots, \gamma_m) \exp(m\gamma_1|t - t_0|) \qquad (3)$$

holds, where

$$\gamma_i = \sup\{\delta_i(f(t, \cdot))\}_{t\in[a,b]} \ \ \text{and} \ \ \gamma_i \in [0, \infty), \quad \forall i \in \{1, \ldots, n+1\}.$$

Let us use Theorem 1 to analyse the following Cauchy problem:

$$\dot{x} = f(t, x), \ \ x(t_0) = x_0, \ \ x \in X, \ \ (t, x) \in \Gamma \text{ -- an open area.} \qquad (4)$$

The following result gives a sufficiant condition imposed on the field $(t, x) \mapsto f(t, x)$ for $\{f(t, \cdot)\}_{t\in[a,b]}$ to be continuous in some $C^n_{Lip}(\mathbb{R}^r)$.

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*The way of the proof permits us to find out the explicit form of $\varphi_{t_0,t}$.

Lemma 1. *Let $r, n \in \mathbb{N}$, there exist $(a,b) \subset \mathbb{R}$ such that $[a,b] \times \mathbb{R}^r \subset \Gamma$, $\{f(t,x)\}_{t\in[a,b]}$ is continuous for every $x \in \mathbb{R}^r$, $\{f(t,\cdot)\}_{t\in[a,b]} \subset C^n(\mathbb{R}^r)$, and*

$$k_i \equiv \sup\left\{\left\|\frac{\partial^i f}{\partial x^\alpha}(t,x)\right\| \middle| x \in R^r, t \in [a,b], |\alpha| = i\right\} < \infty, \tag{5}$$

where $i \in \{1,\dots,n\}$, $\alpha = (\alpha_1,\dots,\alpha_r) \in \mathbb{Z}^r_+$, $|\alpha| = \sum_{i=1}^r \alpha_i$, $\partial x^\alpha = \partial x_1^{\alpha_1} \cdots \partial x_r^{\alpha_r}$.

Then $\{f(t,\cdot)\}_{t\in[a,b]}$ is continuous in $C^{n-1}_{Lip}(\mathbb{R}^r)$ and

$$\sup\{\delta_i(f(t,\cdot))\}_{t\in[a,b]} \le r^i k_i, \quad \forall i \in \{1,\dots,n\}. \tag{6}$$

Scheme of Proof. Take arbitrary $i \in \{1,\dots,n\}$, $t \in [a,b]$ and $x \in \mathbb{R}^r$. The multilinear form $(f(t,\cdot))^{(i)}(x)$ $((f(t,\cdot))^{(i)}$, as usual, denotes the derived function of order i and acts as follows:

$$(\cdots((f(t,\cdot))^{(i)}(x)h_1)h_2\cdots h_i) = \sum_{j_1,\dots,j_i=1}^{r} (h_1)_{j_1}\cdots(h_i)_{j_i}\frac{\partial^i f}{\partial x_{j_1}\cdots\partial x_{j_i}}(t,x),$$

$\forall h_j = ((h_j)_1,\dots,(h_j)_r)^T \in R^r$, $i \in \{1,\dots,n)$. Therefore

$$\sup_{|\alpha|=i}\left\|\frac{\partial^i f}{\partial x^\alpha}(t,x)\right\| \le \|(f(t,\cdot))^{(i)}(x)\|_{\mathcal{L}_i} \le r^i \sup_{|\alpha|=i}\left\|\frac{\partial^i f}{\partial x^\alpha}(t,x)\right\|. \tag{7}$$

(7) and Taylor's formula give:

$$|(f(t,\cdot))^{(i-1)}(x_2) - (f(t,\cdot))^{(i-1)}(x_1)|_{\mathcal{L}_{i-1}} \le$$

$$\le \int_0^1 |(f(t,\cdot))^{(i)}(x_1 + \lambda(x_2 - x_1))|_{\mathcal{L}_i}\|x_2 - x_1\|d\lambda \le r^i k_i\|x_2 - x_1\|,$$

thus $f(t,\cdot) \in C^{n-1}_{Lip}(R^r)$ and (6) holds.

In accordance with the conditions of Lemma 1, $\{(t,x) \mapsto f(t,x)\} : [a,b]\times\mathbb{R}^r \to \mathbb{R}^r$ is a continuous mapping, i.e., it is uniformly continuous on each compact subset of $[a,b]\times\mathbb{R}^r$. Therefore if $\{t_i\}_{i=0}^\infty \subset [a,b]$ and $\lim_{i\to\infty} t_i = t_0$, then

$$d_m(f(t_i,\cdot) - f(t_0,\cdot)) \overset{i\to\infty}{\longrightarrow} 0,$$

$\forall m \in N$, which together with (6) proves that $\{f(t,\cdot)\}_{t\in[a,b]}$ is continuous in $C^{n-1}_{Lip}(\mathbb{R}^r)$. $\square$

Proposition 1. *Let $r, n \in \mathbb{N}$, there exist $(a,b) \subset \mathbb{R}$ such that $[a,b] \times \mathbb{R}^r \subset \Gamma$, $\{f(t,x)\}_{t\in[a,b]}$ is continuous for every $x \in \mathbb{R}^r$, $\{f(t,\cdot)\}_{t\in[a,b]} \subset C^n(\mathbb{R}^r)$, and*

$$k_i = \sup\left\{\left\|\frac{\partial^i f}{\partial x^\alpha}(t,x)\right\| \quad \middle| x \in \mathbb{R}^r,\ t \in [a,b],\ \ \|\alpha\| = i\right\} < \infty.$$

Then there exists a continuous two-parameter family $\{\varphi_{t_0,t}\}_{t_0,t\in[a,b]}$ in $C^{n-1}_{Lip}(R^r)$, such that for each $(t_0,x_0) \in [a,b] \times R^r$ the equalities

$$\begin{cases} \frac{d}{dt}\varphi_{t_0,t}(x_0) = f(t,\varphi_{t_0,t}(x_0)), & \forall t \in (a,b), \\ \varphi_{t_0,t}(x_0) = x_0 \end{cases}$$

are valid, and for each $i \in \{1,\dots,n-1\}$ and every $t \in (a,b)$

$$\sup_{\|\alpha\|=i}\left\|\frac{\partial^i \varphi_{t_0,t}}{\partial x^\alpha}(t,x_0)\right\| \le P_m(|t-t_0|, r^2k_2,\dots,r^ik_i)\exp(irk_1\|t-t_0\|). \tag{8}$$

Proof. By virtue of Lemma 1, $\{f(t,\cdot)\}_{t\in[a,b]}$ is continuous in $C^{n-1}_{Lip}(\mathbb{R}^r)$, therefore we can use Theorem 1. Now we need to show that from (3) it follows (8).

According to (1),

$$\delta_i(\varphi_{t_0,t}) = \sup_{x\in\mathbb{R}^r} \left\| (\varphi_{t_0,t}(\cdot))^{(i)}(x) \right\|_{\mathcal{L}_i}. \tag{9}$$

By virtue of the hypotheses of the theorem we have

$$\gamma_i \le r^i k_i, \quad \forall i \in \{1,\dots,n\}. \tag{10}$$

Let us take into consideration that (9) gives

$$\sup_{|\alpha|=i} \left\| \frac{\partial^i \varphi_{t_0,t}}{\partial x^\alpha}(x) \right\| \le \delta_i(\varphi_{t_0,t}), \quad \forall x \in \mathbb{R}^r, \tag{11}$$

in accordance with (7).

Finally, taking into consideration (9), (10) and (11), we deduce (8) from (3) when $i\in\{1,\dots,n-1\}$, and $t\in(a,b)$. $\square$

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