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ON NONNEGATIVE BOUNDED SOLUTIONS OF SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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1. Statement of the Problem and Formulation of the Main Results

Let $\mathbb{R}$ be the set of real numbers, $C_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ be the space of continuous functions $u:\mathbb{R}\to\mathbb{R}$ with the topology of uniform convergence on every compact interval, $C_{\mathrm{loc}}(\mathbb{R};\mathbb{R}_+)=\{u\in C_{\mathrm{loc}}(\mathbb{R};\mathbb{R}):u(t)\geq 0 \text{ for } t\in\mathbb{R}\}$, $L_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ be the space of locally summable functions $u:\mathbb{R}\to\mathbb{R}$ with the topology of convergence in the mean on every compact interval, and $L_{\mathrm{loc}}(\mathbb{R};\mathbb{R}_+)=\{u\in L_{\mathrm{loc}}(\mathbb{R};\mathbb{R}):u(t)\geq 0 \text{ for almost all } t\in\mathbb{R}\}$. Consider the system of differential equations

$$x_i'(t)=p_i(t)x_i(t)+\sum_{k=1}^n \ell_{ik}(x_k)(t)+q_i(t) \qquad (i=1,\dots,n), \tag{1}$$

where $\ell_{ik}:C_{\mathrm{loc}}(\mathbb{R},\mathbb{R})\to L_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ $(i,k=1,\dots,n)$ are linear continuous operators, p_i and $q_i\in L_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ $(i=1,\dots,n)$. Moreover, there exist linear positive operators $\overline{\ell}_{ik}:C_{\mathrm{loc}}(\mathbb{R},\mathbb{R})\to L_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ $(i,k=1,\dots,n)$ such that for any $u\in C_{\mathrm{loc}}(\mathbb{R},\mathbb{R})$ the inequalities

$$|\ell_{ik}(u)(t)|\leq \overline{\ell}_{ik}(|u|)(t) \qquad (i,k=1,\dots,n)$$

are fulfilled almost everywhere on $\mathbb{R}$.

The simple but important case of (1) is the system of differential equations with deviating arguments

$$x_i'(t)=\sum_{k=1}^n\sum_{j=1}^m p_{ikj}(t)x_k(\tau_{ikj}(t))+q_i(t) \qquad (i=1,\dots,n), \tag{1'}$$

where q_i and $p_{ikj}\in L_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$, $\tau_{ikj}:\mathbb{R}\to\mathbb{R}$ are measurable functions, and $\tau_{ii1}(t)\equiv t$.

A locally absolutely continuous vector function $(x_i)_{i=1}^n:\mathbb{R}\to\mathbb{R}$ is called a nonnegative bounded solution of the system (1) if it satisfies this system almost everywhere on $\mathbb{R}$,

$$\sup\left\{\sum_{i=1}^n |x_i(t)| : t\in\mathbb{R}\right\}<+\infty,$$

and

$$x_i(t)\geq 0 \qquad \text{for } t\in\mathbb{R} \qquad (i=1,\dots,n).$$

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I. Kiguradze [3], [4] has established optimal in some sense sufficient conditions of the existence and uniqueness of nonnegative bounded solutions of the differential system

$$\frac{dx_i(t)}{dt} = \sum_{k=1}^{n} p_{ik}(t)x_k(t) + q_i(t) \qquad (i = 1, \dots, n).$$

In the present paper these results are generalized for the systems (1) and (1′).

Before formulating the main results we want to introduce some notation.

δ_{ik} is Kronecker's symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ik} = 0$ for $i \neq k$.

$A = (a_{ik})_{i,k=1}^n$ is a $n \times n$ matrix with components a_{ik} $(i, k = 1, \dots, n)$.

$r(A)$ is the spectral radius of the matrix A.

$\mathcal{P}_{\mathbb{R}}$ is the set of linear operators mapping $C_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}_+)$ into $L_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}_+)$.

If $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$, then

$$\mathcal{N}_0(t_1, \dots, t_n) = \{i : t_i \in \mathbb{R}\}.$$

If $u \in L_{\mathrm{loc}}(\mathbb{R}, \mathbb{R})$, then

$$\eta(u)(t, s) = \int_t^s u(\xi)\, d\xi \quad \text{for } t \text{ and } s \in \mathbb{R}.$$

For $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ put

$$\sigma_i(t) = \operatorname{sgn}(t - t_i) \qquad \text{if } t_i \in \mathbb{R},$$
$$\sigma_i(t) \equiv 1 \quad \text{if } t_i = -\infty, \qquad \sigma_i(t) \equiv -1 \quad \text{if } t_i = +\infty.$$

Theorem 1. *Let there exist $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$, a matrix $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$, and a nonnegative number a such that*

$$r(A) < 1, \tag{2}$$

$$\left| \int_{t_i}^{t} \exp\left(\int_s^t p_i(\xi) d\xi \right) |\ell_{ik}(1)(s)| ds \right| \leq a_{ik} \quad \textit{for } t \in \mathbb{R} \quad (i, k = 1, \dots, n), \tag{3}$$

$$\sum_{i=1}^{n} \left| \int_{t_i}^{t} \exp\left(\int_s^t p_i(\xi) d\xi \right) |q_i(s)| ds \right| \leq a \qquad \textit{for } t \in \mathbb{R} \tag{4}$$

and

$$\sup \left\{ \int_{t_i}^{t} p_i(\xi) d\xi : t \in \mathbb{R} \right\} < +\infty \qquad \textit{for } i \in \mathcal{N}_0(t_1, \dots, t_n). \tag{5}$$

Let, moreover, $\sigma_i \ell_{ik} \in \mathcal{P}_{\mathbb{R}}$, $\sigma_i q_i \in L_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}_+)$. Then for any $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1, \dots, t_n))$ the system (1) *has at least one nonnegative bounded solution satisfying*

$$x_i(t_i) = c_i \qquad \textit{for } i \in \mathcal{N}_0(t_1, \dots, t_n). \tag{6}$$

Theorem 2. *Let all the assumptions of Theorem* 1 *be fulfilled and*

$$\liminf_{t \to t_i} \int_t^0 p_i(\xi) d\xi = -\infty \qquad \textit{for } i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n).$$

Then for any $c_i \in \mathbb{R}_+$ $i \in \mathcal{N}_0(t_1,\dots,t_n)$ *the system* (1) *has a unique bounded solution satisfying* (6), *and this solution is nonnegative.*

If $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$, then $\mathcal{N}_0(t_1,\dots,t_n)=\varnothing$. In that case in Theorems 1 and 2 the conditions (5) and (6) become unnecessary. Consequently, these theorems are formulated as follows:

Corollary 1. *Let there exist* $t_i \in \{-\infty,+\infty\}$ $(i=1,\dots,n)$, *a matrix* $A=(a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ *and a nonnegative number* a *such that the conditions* (2) – (4) *are fulfilled. Let, moreover,* $\sigma_i\ell_{ik} \in \mathcal{P}_I$, $\sigma_i q_i \in L_{\mathrm{loc}}(\mathbb{R};\mathbb{R}_+)$. *Then the system* (1) *has at least one nonnegative bounded solution.*

Corollary 2. *Let all the assumptions of Corollary* 1 *be fulfilled and*

$$\liminf_{t\to t_i} \int_t^0 p_i(\xi)d\xi = -\infty \qquad (i=1,\dots,n).$$

Then the system (1) *has a unique bounded solution, and this solution is nonnegative.*

The above theorems yield the following statements for the system (5.1′).

Corollary 1′. *Let* $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$,

$$(1-\delta_{ik}\delta_{j1})\sigma_i p_{ikj} \in L_{\mathrm{loc}}(\mathbb{R};\mathbb{R}_+), \qquad \sigma_i q_i \in L_{\mathrm{loc}}(\mathbb{R};\mathbb{R}_+) \tag{7}$$

$$(i,k=1,\dots,n;\ \ m=1,2,\dots),$$

there exist a matrix $A=(a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n\times n}$ *and a nonnegative number* a *such that* $r(A)<1$,

$$\sum_{j=1}^m \int_{t_i}^t \exp\Big(\int_s^t p_{ii1}(\xi)d\xi\Big)(1-\delta_{ik}\delta_{j1})p_{ikj}(s)ds \le a_{ik} \qquad \text{for } t\in\mathbb{R} \tag{8}$$

$$(i,k=1,\dots,n),$$

$$\sum_{i=1}^n \int_{t_i}^t \exp\Big(\int_s^t p_{ii1}(\xi)d\xi\Big)q_i(s)ds \le a \qquad \text{for } t\in\mathbb{R} \tag{9}$$

and

$$\sup\Big\{\int_{t_i}^t p_{ii1}(\xi)d\xi : t\in\mathbb{R}\Big\} < +\infty \qquad \text{for } i\in\mathcal{N}_0(t_1,\dots,t_n).$$

Then for any $c_i \in \mathbb{R}_+$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$ *the system* (5.1′) *has at least one nonnegative bounded solution satisfying the conditions* (6).

Corollary 2′. *Let all the assumptions of Corollary* 1′ *be fulfilled and*

$$\liminf_{t\to t_i} \int_t^0 p_{ii1}(\xi)d\xi = -\infty \qquad \text{for } i\in\{1,\dots,n\}\setminus\mathcal{N}_0(t_1,\dots,t_n).$$

Then for any $c_i \in \mathbb{R}_+$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$ *the system* (5.1′) *has a unique bounded solution satisfying the conditions* (6), *and this solution is nonnegative.*

Corollary 3′. *Let there exist* $t_i \in R \cup \{-\infty, +\infty\}$*,* $b_i \in]0, +\infty[$*,* $b_{ik} \in [0, +\infty[$ $(i, k = 1, \dots, n)$ *such that the condition* (7) *is fulfilled, the real part of every eigenvalue of the matrix* $(-\delta_{ik} b_i + b_{ik})_{i,k=1}^n$ *is negative and the inequalities*

$$\sigma_i(t) p_{ii1}(t) \le -b_i, \quad \sum_{j=1}^m (1 - \delta_{ik}\delta_{j1}) \sigma_i(t) p_{ikj}(t) \le b_{ik} \quad (i, k = 1, \dots, n)$$

hold almost everywhere on $\mathbb{R}$*. Moreover, let*

$$\sup \left\{ \int_t^{t+1} |q_i(s)| ds : t \in R \right\} < +\infty \quad (i = 1, \dots, n). \tag{10}$$

Then for any $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1, \dots, t_n))$ *the system* $(5.1')$ *a unique bounded solution satisfying conditions* (6)*, and this solution is nonnegative.*

Corollary 4′. *Let there exist* $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$*, a matrix* $A = (a_{ik})_{i,k=1}^n \in \mathbb{R}_+^{n \times n}$ *and a nonnegative number* a *such that* $r(A) < 1$ *and the conditions* (7) – (9) *be fulfilled. Then the system* $(5.1')$ *has at least one nonnegative bounded solution.*

Corollary 5′. *Let all the assumptions of Corollary* $4'$ *be fulfilled and*

$$\liminf_{t \to t_i} \int_t^0 p_{ii1}(s)\, ds = -\infty \quad (i = 1, \dots, n).$$

Then the system $(5.1')$ *has a unique bounded solution, and this solution is nonnegative.*

Corollary 6′. *Let there exist* $\sigma_i \in \{-1, 1\}$*,* $b_i \in]0, +\infty[$*,* $b_{ik} \in [0, +\infty[$ $(i, k = 1, \dots, n)$ *such that the condition* (7) *is fulfilled, the real part of every eigenvalue of the matrix* $(-\delta_{ik} b_i + b_{ik})_{i,k=1}^n$ *is negative and the inequalities*

$$\sigma_i p_{ii1}(t) \le -b_i, \qquad \sum_{j=1}^m (1 - \delta_{ik}\delta_{j1}) \sigma_i p_{ikj}(t) \le b_{ik} \qquad (i, k = 1, \dots, n)$$

hold almost everywhere on $\mathbb{R}$*. Moreover, if the conditions* (10) *are fulfilled, then the system* $(5.1')$ *has a unique bounded solution, and this solution is nonnegative.*

2. Proof of the Main Results

Proof of Theorem 1. By Theorem 1.1 in [1] we obtain that under the assumptions of Theorem 1 there exists at least one bounded solution $(x_i)_{i=1}^n$ of the equation (1), which is a uniform limit of the sequence of functions

$$x_{im}(t) = e_{im}(y_{im})(t) \qquad (i = 1, \dots, n; \ m = 1, 2, \dots),$$

where $(y_{im})_{i=1}^n$ is the solution of the problem

$$y_i'(t) = p_{im}(t) y_i(t) + \sum_{k=1}^n \ell_{ikm}(y_k)(t) + q_{im}(t),$$

$$y_i(t_{im}) = c_{im},$$

on the segment $[a_m, b_m]$, $\{a_m\}_{m=1}^{+\infty}$, $\{b_m\}_{m=1}^{+\infty}$ are sequences of real numbers such that $a_m < b_m$, $t_i \in [a_m, b_m]$ for $i \in \mathcal{N}_0(t_1, \dots, t_n)$ $(m = 1, 2, \dots)$,

$$\lim_{m \to +\infty} a_m = -\infty, \qquad \lim_{m \to +\infty} b_m = +\infty,$$

p_{im} and q_{im} are the restrictions of the functions p_i and q_i on the segment $[a_m, b_m]$,

$$\ell_{ikm}(u)(t) \equiv \ell_{ik}(e_m(u))(t),$$

where

$$e_m(u)(t) \stackrel{def}{\equiv} \begin{cases} u(t) & \text{for } a_m \le t \le b_m \\ u(a_m) & \text{for } t < a_m \\ u(b_m) & \text{for } t > b_m \end{cases},$$

$c_{im} = c_i$ if $i \in \mathcal{N}_0(t_1, \dots, t_n)$, $c_{im} = 0$ if $i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n)$, $t_{im} = t_i$ if $t_i \in \mathbb{R}$, $t_{im} = a_m$ if $t_i = -\infty$, $t_{im} = b_m$ if $t_i = +\infty$ $(i, k = 1, \dots, n;\ m = 1, 2, \dots)$.

On the other hand, we have

$$y_{im}(t) \ge 0 \qquad \text{for } t \in [a_m, b_m] \quad (i = 1, \dots, n;\ \ m = 1, 2, \dots).$$

Consequently,

$$x_i(t) \ge 0 \qquad \text{for } t \in \mathbb{R} \quad (i = 1, \dots, n). \quad \square$$

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References

1. R. Hakl, On bounded solutions of systems of linear functional differential equations. *Georgian Math. J.* **6**(1999), No. 5, 429–440.

2. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. *J. Soviet Math.* **43**(1988), No. 2, 2259–2339.

3. I. Kiguradze, Initial and boundary value problems for systems of ordinary differential equations, I. (Russian) *Metsniereba, Tbilisi*, 1997.

5. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional differential equations. *Czechoslovak Math. J.* **47**(1997), No. 2, 341–373.

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