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E. Shargorodsky

AN L_p-ANALOGUE OF THE VISHIK–ESKIN THEORY

Abstract. In the present paper we consider boundary value problems for elliptic pseudodifferential operators (ΨDOs) in Besov and Bessel-potential spaces. The most part of the paper is devoted to ΨDOs not possessing the transmission property. In particular we investigate a special case: the case of boundary value problems on two-dimensional manifolds.

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რეზიუმე. ნაშრომი ეძღვნება სასაზღვრო ამოცანებს ელიფსური ფსევდოდიფერენციალური ოპერატორებისათვის ბესოვისა და ბესელის პოტენციალთა სივრცეებში. ძირითადად განიხილება შემთხვევა, როცა ფსევდოდიფერენციალურ ოპერატორებს არ გააჩნიათ ე.წ. ტრანსმისიის თვისება. შესწავლილია განსაკუთრებული შემთხვევა – სასაზღვრო ამოცანები ორგანზომილებიან მრავალსახეობებზე.

Boundary value problems for elliptic differential equations with pseudodifferential boundary conditions were apparently first considered by A. S. Dynin in [34]. Further, boundary value problems for a more wide class of pseudodifferential equations were investigated by M. S. Agranovich in [3].

General theory of boundary value problems for elliptic ΨDOs was created in a series of works by M. I. Vishik and G. I. Eskin [114]–[119]. The monograph [37] is devoted to the exposition of this theory. The theory developed by Vishik and Eskin is the L_2-theory in which boundary value problems are considered in Sobolev–Slobodeckiĭ spaces H_2^s. Like probably every "elliptic" L_2-theory, it must possess an L_p-analogue. This was understood as early as in the period of origination of the theory of boundary value problems for elliptic ΨDOs. The first results in this direction were announced by A. S. Dynin in [35] (see also [81]). However there is no detailed account of L_p-analogue of the Vishik–Eskin theory so far. This can be apparently explained by the fact that such a description is connected with certain technical difficulties and does not promise the results of principally new character.

The L. Boutet de Monvel theory (see [20]) dealing with boundary value problems for elliptic ΨDOs with transmission property was generalized to the case of Besov–Triebel–Lizorkin spaces in [38], [45], [82, 3.1.1.4].[1]

Multi-dimensional singular integral operators in a half-space and on a manifold with boundary have been investigated in [100] (L_2-theory), [92], [31] (L_p-theory), [33] (the case of L_p spaces with power weights).

The present paper is concerned with boundary value problems for elliptic pseudodifferential operators (ΨDOs) in Besov and Bessel-potential spaces and the most part is devoted to ΨDOs not possessing the transmission property. Let us explain the choice of functional spaces. Bessel-potential space H_p^s is an L_p-analogue of the space H_2^s. This space is most convenient, for, the norm in it is defined by the Fourier transform and we are concerned with ΨDOs (and hence with the Fourier transform). We fail in restricting ourselves only to Bessel-potential spaces, for, the traces of functions from H_p^s on manifolds of lesser dimension belong to Besov spaces. Therefore we investigate boundary value problems whose formulations contain either Bessel-potential and Besov spaces or Besov spaces only. Note that these problems are the generalizations of boundary value problems in the H_2^s spaces, since $H_2^s = B_{2,2}^s$ (see [109, 2.3.3]).

[1] This paper was in print when I received from Dr. J. Johnsen his preprints:

– The stationary Navier–Stokes equations in L_p-related spaces. *Copenh. Univ. Math. Dept. Ph. D. Series* No. 1, 1993.

– Elliptic boundary problems and the Boutet de Monvel calculus in Besov and Triebel–Lizorkin spaces. *Copenh. Univ. Math. Dept. Prepr. Series* No. 25, 1994.

Chapter I is devoted to boundary value problems for anisotropic elliptic ΨDOs with "constant coefficients" in a half-space. When studying these problems one have to overcome principal technical difficulties of the theory, however we need the obtained results mainly for investigation of boundary value problems on compact manifolds. By well understandable reasons we consider these boundary value problems in the isotropic case only (see Ch. II). Therefore in Chapter I we could also confine ourselves to the isotropic case but the final example of Chapter I (Example 1.42) has won us over the anisotropic case. The author wished to show how one, by means of purely "elliptic" theory, could obtain the results on the Cauchy problem for parabolic equations. This subject is not new even in the framework of the theory of ΨDOs (in this connection see [80]). Anisotropic elliptic (= half-elliptic= quasi-elliptic) partial differential operators have been investigated by many authors (see, e.g., [112, §3.8], [54], [9] and references therein). Very interesting results on boundary value problems for a model anisotropic elliptic differential operator in a unit circle have been obtained in [111] (see also 4.8, [89]).

Chapter III deals with boundary value problems on two-dimensional manifolds. From the point of view of the theory of boundary value problems for elliptic (pseudo-)differential operators the two-dimensional case is a particular one (for details see §3.1). Note that examples in §3.5 are given only to show the efficiency of the methods developed in the present paper. One can obtain similar results with the help of more classical means. In analogous situations methods of complex analysis are usually applied. In general, the theory of boundary value problems for elliptic equations in two-dimensional domains (and on the Riemann surfaces) resembles by itself more a part of the complex analysis than a part of the theory of partial differential equations. The approach used by us enables one to consider elliptic boundary value problems in the case of two independent variables in the same way as in the multi-dimensional case, reducing application of the complex analysis to a minimum.

In §3.5 we try to impose minimal restrictions on the smoothness of coefficients. On "freezing" the coefficients in the case of the Nikol'skiĭ spaces $B^{\sigma}_{p,\infty}$, there arise complications. §3.6 is concerned with these difficulties as well as with the ways of their handling. Of course, one could avoid these difficulties by rising slightly the restrictions on the smoothness of coefficients, but sporting excitement did not permit us to make a compromise.

The most important applications of the Vishik–Eskin theory are not considered in the paper. Such in the author's opinion are the applications to the boundary value problems for elliptic differential equations with boundary conditions on open surfaces (see [120], [37], [26], [102], [103], [104], [121], [44] and also [84], [86]). Solution of these problems by the potential method (the method of boundary integral equations) leads to the pseudodifferential equations on manifolds with boundary, ΨDO as a rule being free from transmission property. Scientists working in the theory of boundary value problems are frequently interested in the information on the smoothness of generalized solutions. Unfortunately, it is impossible to get sufficiently exact results from the L_2-theory by means of embedding theorems. In fact, $H_2^s \subset C^\tau$ if $\tau < s - n/2$ (see, e.g., [109, 4.6]), i.e. the difference between exponents of smoothness in Sobolev–Slobodeckiĭ and Hölder spaces must be more than $n/2$. In this respect the theory of boundary value problems for elliptic ΨDOs without transmission property in Hölder spaces (with weight) would be ideal. For the present no such theory is available (as it has been noted above, we have results for ΨDOs with transmission property). The L_p-theory gives satisfactory answers to the requirements of practice. Indeed, for Bessel-potential H_p^s and Besov $B_{p,q}^s$ spaces the embeddings $H_p^s \subset C^\tau$, $B_{p,q}^s \subset C^\tau$ take place if $\tau < s - n/p$ (see [109, 4.6]). Taking $p \in]1, \infty[$ sufficiently large, we can make the difference between s and τ arbitrarily small. Thus we can obtain the exponent of smoothness which is arbitrarily close to the best possible. The L_p-theory of ΨDOs on manifolds with boundary ([31], [94], [95]) has been applied to the problems of elasticity in [32], [72], [97], [73], [71], [51], [23], [24], etc. Note that all this direction was anticipated by the works [92], [93] the importance of which cannot be belittled by the mistakes contained in them.

The present work is a revised version of papers [94]–[96] which were submitted for publication in 1988–1989 but irrespective of the author they have not appeared so far.

The author wishes to express appreciation to T. G. Gegelia and I. T. Kiguradze for leaving it to him to collect the results of papers [94]–[96] and present them in this volume. Most particular thanks are due to my scientific supervisor R. V. Duduchava dealings with whom for almost ten years exercised great influence on me and, in particular, stimulated my interest to the given subject matter.

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1°. Recall some standard notation:
$\mathcal{D}(\mathbf{\Omega})$ is a space of infinitely smooth functions with compact supports belonging to $\Omega \subset \mathbb{R}^n$;
$\mathcal{D}'(\mathbf{\Omega})$ is a corresponding space of distributions;
$S(\mathbb{R}^n)$ is a space of rapidly decreasing infinitely smooth functions;
$S'(\mathbb{R}^n)$ is a corresponding space of tempered distributions;
$F^{\pm 1}$ are direct and inverse Fourier transforms,

$$(F^{\pm 1}\psi)(z) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{\pm izt}\psi(t)\,dt, \quad z \in \mathbb{R}^n, \quad zt = \sum_{k=1}^{n} z_k t_k.$$

Fix an arbitrary vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ such that $a_k > 0$, $k = 1, \dots, n$ and

$$\sum_{k=1}^{n} a_k = n. \tag{1.1}$$

For any $s \in \mathbb{R}$ we put

$$\overline{s} = \left(\frac{s}{a_1}, \dots, \frac{s}{a_n}\right) = (s_1, \dots, s_n) \in \mathbb{R}^n. \tag{1.2}$$

Introduce the following sets:

$$E_j = \left\{\xi \in \mathbb{R}^n \;\middle|\; |\xi_k| \le 2^{ja_k},\ k = 1, \dots, n\right\}, \quad j = 0, 1, 2, \dots, \tag{1.3}$$

$$M_0 = E_1, \quad M_j = E_{j+1} \backslash E_{j-1}, \quad j = 1, 2 \dots . \tag{1.4}$$

For $s \in \mathbb{R}$, $1 \le p \le \infty$, $1 \le q \le \infty$ we put[2]

$$B^{\overline{s}}_{p,q}(\mathbb{R}^n) = \Big\{ f \;\Big|\; f \in S'(\mathbb{R}^n),\ f \underset{S'}{=} \sum_{j=0}^{\infty} f_j,\ \operatorname{supp} Ff_j \subset M_j;$$

$$\|\{f_j\}\|_{l^s_q(L_p)} = \Big(\sum_{j=0}^{\infty} \big(2^{sj}\|f_j\|_{L_p}\big)^q\Big)^{1/q} < \infty \Big\} \tag{1.5}$$

(as usual, the last expression is substituted for $q = \infty$ by $\sup_j 2^{sj}\|f_j\|_{L_p}$).

[2] Definition of Besov spaces differs in form from definitions accepted in the works [74] and [105] we are always referred to. In a standard way one can prove that all these definitions are equivalent (see [110, 2.5.2, 2.5.3], [105, Theorem 2], [74, 5.6]).

We endow the space $B^{\overline{s}}_{p,q}(\mathbb{R}^n)$ with the norm

$$\big\| f \;\big|\; B^{\overline{s}}_{p,q}(\mathbb{R}^n)\big\| = \inf_{f=\sum f_j} \|\{f_j\}\|_{l^s_q(L_p)}. \tag{1.6}$$

Let further

$$\langle y\rangle = (1+|y|^2)^{1/2} = \Big(1+\sum_{k=1}^{l} y_k^2\Big)^{1/2}, \ \forall y\in\mathbb{R}^l\,,\ l=1,\dots,n, \tag{1.7}$$

$$\langle y\rangle_a = (1+|y|_a^2)^{1/2} = \Big(1+\Big(\sum_{k=1}^{l}|y_k|^{2a_1/a_k}\Big)^{1/a_1}\Big)^{1/2}, \tag{1.8}$$

$$\forall y\in\mathbb{R}^l\,,\ \ l=1,\dots,n,$$

$$I_k^\sigma = F^{-1}\langle\xi_k\rangle^\sigma F, \tag{1.9}$$

$$I^{\overline{s}} = F^{-1}\langle\xi\rangle_a^s F \tag{1.10}$$

(see (1.2)). Definition of $|y|_a$ given in (1.8) differs from a more standard one

$$|y|_a = \Big(\sum_{k=1}^{l}|y_k|^{2/a_k}\Big)^{1/2}.$$

Our choice can be explained by the fact that in the case when $a_1 = \cdots = a_{n-1}$ we will have $|\xi'|_a = |\xi'|^{1/a_1}$, $\forall\xi'\in\mathbb{R}^{n-1}$. The most part of §1.4 is devoted to this case.

For $s\in\mathbb{R}^n$, $1<p<\infty$ we put

$$H^{\overline{s}}_p(\mathbb{R}^n) = \big\{f \;\big|\; f\in S'(\mathbb{R}^n),\ \|f\,|H^{\overline{s}}_p(\mathbb{R}^n)\| = \|I^{\overline{s}}f\|_{L_p}<\infty\big\}. \tag{1.11}$$

$B^{\overline{s}}_{p,q}(\mathbb{R}^n)$ is called anisotropic Besov space and $H^{\overline{s}}_p(\mathbb{R}^n)$ is called anisotropic Bessel-potential space (or either the Liouville or the Lebesgue space). For $a=(1,\dots,1)$ we obtain isotropic Besov $B^s_{p,q}(\mathbb{R}^n)$ and isotropic Bessel-potential $H^s_p(\mathbb{R}^n)$ spaces. (In the isotropic case we always write s instead of $\overline{s}=(s,\dots,s)$).

Note that unlike the notations accepted in the given paper, the symbol $H^{\overline{s}}_p$ often denotes the Nikol'skiĭ space $B^{\overline{s}}_{p,\infty}$, while the symbol $L^{\overline{s}}_p$ is often used to denote Bessel-potential spaces (see, e.g., [12], [74]).

Spaces $B^{\overline{s}}_{p,q}(\mathbb{R}^n)$, $H^{\overline{s}}_\rho(\mathbb{R}^n)$, $s\in\mathbb{R}$, $1\le p,q\le\infty$, $1<\rho<\infty$, are Banach spaces and $\mathcal{D}(\mathbb{R}^n)$, $S(\mathbb{R}^n)$ are dense in them for $p,q<\infty$ (see, e.g., [105, Theorem 2], and also [110, 2.3.3]).

We can easily see (see, e.g., Theorem 1.4 below) that for $s\ge 0$

$$\|f|H^{\overline{s}}_p(\mathbb{R}^n)\|^{(1)} = \Big\|\Big(\sum_{k=1}^{n} I_k^{s_k}\Big)f\Big\|_{L_p},\ \ \|f|H^{\overline{s}}_p(\mathbb{R}^n)\|^{(2)} = \sum_{k=1}^{n}\|I_k^{s_k}f\|_{L_p} \tag{1.12}$$

are the equivalent norms in the space $H^{\overline{s}}_p(\mathbb{R}^n)$.

If $\overline{s} = (s_1, \ldots, s_n)$, $s_k \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $k = 1, \ldots, n$, then $H_p^{\overline{s}}(\mathbb{R}^n)$ coincides with an anisotropic Sobolev space $W_p^{\overline{s}}(\mathbb{R}^n)$:

$$H_p^{\overline{s}}(\mathbb{R}^n) = W_p^{\overline{s}}(\mathbb{R}^n) = \Big\{ f | f \in S'(\mathbb{R}^n),\ \|f | W_p^{\overline{s}}(\mathbb{R}^n)\| = $$

$$= \|f\|_{L_p} + \sum_{k=1}^{n} \Big\| \frac{\partial^{s_k} f}{\partial x_k^{s_k}} \Big\|_{L_p} < \infty \Big\} =$$

$$= \Big\{ f | f \in S'(\mathbb{R}^n),\ \|f | W_p^{\overline{s}}(\mathbb{R}^n)\|^{(1)} = \sum_{0 \le m_k \le s_k} \Big\| \frac{\partial^{m_k} f}{\partial x_k^{m_k}} \Big\|_{L_p} < \infty \Big\} \quad (1.13)$$

(see [74, 9.1]).

Let f be an arbitrary function on $\mathbb{R}^n$, $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$. Introduce the notation

$$(\Delta_h^1 f)(x) = f(x+h) - f(x),$$
$$(\Delta_h^l f)(x) = \Delta_h^1 (\Delta_h^{l-1} f)(x), \quad l = 2, 3, \ldots,$$
$$(\Delta_{h_k}^1 f)(x) = f(x_1, \ldots, x_{k-1}, x_k + h_k, x_{k+1}, \ldots, x_n) - f(x),$$
$$(\Delta_{h_k}^l f)(x) = \Delta_{h_k}^1 (\Delta_{h_k}^{l-1} f)(x), \quad l = 2, 3, \ldots .$$

Suppose $\overline{s} = (s_1, \ldots, s_n) \in \mathbb{R}^n$, $s_k > 0$, $l_k \in \mathbb{N}$, $m_k \in \mathbb{Z}_+$, $l_k > s_k - m_k > 0$, $k = 1, \ldots, n$. Then (see [74, 4.3 and 5.6])

$$\|f | B_{p,q}^{\overline{s}}(\mathbb{R}^n)\|^{(1)} = \|f\|_{L_p} + \sum_{k=1}^{n} \left(\int_{-1}^{1} \left(\frac{\Big\| \Delta_{h_k}^{l_k} \frac{\partial^{m_k} f}{\partial x_k^{m_k}} \Big\|_{L_p}}{|h_k|^{s_k - m_k}} \right)^{q} \frac{dh_k}{|h_k|} \right)^{1/q} \quad (1.14)$$

is an equivalent norm in the space $B_{p,q}^{\overline{s}}(\mathbb{R}^n)$ (in the case $q = \infty$ the last sum is substituted by

$$\sum_{k=1}^{n} \sup_{0 < |h_k| < 1} |h_k|^{m_k - s_k} \Big\| \Delta_{h_k}^{l_k} \frac{\partial^{m_k} f}{\partial x_k^{m_k}} \Big\|_{L_p} \Big).$$

In the isotropic case $\overline{s} = (s, \ldots, s)$, $s > 0$, we have (see [74, 4.3, 5.6] or [110, 2.3.8, 2.5.12]):

$$\|f | B_{p,q}^{s}(\mathbb{R}^n)\|^{(2)} = \|f\|_{L_p} + \sum_{k=1}^{n} \left(\int_{\mathbb{R}^n} |h|^{(m-s)q} \Big\| \Delta_h^l \frac{\partial^m f}{\partial x_k^m} \Big\|_{L_p}^{q} \frac{dh}{|h|^n} \right)^{1/q}, \quad (1.15)$$

where $l \in \mathbb{N}$, $m \in \mathbb{Z}_+$, $l > s - m > 0$, is an equivalent norm in $B_{p,q}^{s}(\mathbb{R}^n)$ (in the case $q = \infty$ the last sum is, as usual, replaced by

$$\sum_{k=1}^{n} \sup_{h \in \mathbb{R}^n \setminus \{0\}} |h|^{m-s} \Big\| \Delta_h^l \frac{\partial^m f}{\partial x_k^m} \Big\|_{L_p} \Big).$$

In (1.14) we can replace

$$\int_{-1}^{1}\cdots \qquad \Big(\sup_{0<|h_k|<1}\cdots\Big) \quad \text{by} \quad \int_{\mathbb{R}}\cdots \qquad \Big(\sup_{h_k\in\mathbb{R}\setminus\{0\}}\cdots\Big).$$

Similarly, in (1.15) we can replace

$$\int_{\mathbb{R}^n}\cdots \qquad \Big(\sup_{h\in\mathbb{R}^n\setminus\{0\}}\cdots\Big) \quad \text{by} \quad \int_{|h|<1}\cdots \qquad \Big(\sup_{0<h<1}\cdots\Big).$$

In the sequel for an arbitrary $s\in\mathbb{R}$ we shall use the following representations:

$$s=[s]+\{s\},\quad [s]\in\mathbb{Z},\quad 0\le\{s\}<1, \tag{1.16}$$

$$s=[s]^-+\{s\}^+,\quad [s]^-\in\mathbb{Z},\quad 0<\{s\}^+\le 1. \tag{1.17}$$

It is clear that in (1.14) we can take $m_k=[s_k]^-$, $l_k=2$ and when $s_k\notin\mathbb{Z}$ we can take $m_k=[s_k]$, $l_k=1$. The same is true for the formula (1.15). In particular, $B^s_{\infty,\infty}(\mathbb{R}^n)$ coincides with the Zygmund space $Z^s(\mathbb{R}^n)$

$$B^s_{\infty,\infty}(\mathbb{R}^n)=Z^s(\mathbb{R}^n)=\Big\{f|f\in C^{[s]^-}(\mathbb{R}^n),\ \|f|Z^s(\mathbb{R}^n)\|=\|f|C^{[s]^-}(\mathbb{R}^n)\|+ \\ +\sum_{|\alpha|=[s]^-}\sup_{h\in\mathbb{R}^n\setminus\{0\}}|h|^{-\{s\}^+}\|\Delta^2_h\partial^\alpha f|C(\mathbb{R}^n)\|<\infty\Big\} \tag{1.18}$$

for $s>0$ and with the Hölder space $C^s(\mathbb{R}^n)$

$$B^s_{\infty,\infty}(\mathbb{R}^n)=Z^s(\mathbb{R}^n)=C^s(\mathbb{R}^n)=\Big\{f|f\in C^{[s]}(\mathbb{R}^n),\ \|f|C^s(\mathbb{R}^n)\|= \\ =\|f|C^{[s]}(\mathbb{R}^n)\|+\sum_{|\alpha|=[s]}\sup_{h\in\mathbb{R}^n\setminus\{0\}}|h|^{-\{s\}}\|\Delta^1_h\partial^\alpha f|C(\mathbb{R}^n)\|<\infty\Big\} \tag{1.19}$$

for $s>0$, $s\notin\mathbb{N}$ (note that $[s]=[s]^-$, $\{s\}=\{s\}^+$ for $s\notin\mathbb{Z}$).

In (1.18),(1.19) we have used the following standard notation:

$$|\alpha|=\alpha_1+\cdots+\alpha_n,\quad \forall\alpha\in\mathbb{Z}^n_+=(\mathbb{Z}_+)^n,\quad \partial^\alpha=\partial^\alpha_x=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},$$

$C(\mathbb{R}^n)$ is the space of bounded uniformly continuous on $\mathbb{R}^n$ functions,

$$\|f|C(\mathbb{R}^n)\|=\sup_{x\in\mathbb{R}^n}|f(x)|,$$

$$C^m(\mathbb{R}^n)=\big\{f|\partial^\alpha f\in C(\mathbb{R}^n)\text{ for }|\alpha|\le m\big\},\quad \forall m\in\mathbb{Z}_+,$$

$$\|f|C^m(\mathbb{R}^n)\|=\sum_{|\alpha|\le m}\|\partial^\alpha f|C(\mathbb{R}^n)\|.$$

2°. We present here some well-known facts from the theory of function spaces. Not trying to attain maximal generality, we shall formulate them in a form more convenient for us. In particular, we shall consider Besov spaces for $1<p<\infty$, though great many assertions are also true for $p=1,\infty$.

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$. *Then* $\left(H_p^{\overline{s}}(\mathbb{R}^n)\right)^* = H_{p'}^{-\overline{s}}(\mathbb{R}^n)$, $\left(B_{p,q}^{\overline{s}}(\mathbb{R}^n)\right)^* = B_{p',q'}^{-\overline{s}}(\mathbb{R}^n)$.

Proof. For Bessel-potential spaces the proof is completely similar to that of Theorem 2.6.1-(a) in [109], and for Besov spaces to that of Theorem 2.11.2-(i) in [110]. ■

Let $s, \sigma \in \mathbb{R}$, $1 < p_0, p_1, p_2 < \infty$, $1 \le q_0, q_1, q_2 \le \infty$, $0 < \theta < 1$, $r = (1-\theta)s + \theta\sigma$, $\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$, $\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}$. *Then (see* (1.2)*)*

a) $[B_{p_1,q_1}^{\overline{s}}(\mathbb{R}^n), B_{p_2,q_2}^{\overline{\sigma}}(\mathbb{R}^n)]_\theta = B_{p,q}^{\overline{r}}(\mathbb{R}^n)$ *if at least one of the numbers* q_1, q_2 *does not equal* ∞;

b) $(B_{p_0,q_1}^{\overline{s}}(\mathbb{R}^n), B_{p_0,q_2}^{\overline{\sigma}}(\mathbb{R}^n))_{\theta,q_0} = B_{p_0,q_0}^{\overline{r}}(\mathbb{R}^n)$ *if* $s \ne \sigma$;

c) $[H_{p_1}^{\overline{s}}(\mathbb{R}^n), H_{p_2}^{\overline{\sigma}}(\mathbb{R}^n)]_\theta = H_p^{\overline{r}}(\mathbb{R}^n)$;

d) $(H_{p_1}^{\overline{s}}(\mathbb{R}^n), H_{p_2}^{\overline{s}}(\mathbb{R}^n))_{\theta,p} = H_p^{\overline{s}}(\mathbb{R}^n)$;

e) $(H_{p_0}^{\overline{s}}(\mathbb{R}^n), H_{p_0}^{\overline{\sigma}}(\mathbb{R}^n))_{\theta,q_0} = B_{p_0,q_0}^{\overline{r}}(\mathbb{R}^n)$ *if* $s \ne \sigma$.

Proof. The proof is completely similar to that of Theorems 2.4.1, 2.4.2 in [109] (see also [105, Theorem 7], [110, Theorem 2.11.2-(ii)], and Theorem 1.1 above). ■

Let $s, \sigma \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. *Then the mappings*

$$I^{\overline{s}} : H_p^{\overline{s}}(\mathbb{R}^n) \to H_p^{\overline{\sigma}-\overline{s}}(\mathbb{R}^n), \quad B_{p,q}^{\overline{\sigma}}(\mathbb{R}^n) \to B_{p,q}^{\overline{\sigma}-\overline{s}}(\mathbb{R}^n)$$

are the (continuous) isomorphisms.

Proof. The assertion in the case of Bessel-potential spaces is the direct consequence of the definition (1.11). In the case of Besov spaces it suffices to apply interpolation (see Theorem 1.2-e) and [109, 1.3.3]). ■

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $X(\mathbb{R}^n) = H_p^{\overline{s}}(\mathbb{R}^n)$ *or* $B_{p,q}^{\overline{s}}(\mathbb{R}^n)$,

$$\|A\|_* = \sum_{|\alpha| \le [n/2]+1} \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^n} |\xi^\alpha \partial_\xi^\alpha A(\xi)| < +\infty.$$

Then the function A *is Fourier* $X(\mathbb{R}^n)$*-multiplier and*

$$\|F^{-1} A F | X(\mathbb{R}^n) \to X(\mathbb{R}^n)\| \le C \|A\|_*,$$

where $C < +\infty$ *depends only on* n, p *and* q.

Proof. For $X(\mathbb{R}^n) = L_p(\mathbb{R}^n)$ the above assertion is a variant of the Mikhlin–Hörmander–Lizorkin theorem on Fourier multipliers. In such a form it has been proved in [91]. By means of definition (1.11) it can be transferred to Bessel-potential spaces, while by interpolation (see Theorem 1.2-e)) or by definitions (1.5),(1.6) it can be transferred to Besov spaces. ■

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $m \in \mathbb{Z}_+$, $s_n > m + 1/p$ (*see* (1.2)),

$$\tau_j = 1 - \frac{1}{s_n}(j + 1/p), \quad j = 0, \dots, m. \tag{1.20}$$

Then the mapping given by

$$\pi_0^m f = \left\{ f(x', 0), \frac{\partial f}{\partial x_n}(x', 0), \dots, \frac{\partial^m f}{\partial x_n^m}(x', 0) \right\}, \tag{1.21}$$

$$x' = (x_1, \dots, x_{n-1}), \tag{1.22}$$

is a continuous invertible from the right (*and hence surjective*) *operator from* $H_p^{\overline{s}}(\mathbb{R}^n)$ *to* $\prod_{j=0}^{m} B_{p,p}^{\tau_j \overline{s}}(\mathbb{R}^{n-1})$ *and from* $B_{p,q}^{\overline{s}}(\mathbb{R}^n)$ *to* $\prod_{j=0}^{m} B_{p,q}^{\tau_j \overline{s}}(\mathbb{R}^{n-1})$.

Proof. See, e.g., [74, 6.4, 6.7, 6.8, 9.5]. ■

Let $s, \sigma \in \mathbb{R}$, $1 < p, p_1, p_2 < \infty$, $p_1 \le p_2$, $s - n/p_1 \ge \sigma - n/p_2$, $1 \le q, q_1, q_2 \le \infty$, $q_1 \le q_2$, $\varepsilon > 0$. *Then*

a) $B_{p,\infty}^{\overline{s}+\overline{\varepsilon}}(\mathbb{R}^n) \subset B_{p,1}^{\overline{s}}(\mathbb{R}^n) \subset B_{p,q_1}^{\overline{s}} \subset B_{p,q_2}^{\overline{s}}(\mathbb{R}^n) \subset B_{p,\infty}^{\overline{s}}(\mathbb{R}^n) \subset B_{p,1}^{\overline{s}-\overline{\varepsilon}}(\mathbb{R}^n)$;

b) $B_{p,\min(2,p)}^{\overline{s}}(\mathbb{R}^n) \subset H_p^{\overline{s}}(\mathbb{R}^n) \subset B_{p,\max(2,p)}^{\overline{s}}(\mathbb{R}^n)$;

c) $B_{p_1,q}^{\overline{s}}(\mathbb{R}^n) \subset B_{p_2,q}^{\overline{\sigma}}(\mathbb{R}^n)$;

d) $H_{p_1}^{\overline{s}}(\mathbb{R}^n) \subset H_{p_2}^{\overline{\sigma}}(\mathbb{R}^n)$.

Proof. The proof of points c) and d) for positive s and σ may be found in [74, 6.3 and 9.6] (see also (1.1),(1.2)). The general case is reduced to that by Theorem 1.3. Point a) is proved, e.g., in [74], 6.2. Point b) can be proved in exactly the same way as in the isotropic case (see, e.g., [110, Proposition 2.3.2-2, (iii)] and also [105, Theorem 7], [110, Theorem 2.11.2-(ii)] and Theorem 1.1 above). As for point b), see also [74, 9.3]. ■

Introduce the notation

$$\mathbb{R}_{\pm}^n = \{x | x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n, \ \pm x_n > 0\}. \tag{1.23}$$

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $1/p - 1 < s_n < 1/p$ (*see* (1.2)). *Then* $\chi_+ I$, *the operator of multiplication by the characteristic function of the upper half-space* $\mathbb{R}_+^n$, *is continuous both in* $H_p^{\overline{s}}(\mathbb{R}^n)$ *and in* $B_{p,q}^{\overline{s}}(\mathbb{R}^n)$.

Proof. In the case of the spaces $H_p^{\overline{s}}(\mathbb{R}^n)$, $0 \le s_n < 1/p$, using the second norm in (1.12), we can easily reduce the problem to one-dimension, and hence, to isotropic case. The assertion in this case has been proved in [90], [106] (see also [110], 2.8.7). The proof can be completed by means of duality (see Theorem 1.1) and by interpolation (see Theorems 1.2-e)). ■

§ $\mathbb{R}^n_+$

1°. Let $X(\mathbb{R}^n) = H_p^{\overline{s}}(\mathbb{R}^n)$ or $B_{p,q}^{\overline{s}}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$. In the sequel the following spaces will play an important role:

$$\widetilde{X}(\overline{\mathbb{R}^n_\pm}) = \big\{u | u \in X(\mathbb{R}^n), \ \ \operatorname{supp} u \subset \overline{\mathbb{R}^n_\pm}\big\} \tag{1.24}$$

and $X(\overline{\mathbb{R}^n_\pm})$ – the space of all restrictions on $\mathbb{R}^n_\pm$ of elements from $X(\mathbb{R}^n)$ endowed with the norm

$$\|u|X(\overline{\mathbb{R}^n_\pm})\| = \inf\big\{\|u_0|X(\mathbb{R}^n)\| \,\big|\, u_0 \in X(\mathbb{R}^n), \ u_0|_{\mathbb{R}^n_\pm} = u\big\}. \tag{1.25}$$

The spaces $\widetilde{X}(\overline{\mathbb{R}^n_\pm})$ and $X(\overline{\mathbb{R}^n_\pm})$ are Banach ones. Clearly

$$X(\overline{\mathbb{R}^n_\pm}) = X(\mathbb{R}^n)/\widetilde{X}(\mathbb{R}^n_\mp). \tag{1.26}$$

$\mathcal{D}(\mathbb{R}^n_\pm)$ is dense in $\widetilde{X}(\overline{\mathbb{R}^n_\pm})$ for $q < \infty$.

Proof. For an arbitrary $h \in \mathbb{R}^n$ we put $(\tau_h f)(x) = f(x-h)$, $\forall x \in \mathbb{R}^n$. Then if $q < \infty$, we have

$$\|f - \tau_h f|X(\mathbb{R}^n)\| \to 0 \quad \text{for} \ \ h \to 0, \ \ \forall f \in X(\mathbb{R}^n). \tag{1.27}$$

Indeed: first, $S(\mathbb{R}^n)$ is dense in $X(\mathbb{R}^n)$; second, for any $\varphi \in S(\mathbb{R}^n)$ the function $\tau_h\varphi$ tends to φ in $S(\mathbb{R}^n)$ for $h \to 0$; third,

$$\|\tau_h f|X(\mathbb{R}^n)\| = \|f|X(\mathbb{R}^n)\|, \quad \forall h \in \mathbb{R}^n, \ \ \forall f \in X(\mathbb{R}^n).$$

The operator of multiplication by the function from $\mathcal{D}(\mathbb{R}^n_\pm)$ is continuous in $X(\mathbb{R}^n)$. This can be proved by the same scheme as Theorem 1.7 (the only difference is that instead of [110, 2.8.7] we have to refer to [110, Theorem 2.8.2]). From this and (1.27) (for $h = (0, \ldots, 0, h_n)$, $\pm h_n > 0$) we can easily complete the proof of the assertion, taking into account that $\mathcal{D}(\mathbb{R}^n)$ is dense in $X(\mathbb{R}^n)$. ■

Pseudodifferential operator (ΨDO) with the symbol $A(\xi)$ will be denoted by $A(D)$, i.e.

$$A(D) = F^{-1}A(\xi)F. \tag{1.28}$$

Let $X(\mathbb{R}^n)$ and $Y(\mathbb{R}^n)$ be arbitrary spaces from the scale of spaces $H_p^{\overline{s}}(\mathbb{R}^n)$, $B_{p,q}^{\overline{s}}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, a ΨDO with the symbol $A(\xi)$ being bounded from $X(\mathbb{R}^n)$ to $Y(\mathbb{R}^n)$ and let the symbol $A(\xi', \xi_n)$ admit for almost all $\xi' \in \mathbb{R}^{n-1}$ an analytic with respect to ξ_n continuation to the upper (lower) complex half-plane such that

$$|A(\xi', \xi_n + i\tau)| \leq C(|\xi| + |\tau| + 1)^N, \quad \pm\tau \geq 0, \tag{1.29}$$

where C and N are some constants. Then the operator $A(D)$ is continuous from $\widetilde{X}(\overline{\mathbb{R}^n_\pm})$ to $\widetilde{Y}(\overline{\mathbb{R}^n_\pm})$.

Proof. Let us take $\forall u \in \mathcal{D}(\mathbb{R}^n_\pm)$. It is easily seen that $(Fu)(\xi', \xi_n)$ continues analytically with respect to ξ_n to the upper (lower) half-plane and admits the estimate

$$|(Fu)(\xi', \xi_n + i\tau)| \le C_M(|\xi| + |\tau| + 1)^{-M}, \quad \pm\tau \ge 0, \quad \forall M > 0.$$

Using the Paley–Wiener theorem (see, e.g., [37, Theorem 4.5]), it is easy to show that $A(D)u \in L_2(\mathbb{R}^n)$, $\operatorname{supp} A(D)u \subset \overline{\mathbb{R}^n_\pm}$. On the other hand, $A(D)u \in Y(\mathbb{R}^n)$. Taking into account that $\mathcal{D}(\mathbb{R}^n_\pm)$ is dense in $\widetilde{X}(\overline{\mathbb{R}^n_\pm})$ for $q < \infty$ (see Lemma 1.8), due to the continuity we obtain that $A(D)$ is bounded from $\widetilde{X}(\overline{\mathbb{R}^n_\pm})$ to $\widetilde{Y}(\overline{\mathbb{R}^n_\pm})$ for $q < \infty$.

It remains for us to consider the case $X(\mathbb{R}^n) = X^{\overline{s}}(\mathbb{R}^n) = B^{\overline{s}}_{p,\infty}(\mathbb{R}^n)$. For the sake of convenience we will write $Y^{\overline{\sigma}}(\mathbb{R}^n)$ instead of $Y(\mathbb{R}^n)$. By virtue of Theorem 1.3, the boundedness of the operator $A(D) : X^{\overline{s}}(\mathbb{R}^n) \to Y^{\overline{\sigma}}(\mathbb{R}^n)$ implies that of $A(D)$ from $X^{\overline{s}\pm\overline{\varepsilon}}(\mathbb{R}^n)$ to $Y^{\overline{\sigma}\pm\overline{\varepsilon}}(\mathbb{R}^n)$, $\varepsilon > 0$. Using interpolation (see Theorem 1.2-b)), we get that $A(D)$ is bounded from $B^{\overline{s}-\frac{1}{2}\overline{\varepsilon}}_{p,q_0}(\mathbb{R}^n)$, $1 \le q_0 < \infty$, to the corresponding function space. Now the assertion of the theorem (for $q = \infty$) follows from already proven and from the embedding Theorem 1.6-a). ■

Denote by $\pi_\pm$ the restriction operator from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$:

$$\pi_\pm : X(\mathbb{R}^n) \to X(\overline{\mathbb{R}^n_\pm}). \tag{1.30}$$

For the function $u \in X(\overline{\mathbb{R}^n_\pm})$ an extension on $\mathbb{R}^n$ will be denoted by $\ell u \in X(\mathbb{R}^n)$: $\pi_\pm \ell u = u$.

Let the conditions of the previous theorem be fulfilled. Then the operator $\pi_\mp A(D)\ell$ does not depend on the choice of the extension ℓ and is continuous from $X(\overline{\mathbb{R}^n_\mp})$ to $Y(\overline{\mathbb{R}^n_\mp})$.

Proof. Let us take an arbitrary $u \in X(\overline{\mathbb{R}^n_\mp})$ and its arbitrary extensions $\ell_1 u, \ell_2 u \in X(\mathbb{R}^n)$. Clearly $\pi_\mp(\ell_1 u - \ell_2 u) = 0$, i.e. $\ell_1 u - \ell_2 u \in \widetilde{X}(\overline{\mathbb{R}^n_\pm})$. Then according to Theorem 1.9, $A(D)(\ell_1 u - \ell_2 u) \in \widetilde{Y}(\mathbb{R}^n_\pm)$, i.e. $\pi_\mp A(D)(\ell_1 u - \ell_2 u) = 0$, i.e. $\pi_\mp A(D)\ell_1 u = \pi_\mp A(D)\ell_2 u$.

Continuity of the operator $\pi_\mp A(D)\ell$ follows from the fact that we can always choose ℓu so that the inequality $\|\ell u|X(\mathbb{R}^n)\| \le 2\|u|X(\overline{\mathbb{R}^n_\mp})\|$ be fulfilled (see (1.25)). ■

Remark. It follows from Theorem 1.10 that if the pseudodifferential operator $A(D)$ satisfies the conditions of Theorem 1.9, then $(\pi_\mp A(D)\ell)\pi_\mp = \pi_\mp A(D)$. We shall use this fact in §1.4.

In the sequel for function spaces on $\mathbb{R}^n_\pm$ we shall need an analogue of Theorem 1.3.

Introduce the operators

$$I^{\overline{s}}_\pm = F^{-1}(\xi_n \pm i\langle\xi'\rangle_a^{a_n})^{s/a_n} F \tag{1.31}$$

(see (1.2),(1.8),(1.23)).

Let $X^{\overline{\sigma}}(\mathbb{R}^n) = H_p^{\overline{\sigma}}(\mathbb{R}^n)$ *or* $B_{p,q}^{\overline{\sigma}}\mathbb{R}^n)$, $\sigma \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, $s \in \mathbb{R}$. *Then the mappings*

$$I_{\pm}^{\overline{s}} : X^{\overline{\sigma}}(\mathbb{R}^n) \to X^{\overline{\sigma}-\overline{s}}(\mathbb{R}^n), \quad \widetilde{X}^{\overline{\sigma}}(\overline{\mathbb{R}^n_{\pm}}) \to \widetilde{X}^{\overline{\sigma}-\overline{s}}(\overline{\mathbb{R}^n_{\pm}}),$$
$$\pi_{\mp} I_{\pm}^{\overline{s}}\ell : X^{\overline{\sigma}}(\overline{\mathbb{R}^n_{\mp}}) \to X^{\overline{\sigma}-\overline{s}}(\overline{\mathbb{R}^n_{\mp}})$$

are (*continuous*) *isomorphisms.*

Proof. To prove this it suffices to refer to Theorems 1.3, 1.4, 1.9, 1.10. ■

2°. In the assertions given below by $\delta^{(k)}$ will be denoted the k-th derivative of the Dirac δ-function $\delta \in \mathcal{D}'(\mathbb{R})$.

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $s_n < \frac{1}{p} - 1$, $v_j \in B_{p,p}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1})$ $(B_{p,q}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1}))$,

$$\lambda_j = 1 + \frac{1}{s_n}\,(j - 1/p), \quad j = 1, \ldots, [1/p - s_n]^- \tag{1.32}$$

(*see* (1.2), (1.17)). *Then*

$$u = \sum_{j=1}^{[1/p-s_n]^-} v_j(x') \times \delta^{(j-1)}(x_n) \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}^n_{\pm}})\ \big(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_{\pm}})\big). \tag{1.33}$$

Proof. Using Theorems 1.1 and 1.5, we obtain that u is a continuous linear functional on $H_{p'}^{-\overline{s}}(\mathbb{R}^n)$ $(B_{p',q'}^{-\overline{s}}(\mathbb{R}^n)$ for $1 < q \leq \infty)$. Therefore, by virtue of Theorem 1.1, $u \in H_p^{\overline{s}}(\mathbb{R}^n)$ $(B_{p,q}^{\overline{s}}(\mathbb{R}^n)$, $1 < q \leq \infty)$. To prove the last relation for $q = 1$, it suffices to apply interpolation Theorem 1.2-b) to the operator $(v_k) \longmapsto u$. Now (1.33) follows from the obvious fact that $\operatorname{supp} u \in \overline{\mathbb{R}^n_+} \cap \overline{\mathbb{R}^n_-}$. ■

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $s_n > m - 1 + 1/p$, $m \in \mathbb{N}$, $f \in H_p^{\overline{s}}(\mathbb{R}^n)$ $(B_{p,q}^{\overline{s}}(\mathbb{R}^n))$. *Then*

$$\partial_{x_n}^k(\chi_+ f)(x) = (\chi_+ \partial_{x_n}^k f)(x) + \sum_{j=0}^{k-1} (\partial_{x_n}^j f)(x', 0) \times \delta^{(k-1-j)}(x_n), \tag{1.34}$$
$$k = 1, \ldots, m,$$

where χ_+ *is a characteristic function of the upper half-space* $\mathbb{R}^n_+$.

Proof. For an arbitrary function $f \in S(\mathbb{R}^n)$ the formula (1.34) can be easily obtained from the definition of a generalized derivative by integration by parts. $S(\mathbb{R}^n)$ is dense in $H_p^{\overline{s}}(\mathbb{R}^n)$ $(B_{p,q}^{\overline{s}}(\mathbb{R}^n)$ for $q < \infty)$. Hence using Theorems 1.5 and 1.7 we can prove (1.34) by simple passage to the limit. (Convergence of the right- and left-hand sides of the corresponding equalities of the type (1.34) occurs, for example, in $S'(\mathbb{R}^n)$). In the case of spaces

$B^{\overline{s}}_{p,\infty}(\mathbb{R}^n)$ formula (1.34) follows from the already proven and the embedding Theorem 1.6-a). ■

Before going further on, it should be noted that Theorem 1.5 remains valid if we replace the spaces $H^{\overline{s}}_p(\mathbb{R}^n)$ and $B^{\overline{s}}_{p,q}(\mathbb{R}^n)$ by $H^{\overline{s}}_p(\overline{\mathbb{R}^n_\pm})$ and $B^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_\pm})$, respectively.

Let $1 < p < \infty$, $1 \le q \le \infty$, $m + 1/p - 1 < s_n < m + 1/p$, $m \in \mathbb{N}$, $f \in H^{\overline{s}}_p(\overline{\mathbb{R}^n_+})$ $(B^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_+}))$, f° *is the extension of* f *by zero from* $\mathbb{R}^n_+$ *onto* $\mathbb{R}^n$: $f^\circ\big|_{\mathbb{R}^n_+} = f$, $f^\circ\big|_{\overline{\mathbb{R}^n_-}} = 0$. *Then* $f^\circ \in \widetilde{H}^{\overline{s}}_p(\overline{\mathbb{R}^n_+})$ $(\widetilde{B}^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_+}))$ *if and only if* $\pi_0^{m-1} f = 0$ (*see* (1.21)). *In this case*

$$\|f^\circ|\widetilde{H}^{\overline{s}}_p(\overline{\mathbb{R}^n_+})\| \le C\|f|H^{\overline{s}}_p(\overline{\mathbb{R}^n_+})\| \quad \big(\, \|f^\circ|\widetilde{B}^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_+})\| \le C\|f|B^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_+})\|\big),$$

where $C < +\infty$ *is a constant depending only on* p, q, s, a *and* n.

Proof. Assume $f^\circ \in \widetilde{H}^{\overline{s}}_p(\overline{\mathbb{R}^n_+})$ $(\widetilde{B}^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_+}))$. Then

$$\pi_0^{m-1} f = \pi_0^{m-1}\big(f^\circ\big|_{\mathbb{R}^n_+}\big) = \pi_0^{m-1} f^\circ = \pi_0^{m-1}\big(f^\circ\big|_{\mathbb{R}^n_-}\big) = 0$$

(see Theorem 1.5).

Let now $\pi_0^{m-1} f = 0$. Take an arbitrary extension $f_0 \in H^{\overline{s}}_p(\mathbb{R}^n)$ $(B^{\overline{s}}_{p,q}(\mathbb{R}^n))$ of the function f. It is clear that

$$f^\circ = \chi_+ f_0, \quad \pi_0^{m-1} f_0 = 0. \tag{1.35}$$

By virtue of (1.31) we have

$$I_+^{a_n \overline{m}} = i^m \sum_{k=0}^{m} \binom{m}{k} \partial^k_{x_n} I_0^{a_n(\overline{m}-\overline{k})}, \tag{1.36}$$

where

$$I_0^{\overline{\sigma}} = F^{-1}\langle\xi'\rangle^\sigma_a F, \quad \xi' = (\xi_1, \ldots, \xi_{n-1}). \tag{1.37}$$

From (1.35) and Lemma 1.14 we obtain $\partial^k_{x_n} f^\circ = \chi_+ \partial^k_{x_n} f_0$, $k = 1, \ldots, m$. It is also easily seen that $I_0^{a_n(\overline{m}-\overline{k})} f^\circ = \chi_+ I_0^{a_n(\overline{m}-\overline{k})} f_0$. Therefore

$$I_+^{a_n\overline{m}} f^\circ = \chi_+ I_+^{a_n\overline{m}} f_0. \tag{1.38}$$

Put $\sigma = s - a_n m$. The component $\sigma_n = s_n - m$ of the vector $\overline{\sigma} = \overline{s} - a_n\overline{m}$ (see (1.2)) satisfies the inequality $\frac{1}{p} - 1 < \sigma_n < \frac{1}{p}$ whence and from (1.38) and Theorems 1.7, 1.12 it follows that

$$\begin{aligned}\|f^\circ|\widetilde{H}^s_p(\overline{\mathbb{R}^n_+})\| &\le C_1\|I_+^{a_n\overline{m}} f^\circ|\widetilde{H}^{\overline{\sigma}}_p(\overline{\mathbb{R}^n_+})\| = C_1\|\chi_+ I_+^{a_n\overline{m}} f_0|\widetilde{H}^{\overline{\sigma}}_p(\overline{\mathbb{R}^n_+})\| \le \\ &\le C_2\|I_+^{a_n\overline{m}} f_0|H^{\overline{\sigma}}_p(\mathbb{R}^n)\| \le C_3\|f_0|H^{\overline{s}}_p(\mathbb{R}^n)\|.\end{aligned}$$

Similar inequalities are also valid in the case of Besov spaces. To complete the proof it suffices to refer to the definition of the norm in $X(\overline{\mathbb{R}^n_+})$ (see (1.25)). ■

Remark. The above-proven theorem can be formulated quite differently: the kernel of the operator (see Theorem 1.5)

$$\pi_0^{m-1} : H_p^{\overline{s}}(\overline{\mathbb{R}_+^n}) \to \prod_{j=0}^{m-1} B_{p,p}^{\tau_j \overline{s}}(\mathbb{R}^{n-1}) \ \left(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}) \to \prod_{j=0}^{m-1} B_{p,q}^{\tau_j \overline{s}}(\mathbb{R}^{n-1}) \right)$$

satisfies the equality

$$\operatorname{Ker} \pi_0^{m-1} = \pi_+ \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n}) \ \left(\pi_+ \widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}) \right) \tag{1.39}$$

for $m + 1/p - 1 < s_n < m + 1/p$.

Let $1 < p < \infty$, $1 \le q \le \infty$, $s_n < 1/p - 1$, $s_n - 1/p \notin \mathbb{Z}$, $u \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n}) \cap \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_-^n})$ $(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}) \cap \widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_-^n}))$. *Then*

$$u = \sum_{j=1}^{[1/p - s_n]} v_j(x') \times \delta^{(j-1)}(x_n), \tag{1.40}$$

where $v_j \in B_{p,p}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1})$ $\left(B_{p,q}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1}) \right)$ *(see* (1.32)).

Proof. Consider first the case of Bessel-potential spaces. In view of Theorem 1.1, u is a continuous linear functional on $H_{p'}^{-\overline{s}}(\mathbb{R}^n)$. Moreover, $\operatorname{supp} u \subset \overline{\mathbb{R}_+^n} \cap \overline{\mathbb{R}_-^n}$. Hence u considered as a functional on $H_{p'}^{-\overline{s}}(\mathbb{R}^n)$ vanishes on $\widetilde{H}_{p'}^{-\overline{s}}(\overline{\mathbb{R}_-^n})$ (see Lemma 1.8). Therefore u can be considered as a functional on $H_{p'}^{-\overline{s}}(\overline{\mathbb{R}_+^n}) = H_{p'}^{-\overline{s}}(\mathbb{R}^n) / \widetilde{H}_{p'}^{-\overline{s}}(\overline{\mathbb{R}_-^n})$ (see (1.26)). But u equals zero on $\widetilde{H}_{p'}^{-\overline{s}}(\overline{\mathbb{R}_+^n})$ as well. Thus we can consider it as a functional on

$$H_{p'}^{-\overline{s}}(\overline{\mathbb{R}_+^n}) / \pi_+ \widetilde{H}_{p'}^{-\overline{s}}(\overline{\mathbb{R}_+^n}). \tag{1.41}$$

From the condition $s_n < 1/p - 1$ it follows that $[1/p - s_n] \ge 1$, and from the condition $s_n - 1/p \notin \mathbb{Z}$ we have $0 < \frac{1}{p} - s_n - \left[\frac{1}{p} - s_n\right] < 1$, i.e. $\left[\frac{1}{p} - s_n\right] - 1 + \frac{1}{p'} < -s_n < \left[\frac{1}{p} - s_n\right] + \frac{1}{p'}$. Then from Theorem 1.5 and Remark 1.16 we obtain that π_0^{m-1} induces an isomorphism of the space (1.41) on $\prod_{j=1}^{[1/p - s_n]} B_{p',p'}^{-\lambda_j \overline{s}}(\mathbb{R}^{n-1})$.

Taking into account the form of this isomorphism and using Theorem 1.1, we arrive at (1.40).

Exactly in the same way we can prove the assertion of the lemma in the case of the spaces $B_{p,q}^{\overline{s}}$ for $1 < q \le \infty$.

In a general case the assertion of the lemma for the spaces $B_{p,q}^{\overline{s}}$, $1 \le q \le \infty$, can be obtained from the already proven part of the theorem by applying interpolation to the operator $u \longmapsto (v_k)$ (see points b) and e) of Theorem 1.2). ■

Consider the equation

$$\pi_+ f = g, \tag{1.42}$$

where $g \in H_p^{\overline{s}}(\overline{\mathbb{R}_+^n})$ $(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}))$ is a given function and $f \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n})$ $(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}))$ is an unknown function.

Let $1 < p < \infty$, $1 \le q \le \infty$, $m + 1/p - 1 < s_n < m + 1/p$, $m \in \mathbb{Z}$. *Then*

a) *if* $m = 0$, *then equation* (1.42) *has a unique solution and this solution is equal to* $\chi_+ \ell g$, *where* $\ell g \in H_p^{\overline{s}}(\mathbb{R}^n)$ $(B_{p,q}^{\overline{s}}(\mathbb{R}^n))$ *is an arbitrary extension of the function* g;

b) *if* $m > 0$, *then for equation* (1.42) *to be solvable it is necessary and sufficient that the equality* $\pi_0^{m-1} g = 0$ *be fulfilled* (*see* (1.21)); *moreover, the solution is unique and equals* $\chi_+ \ell g$;

c) *if* $m < 0$, *then equation* (1.42) *is solvable and its arbitrary solution is given by* (*see* (1.31))

$$f = I_+^{-a_n \overline{m}} \chi_+ I_+^{a_n \overline{m}} \ell g + \sum_{j=1}^{|m|} v_j(x') \times \delta^{(j-1)}(x_n), \tag{1.43}$$

where $v_j \in B_{p,p}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1})$ $(B_{p,q}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1}))$, $j = 1, \dots, |m|$ (*see* (1.32)).

Proof. a) Let $1/p - 1 < s_n < 1/p$. Using Theorem 1.7, we readily obtain that $\chi_+ \ell g \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n})$ $(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}))$. It is clear that $\pi_+ \chi_+ \ell g = g$, i.e., $f = \chi_+ \ell g$ is in fact a solution of (1.42). Let us prove its uniqueness.

Assume $u \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n})$ $(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}))$ and $\pi_+ u = 0$. In this case $\operatorname{supp} u \subset \overline{\mathbb{R}_+^n} \setminus \mathbb{R}_+^n = \overline{\mathbb{R}_+^n} \cap \overline{\mathbb{R}_-^n}$, i.e. $u \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n}) \cap \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_-^n})$ $(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}) \cap \widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_-^n}))$.

Check that

$$\chi_\pm w = 0, \quad \forall w \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_\mp^n}) \ \left(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_\mp^n})\right), \tag{1.44}$$

where $\chi_- = 1 - \chi_+$. If $q < \infty$, then according to Lemma 1.8 w can be approximated by functions from $\mathcal{D}(\mathbb{R}_\mp^n)$ for which (1.44) is obvious. Taking into account Theorem 1.7, due to continuity we get (1.44). For $q = \infty$ (1.44) is obtained from the already proven and from the embedding Theorem 1.6-a).

Apply (1.44) to the function u: $u = \chi_+ u + \chi_- u = 0$. Thus the uniqueness of the solution of equation (1.42) in the case under consideration is proved.

b) For $m > 0$ our assertion follows from the already proven part of the theorem (the uniqueness) and from Lemma 1.15.

c) By virtue of Theorems 1.7 and 1.12, $w = I_+^{-a_n \overline{m}} \chi_+ I_+^{a_n \overline{m}} \ell g \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n})$ $(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}))$. Moreover,

$$\begin{aligned} \pi_+ w &= \pi_+ I_+^{-a_n \overline{m}} I_+^{a_n \overline{m}} \ell g - \pi_+ I_+^{-a_n \overline{m}} \chi_- I_+^{a_n \overline{m}} \ell g = \\ &= g - \pi_+ I_+^{-a_n \overline{m}} \chi_- I_+^{a_n \overline{m}} \ell g. \end{aligned} \tag{1.45}$$

Using again Theorems 1.7 and 1.12, we obtain

$$\chi_- I_+^{a_n\overline{m}} \ell g \in \widetilde{H}_p^{\overline{s}-a_n\overline{m}}(\overline{\mathbb{R}^n_-}) \ \left(\widetilde{B}_{p,q}^{\overline{s}-a_n\overline{m}}(\overline{\mathbb{R}^n_-})\right). \tag{1.46}$$

On the other hand, $-m = |m| > 0$. Therefore the function $(\xi_n + i\langle\xi'\rangle_a^{a_n})^{-m}$ admits analytic with respect to ξ_n continuation to the lower half-plane satisfying the estimate of type (1.29).

Then, according to Theorem 1.9, $I_+^{-a_n\overline{m}} \chi_- I_+^{a_n\overline{m}} \ell g \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}^n_-}) \ \left(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_-})\right)$ (see (1.46)) and hence $\pi_+ I_+^{-a_n\overline{m}} \chi_- I_+^{a_n\overline{m}} \ell g = 0$.

Thus $\pi_+ w = g$ (see (1.45)), i.e. w is a solution of (1.42). It remains to notice that the kernel of the operator π_+ in the case under consideration is a set of distributions of the type $\sum\limits_{j=1}^{|m|} v_j(x') \times \delta^{(j-1)}(x_n)$ where $v_j \in B_{p,p}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1})$ $(B_{p,q}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1}))$, $j = 1, \ldots, |m|$, (see Theorem 1.13 and Lemma 1.17). ■

§ a

Let $r \in \mathbb{N}$, $\mu \in \mathbb{C}$, $b = (b_1, \ldots, b_{n-1})$, $b_k > 0$, $k = 1, \ldots, n-1$, a be the same vector as in §§1.1, 1.2. Then $O_{b,r}^{a,\mu}$ denotes an algebra of continuous on $\mathbb{R}^n \backslash \{0\}$ functions satisfying the following conditions:

i) $g(t^{a_1}\xi_1, \ldots, t^{a_n}\xi_n) = t^\mu g(\xi_1, \ldots, \xi_n)$, $\forall t > 0$, $\forall \xi \in \mathbb{R}^n \backslash \{0\}$, (1.47)

ii) for any of 2^{n-1} collections $e = (e_1, \ldots, e_{n-1})$ of numbers $e_k = \pm 1$

$$g_{b,e} \in C^r\left(((\overline{\mathbb{R}_+})^{n-1} \times \mathbb{R}) \backslash \{0\}\right), \tag{1.48}$$

where

$$g_{b,e}(t_1, \ldots, t_{n-1}, t_n) = g(e_1 t_1^{b_1}, \ldots, e_{n-1} t_{n-1}^{b_{n-1}}, t_n), \tag{1.49}$$
$$t_k \geq 0, \quad k = 1, \ldots, n-1, \ \ t_n \in \mathbb{R}.$$

Explain the definition by several examples. In case $a = (1, \ldots, 1)$, $b = (1, \ldots, 1)$ any positively homogeneous of order μ function whose restriction on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ belongs to $C^r(S^{n-1})$ is an element of the algebra $O_{b,r}^{a,\mu}$.

If the numbers a_k, $k = 1, \ldots, n$, are rational and $m \in \mathbb{N}$ is such that the set of multiindices $\alpha \in \mathbb{Z}_+^n = (\mathbb{Z}_+)^n$ satisfying the equality $\alpha a \equiv \sum\limits_{k=1}^n \alpha_k a_k = m$ is nonempty, then any polynomial $\mathcal{P}$ of the type $\mathcal{P}(\xi) = \sum\limits_{\alpha a = m} c_\alpha \xi_\alpha$, $c_\alpha \in \mathbb{C}$, belongs to the algebra $O_{b,r}^{a,m}$, $\forall r \in \mathbb{N}$, where $b = (1, \ldots, 1)$.

The functions $\left(\xi_n \pm i|\xi'|_a^{a_n}\right)^{\mu/a_n}$, $\xi' = (\xi_1, \ldots, \xi_{n-1})$, (see (1.8)) belong to $O_{b,r}^{a,\mu}$ when $\min\{a_n, 2a_1\} \min\limits_{k=1,\ldots,n-1} \frac{b_k}{a_k} \geq r$.

We shall denote by $(O_{b,r}^{a,\mu})^{N \times N}$ the algebra of matrix functions of type $A = \|A_{jk}\|_{j,k=1}^N$, $A_{jk} \in O_{b,r}^{a,\mu}$.

The symbol $A \in (O_{b,r}^{a,\mu})^{N \times N}$ is said to be a-elliptic if $\det A(\xi) \neq 0$ for $\xi \neq 0$ (in scalar case $A(\xi) \neq 0$ for $\xi \neq 0$).

Using the results from [37, §6] and [37, Lemma 17.1], it is not difficult to prove (see also [92] and [31]) that for an a-elliptic symbol $A \in O^{a,\mu}_{b,[n/2]+2}$ the representation

$$A(\xi) = (\xi_n - i|\xi'|_a^{a_n})^{\mu/2a_n+\varkappa(\omega)+\delta} A_0^-(\xi) A_0^+(\xi) \times \\ \times (\xi_n + i|\xi'|_a^{a_n})^{\mu/2a_n-\varkappa(\omega)-\delta} \tag{1.50}$$

is valid, where

$$\delta = \frac{1}{2\pi i} \log \frac{A(0,\dots,0,+1)}{A(0,\dots,0,-1)},$$

$\varkappa(\omega) \in \mathbb{Z}, \omega = (\xi_1/|\xi'|_a^{a_1}, \dots, \xi_{n-1}/|\xi'|_a^{a_{n-1}})$, $\varkappa(\omega) = \frac{1}{2\pi}\Delta \arg \big(A(\xi',\xi_n)(\xi_n^2 + |\xi'|_a^{2a_n})^{-\mu/2a_n} \big(\frac{\xi_n - i|\xi'|_a^{a_n}}{\xi_n + i|\xi'|_a^{a_n}}\big)^{-\delta}\big)\big|_{\xi_n=-\infty}^{+\infty}$, $\varkappa(\omega)$ depends continuously on $\omega \in S_a^{n-2}$,

$$S_a^{n-2} = \{\xi' \in \mathbb{R}^{n-1} \mid |\xi'|_a = 1\}; \tag{1.51}$$

$(A_0^-)^{\pm 1}$ (respectively $(A_0^+)^{\pm 1}$) is an a-homogeneous of order 0 function (i.e. for it the equality of type (1.47) with $\mu = 0$ is fulfilled) satisfying the conditions of Theorem 1.4 and admitting bounded analytic with respect to ξ_n continuation to the lower (respectively, upper) complex half-plane.

Let $A \in (O^{a,\mu}_{b,[n/2]+3})^{N\times N}$ be an a-elliptic symbol; let $\lambda_1, \dots, \lambda_l$ be eigenvalues of the matrix $A^{-1}(0,\dots,0,-1)A(0,\dots,0,+1)$ to which there correspond Jordan blocks of dimensions $m_1, \dots, m_l$, respectively, $\sum\limits_{j=1}^{l} m_j = N$.

Consider the matrices

$$B^m(z) = \|B_{\nu k}(z)\|_{\nu,k=1}^m, \quad B_{\nu k}(z) = \begin{cases} 0, & \nu < k, \\ 1, & \nu = k, \\ \frac{z^{\nu-k}}{(\nu-k)!}, & \nu > k. \end{cases} \tag{1.52}$$

They possess the following properties: $B^m(z_1 + z_2) = B^m(z_1)B^m(z_2)$, $B^m(0) = I$ and hence $B^m(-z_1) = \big(B^m(z_1)\big)^{-1}$.

Introduce the notation

$$B_\pm(t) = \operatorname{diag}\Big[B^{m_1}\Big(\frac{1}{2\pi i}\log(t\pm i)\Big), \dots, B^{m_l}\Big(\frac{1}{2\pi i}\log(t\pm i)\Big)\Big], \tag{1.53}$$

$\delta'_j = \frac{\log \lambda_j}{2\pi i}$ (the branch of the logarithm is chosen arbitrarily), $\delta_k = \delta'_j$ for $\sum\limits_{\nu=1}^{j-1} m_\nu < k \le \sum\limits_{\nu=1}^{j} m_\nu$, $k = 1, \dots, N$.

Consider the matrix function $A_0(\xi) = (\xi_n^2 + |\xi'|_a^{2a_n})^{-\mu/2a_n} A(\xi)$, which is a-homogeneous of zero order. In exactly similar way as in [92], [31] we can prove that for the symbol $A_{0\omega}(\xi) = A_0(\omega, \xi_n/|\xi'|_a^{a_n})$ ($\omega \in S_a^{n-2}$ is fixed) we have the factorization:

$$A_{0\omega}(\xi)=c\big(A_{0\omega}^{-}(\xi)\big)^{-1}\mathrm{diag}\left[\Big(\frac{\xi_n-i|\xi'|_a^{a_n}}{\xi_n+i|\xi'|_a^{a_n}}\Big)^{\varkappa_k(\omega)+\delta_k}\right]_{k=1}^{N} A_{0\omega}^{+}(\xi), \quad (1.54)$$

where $A_{0\omega}^{\pm}(\xi) = A_0^{\pm}(\omega, \xi_n/|\xi'|_a^{a_n})$, $A_0^{\pm}(\omega,t) = A_1^{\pm}(\omega,t)B_{\pm}^{-1}(t)g^{-1}$, $(A_1^{-}(\omega,t))^{\pm 1}$ (respectively, $(A_1^{+}(\omega,t))^{\pm 1}$) admits bounded analytic with respect to t continuation into the lower (upper) complex half-plane and the elements of the matrix $A_1^{\pm}(\omega,t) - I$ satisfy the inequalities

$$\begin{gathered}\big|\partial_t^m (A_1^{\pm}(\omega,t)-I)_{jk}\big| \le const(1+|t|)^{-\sigma-m}, \\ m=0,1,\ldots,[n/2]+1, \quad \sigma>0;\end{gathered} \quad (1.55)$$

c, g are constant non-degenerate matrices; $\varkappa_1(\omega) \geq \cdots \geq \varkappa_N(\omega)$, $\varkappa_k(\omega) \in \mathbb{Z}$, the integer $\varkappa(\omega) = \sum_{k=1}^{N} \varkappa_k(\omega)$ depends continuously on ω while partial sums $\sum_{k=1}^{r} \varkappa_k(\omega)$, $1 \le r < N$, are upper semicontinuous, i.e. they do not increase for small variations of ω.

Transform the symbols $A_0^{\pm}(\omega,t)$:

$$\begin{aligned} A_0^{\pm}(\omega,t) &= \big((A_1^{\pm}(\omega,t)-I)+I\big)B_{\pm}^{-1}(t)g^{-1} = \\ &= B_{\pm}^{-1}\big[B_{\pm}(t)\big(A_1^{\pm}(\omega,t)-I\big)B_{\pm}^{-1}(t)+I\big]g^{-1} \equiv \\ &\equiv B_{\pm}^{-1}(t)A_2^{\pm}(\omega,t)g^{-1}. \end{aligned} \quad (1.56)$$

From (1.53) and (1.55) it easily follows that matrices $A_2^{\pm}(\omega,t)$ possess the same properties as $A_1^{\pm}(\omega,t)$ with the only difference that in (1.55) one should replace σ by an arbitrary $\sigma' \in (0,\sigma)$.

Using properties of block-diagonal matrices, we obtain from (1.54) and (1.56) that

$$\begin{gathered} A_0(\omega,t) = cg\big(A_2^{-}(\omega,t)\big)^{-1}B_{-}(t)\,\mathrm{diag}\left[\Big(\frac{t-i}{t+i}\Big)^{\varkappa_k(\omega)+\delta_k}\right]_{k=1}^{N} \times \\ \times B_{+}^{-1}(t)A_2^{+}(\omega,t)g^{-1} = cg\big(A_2^{-}(\omega,t)\big)^{-1}\,\mathrm{diag}\left[\Big(\frac{t-i}{t+i}\Big)^{\varkappa_k(\omega)+\delta_k}\right]_{k=1}^{N} \times \\ \times\,\mathrm{diag}\Big[B^{m_1}\Big(\frac{1}{2\pi i}\log\frac{t-i}{t+i}\Big),\ldots,B^{m_l}\Big(\frac{1}{2\pi i}\log\frac{t-i}{t+i}\Big)\Big]A_2^{+}(\omega,t)g^{-1} \equiv \\ \equiv cg\big(A_2^{-}(\omega,t)\big)^{-1}d(\omega,t)A_2^{+}(\omega,t)g^{-1}, \end{gathered}$$

where $d(\omega,t)$ is a lower triangular matrix with elements $(\frac{t-i}{t+i})^{\varkappa_k(\omega)+\delta_k}$ lying on its diagonal.

Summarizing the above-said, we come to the following statement

Let $A \in (O_{b,[n/2]+3}^{a,\mu})^{N\times N}$ be an a-elliptic symbol. Then $A_\omega(\xi) = A(|\xi'|_a^{a_1}\omega_1,\ldots,|\xi'|_a^{a_{n-1}}\omega_{n-1},\xi_n)$ admits the factorization

$$A_\omega(\xi)=(\xi_n-i|\xi'|_a^{a_n})^{\mu/2a_n}A_\omega^{-}(\xi)\mathcal{D}(\omega,\xi)A_\omega^{+}(\xi)(\xi_n+i|\xi'|_a^{a_n})^{\mu/2a_n}, \quad (1.57)$$

where $(A_\omega^-(\xi))^{\pm 1}$ $((A_\omega^+(\xi))^{\pm 1})$ *is an* a*-homogeneous of zero order matrix function* (*i.e. for its components the equality of type* (1.47) *with* $\mu = 0$ *is fulfilled*) *satisfying the conditions of Theorem* 1.4 *and admitting bounded analytic with respect to* ξ_n *continuation into the lower* (*upper*) *complex half-plane;* $\mathcal{D}(\omega, \xi)$ *is a lower triangular matrix with elements*

$$\left(\frac{\xi_n - i|\xi'|_a^{a_n}}{\xi_n + i|\xi'|_a^{a_n}}\right)^{\varkappa_k(\omega)+\delta_k}$$

lying on its diagonal and with a*-homogeneous of zero order functions lying under it and satisfying the conditions of Theorem* 1.4*;* $\varkappa_1(\omega) \geq \cdots \geq \varkappa_N(\omega)$, $\varkappa_k(\omega) \in \mathbb{Z}$, *the integer*

$$\varkappa(\omega) = \sum_{k=1}^{N} \varkappa_k(\omega) =$$

$$= \frac{1}{2\pi} \Delta \arg\det\left[(\xi_n^2 + |\xi'|_a^{2a_n})^{-\mu/2a_n} A_\omega(\xi', \xi_n)\right]\Big|_{\xi_n=-\infty}^{+\infty} - \sum_{k=1}^{N} \operatorname{Re}\delta_k$$

depends continuously on $\omega \in S_a^{n-2}$ *while partial sums* $\sum_{k=1}^{r} \varkappa_k(\omega)$, $1 \leq r < N$, *are upper semicontinuous;*

$$\delta_k = \frac{\log \lambda_j}{2\pi i} \quad \textit{for} \quad \sum_{\nu=1}^{j-1} m_\nu < k \leq \sum_{\nu=1}^{j} m_\nu, \quad k = 1, \ldots, N,$$

λ_j *are eigenvalues of the matrix* $A^{-1}(0, \ldots, 0, -1)A(0, \ldots, 0, +1)$ *to which there correspond Jordan blocks of dimension* m_j.

Both in matrix and scalar cases $\varkappa(\omega)$ depends continuously on $\omega \in S_a^{n-2}$ (see (1.51)) and takes integer values. For $n \geq 3$ an "a-sphere" S_a^{n-2} is connected. Hence $\varkappa(\omega) = \varkappa = const$.

Throughout this chapter we shall additionally assume that

$$\varkappa(-1) = \varkappa(+1) = \varkappa = const \tag{1.58}$$

for $n = 2$ (when $S_a^{n-2} = S_a^0 = \{\pm 1\}$).

The case when (1.58) is not fulfilled will be considered in Chapter III.

§ a

The most part of this section is devoted to the investigation of the boundary value problem for an a-elliptic system of pseudodifferential equations. Moreover, unless otherwise stated, we shall assume that

$$a_1 = a_2 = \cdots = a_{n-1}. \tag{1.59}$$

In this case $|\xi'|_a = |\xi|^{1/a_1}$ (see (1.8)) and $B^{\overline{\sigma}}_{p,q}(\mathbb{R}^{n-1}) = B^{\sigma/a_1}_{p,q}(\mathbb{R}^{n-1})$ (see (1.2)). Nevertheless we shall use anisotropic notation to make them applicable to the case of one equation when (1.59) is not required to be fulfilled (see Remark 1.26). When studying a system of equations we can do without restriction (1.59) only if $p = 2$ (see Remark 1.27).

In this chapter $\widehat{G}$ will denote the following:

$$\widehat{G}(\xi) = G\Big(\langle\xi'\rangle_a^{a_1}\frac{\xi_1}{|\xi'|_a^{a_1}}, \dots, \langle\xi'\rangle_a^{a_{n-1}}\frac{\xi_{n-1}}{|\xi'|_a^{a_{n-1}}}, \xi_n\Big). \tag{1.60}$$

Let $A \in (O^{a,\mu}_{b,[n/2]+3})^{N\times N}$ be an a-elliptic symbol, $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$.

Consider the boundary value problem (see (1.28))

$$\pi_+\widehat{A}(D)u_+ + \sum_{k=1}^{m_-}\pi_+\widehat{C}_k(D)\big(w_k(x')\times\delta(x_n)\big) = f(x), \tag{1.61}$$

$$\pi_0\widehat{B}_j(D)u_+ + \sum_{k=1}^{m_-}\widehat{E}_{jk}(D')w_k(x') = g_j(x'), \quad 1 \le j \le m_+, \tag{1.62}$$

where B_j, C_k are N-dimensional vector functions and E_{jk} are scalar functions satisfying the following conditions:

$$B_j(\xi) = |\xi'|_a^{\beta_j-\beta_{1j}}\big(\xi_n^2 + |\xi'|_a^{2a_n}\big)^{\beta_{1j}/2a_n}B_{0j}(\xi), \tag{1.63}$$

$$\operatorname{Re}\beta_{1j} < s - \frac{a_n}{p}, \quad 1 \le j \le m_+, \tag{1.64}$$

$$C_k(\xi) = |\xi'|_a^{\gamma_k-\gamma_{1k}}\big(\xi_n^2 + |\xi'|_a^{2a_n}\big)^{\gamma_{1k}/2a_n}C_{0k}(\xi), \tag{1.65}$$

$$\operatorname{Re}\gamma_{1k} < -s + \operatorname{Re}\mu - a_n\big(1 - \frac{1}{p}\big), \quad 1 \le k \le m_-, \tag{1.66}$$

$$E_{jk}(\xi') = |\xi'|_a^{\rho_{jk}}E_{0jk}(\xi'), \tag{1.67}$$

$$\rho_{jk} = \beta_j + \gamma_k - \mu + 1, \quad 1 \le j \le m_+, \ \ 1 \le k \le m_-, \tag{1.68}$$

B_{0j}, C_{0k}, E_{0jk} are a-homogeneous of zero order (vector) functions such that the components of vector functions $\widehat{B}_{0j}$, $\widehat{C}_{0k}$ satisfy the conditions of Theorem 1.4, while the functions $\widehat{E}_{0jk}$ satisfy the conditions obtained from those of Theorem 1.4 by substituting n and ξ by $n-1$ and ξ', respectively;

$$f \in H^{\overline{r}}_p(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \ \ \big(B^{\overline{r}}_{p,q}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N)\big), \ \ r = s - \operatorname{Re}\mu, \tag{1.69}$$

$$g_j \in B^{\overline{r^{(j)}}}_{p,p}(\mathbb{R}^{n-1}) \ \ \big(B^{\overline{r^{(j)}}}_{p,q}(\mathbb{R}^{n-1})\big), \ \ r^{(j)} = s - \operatorname{Re}\beta_j - a_n/p, \tag{1.70}$$

are given functions;

$$\begin{gathered} u_+ \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \ \big(\widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_+}), \mathbb{C}^N)\big), \\ w_k \in B_{p,p}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \ \big(B_{p,q}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1})\big), \\ s^{(k)} = s - \operatorname{Re}\mu + \operatorname{Re}\gamma_k + a_n\big(1 - \frac{1}{p}\big) \end{gathered} \tag{1.71}$$

are the unknown functions;

$\pi_0 = \pi_0^0$ is an operator of restriction to $\mathbb{R}^{n-1}$ (see (1.21)).

The left-hand sides of equations (1.61) and (1.62) define the continuous operator

$$\begin{gathered} U = \begin{pmatrix} \pi_+\widehat{A}(D) & \pi_+\widehat{C}(D)(\cdot \times \delta(x_n)) \\ \pi_0\widehat{B}(D) & \widehat{E}(D') \end{pmatrix} : H_1(s,p) = \\ = \begin{matrix} \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \\ \oplus \\ \overset{m_-}{\underset{k=1}{\oplus}} B_{p,p}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \end{matrix} \longrightarrow H_2(s,p) = \begin{matrix} H_p^{\overline{r}}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \\ \oplus \\ \overset{m_+}{\underset{j=1}{\oplus}} B_{p,p}^{\overline{r^{(j)}}}(\mathbb{R}^{n-1}) \end{matrix}, \\ \big(U \ : \ B_1(s,p,q) = \widetilde{B}_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \oplus \overset{m_-}{\underset{k=1}{\oplus}} B_{p,q}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \to \\ \to B_2(s,p,q) = B_{p,q}^{\overline{r}}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \oplus \overset{m_+}{\underset{j=1}{\oplus}} B_{p,q}^{\overline{r^{(j)}}}(\mathbb{R}^{n-1})\big) \end{gathered} \tag{1.72}$$

(see (1.69)–(1.71)), where

$$\begin{aligned} &\widehat{B}(D) = \big(\widehat{B}_j(D)\big)_{j=1}^{m_+} \text{ – is a matrix } m_+ \times N \ \Psi\text{DO}, \\ &\widehat{C}(D) = \big(\widehat{C}_k(D)\big)_{k=1}^{m_-} \text{ – is a matrix } N \times m_- \ \Psi\text{DO}, \\ &\widehat{E}(D') = \big(\widehat{E}_{jk}(D')\big)_{\substack{j=1,\dots,m_+ \\ k=1,\dots,m_-}} \text{ – is a matrix } m_+ \times m_- \ \Psi\text{DO}. \end{aligned}$$

The proof of this fact goes in a standard way (see Theorems 1.3 and 1.4). Note only that conditions (1.64) allow us to use Theorem 1.5, while Theorem 1.13 is used in the case of conditions (1.66).

Fix an arbitrary $\omega \in S_a^{n-2}$.(Note that by virtue of (1.59) S_a^{n-2} (see (1.8) and (1.51)) coincides with ordinary unit sphere $S^{n-2} = \{\xi' \in \mathbb{R}^{n-1} \,\big|\, |\xi'| = 1\}$. Nevertheless by the above mentioned arguments we prefer anisotropic notation). Introduce the notation (see Lemma 1.19)

$$A_\omega(\xi) = A(|\xi'|_a^{a_1}\omega_1, \dots, |\xi'|_a^{a_{n-1}}\omega_{n-1}, \xi_n). \tag{1.73}$$

The notations $B_{\omega j}(\xi)$, $C_{\omega k}(\xi)$, $E_{\omega jk}(\xi')$ are treated analogously. The operator corresponding to these symbols we denote by

$$U_\omega : H_1(s,p) \to H_2(s,p) \ \big(B_1(s,p,q) \to B_2(s,p,q)\big). \tag{1.74}$$

The operator U is invertible if and only if the operators U_ω, $\forall\omega \in S_a^{n-2}$, are invertible.

Proof. Let $\sigma : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, $|\sigma\xi'| = |\xi'|$, be a rotation of the space $\mathbb{R}^{n-1}$ about origin and $\sigma_*\varphi(\xi') = \varphi(\sigma\xi')$.

Consider the function

$$\chi^1_+(\xi) = \chi^1_+(\xi') = \chi_+(\xi_1) = \frac{1}{2}(1 + \operatorname{sgn}\xi_1). \tag{1.75}$$

It is well known (see, e.g., [37, Lemma 5.2]) that (see (1.28))

$$\big(\chi^1_+(D)\varphi\big)(x_1,\dots,x_n) = \frac{1}{2}\varphi(x_1,\dots,x_n) - \\ -\frac{1}{2\pi i}\int_{\mathbb{R}} \frac{\varphi(t_1,x_2,\dots,x_n)}{t_1 - x_1}\,dt_1, \quad \forall\varphi \in S(\mathbb{R}^n).$$

Therefore χ^1_+ is bounded in $L_p(\mathbb{R}^n)$ (see, e.g., [55, Ch. VI, point D]) and hence in the spaces $H^{\overline{s}}_p(\mathbb{R}^n)$, $B^{\overline{s}}_{p,q}(\mathbb{R}^n)$ (see the last phrase in the proof of Theorem 1.4). Exactly in the same way $\chi^1_+(D')$ is also bounded in the spaces $H^{\overline{s}}_p(\mathbb{R}^{n-1})$, $B^{\overline{s}}_{p,q}(\mathbb{R}^{n-1})$. Moreover,

$$\|(\sigma_*\chi^1_+)(D)|L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| = \|\sigma_*(\chi^1_+(D))\sigma'_*|L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| = \\ = \|\chi^1_+(D)|L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| \tag{1.76}$$

since σ_* and σ'_* (where σ' is a matrix conjugate to σ) are isometric isomorphisms in $L_p(\mathbb{R}^n)$. Therefore the norm of the operator $(\sigma_*\chi^1_+)(D)$ in $H^{\overline{s}}_p(\mathbb{R}^n)$ and $B^{\overline{s}}_{p,q}(\mathbb{R}^n)$ is majorized by the value independent of the rotation σ. Similar arguments are valid for the operator $(\sigma_*\chi^1_+)(D')$.

Denote by Σ the set of all functions of the type

$$\chi(\xi) = \chi(\xi') = \prod_{k=1}^{n-1}(\sigma^{(k)}_*\chi^1_+)(\xi'), \tag{1.77}$$

where $\sigma^{(1)},\dots,\sigma^{(n-1)}$ are certain rotations.

It follows from the above-said and (1.77) that norms of ΨDOs with symbols from Σ in $H^{\overline{s}}_p$ and $B^{\overline{s}}_{p,q}$ are uniformly bounded:

$$\|\chi(D)\| \le C < +\infty, \quad \|\chi(D')\| \le C' < +\infty, \quad \forall\chi \in \Sigma. \tag{1.78}$$

Let

$$\bigcup_{k=1}^{m} \operatorname{supp}\chi^{(k)} \bigcap S^{n-2}_a = S^{n-2}_a, \quad g = \sum_{k=1}^{m}\chi^{(k)}, \quad \chi^{(k)} \in \Sigma.$$

Then the operator $g(D)$ $(g(D'))$ is invertible in spaces $H^{\overline{s}}_p$, $B^{\overline{s}}_{p,q}$. Indeed, the function g^{-1} can be represented as a linear combination of products of functions from Σ, and $(g^{-1})(D)$ $((g^{-1})(D'))$ will be the inverse operator.

Consider the operators (see Theorem 1.10)

$$\mathcal{K}^1_\chi = \chi(D) \oplus \chi(D') : H_1(s,p) \to H_1(s,p) \ \big(B_1(s,p,q) \to B_1(s,p,q)\big),$$
$$\mathcal{K}^2_\chi = \pi_+\chi(D)\ell \oplus \chi(D') : H_2(s,p) \to H_2(s,p) \ \big(B_2(s,p,q) \to B_2(s,p,q)\big),$$

where $\chi \in \Sigma$. We can easily see that

$$\mathcal{K}_\chi^2 U = \begin{pmatrix} \pi_+\chi(D)\widehat{A}(D) & \pi_+\chi(D)\widehat{C}(D)(\cdot\ \times \delta(x_n)) \\ \pi_0\chi(D)\widehat{B}(D) & \chi(D')\widehat{E}(D') \end{pmatrix} = U\mathcal{K}_\chi^1.$$

Choosing $\sigma^{(1)}, \dots, \sigma^{(n-1)}$ properly, we can make $\chi|_{S_a^{n-2}}$ (see (1.77)) to be the characteristic function of arbitrarily small neighbourhood of any point $\omega \in S_a^{n-2} = S^{n-2}$. Thus the operator

$$\mathcal{K}_\chi^2(U - U_\omega) : H_1(s,p) \to H_2(s,p) \ \big(B_1(s,p,q) \to B_2(s,p,q)\big)$$

can be achieved to be arbitrarily small in norm. Really, ΨDOs contained in this operator have small norms in the appropriate pairs of spaces $H_2^{\overline{\sigma}}$. Moreover, they are uniformly bounded in the appropriate pairs of spaces $H_p^{\overline{\sigma}}$, $B_{p,q}^{\overline{\sigma}}$ (see (1,78)). Therefore to prove the statement it suffices to use the interpolation (see Theorem 1.2).

Let the operators U_ω be invertible $\forall \omega \in S_a^{n-2}$. Take $\chi_{_\omega} \in \Sigma$ such that $\|\mathcal{K}_{\chi_\omega}^2(U - U_\omega\| < \|U_\omega^{-1}\|^{-1}$. Then the operator $U_\omega + \mathcal{K}_{\chi_\omega}^2(U - U_\omega)$ has an inverse $\mathcal{R}_\omega$.

Choose from the family $\{\chi_{_\omega}\}_{\omega \in S_a^{n-2}}$ a finite subfamily $\{\chi_{_k}\}_{k=1,\dots,m}$ such that $\bigcup\limits_{k=1}^{m} \operatorname{supp} \chi_{_k} \cap S_a^{n-2} = S_a^{n-2}$.

Denote $g = \sum_{k=1}^m \chi_{_k}$. Consider the operator

$$\mathcal{R} = \mathcal{K}_{g^{-1}}^1 \sum_{k=1}^{m} \mathcal{K}_{\chi_k}^1 \mathcal{R}_k \mathcal{K}_{\chi_k}^2. \tag{1.79}$$

Note that $\chi^2 = \chi$, $\forall \chi \in \Sigma$. Hence

$$\begin{aligned} &(\mathcal{K}_\chi^1)^2 = \mathcal{K}_\chi^1, \quad (\mathcal{K}_\chi^2)^2 = \mathcal{K}_\chi^2; \\ \mathcal{R}U &= \mathcal{K}_{g^{-1}}^1 \sum_{k=1}^{m} \mathcal{K}_{\chi_k}^1 \mathcal{R}_k \mathcal{K}_{\chi_k}^2 U = \\ &= \mathcal{K}_{g^{-1}}^1 \sum_{k=1}^{m} \mathcal{K}_{\chi_k}^1 \mathcal{R}_k \mathcal{K}_{\chi_k}^2 (U_k + \mathcal{K}_{\chi_k}^2(U - U_k)) = \\ &= \mathcal{K}_{g^{-1}}^1 \sum_{k=1}^{m} \mathcal{K}_{\chi_k}^1 \mathcal{R}_k (U_k + \mathcal{K}_{\chi_k}^2(U - U_k)) \mathcal{K}_{\chi_k}^1 = \\ &= \mathcal{K}_{g^{-1}}^1 \sum_{k=1}^{m} \mathcal{K}_{\chi_k}^1 I_{H_1(s,p)} \mathcal{K}_{\chi_k}^1 = \mathcal{K}_{g^{-1}}^1 \sum_{k=1}^{m} \mathcal{K}_{\chi_k}^1 = \\ &= \mathcal{K}_{g^{-1}}^1 \mathcal{K}_g^1 = I_{H_1(s,p)}, \quad \big(\cdots = I_{B_1(s,p,q)}\big). \end{aligned}$$

Analogously we obtain $U\mathcal{R} = I_{H_2(s,p)}$ $(I_{B_2(s,p,q)})$.

The sufficiency is proved. Let us prove the necessity.

Let U be invertible. Take $\forall\omega \in S_a^{n-2} = S^{n-2}$ and choose $\chi_\omega \in \Sigma$ such that $\|\mathcal{K}^2_{\chi_\omega}(U - U_\omega)\| < \|U^{-1}\|^{-1}$. Then the operator $U + \mathcal{K}^2_{\chi_\omega}(U_\omega - U)$ has its inverse $\mathcal{R}^\omega$.

$$\mathcal{K}^2_{\chi_\omega} U_\omega \mathcal{R}^\omega = \mathcal{K}^2_{\chi_\omega}\big(U + \mathcal{K}^2_{\chi_\omega}(U_\omega - U)\big)\mathcal{R}^\omega = \mathcal{K}^2_\omega. \tag{1.80}$$

Similarly

$$\mathcal{R}^\omega U_\omega \mathcal{K}^1_{\chi_\omega} = \mathcal{K}^1_{\chi_\omega}. \tag{1.81}$$

Obviously $\sigma_* U_\omega = U_\omega \sigma_*$ for any rotation of space $\mathbb{R}^{n-1}$ (see (1.73) and (1.8)).

Apply rotations to equalities (1.80) and (1.81). We conclude that there exist bounded operators $\mathcal{R}_1, \dots, \mathcal{R}_m$ and functions $\chi_1, \dots, \chi_m \in \Sigma$ such that

$$\mathcal{K}^2_{\chi_k} U_\omega \mathcal{R}_k = \mathcal{K}^2_{\chi_k}, \quad \mathcal{R}_k U_\omega \mathcal{K}^1_{\chi_k} = \mathcal{K}^1_{\chi_k}, \quad k = 1, \dots, m,$$
$$g = \sum_{k=1}^m \chi_k \ \text{ is an invertible symbol.}$$

Consider the operator $\mathcal{R}(\omega) = \mathcal{K}^1_{g^{-1}} \sum\limits_{k=1}^m \mathcal{K}^1_{\chi_k} \mathcal{R}_k \mathcal{K}^2_{\chi_k}$. As above we can prove that

$$\mathcal{R}(\omega) U_\omega = I_{H_1(s,p)} \ \big(I_{B_1(s,p,q)}\big), \quad U_\omega \mathcal{R}(\omega) = I_{H_2(s,p)} \ \big(I_{B_2(s,p,q)}\big),$$
$$\forall\omega \in S_a^{n-2} = S^{n-2}. \qquad \blacksquare$$

Consider now the boundary value problem

$$\pi_+ \widehat{A}_\omega(D) u_+ + \pi_+ \widehat{C}_\omega(D)\big(w(x') \times \delta(x_n)\big) = f(x), \tag{1.82}$$
$$\pi_0 \widehat{B}_\omega(D) u_+ + \widehat{E}_\omega(D') w(x') = g(x') \tag{1.83}$$

corresponding to the operator U_ω, where $w = (w_1, \dots, w_{m_-})$, $g = (g_1, \dots, g_{m_+})$.

Apply the operator

$$\pi_+ \widehat{\Lambda}^-_\omega(D)\ell \equiv \pi_+ (\widehat{A}^-_\omega)^{-1} \operatorname{diag}\left[I_-^{\overline{\mu_-^{(k)}(\omega)}} \right]_{k=1}^N \ell, \tag{1.84}$$
$$\mu^{(k)}_\pm(\omega) = \pm\mu/2 - \big(\varkappa_k(\omega) + \delta_k\big) a_n \tag{1.85}$$

(see (1.31), Theorem 1.10 and Lemma 1.19) to (1.82). We obtain (see Remark 1.11)

$$\pi_+ \widehat{G}_\omega(D) v_+ = f_0 - \pi_+ \widehat{Q}_\omega(D)\big(w(x') \times \delta(x_n)\big), \tag{1.86}$$

where (see (1.84), (1.85))

$$v_+ = \widehat{\Lambda}^+_\omega(D)u_+ \equiv \operatorname{diag}\left[I_+^{\overline{\mu_+^{(k)}(\omega)}}\right]_{k=1}^N \widehat{A}^+_\omega(D)u_+, \tag{1.87}$$

$$f_0 = \pi_+\widehat{\Lambda}^-_\omega(D)\ell f, \tag{1.88}$$

$$\widehat{Q}_\omega(D) = \widehat{\Lambda}^-_\omega(D)\widehat{C}_\omega(D), \tag{1.89}$$

$$G_\omega = \|G_{\omega jk}\|_{N\times N}, \quad G_{\omega jk}(\xi) = \begin{cases} 0 & \text{for } j<k, \\ 1 & \text{for } j=k, \\ |\xi|^{\varkappa_{jk}(\omega)} G^0_{\omega jk}(\xi) & \text{for } j>k, \end{cases} \tag{1.90}$$

$$\varkappa_{jk}(\omega) = \varkappa_k(\omega) + \delta_k - \big(\varkappa_j(\omega) + \delta_j\big), \tag{1.91}$$

$G^0_{\omega jk}$ are a-homogeneous of zero order functions satisfying the conditions of Theorem 1.4.

By means of Theorems 1.3, 1.4, 1.9 and 1.10 we obtain that

$$v_+ \in \prod_{k=1}^N \widetilde{H}_p^{\overline{s}-\overline{\operatorname{Re}\mu_+^{(k)}(\omega)}}(\overline{\mathbb{R}^n_+}) \quad \Big(\prod_{k=1}^N \widetilde{B}_{p,q}^{\overline{s}-\overline{\operatorname{Re}\mu_+^{(k)}(\omega)}}(\overline{\mathbb{R}^n_+})\Big), \tag{1.92}$$

$$f_0 \in \prod_{j=1}^N H_p^{\overline{r}-\overline{\operatorname{Re}\mu_-^{(j)}(\omega)}}(\overline{\mathbb{R}^n_+}) \quad \Big(\prod_{j=1}^N B_{p,q}^{\overline{r}-\overline{\operatorname{Re}\mu_-^{(j)}(\omega)}}(\overline{\mathbb{R}^n_+})\Big), \tag{1.93}$$

where

$$s - \operatorname{Re}\mu_+^{(k)}(\omega) = r - \operatorname{Re}\mu_-^{(k)}(\omega) = s - \frac{1}{2}\operatorname{Re}\mu + \big(\varkappa_k(\omega) + \operatorname{Re}\delta_k\big)a_n \tag{1.94}$$

by virtue of (1.69) and (1.85).

Assume (see (1.2))

$$s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m - 1/p \notin \mathbb{Z}, \quad m = 1, \dots, N. \tag{1.95}$$

Then

$$\begin{gathered} s_n - \operatorname{Re}\mu/2a_n + \varkappa_m(\omega) + \operatorname{Re}\delta_m = d_m(\omega) + \nu_m, \\ d_m(\omega) \in \mathbb{Z}, \ \ \frac{1}{p} - 1 < \nu_m < \frac{1}{p}. \end{gathered} \tag{1.96}$$

Consequently $0 < \nu_m + 1 - 1/p < 1$,

$$\begin{aligned} \nu_m + 1 - \frac{1}{p} &= \big\{s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m + 1 - 1/p\big\} = \\ &= \{s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m - 1/p\}, \end{aligned} \tag{1.97}$$

that is

$$\nu_m = 1/p - 1 + \big\{s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m - 1/p\big\} \tag{1.98}$$

and ν_m does not depend on ω.

Obviously $\varkappa_1(\omega) \geq \cdots \geq \varkappa_N(\omega) \Rightarrow d_1(\omega) \geq \cdots \geq d_N(\omega)$.

Let $d_m(\omega) > 0$ for $m = 1, \ldots, m_1(\omega)$, $d_m(\omega) = 0$ for $m = m_1(\omega) + 1, \ldots, m_2(\omega)$ and $d_m(\omega) < 0$ for $m = m_2(\omega)+1, \ldots, N$. It may happen that $m_1(\omega) = 0$ or $m_1(\omega) = N$ or $m_1(\omega) = m_2(\omega)$, etc.

Keep in mind (1.90) and rewrite (1.86) in a scalar form,

$$\pi_+ v_{+m} + \pi_+ \sum_{l=1}^{m-1} \widehat{G}_{\omega ml}(D) v_{+l} =$$
$$= f_{0m} - \pi_+ \sum_{k=1}^{m_-} \widehat{Q}_{\omega mk}(D)(w_k \times \delta), \quad m = 1, \ldots, N. \tag{1.99}$$

We shall act as follows. Using Theorem 1.18, express v_{+1} from the first equation in (1.99) by f_{01} and w and substitute the result in the second equation. Apply again Theorem 1.18 and express v_{+2} by f_{02}, f_{01} and w. The obtained expressions for v_{+1} and v_{+2} we substitute in the third equation in (1.99), etc.

From point b) of Theorem 1.18 we find that the first $m_1(\omega)$ equations of (1.99) yield

$$M_+(\omega) = \sum_{m=1}^{m_1(\omega)} d_m(\omega)$$

equations with respect to $w = (w_1, \ldots, w_{m_-})$.

Apply point c) of Theorem 1.18 to obtain that from equations (1.99) for $m = m_2(\omega) + 1, \ldots, N$ there arise

$$M_-(\omega) = \sum_{m=m_2(\omega)+1}^{N} |d_m(\omega)|$$

new functions of the variable x' by which v_{+m}, $m = m_2(\omega) + 1, \ldots, N$, can be expressed. Denote these functions by $w_{m_-+1}, \ldots, w_{m_-+M_-(\omega)}$.

Introduce the notation

$$\begin{gathered} s^{(k)} = s - \frac{1}{2}\operatorname{Re}\mu + \bigl(\varkappa_m(\omega) + \operatorname{Re}\delta_m + t - \frac{1}{p}\bigr)a_n \\ \text{for } \ k = m_- + \sum_{e=m_2(\omega)+1}^{m-1} |d_e(\omega)| + t, \\ m = m_2(\omega) + 1, \ldots, N, \quad t = 1, \ldots, |d_m(\omega)|. \end{gathered} \tag{1.100}$$

By virtue of point c) of Theorem 1.18

$$w_k \in B_{p,p}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \ \bigl(B_{p,q}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1})\bigr), \ \ k = m_-+1, \ldots, m_- + M_-(\omega). \tag{1.101}$$

It follows from point a) of Theorem 1.18 that the group of equations (1.99) for $m = m_1(\omega) + 1, \dots, m_2(\omega)$ generates neither new functions nor new equations.

Thus from equations (1.99) we have expressed v_+ and hence u_+ (see (1.87)) by f and $w_0 = (w_1, \dots, w_{m_-}, w_{m_-+1}, \dots, w_{m_-+M_-(\omega)})$. Moreover, we have obtained $M_+(\omega)$ equations with respect to $w = (w_1, \dots, w_{m_-})$. Substitute the obtained expression for u_+ into the boundary condition (1.83) to obtain m_+ more equations with respect to w_0.

Introduce the notation

$$\begin{gathered} r^{(j)} = s - \frac{1}{2}\operatorname{Re}\mu + \big(\varkappa_m(\omega) + \operatorname{Re}\delta_m - t + 1 - \frac{1}{p}\big)a_n \\ \text{for } j = m_+ + \sum_{e=1}^{m-1} d_e(\omega) + t,\ m = 1, \dots, m_1(\omega),\ t = 1, \dots, d_m(\omega). \end{gathered} \tag{1.102}$$

The obtained system of equations with respect to w_0 is of the form

$$T_\omega w_0 = g_0, \tag{1.103}$$

where

$$w_0 \in \bigoplus_{k=1}^{m_-+M_-(\omega)} B_{p,p}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \ \Big(\bigoplus_{k=1}^{m_-+M_-(\omega)} B_{p,q}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1})\Big), \tag{1.104}$$

$$g_0 \in \bigoplus_{j=1}^{m_++M_+(\omega)} B_{p,p}^{\overline{r^{(j)}}}(\mathbb{R}^{n-1}) \ \Big(\bigoplus_{j=1}^{m_++M_+(\omega)} B_{p,q}^{\overline{r^{(j)}}}(\mathbb{R}^{n-1})\Big) \tag{1.105}$$

and T_ω is an operator bounded in the appropriate spaces.

It is not difficult to verify that T_ω is a translation invariant operator. Moreover, $\sigma_* T_\omega = T_\omega \sigma_*$ for any rotation σ of space $\mathbb{R}^{n-1}$. Therefore T_ω is a pseudodifferential operator:

$$T_\omega = Z_\omega(D') \tag{1.106}$$

(see [48] or the proof of [101, Ch.I, Theorem 3.16]), the matrix function Z_ω being dependent on $|\xi'|_a$ and not on $(\frac{\xi_1}{|\xi'|_a^{a_1}}, \dots, \frac{\xi_{n-1}}{|\xi'|_a^{a_{n-1}}}) \in S_a^{n-2}$. Analysing the deduction of system (1.103) convinces us that

$$Z_\omega = \|Z_{\omega jk}\|, \quad Z_{\omega jk}(\xi') = \langle \xi' \rangle_a^{s_c^{(k)} - r_c^{(j)}} Z_{jk}(\omega), \tag{1.107}$$

where $s_c^{(k)}$ and $r_c^{(j)}$ are the numbers obtained from formulas (1.71), (1.100) and, respectively, (1.70), (1.102) by omitting the sign "Re"; $Z_{jk}(\omega)$ are some constants (for $\omega \in S_a^{n-2}$ fixed), and moreover,

$$\begin{gathered} Z_{jk}(\omega) = 0 \ \text{ for } \ j = m_+ + 1, \dots, m_+ + M_+(\omega), \\ k = m_- + 1, \dots, m_- + M_-(\omega). \end{gathered} \tag{1.108}$$

Hence

$$\|Z_{\omega jk}(\xi')\| = \operatorname{diag}\big[\langle \xi' \rangle_a^{-r_c^{(j)}}\big]_{j=1}^{m_++M_+(\omega)} \|Z_{jk}(\omega)\| \operatorname{diag}\big[\langle \xi' \rangle_a^{s_c^{(k)}}\big]_{k=1}^{m_-+M_-(\omega)}.$$

Thus the unique solvability of the boundary value problem (1.82), (1.83) for any right-hand sides is equivalent to that of the system (1.103) for any right-hand sides which in turn is equivalent to the invertibility of the operator of multiplication by a constant matrix $\|Z_{jk}(\omega)\|$ (see (1.106), (1.107)) acting from the space $B^0_{p,p}(\mathbb{R}^{n-1}, \mathbb{C}^{m_-+M_-(\omega)})$ $(B^0_{p,q}(\mathbb{R}^{n-1}, \mathbb{C}^{m_-+M_-(\omega)}))$ into the space $B^0_{p,p}(\mathbb{R}^{n-1}, \mathbb{C}^{m_++M_+(\omega)})$ $(B^0_{p,q}(\mathbb{R}^{n-1}, \mathbb{C}^{m_++M_+(\omega)}))$ since $\operatorname{Re} s_c^{(k)} = s^{(k)}$, $\operatorname{Re} r_c^{(j)} = r^{(j)}$ (see Theorem 1.3). Hence the invertibility of the operator U_ω (see (1.74)) is equivalent to that of the matrix $\|Z_{jk}(\omega)\|$ $\big(j = 1, \dots, m_+ + M_+(\omega),\ k = 1, \dots, m_- + M_-(\omega)\big)$.

For the matrix to be invertible we need, first of all, it to be quadratic. Thus we arrive at the condition

$$m_- + M_-(\omega) = m_+ + M_+(\omega), \tag{1.109}$$

that is

$$m_- + \sum_{m=m_2(\omega)+1}^{N} |d_m(\omega)| = m_+ + \sum_{m=1}^{m_1(\omega)} d_m(\omega).$$

From (1.96), (1.97) and said in §1.3 we have

$$\begin{aligned}
&\sum_{m=1}^{m_1(\omega)} d_m(\omega) - \sum_{m=m_2(\omega)+1}^{N} |d_m(\omega)| = \sum_{m=1}^{N} d_m(\omega) = \\
&= \sum_{m=1}^{N} \varkappa_m(\omega) + \sum_{m=1}^{N} \big(s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m\big) - \sum_{m=1}^{N} \nu_m = \\
&= \varkappa(\omega) + \sum_{m=1}^{N} \big(s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m + 1 - 1/p\big) - \sum_{m=1}^{N} (\nu_m + 1 - 1/p) = \\
&= \varkappa + \sum_{m=1}^{N} \big(s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m + 1 - 1/p\big) - \\
&- \sum_{m=1}^{N} \big\{s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m + 1 - 1/p\big\} = \\
&= \varkappa + \sum_{m=1}^{N} \big[s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m + 1 - 1/p\big] = \\
&= \varkappa + N + \sum_{m=1}^{N} \big[s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m - 1/p\big].
\end{aligned}$$

Hence we can choose the integers m_+ and m_- not depending on ω such that (1.109) holds for any $\omega \in S_a^{n-2}$ (see (1.58)).

Thus for the matrix $\|Z_{jk}(\omega)\|$ to be invertible it is necessary that the equality

$$m_- - m_+ = \varkappa + N + \sum_{m=1}^{N} \left[s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m - 1/p \right] \quad (1.110)$$

be fulfilled.

Consider now the boundary value problem on a semi-axis

$$\pi_+ A(\omega, D_n) u_+(x_n) + \sum_{k=1}^{m_-} w_k \pi_+ C_k(\omega, D_n) \delta(x_n) = f(x_n), \quad (1.111)$$

$$\pi_0 B_j(\omega, D_n) u_+(x_n) + \sum_{k=1}^{m_-} E_{jk}(\omega) w_k = g_j, \quad j = 1, \dots, m_+, \quad (1.112)$$

where

$$f \in H_p^{(s-\operatorname{Re}\mu)/a_n}(\overline{\mathbb{R}^1_+}, \mathbb{C}^N) \ \left(B_{p,q}^{(s-\operatorname{Re}\mu)/a_n}(\overline{\mathbb{R}^1_+}, \mathbb{C}^N)\right),$$
$$u_+ \in \widetilde{H}_p^{s/a_n}(\overline{\mathbb{R}^1_+}, \mathbb{C}^N) \ \left(\widetilde{B}_{p,q}^{s/a_n}(\overline{\mathbb{R}^1_+}, \mathbb{C}^N)\right),$$

w_k, g_j are complex numbers, $A(\omega, D_n)$, $C_k(\omega, D_n)$, $B_j(\omega, D_n)$ are the ΨDOs with respect to x_n depending on $\omega \in S_a^{n-2}$ with the symbols $A(\omega, \xi_n)$, $C_k(\omega, \xi_n)$, $B_j(\omega, \xi_n)$, respectively.

Repeate almost word for word the investigation of boundary value problem (1.82), (1.83) and take into account the form of factors in Lemma 1.19 (see [92], [31]) to see that the unique solvability of the system (1.111), (1.112) for any right-hand sides is equivalent to the invertibility of the matrix $\|Z_{jk}(\omega)\|$ when (1.95) is fulfilled.

Let (1.95) *be fulfilled. Then the following statements are equivalent:*

a) *the operator* U_ω *is invertible;*

b) *boundary value problem* (1.111), (1.112) *is uniquely solvable for any right-hand sides;*

c) *the matrix* $\|Z_{jk}(\omega)\|$ *is invertible.*

For a)–c) *to be fulfilled it is necessary that the equality* (1.110) *hold.*

Assume that the condition (1.95) *is not fulfilled. Then the operator* U *is not invertible.*

Proof. Assume the contrary: the operator U is invertible. Denote by $U_{\pm\varepsilon}$ the operator obtained from U by substitution ΨDO $\widehat{A}(D)$ by ΨDO with the symbol

$$\left(\frac{\xi_n - i\langle \xi' \rangle_a^{a_n}}{\xi_n + i\langle \xi' \rangle_a^{a_n}} \right)^{\pm\varepsilon} \widehat{A}(\xi), \quad \varepsilon > 0.$$

In a standard way (as in proving Lemma 1.20) we can easily ascertain that the operators U and $U_{\pm\varepsilon}$ may be made arbitrarily close in norm by reducing

ε. Let us take sufficiently small $\varepsilon > 0$, such that $U_{\pm\varepsilon}$ are invertible and for them the conditions of the form (1.95) are fulfilled.

Denote by l the number of values of the index m for which the condition (1.95) is violated for the operator U. Apply Lemmas 1.20–1.21 to operators $U_{\pm\varepsilon}$ and write for them the equalities of type (1.110). We obtain

$$m_- - m_+ = \varkappa + N + \sum_{m=1}^{N} \big[s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m - 1/p\big],$$

$$m_- - m_+ = \varkappa + N + \sum_{m=1}^{N} \big[s_n - \operatorname{Re}\mu/2a_n + \operatorname{Re}\delta_m - 1/p\big] - l.$$

The obtained contradiction proves the lemma. ■

The necessity of the condition (1.95) for the operator U_ω to be invertible and for system (1.111), (1.112) to be uniquely solvable for any right-hand sides can be proved similarly.

Analogously to Lemma 1.20 one can prove that the Noetherity of the operator U is equivalent to that of the operators U_ω, $\forall\omega \in S_a^{n-2}$. If the condition (1.95) is fulfilled, we can easily see that the operator U_ω has infinite dimensional kernel or cokernel when the matrix $\|Z_{jk}(\omega)\|$ is non-invertible. Hence when the condition (1.95) is fulfilled the Noetherity of U_ω is equivalent to its invertibility. As in the proof of Lemma 1.22 one can show that when (1.95) is not fulfilled, the operator U_ω is non-Noetherian.

The Noetherity of the operator U (U_ω, $\omega \in S_a^{n-2}$) is equivalent to its invertibility.

Introduce the following notation:

$$\mathbb{Z}(A) = \big\{ \operatorname{Re}\mu/2a_n - \operatorname{Re}\delta_m + \ell \,|\, \ell \in \mathbb{Z},\, m = 1, \dots, N \big\}, \tag{1.113}$$

$$\begin{aligned} s_+ = \min \big\{ &\operatorname{Re}\mu/a_n - \operatorname{Re}\gamma_{1k} - 1, t \,|\, t \in \mathbb{Z}(A), \\ &t \ge s_n - \frac{1}{p},\ k = 1, \dots, m_- \big\}, \end{aligned} \tag{1.114}$$

$$\begin{aligned} s_- = \max \big\{ &\operatorname{Re}\beta_{1j}/a_n, t \,|\, t \in \mathbb{Z}(A), \\ &t \le s_n - \frac{1}{p},\ j = 1, \dots, m_+ \big\} \end{aligned} \tag{1.115}$$

(see (1.63)–(1.66)). Clearly if (1.95) is fulfilled, then $s_- < s_n - \frac{1}{p} < s_+$.

From the proof of Lemma 1.21 we easily get that the invertibility of the operator $U_\omega : H_1(s,p) \to H_2(s,p)$ is equivalent to that of the operator $U_\omega : B_1(s,p,q) \to B_2(s,p,q)$, $\forall q \in [1, +\infty]$. Similarly, the unique solvability of system (1.111), (1.112) for any right-hand sides in the case of H_p^σ scale is equivalent to that in the case of $B^{\sigma}_{p,q}$ scale. Moreover, let $1 < p^* < \infty$, $s^* \in \mathbb{R}$ and

$$s_- < s^*_n - \frac{1}{p^*} < s_+ \tag{1.116}$$

(see (1.2)). Then the invertibility of the operator $U_\omega : H_1(s,p) \to H_2(s,p)$ $(B_1(s,p,q) \to B_2(s,p,q))$ is equivalent to that of the operator $U_\omega : H_1(s^*,p^*) \to H_2(s^*,p^*)$ $(B_1(s^*,p^*,q^*) \to B_2(s^*,p^*,q^*))$. Indeed, from (1.96), (1.113)–(1.116) we have

$$\begin{aligned} d_m(\omega) &= \big[s_n - 1/p - \operatorname{Re}\mu/2a_n + \varkappa_m(\omega) + \operatorname{Re}\delta_m + 1\big] = \\ &= \big[s^*_n - 1/p^* - \operatorname{Re}\mu/2a_n + \varkappa_m(\omega) + \operatorname{Re}\delta_m + 1\big]. \end{aligned}$$

Let us summarize the results obtained in this section.

The following statements are equivalent:

a) *the operator* $U : H_1(s,p) \to H_2(s,p)$ *is Noetherian;*

b) *the operator* $U : H_1(s,p) \to H_2(s,p)$ *is invertible;*

c) *operators* $U_\omega : H_1(s,p) \to H_2(s,p)$ *are Noetherian for any* $\omega \in S^{n-2}_a$*;*

d) *operators* $U_\omega : H_1(s,p) \to H_2(s,p)$ *are invertible for any* $\omega \in S^{n-2}_a$*;*

e) *boundary value problem* (1.111), (1.112) *is uniquely solvable for any right-hand sides and any* $\omega \in S^{n-2}_a$*;*

f) *the matrix* $\|Z_{jk}(\omega)\|$ *is invertible for any* $\omega \in S^{n-2}_a$.

In any of the points a)–d) *we can substitute* $H_i(s,p)$ *by* $B_i(s,p,q)$, $H_i(s^*,p^*)$ *or* $B_i(s^*,p^*,q)$, $i = 1,2$, *if* (1.116) *is fulfilled. Analogous is also valid for the point* e).

For the points a)–f) *to be fulfilled, it is necessary that the relations* (1.95) *and* (1.110) *take place.*

Remark. The above proven theorem allows one to reduce the investigation of boundary value problems for a-elliptic ΨDOs in Besov and Bessel-potential spaces to their investigation in the H^{σ}_2 spaces. To this end it suffices to replace p by 2 and s by $s - a_n/p + a_n/2$ in the exponents of the corresponding spaces (see (1.116)).

Remark. In the scalar case we can do without the localization with respect to

$$\Big(\frac{\xi_1}{|\xi'|^{a_1}_a}, \dots, \frac{\xi_{n-1}}{|\xi'|^{a_{n-1}}_a}\Big) \in S^{n-2}_a \tag{1.117}$$

(see Lemma 1.20) and hence without the restriction (1.59) which was necessary only in proving Lemma 1.20.

Indeed, using the factorization (1.50), it is not difficult to reduce a boundary value problem of type (1.61), (1.62) for one scalar a-elliptic pseudodifferential equation to the equivalent system of type (1.103)–(1.107) $\widehat{Z}(D')w_0 = g_0$. The unique solvability of this system for any right-hand sides is equivalent to the invertibility of the corresponding ΨDO with the symbol

Z_0 being an a-homogeneous matrix function of zero order (see (1.107) and Theorem 1.3) whose components satisfy the conditions of Theorem 1.4 (with ξ' and $n-1$ instead of ξ and n). Note that in the case under consideration the matrix function Z_0, unlike (1.107), is not, in general, constant and can depend on the variable (1.117).

For the pseudodifferential operator $Z_0(D)$ to be invertible, it is necessary and sufficient that the matrix function Z_0 have its inverse $Z_0^{-1} \in L_\infty(\mathbb{R}^{n-1})$. Really, if this condition is fulfilled, then by means of Theorem 1.4 we can see that the pseudodifferential operator $Z_0^{-1}(D')$ is inverse to $Z_0(D')$. Let now the pseudodifferential operator $Z_0(D')$ be invertible. Then it is easy to see that the inverse operator $(Z_0(D'))^{-1}$ is translation invariant and hence can be represented as a pseudodifferential operator: $(Z_0(D'))^{-1} = Z_0^*(D')$ (see [48] or the proof of [101, Ch. I, Theorem 3.16]). It follows from the boundedness of the pseudodifferential operator $Z_0^*(D')$ that $Z_0^* \in L_\infty(\mathbb{R}^{n-1})$ (see, e.g., [110, Theorem 2.6.3]) and from the equalities $Z_0^*(D')Z_0(D') = I$, $Z_0(D')Z_0^*(D') = I$ there follow the equalities $Z_0^*Z_0 = I$, $Z_0Z_0^* = I$ (almost everywhere in $\mathbb{R}^{n-1}$). Hence $Z_0^{-1} = Z_0^* \in L_\infty(\mathbb{R}^{n-1})$.

Thus the unique solvability of a boundary value problem of type (1.61), (1.62) for any right-hand sides in the scalar case is equivalent to the invertibility of the matrix function Z_0 in $L_\infty(\mathbb{R}^{n-1})$. As above, this condition is likewise necessary and sufficient for the unique solvability for any right-hand sides and any $\omega \in S_a^{n-2}$ of a boundary value problem on semi-axis of type (1.111), (1.112).

Remark. If $p = 2$, we can determine a sufficient condition for the operator U to be invertible (see (1.72)) in the case when the condition (1.59) is not fulfilled. Indeed, using instead of the functions of type (1.77) the functions χ_ω, $\chi_\omega(\xi) = \chi_\omega(\xi') = \chi^\omega(\frac{\xi_1}{|\xi'|_a^{a_1}}, \dots, \frac{\xi_{n-1}}{|\xi'|_a^{a_{n-1}}})$, where $\chi^\omega : S_a^{n-2} \to \mathbb{R}$ is a characteristic function of sufficiently small neighbourhood $W \subset S_a^{n-2}$ of the point $\omega \in S_a^{n-2}$, and repeating the arguments from the proof of Lemma 1.20, show us that the invertibility of the operators U_ω, $\forall\omega \in S_a^{n-2}$ (see (1.74)) is sufficient for the operator

$$U : H_1(s,2) \to H_2(s,2) \ \big(B_1(s,2,q) \to B_2(s,2,q)\big) \tag{1.118}$$

to be invertible. Here the fact that the pseudodifferential operator $\chi_\omega(D)$ ($\chi_\omega(D')$) is bounded in $L_2(\mathbb{R}^n)$ ($L_2(\mathbb{R}^{n-1})$), and hence (see Theorem 1.3 as well as point e) of Theorem 1.2) in the spaces $H_2^{\overline{\sigma}}(\mathbb{R}^n)$, $B_{2,q}^{\overline{\sigma}}(\mathbb{R}^n)$ ($H_2^{\overline{\sigma}}(\mathbb{R}^{n-1})$, $B_{2,q}^{\overline{\sigma}}(\mathbb{R}^{n-1})$) plays an essential role.

Note that in the case under consideration the second part of the proof of Lemma 1.20, i.e. the proof that the invertibility of operators U_ω, $\forall\omega \in S_a^{n-2}$, is necessary for the invertibility of U, fails since we cannot use the rotation when (1.59) is not fulfilled.

In investigating the operator U_ω, i.e. in proving Lemma 1.21 we do not use (1.59). Thus we obtain that for operator (1.118) to be invertible, it is sufficient that the matrix $\|Z_{jk}(\omega)\|$, $\forall\omega \in S_a^{n-2}$, be invertible.

Introduce the notation

$$L(\xi) = \big(\xi_n - i|\xi'|_a^{a_n}\big)^{(s-\mu)/a_n - 1/p+1/2} A(\xi) \times \\ \times \big(\xi_n + i|\xi'|_a^{a_n}\big)^{-(s/a_n - 1/p+1/2)}, \tag{1.119}$$

λ_l^0, $l = 1, \ldots, N$, are eigenvalues of the matrix $\big(L(0, \ldots, 0, -1)\big)^{-1} \times L(0, \ldots, 0, +1)$.

It is not difficult to see that (1.95) is equivalent to

$$\frac{1}{2\pi} \arg \lambda_l^0 - \frac{1}{2} \notin \mathbb{Z}, \quad l = 1, \ldots, N. \tag{1.120}$$

Use Remark 1.25 and the results from [37, §16] to obtain the following statement.

Let (1.95) *hold. For the symbols B_j, C_k, E_{jk} ensuring unique solvability of the boundary value problem* (1.61), (1.62) *for any right-hand sides to exist for sufficiently large m_+ and m_-, it is necessary and sufficient that for sufficiently large $m \in N$ the matrix function*

$$\begin{Vmatrix} L(\xi', \xi_n) & 0 \\ 0 & I_m \end{Vmatrix}$$

(*where I_m is the unit $m \times m$-matrix*) *be homotopic to the matrix function*

$$\begin{Vmatrix} \left(\frac{\xi_n - i|\xi'|_a^{a_n}}{\xi_n + i|\xi'|_a^{a_n}}\right)^{m_- - m_+} & 0 \\ 0 & I_{m+N-1} \end{Vmatrix}$$

in the class of a-elliptic matrix functions satisfying (1.120).

§ a Ψ

An intersection of a finite number of half-spaces will be called a polyhedron. A polyhedron is said to be conic if boundaries of all half-spaces taking part in its definition pass through the origin.

Let $\Omega_1, \ldots, \Omega_l \subset \mathbb{R}^{n-1}$ be open polyhedra such that

$$\mathbb{R}^{n-1} = \bigcup_{m=1}^{l} \overline{\Omega}_m, \quad \Omega_m \cap \Omega_k = \varnothing \text{ for } m \neq k. \tag{1.121}$$

Assume that the condition (1.59) is fulfilled and $U_1, \ldots, U_l$ are operators of type (1.72). Consider the operator (see proof of Lemma 1.20)

$$U = \sum_{m=1}^{l} \mathcal{K}^2_{\chi_m} U_m = \sum_{m=1}^{l} U_m \mathcal{K}^1_{\chi_m}, \tag{1.122}$$

where χ_m is a characteristic function of the polyhedron Ω_m.

For the operator $U : H_1(s,p) \to H_2(s,p)$ $(B_1(s,p,q) \to B_2(s,p,q))$ of type (1.122) *to be invertible, it is sufficient, and if $\Omega_1, \dots, \Omega_l$ are conic polyhedra, it is also necessary that the operators*

$$\begin{gathered} U_{m\omega} : H_1(s,p) \to H_2(s,p) \ \big(B_1(s,p,q) \to B_2(s,p,q)\big), \\ \forall \omega \in \overline{\Omega}^*_m \cap S^{n-2}_a, \quad \forall m = 1, \dots, l, \end{gathered} \tag{1.123}$$

where $\Omega^* = \{(t^{a_1}\xi_1, \dots, t^{a_{n-1}}\xi_{n-1}) \mid (\xi_1, \dots, \xi_{n-1}) \in \Omega,\ t > 0\}$ for any $\Omega \subset \mathbb{R}^{n-1}$, be invertible (*see* (1.74)).

Proof. Let the operators (1.123) be invertible. They depend continuously on ω (see the proof of Lemma 1.20). Therefore inverse operators also depend continuously on ω. Since the set $\overline{\Omega}^*_m \cap S^{n-2}_a$ is compact, we obtain

$$\max_{m=1,\dots,l} \sup_{\omega \in \overline{\Omega}^*_m \cap S^{n-2}_a} \|U^{-1}_{m\omega}\| = M < +\infty. \tag{1.124}$$

Introduce the notation $(\tau_{h'} g)(\xi') = g(\xi' - h')$, $\forall \xi', h' \in \mathbb{R}^{n-1}$. We easily see that (see (1.28)) $(\tau_{h'} g)(D') = e_{-h'} g(D') e_{h'} I$, $\forall h' \in \mathbb{R}^{n-1}$, where $e_{\pm h'}(x) = \exp(\pm i h' x')$, $\forall x' \in \mathbb{R}^{n-1}$. Hence if g is a Fourier $L_p(\mathbb{R}^{n-1})$-multiplier, then $\tau_{h'} g$ is likewise a Fourier $L_p(\mathbb{R}^{n-1})$-multiplier. Moreover, their norms coincide. The same is true for the operators $g(D)$ and $(\tau_{h'} g)(D)$ in the space $L_p(\mathbb{R}^n)$. From this and from the first part of the proof of Lemma 1.20 (see (1.76), (1.78)) it follows that for any polyhedron $\Omega \subset \mathbb{R}^{n-1}$ the norm of the operators $\chi(\Omega, D')$, $\chi(\Omega, D)$ in the corresponding Besov and Bessel-potential spaces has a majorant which depends only on the number of half-spaces taking part in the definition of Ω. (Throughout the rest of the paper $\chi(\Omega) = \chi(\Omega, \cdot)$ is the characteristic function of Ω).

It is not difficult to see now that we can break up $\mathbb{R}^{n-1}$ into conic polyhedra $\Gamma_1, \dots, \Gamma_k$ such that

$$\begin{gathered} \mathbb{R}^{n-1} = \bigcup_{j=1}^{k} \overline{\Gamma}_j, \quad \Gamma_j \cap \Gamma_m = \varnothing \quad \text{for } j \neq m, \\ \|\mathcal{K}^2_{\chi(\Omega_m \cap \Gamma_j)}(U - U_{m\omega})\| < M^{-1}, \quad \forall \omega \in \overline{\Omega}^*_m \cap \overline{\Gamma}_j \cap S^{n-2}_a. \end{gathered}$$

Then the operator $U_{m\omega} + \mathcal{K}^2_{\chi(\Omega_m \cap \Gamma_j)}(U - U_{m\omega})$ has an inverse operator $\mathcal{R}_\omega$ (see (1.124)).

Choose arbitrary $\omega_{mj} \in \overline{\Omega}^*_m \cap \overline{\Gamma}_j \cap S^{n-2}_a$ and consider the operator

$$\mathcal{R} = \sum_{m,j} \mathcal{K}^1_{\chi(\Omega_m \cap \Gamma_j)} \mathcal{R}_{\omega_{mj}} \mathcal{K}^2_{\chi(\Omega_m \cap \Gamma_j)}.$$

As in proving Lemma 1.20 we obtain

$$\mathcal{R}U = \sum_{m,j} \mathcal{K}^1_{\chi(\Omega_m \cap \Gamma_j)} = I_{H_1(s,p)} \ \big(I_{B_1(s,p,q)}\big).$$

The second equality holds since $\sum\limits_{m,j} \chi(\Omega_m \cap \Gamma_j, \cdot) = 1$ almost everywhere.

Analogously we obtain $U\mathcal{R} = I_{H_2(s,p)}$ $\big(I_{B_2(s,p,q)}\big)$.

Thus the operator U is invertible, and the sufficiency is proved.

The necessity can be proved modifying analogously the reasonings from the proof of Lemma 1.20. ■

Remark. In the same way as Lemma 1.23 we can prove that if $\Omega_1, \dots, \Omega_l$ are conic polyhedra, then the invertibility of the operator $U : H_1(s,p) \to H_2(s,p)$ $(B_1(s,p,q) \to B_2(s,p,q))$ of type (1.122) is equivalent to its Noetherity.

Remark. The invertibility of operators (1.123) for $p = 2$ is sufficient for the operator $U : H_1(s,2) \to H_2(s,2)$ $(B_1(s,2,q) \to B_2(s,2,q))$ to be invertible even in the case when (1.59) is not fulfilled and $\Omega_1, \dots, \Omega_l$ are arbitrary measurable sets (see Remark 1.27 and the proof of Theorem 1.29).

§ a Ψ

1^0. Suppose $A \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}\backslash\{0\} \times \mathbb{R})$ is an a-homogeneous function of order $\mu \in \mathbb{C}$ (see (1.47)). Introduce the notation

$$\rho = |\xi'|_a, \quad \omega = \left(\frac{\xi_1}{|\xi'|_a^{a_1}}, \dots, \frac{\xi_{n-1}}{|\xi'|_a^{a_{n-1}}} \right) \in S_a^{n-2}, \tag{1.125}$$

$$A^0(\omega, \rho, \xi_n) = A(\xi', \xi_n), \tag{1.126}$$

$$A^*(\omega, \rho, \xi_n) = A^0(\omega, \rho^{1/a_n}, \xi_n). \tag{1.127}$$

We shall say that A belongs to the class $\mathcal{D}_{a,\mu}$ if

$$\begin{gathered} A^* \in C^\infty(S_a^{n-2} \times [0, +\infty[\times \mathbb{R} \backslash S_a^{n-2} \times \{0\} \times \{0\}) \text{ and} \\ \frac{\partial^k A^*(\omega, 0, -1)}{\partial \rho^k} = e^{i\pi(k - \mu/a_n)} \frac{\partial^k A^*(\omega, 0, +1)}{\partial \rho^k}, \quad \forall k \in \mathbb{Z}_+. \end{gathered} \tag{1.128}$$

The condition (1.128) is called the transmission condition.

Note that we can differentiate the equality (1.128) with respect to $\omega \in S_a^{n-2}$.

It is not difficult to see that $(\xi_n - i|\xi'|_a^{a_n})^{\mu/a_n} \in \mathcal{D}_{a,\mu}$, $\forall \mu \in \mathbb{C}$, $(\xi_n + i|\xi'|_a^{a_n})^{\mu/a_n} \in \mathcal{D}_{a,\mu}$ if $\mu/a_n \in \mathbb{Z}$. It is also evident that if $A_j \in \mathcal{D}_{a,\mu_j}$, $j = 1, 2$, then $A_1 A_2 \in \mathcal{D}_{a,\mu}$ where $\mu = \mu_1 + \mu_2$.

As usual, $\mathcal{D}_{a,\mu}^{N \times N}$ denotes a class of $N \times N$-matrices with components from $\mathcal{D}_{a,\mu}$.

Let $A \in \mathcal{D}_{a,0}$. In A^* let us make change of variables

$$\zeta = \frac{\xi_n - i\rho}{\xi_n + i\rho}, \quad \rho > 0. \tag{1.129}$$

This transformation maps (for fixed ρ) the upper complex half-plane $\operatorname{im} \xi_n \geq 0$ onto the unit circle $|\zeta| \leq 1$. When ξ_n runs through the real axis, $\zeta = e^{i\varphi}$,

$\varphi \in]0, 2\pi[$, runs through the unit circumference S^1. Moreover,

$$\xi_n = i\rho\,\frac{1+\zeta}{1-\zeta} = i\rho\,\frac{1+e^{i\varphi}}{1-e^{i\varphi}} = -\rho\,\mathrm{ctg}\,\frac{\varphi}{2}.$$

Hence

$$A^*(\omega,\rho,\xi_n) = A^*\Big(\omega,\rho,i\rho\,\frac{1+\zeta}{1-\zeta}\Big) = A^*\Big(\omega,1,-\mathrm{ctg}\,\frac{\varphi}{2}\Big),\quad \varphi\in]0,2\pi[. \tag{1.130}$$

Consider the function

$$A_*(\omega,\zeta) = A_*(\omega,e^{i\varphi}) = A^*\big(\omega,1,-\,\mathrm{ctg}\,\frac{\varphi}{2}\big). \tag{1.131}$$

Since the function A is a-homogeneous, we obtain

$$A_*(\omega,e^{i\varphi}) = A^*\big(\omega,\mathrm{tg}\,\frac{\varphi}{2},-1\big)\quad\text{for}\;\; 0<\varphi<\pi,$$
$$A_*(\omega,e^{i\varphi}) = A^*\big(\omega,-\,\mathrm{tg}\,\frac{\varphi}{2},1\big)\quad\text{for}\;\; \pi<\varphi<2\pi.$$

From (1.128) for $\mu = 0$ it follows that

$$\frac{\partial^k}{\partial\varphi^k}\,A^*\big(\omega,\mathrm{tg}\,\frac{\varphi}{2},-1\big)\Big|_{\varphi=+0} = \frac{\partial^k}{\partial\varphi^k}A^*\big(\omega,-\mathrm{tg}\,\frac{\varphi}{2},1\big)\Big|_{\varphi=2\pi-0},\ \forall k\in\mathbb{Z}_+. \tag{1.132}$$

We can differentiate these equalities with respect to $\omega\in S_a^{n-2}$. Thus $A_*\in C^\infty(S_a^{n-2}\times S^1)$. Conversely if $A_*\in C^\infty(S_a^{n-2}\times S^1)$, then the corresponding symbol A belongs to $\mathcal{D}_{a,0}$ (see (1.125)–(1.127), (1.131)).

For any $m\in\mathbb{Z}_+$ a symbol $A\in\mathcal{D}_{a,\mu}$ admits the representation

$$A(\xi) = A_m^-(\xi) + R_m(\xi) \tag{1.133}$$

with

$$A_m^-(\xi',\xi_n) = \big(\xi_n - i|\xi'|_a^{a_n}\big)^{\mu/a_n} A_{m,0}^-(\xi',\xi_n),$$
$$R_m(\xi',\xi_n) = |\xi'|_a^{(m+1)a_n}\big(\xi_n - i|\xi'|_a^{a_n}\big)^{\mu/a_n-m-1} R_{m,0}(\xi',\xi_n),$$

$A_{m,0}^-$, $R_{m,0}\in\mathcal{D}_{a,0}$ *and* $A_{m,0}^-$ *admits bounded analytic with respect to* ξ_n *continuation into the lower complex half-plane.*

Proof.

$$A(\xi',\xi_n) = \big(\xi_n - i|\xi'|_a^{a_n}\big)^{\mu/a_n}\big(\xi_n - i|\xi'|_a^{a_n}\big)^{-\mu/a_n} A(\xi',\xi_n),$$
$$\big(\xi_n - i|\xi'|_a^{a_n}\big)^{-\mu/a_n} A(\xi',\xi_n)\in\mathcal{D}_{a,0}.$$

Therefore it suffices to prove the theorem for the class $\mathcal{D}_{a,0}$. Thus we assume that $A\in\mathcal{D}_{a,0}$. Then $A_*\in C^\infty(S_a^{n-2}\times S^1)$.

Consider the function

$$b(\omega, e^{i\varphi}) \equiv A_*(\omega, e^{i\varphi}) - \sum_{k=0}^{m} \frac{1}{k!} \Big[\Big(- ie^{i\varphi} \frac{\partial}{\partial\varphi} \Big)^k A_*(\omega, e^{i\varphi}) \Big]_{\varphi=0} (1 - e^{-i\varphi})^k.$$

It has a zero of order $(m+1)$ at the point $e^{i\varphi} = 1$. Hence $\exists c \in C^\infty(S_a^{n-2} \times S^1)$: $b(\omega, e^{i\varphi}) = (1 - e^{-i\varphi})^{m+1} c(\omega, e^{i\varphi})$.

Thus

$$A_*(\omega, e^{i\varphi}) = \sum_{k=0}^{m} \frac{c_k(\omega)}{k!} (1 - e^{-i\varphi})^k + (1 - e^{-i\varphi})^{m+1} c(\omega, e^{i\varphi}),$$

where

$$c_k = \Big[\Big(- ie^{i\varphi} \frac{\partial}{\partial\varphi} \Big)^k A_*(\cdot, e^{i\varphi}) \Big]_{\varphi=0} \in C^\infty(S_a^{n-2}).$$

For the symbol A we obtain the representation

$$A(\xi', \xi_n) = \sum_{k=0}^{m} \frac{(-2i)^k}{k!} c_k(\omega) \frac{|\xi'|_a^{k a_n}}{(\xi_n - i|\xi'|_a^{a_n})^k} + \\ + (-2i)^{m+1} \frac{|\xi'|_a^{(m+1)a_n}}{(\xi_n - i|\xi'|_a^{a_n})^{m+1}} R_m^0(\xi', \xi_n),$$

where $R_m^0 \in \mathcal{D}_{a,0}$ (see (1.125)–(1.127), (1.131)). ■

2^0. Thorough examination of the proof of the results given in §1.3 (see [37, §6 and Lemma 17.1], [92], [31]) shows that they are also true for the symbols from $\mathcal{D}_{a,\mu}^{N\times N}$. For such symbols we have more exact results. The following two theorems are devoted to them.

Let $A \in \mathcal{D}_{a,\mu}$ *be an a-elliptic symbol. Then in the representation* (1.50) $A_0^\pm \in \mathcal{D}_{a,0}$ *and*

$$\mu/2a_n - \delta \in \mathbb{Z}. \tag{1.134}$$

Proof. (1.134) follows from (1.128) for $k = 0$. This means that the factor $(\xi_n + i|\xi'|_a^{a_n})$ contained in (1.50) has an integer power.

Show that $A_0^\pm \in \mathcal{D}_{a,0}$. As in proving Theorem 1.32 we can restrict ourselves to case $A \in \mathcal{D}_{a,0}$.

Introduce the notation

$$m = \operatorname{ind} A_*(\omega, \cdot) = \frac{1}{2\pi} \Delta \arg A_*(\omega, e^{i\varphi}) \Big|_{\varphi=0}^{2\pi} \in \mathbb{Z}.$$

(This is an integer, since $A_* \in C^\infty(S_a^{n-2} \times S^1) \subset C(S_a^{n-2} \times S^1)$).

Consider the function b, $b(\omega, \zeta) = \zeta^{-m} A_*(\omega, \zeta)$. Obviously $\log b \in C^\infty(S_a^{n-2} \times S^1)$. Therefore

$$A_*(\omega, \zeta) = c^-(\omega, \zeta) \zeta^m c^+(\omega, \zeta), \tag{1.135}$$

where $c^{\pm}(\omega,\zeta) = \exp\big((P_{\pm}\log b)(\omega,\zeta)\big)$, $P_{\pm}$ are analytic projectors: $P_{\pm} = \frac{1}{2}(I \pm S)$, I is the unit operator,

$$(Sf)(\zeta) = \frac{1}{\pi i}\int_{|z|=1} \frac{f(z)}{z-\zeta}\,dz, \quad |\zeta| = 1, \tag{1.136}$$

the integral in (1.136) is understood in the sense of the Cauchy principal value.

From $\log b \in C^{\infty}(S_a^{n-2} \times S^1)$ it follows that $P_{\pm}\log b \in C^{\infty}(S_a^{n-2} \times S^1)$ (see, e.g., [40, 4.4]). Thus, $c^{\pm} \in C^{\infty}(S_a^{n-2} \times S^1)$.

Returning to the symbol A (see (1.125)–(1.127), (1.131)), we obtain from (1.135)

$$A(\xi',\xi_n) = (\xi_n - i|\xi'|_a^{a_n})^m A_1^-(\xi',\xi_n)A_1^+(\xi',\xi_n)(\xi_n + i|\xi'|_a^{a_n})^{-m},$$

where $A_1^{\pm}$, $(A_1^{\pm})^{-1} \in \mathcal{D}_{a,0}$ admit bounded analytic with respect to ξ_n continuation to the corresponding complex half-plane (upper for the sign "+" and lower for the sign "−").

It remains for us to note that by virtue of the uniqueness of the factorization (see [37], the proof of Theorem 6.1), $A_0^{\pm} = A_1^{\pm}$. ■

Let $A \in \mathcal{D}_{a,\mu}^{N\times N}$ be an a-elliptic symbol. Then for any $\omega \in S_a^{n-2}$ the symbol $A_\omega(\xi) = A(|\xi'|_a^{a_1}\omega_1,\ldots,|\xi'|_a^{a_{n-1}}\omega_{n-1},\xi_n)$ admits the factorization

$$\begin{aligned} A_\omega(\xi) = (\xi_n - i|\xi'|_a^{a_n})^{\mu/a_n} A_\omega^-(\xi) \times& \\ \times \operatorname{diag}\left[\left(\frac{\xi_n - i|\xi'|_a^{a_n}}{\xi_n + i|\xi'|_a^{a_n}}\right)^{\varkappa_k(\omega)}\right]_{k=1}^{N} A_\omega^+(\xi),& \end{aligned} \tag{1.137}$$

where $(A_\omega^-)^{\pm 1} \in \mathcal{D}_{a,0}^{N\times N}$ $((A_\omega^+)^{\pm 1} \in \mathcal{D}_{a,0}^{N\times N})$ admits bounded analytic with respect to ξ_n continuation into the lower (upper) complex half-plane;

$$\varkappa_1(\omega) \ge \cdots \ge \varkappa_N(\omega), \quad \varkappa_k(\omega) \in \mathbb{Z},$$

$$\varkappa(\omega) \equiv \sum_{k=1}^{N} \varkappa_k(\omega) = \frac{1}{2\pi}\Delta \arg\det\left[(\xi_n - i|\xi'|_a^{a_n})^{-\mu/a_n} A_\omega(\xi',\xi_n)\right]\Big|_{\xi_n=-\infty}^{+\infty}.$$

Proof. Denote by W_m the set of all functions $f \in C(S^1)$ whose Fourier series $f(\zeta) = \sum\limits_{j\in\mathbb{Z}} f_j\zeta^j$ satisfy the condition $\sum_{j\in\mathbb{Z}} j^m|f_j| < \infty$, $m \in \mathbb{Z}_+$. Clearly, $W_m \subset C^m(S^1)$, $W_l \subset W_m$ for $l > m$ and $C^{\infty}(S^1) = \bigcap\limits_{m\in\mathbb{Z}_+} W_m$.

It is not difficult to see that W_m is a decomposing R-algebra (see [42, Ch. I, §5]).

As above we can restrict ourselves to the case $\mu = 0$. It follows from $A_\omega \in \mathcal{D}_{a,0}^{N\times N}$ that $b \in (C^{\infty}(S^1))^{N\times N}$, where $b(\zeta) = A_*(\omega,\zeta)$.

Consider the matrix function b as an element of $W_0^{N\times N}$ and apply [42, Ch. VIII, Theorem 2.1]:

$$b(\zeta) = b_-(\zeta)\,\mathrm{diag}[\zeta^{\varkappa_k}]_{k=1}^N b_+(\zeta), \tag{1.138}$$

where $b_\pm, b_\pm^{-1} \in (W_0^\pm)^{N\times N}$, $W_m^\pm = \left\{ f \in W_m | f(\zeta) = \sum\limits_{\pm j=0}^{\infty} f_j \zeta^j \right\}$, $\varkappa_k = \varkappa_k(\omega) \in \mathbb{Z}$.

Take an arbitrary $m \in \mathbb{N}$. Regarding b as an element of $W_m^{N\times N}$, we obtain, in general, the other factorization: $b(\zeta) = \widetilde{b}_-(\zeta)\,\mathrm{diag}[\zeta^{\widetilde{\varkappa}_k}]_{k=1}^N \widetilde{b}_+(\zeta)$ where $\widetilde{b}_\pm, \widetilde{b}_\pm^{-1} \in (W_m^\pm)^{N\times N}$, $\widetilde{\varkappa}_k \in \mathbb{Z}$ (see [42, Ch. VIII, Theorem 2.1]). It is clear that the above equality can be treated as a factorization in $W_0^{N\times N}$. Therefore from [42, Ch. VIII, Theorem 1.1] it follows that $\widetilde{\varkappa}_k = \varkappa_k$, $k = 1,\dots,N$, while from [42, Ch. VIII, Theorem 1.2] it follows that $b_\pm, b_\pm^{-1} \in (W_m^\pm)^{N\times N}$. Since $m \in \mathbb{N}$ is arbitrary, we have $b_\pm$, $b_\pm^{-1} \in \big(C^\infty(S^1)\big)^{N\times N}$.

Return now to the symbol A (see (1.125)–(1.127), (1.131)) and obtain (1.137) from (1.138). ■

3^0. Suppose $u \in H_p^{\overline{s}}(\overline{\mathbb{R}^n_+})$ $(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_+}))$, $1 < p < \infty$, $1 \le q \le \infty$, $s \in \mathbb{R}$, $s_n > 1/p - 1$ (see (1.2)). Denote by $\ell_0 u$ the extension of the function u by zero into the lower half-space. It is not difficult to deduce from Theorem 1.7 that $\ell_0 u \in H_p^{\overline{t}}(\mathbb{R}^n)$ $(B_{p,q}^{\overline{t}}(\mathbb{R}^n))$, where $t = \min\{s, 0\}$.

For an arbitrary $A \in \mathcal{D}_{a,\mu}$ let us define the ΨDO $\widehat{A}(D)$ (see (1.60)) on a space $H_p^{\overline{s}}(\overline{\mathbb{R}^n_+})$ $(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_+}))$ as follows:

$$\widehat{A}(D)u = \widehat{A}(D)\ell_0 u, \quad \forall u \in H_p^{\overline{s}}(\overline{\mathbb{R}^n_+}) \ \big(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_+})\big). \tag{1.139}$$

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $s_n > 1/p - 1$, $A \in \mathcal{D}_{a,\mu}$. Then the operator

$$\pi_+ \widehat{A}(D) : H_p^{\overline{s}}(\overline{\mathbb{R}^n_+}) \to H_p^{\overline{r}}(\overline{\mathbb{R}^n_+}) \ \big(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}^n_+}) \to B_{p,q}^{\overline{r}}(\overline{\mathbb{R}^n_+})\big),$$

where $r = s - \mathrm{Re}\,\mu$, is bounded.

Proof. In the case $1/p - 1 < s_n < 1/p$ we can easily get the assertion from Theorems 1.3, 1.4 and 1.7. Therefore we shall assume that $s \ge 0$.

Make the use of Theorem 1.32:

$$\pi_+ \widehat{A}(D)u = \pi_+ \widehat{A}_m^-(D)\ell_0 u + \pi_+ \widehat{R}_m(D)\ell_0 u.$$

Let $\ell u \in H_p^{\overline{s}}(\mathbb{R}^n)$ $(B_{p,q}^{\overline{s}}(\mathbb{R}^n))$ be an extension of the function u. Then $\ell_0 u - \ell u = 0$ for $x_n > 0$. Hence $\pi_+ \widehat{A}_m^-(D)\ell_0 u = \pi_+ \widehat{A}_m^-(D)\ell u$ (see Theorem 1.9). Therefore

$$\begin{aligned} \|\pi_+ \widehat{A}_m^-(D)\ell_0 u | H_p^{\overline{r}}(\overline{\mathbb{R}^n_+})\| = \|\pi_+ \widehat{A}_m^-(D)\ell u | H_p^{\overline{r}}(\overline{\mathbb{R}^n_+})\| \le \\ \le const \|\ell u | H_p^{\overline{s}}(\mathbb{R}^n)\| \le const \|u | H_p^{\overline{s}}(\overline{\mathbb{R}^n_+})\|. \end{aligned}$$

Similarly

$$\|\pi_+ \widehat{A}^-_m(D)\ell_0 u | B^{\overline{r}}_{p,q}(\overline{\mathbb{R}^n_+})\| \le const \|u|B^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_+})\|.$$

It remains for us to estimate $\pi_+\widehat{R}_m(D)\ell_0 u$. Let $(m+1)a_n \ge s$. Represent $\widehat{R}_m$ as follows (see Theorem 1.32) $\widehat{R}_m(\xi',\xi_n) = \widehat{R}_{m1}(\xi',\xi_n)\langle\xi'\rangle^s_a$, where

$$\widehat{R}_{m1}(\xi',\xi_n) = \langle\xi'\rangle_a^{(m+1)a_n - s}(\xi_n - i\langle\xi'\rangle_a^{a_n})^{\mu/a_n - m - 1}\widehat{R}_{m0}(\xi',\xi_n).$$

For any $\sigma \in \mathbb{R}$ the operator

$$\widehat{R}_{m1}(D) : H^{\overline{\sigma}}_p(\mathbb{R}^n) \to H^{\overline{\sigma}+\overline{r}}_p(\mathbb{R}^n) \ \big(B^{\overline{\sigma}}_{p,q}(\mathbb{R}^n) \to B^{\overline{\sigma}+\overline{r}}_{p,q}(\mathbb{R}^n)\big)$$

is bounded, since $(m+1)a_n - s \ge 0$ (see Theorems 1.3 and 1.4).

It is easy to see that $I^{\overline{s}}_0 \ell_0 u = \chi_+ I^{\overline{s}}_0 \ell u$ where

$$I^{\overline{s}}_0 = F^{-1}\langle\xi'\rangle^s_a F \tag{1.140}$$

(cf. the proof of equality (1.44)).

By Theorems 1.3 and 1.4, $I^{\overline{s}}_0\ell u \in L_p(\mathbb{R}^n)$ $(B^{\overline{0}}_{p,q}(\mathbb{R}^n))$. Therefore (see Theorem 1.7) $\chi_+ I^{\overline{s}}_0 \ell u \in L_p(\mathbb{R}^n)$ $(B^{\overline{0}}_{p,q}(\mathbb{R}^n))$ and

$$\|\pi_+\widehat{R}_m(D)\ell_0 u|H^{\overline{r}}_p(\overline{\mathbb{R}^n_+})\| \le const\|I^{\overline{s}}_0\ell_0 u|L_p(\mathbb{R}^n)\| =$$
$$= const\|\chi_+ I^{\overline{s}}_0 \ell u|L_p(\mathbb{R}^n)\| \le \|\ell u|H^{\overline{s}}_p(\mathbb{R}^n)\| \le const\|u|H^{\overline{s}}_p(\overline{\mathbb{R}^n_+})\|.$$

Analogously

$$\|\pi_+\widehat{R}_m(D)\ell_0 u|B^{\overline{r}}_{p,q}(\overline{\mathbb{R}^n_+})\| \le const\|u|B^{\overline{s}}_{p,q}(\overline{\mathbb{R}^n_+})\|. \quad \blacksquare$$

Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, $A \in \mathcal{D}_{a,\mu}$. Then the operator $v \longmapsto \pi_+\widehat{A}(D)\big(v(x') \times \delta(x_n)\big)$ is bounded from (see (1.32)) $B^{\lambda_1\overline{s}}_{p,p}(\mathbb{R}^{n-1})$ $(B^{\lambda_1\overline{s}}_{p,q}(\mathbb{R}^{n-1}))$ to $H^{\overline{r}}_p(\overline{\mathbb{R}^n_+})$ $(B^{\overline{r}}_{p,q}(\overline{\mathbb{R}^n_+}))$, where $r = s - \operatorname{Re}\mu$.

Proof. By Theorem 1.32

$$\pi_+\widehat{A}(D)(v\times\delta) = \pi_+\widehat{A}^-_m(D)(v\times\delta) + \pi_+\widehat{R}_m(D)(v\times\delta).$$

It follows from Theorem 1.13 that $v\times\delta \in \widetilde{H}^{\overline{t}}_p(\overline{\mathbb{R}^n_-})$ $(\widetilde{B}^{\overline{t}}_{p,q}(\overline{\mathbb{R}^n_-}))$, where $t = \min\{s, -a_n\}$. Therefore $\pi_+\widehat{A}^-_m(D)(v\times\delta) = 0$ (see Theorem 1.9).

Let $ma_n \ge s$. Represent $\widehat{R}_m$ as follows (see Theorem 1.32)

$$\widehat{R}_m(\xi',\xi_n) = \widehat{R}_{m2}(\xi',\xi_n)\langle\xi'\rangle_a^{s+a_n},$$

where $\widehat{R}_{m2} = \langle\xi'\rangle_a^{ma_n - s}\big(\xi_n - i\langle\xi'\rangle^{a_n}_a\big)^{\mu/a_n - m - 1}\widehat{R}_{m0}(\xi',\xi_n)$.

For any $\sigma\in\mathbb{R}$ the operator

$$\widehat{R}_{m2}(D) : H^{\overline{\sigma}}_p(\mathbb{R}^n) \to H^{\overline{\sigma}+\overline{r}+\overline{a_n}}_p(\mathbb{R}^n) \ \big(B^{\overline{\sigma}}_{p,q}(\mathbb{R}^n) \to B^{\overline{\sigma}+\overline{r}+\overline{a_n}}_{p,q}(\mathbb{R}^n)\big)$$

is bounded since $ma_n - s \ge 0$ (see Theorems 1.3 and 1.4).

It is easily seen that (see (1.140))

$$I^{\overline{s}+\overline{a_n}}_0(v\times\delta) = (I^{\overline{s}+\overline{a_n}}_0 v)\times\delta, \ \ I^{\overline{s}+\overline{a_n}}_0 v \in B^{\overline{\tau}}_{p,p}(\mathbb{R}^{n-1}) \ \big(B^{\overline{\tau}}_{p,q}(\mathbb{R}^{n-1})\big),$$

where $\tau = -a_n/p$ (see Theorem 1.3). Hence by virtue of Theorem 1.13

$$I_0^{\overline{s}+\overline{a_n}}(v \times \delta) \in H_p^{-\overline{a_n}}(\mathbb{R}^n) \ \ \big(B_{p,q}^{-\overline{a_n}}(\mathbb{R}^n)\big).$$

The subsequent is obvious:

$$\|\pi_+ \widehat{R}_m(D)(v \times \delta)|H_p^{\overline{r}}(\overline{\mathbb{R}_+^n})\| \leq const \|I_0^{\overline{s}+\overline{a_n}}(v \times \delta)|H_p^{-\overline{a_n}}(\mathbb{R}^n)\| \leq \\ \leq const\|v|B_{p,p}^{\lambda_1 \overline{s}}(\mathbb{R}^{n-1})\|.$$

Similarly

$$\|\pi_+ \widehat{R}_m(D)(v \times \delta)|B_{p,q}^{\overline{r}}(\overline{\mathbb{R}_+^n})\| \leq const\|v|B_{p,q}^{\lambda_1 \overline{s}}(\mathbb{R}^{n-1})\|. \qquad \blacksquare$$

4^0. We say that the symbol B belongs to the class $\mathcal{D}_{a,\beta}^{\lambda}$ if

$$B(\xi', \xi_n) = \sum_{m=1}^{m_0} B_0^{(m)}(\xi') B_1^{(m)}(\xi', \xi_n), \quad m_0 \in \mathbb{N}, \tag{1.141}$$

where $B_1^{(m)} \in \mathcal{D}_{a,\beta^{(m)}}$, $\operatorname{Re}\beta^{(m)} \leq \lambda$, $B_0^{(m)}(\xi') = |\xi'|_a^{\beta-\beta^{(m)}} B_{00}^{(m)}(\xi')$, $B_{00}^{(m)}$ is an a-homogeneous of zero order function such that $\widehat{B}_{00}^{(m)}$ satisfies the conditions which are obtained from those of Theorem 1.4 by substituting n and ξ by $(n-1)$ and ξ', respectively. Introduce also the set $\mathcal{D}_{a,\beta}^{\infty} = \bigcup_{\lambda \in \mathbb{R}} \mathcal{D}_{a,\beta}^{\lambda}$.

Consider now a boundary value problem of type (1.61), (1.62), where $A \in \mathcal{D}_{a,\mu}^{N \times N}$ is an a-elliptic symbol, B_j, C_k are N-dimensional vector functions whose components belong to the sets $\mathcal{D}_{a,\beta_j}^{\lambda_j}$ and $\mathcal{D}_{a,\gamma_k}^{\infty}$, respectively, E_{jk} and f, g_j, w_k are the same functions as in (1.67)–(1.68) and (1.69)–(1.71), respectively,

$$u_+ \in H_p^{\overline{s}}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N) \ \ \big(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N)\big), \ \ 1 < p < \infty, \ \ 1 \leq q \leq \infty, \ \ s \in \mathbb{R}, \\ s > \max_{1 \leq j \leq m_+} \lambda_j + \frac{a_n}{p}, \ \ s > a_n\big(\frac{1}{p} - 1\big). \tag{1.142}$$

Use Theorems 1.35 and 1.36 (see also Theorems 1.3-1.5) to obtain that in the case under consideration the left-hand sides of equations (1.61), (1.62) define the continuous operator (see (1.72))

$$U' : H_1'(s,p) = H_p^{\overline{s}}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N) \oplus \bigoplus_{k=1}^{m_-} B_{p,p}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \to H_2(s,p) \tag{1.143}$$

$$\big(U' : B_1'(s,p,q) = B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N) \oplus \bigoplus_{k=1}^{m_-} B_{p,q}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \to B_2(s,p,q)\big).$$

By analogy with (1.73), (1.74) let us introduce the operators

$$U_\omega' : H_1'(s,p) \to H_2(s,p) \ \ \big(B_1'(s,p,q) \to B_2(s,p,q)\big). \tag{1.144}$$

Quite similarly to Lemma 1.20 we can prove that if the condition (1.59) is fulfilled, then the operator (1.143) is invertible (Noetherian) if and only if the operators (1.144) are invertible (Noetherian) for any $\omega \in S_a^{n-2} = S^{n-2}$.

Thus all the above is reduced to the investigation of a boundary value problem of type (1.82), (1.83), where $(u_+, w) \in H_1'(s,p)$ $(B_1'(s,p,q))$.

Let us consider an auxiliary space

$$\widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n}) = \left\{ u_+ \in \widetilde{H}_p^{\overline{t}}(\overline{\mathbb{R}_+^n}) | \pi_+ u_+ \in H_p^{\overline{s}}(\overline{\mathbb{R}_+^n}),\ \exists \sigma \geq 0 : I_0^{\overline{\sigma}} u_+ \in \widetilde{H}_p^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n}) \right\},$$

$t \leq s$ (see (1.140)). The space $\widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n})$ is defined analogously.

Suppose that $u_+ \in \widetilde{H}_p^{\overline{t}}(\overline{\mathbb{R}_+^n})$ $(\widetilde{B}_{p,q}^{\overline{t}}(\overline{\mathbb{R}_+^n}))$, $t > a_n(1/p - 1)$, and $\pi_+ u_+ \in H_p^{\overline{s}}(\overline{\mathbb{R}_+^n})$ $(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}))$. Then $u_+ \in \widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n})$ $(\widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n}))$. Indeed, $u_+ = \chi_+ u$, where $u \in H_p^{\overline{s}}(\mathbb{R}^n)$ $(B_{p,q}^{\overline{s}}(\mathbb{R}^n))$ is an extension of $\pi_+ u_+$ to $\mathbb{R}^n$ (see the proof of point a) of Theorem 1.18). Letting $\sigma = \max\{s, 0\}$, we can easily get $I_0^{\overline{\sigma}} u_+ \in \widetilde{H}_p^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n})$ $\big(\widetilde{B}_{p,q}^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n})\big)$ (see the proof of Theorem 1.35).

Let u_+ and u be as in the previous section with the only difference that now $m + \frac{1}{p} - 1 < \frac{t}{a_n} < m + \frac{1}{p}$, $m < 0$, $m \in \mathbb{Z}$.

According to point c) of Theorem 1.18 we have

$$u_+ = I_+^{-a_n \overline{m}} \chi_+ I_+^{a_n \overline{m}} u + \sum_{j=1}^{|m|} v_j(x') \times \delta^{(j-1)}(x_n). \tag{1.145}$$

Use the arguments from the proof of Theorem 1.35 to see that

$$I_0^{\overline{\sigma}} I_+^{-a_n \overline{m}} \chi_+ I_+^{a_n \overline{m}} u \in \widetilde{H}_p^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n}) \ \ \big(\widetilde{B}_{p,q}^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n})\big)$$

if $\sigma \geq s - t$. Therefore the condition

$$\exists \sigma \geq 0 : I_0^{\overline{\sigma}} u_+ \in \widetilde{H}_p^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n}) \ \ \big(\widetilde{B}_{p,q}^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n})\big)$$

is equivalent to

$$\exists \sigma \geq 0 : u_\sigma = I_0^{\overline{\sigma}} \Big(\sum_{j=1}^{|m|} v_j \times \delta^{(j-1)} \Big) = \sum_{j=1}^{|m|} (I_0^{\overline{\sigma}} v_j) \times \delta^{(j-1)} \in$$
$$\in \widetilde{H}_p^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n}) \ \ \big(\widetilde{B}_{p,q}^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}_+^n})\big).$$

By virtue of Theorem 1.3 and Lemma 1.17 the last condition is fulfilled if and only if

$$v_j \in B_{p,p}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1}) \ \ \big(B_{p,q}^{\lambda_j \overline{s}}(\mathbb{R}^{n-1})\big) \tag{1.146}$$

(see (1.32)) for, $\operatorname{supp} u_\sigma \subset \overline{\mathbb{R}_+^n} \cap \overline{\mathbb{R}_-^n}$. Thus in the case under consideration $u_+ \in \widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n})$ $(\widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n}))$ if and only if the condition (1.146) is fulfilled (see (1.145)).

Clearly, $u_+ = u - u_-$ where $u_- \in \widetilde{H}_p^{\overline{t}}(\overline{\mathbb{R}_-^n})$ $(\widetilde{B}_{p,q}^{\overline{t}}(\overline{\mathbb{R}_-^n}))$, u_+ and u are the same as above, $t, s \in \mathbb{R}$ are arbitrary numbers satisfying the inequality $t \leq s$. Therefore it follows from the proof of Theorem 1.35 that for any $\Lambda \in \mathcal{D}_{a,\mu}$ the operator

$$\pi_+ \widehat{\Lambda}(D) : \widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n}) \to H_p^{\overline{s}-\overline{\operatorname{Re}\mu}}(\overline{\mathbb{R}_+^n}) \ \ \big(\widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}_+^n}) \to B_{p,q}^{\overline{s}-\overline{\operatorname{Re}\mu}}(\overline{\mathbb{R}_+^n})\big)$$

is continuous (see (1.60)). In addition, if $\widehat{\Lambda}$ satisfies the conditions of Theorem 1.9 in the case of sign "+", then the operator

$$\widehat{\Lambda}(D) : \widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+}) \to \widetilde{H}_p^{\overline{t}-\overline{\operatorname{Re}\mu},\overline{s}-\overline{\operatorname{Re}\mu}}(\overline{\mathbb{R}^n_+}) \ \left(\widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+}) \to \widetilde{B}_{p,q}^{\overline{t}-\overline{\operatorname{Re}\mu},\overline{s}-\overline{\operatorname{Re}\mu}}(\overline{\mathbb{R}^n_+})\right)$$

is likewise continuous. (Nothing have been said above on the topology in the space $\widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+})$. We assume that the topology in that space is introduced by means of the following notion of convergence: a sequence $u_+^{(k)}$, $k \in \mathbb{N}$, converges to u_+ if

$$\|u_+ - u_+^{(k)} | \widetilde{H}_p^{\overline{t}}(\overline{\mathbb{R}^n_+})\| \to 0 \quad \text{for} \quad k \to \infty,$$
$$\|\pi_+ u_+ - \pi_+ u_+^{(k)} | \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}^n_+})\| \to 0 \quad \text{for} \quad k \to \infty,$$
$$\exists \sigma \geq 0 : \|I_0^{\overline{\sigma}} u_+ - I_0^{\overline{\sigma}} u_+^{(k)} | \widetilde{H}_p^{\overline{s}-\overline{\sigma}}(\overline{\mathbb{R}^n_+})\| \to 0 \quad \text{for} \quad k \to \infty.$$

The topology in the space $\widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+})$ is defined similarly).

Return now to the operators (1.143), (1.144). Put $t = \min\{s, 0\}$. From the second inequality in (1.142) it follows that (see (1.2))

$$t_n \in \left]\frac{1}{p} - 1, 0\right] \subset \left]\frac{1}{p} - 1, \frac{1}{p}\right[. \tag{1.147}$$

Therefore by virtue of Theorem 1.7 we can identify the operators (1.143), (1.144) with

$$\begin{aligned} U :& H_1''(s,p) = \widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \oplus \bigoplus_{k=1}^{m_-} B_{p,p}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \to H_2(s,p) \\ &\left(B_1''(s,p,q) = \widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+}, \mathbb{C}^N) \oplus \bigoplus_{k=1}^{m_-} B_{p,q}^{\overline{s^{(k)}}}(\mathbb{R}^{n-1}) \to B_2(s,p,q)\right), \\ U_\omega :& H_1''(s,p) \to H_2(s,p) \ \left(B_1''(s,p,q) \to B_2(s,p,q)\right) \end{aligned}$$

(see (1.139)). As it have been said above, it suffices to consider boundary value problem (1.82), (1.83), where now $(u_+, w) \in H_1''(s,p)$ $(B_1''(s,p,q))$.

With the help of Theorem 1.34 equation (1.82) is reduced equivalently to that of type (1.86) in which G_ω is the unit matrix,

$$\begin{aligned} v_+ \in \prod_{k=1}^N \widetilde{H}_p^{\overline{t}+a_n\overline{\varkappa_k(\omega)},\overline{s}+a_n\overline{\varkappa_k(\omega)}}(\overline{\mathbb{R}^n_+}) \subset \prod_{k=1}^N \widetilde{H}_p^{\overline{t}+a_n\overline{\varkappa_k(\omega)}}(\overline{\mathbb{R}^n_+}) \\ \left(\prod_{k=1}^N \widetilde{B}_{p,q}^{\overline{t}+a_n\overline{\varkappa_k(\omega)},\overline{s}+a_n\overline{\varkappa_k(\omega)}}(\overline{\mathbb{R}^n_+}) \subset \prod_{k=1}^N \widetilde{B}_{p,q}^{\overline{t}+a_n\overline{\varkappa_k(\omega)}}(\overline{\mathbb{R}^n_+})\right). \end{aligned}$$

For the obtained equation the condition (1.95) is of the form

$$t_n - \frac{1}{p} \notin \mathbb{Z}. \tag{1.148}$$

According to (1.147) this condition is fulfilled. Use Theorem 1.18 and the above given properties of spaces $\widetilde{H}_p^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+})$ $\widetilde{B}_{p,q}^{\overline{t},\overline{s}}(\overline{\mathbb{R}^n_+})$ and repeat without principal changes the reasonings from the proof of Lemma 1.21 (see also Theorems 1.32 and 1.34–1.36).

As a result we obtain that boundary value problem (1.82), (1.83), where $(u_+, w) \in H_1''(s,p)$ $(B_1''(s,p,q))$, is uniquely solvable for any $(f,g) \in H_2(s,p)$ $(B_2(s,p,q))$ if and only if the corresponding matrix $\|Z_{jk}(\omega)\|$ is invertible (see (1.103)–(1.107)). In other words, the invertibility of the operator (1.144) is equivalent to that of the matrix $\|Z_{jk}(\omega)\|$ which is independent of s. To construct it we have, roughly speaking, to pretend as if we were searching for a solution of boundary value problem (1.82), (1.83) in the space $H_1(t,p)$ $(B_1(t,p,q))$.

Note that when calculating the matrix $\|Z_{jk}(\omega)\|$ we have to calculate integrals with respect to ξ_n. For these integrals to be absolutely convergent, the use should be made of the decomposition of type (1.133). Integrals corresponding to the first summands in (1.133) will be equal to 0, while integrals corresponding to the second summands in (1.133) will absolutely converge for sufficiently large m (in this connection see [37, §§11, 12]).

We shall say that for the operator (1.143) the Shapiro–Lopatinskiĭ condition is fulfilled if the corresponding matrix $\|Z_{jk}(\omega)\|$ is invertible for any $\omega \in S_a^{n-2}$. For the Shapiro–Lopatinskiĭ condition to be fulfilled, it is necessary that the equality

$$m_- - m_+ = \varkappa \tag{1.149}$$

take place, which is obtained from (1.110) by substituting s by $t = \min\{s, 0\}$ and taking into account Theorem 1.34 (see likewise (1.147)).

Further investigation of the operator (1.143) is similar to that of the operator (1.72). The analogue of Theorem 1.24 completes the investigation.

Let the above-stated conditions as well as the condition (1.59) *be fulfilled. Then the following statements are equivalent:*

(a) *the operator* $U' : H_1'(s,p) \to H_2(s,p)$ *is Noetherian;*

(b) *the operator* $U' : H_1'(s,p) \to H_2(s,p)$ *is invertible;*

(c) *operators* $U'_\omega : H_1'(s,p) \to H_2(s,p)$ *are Noetherian for any* $\omega \in S_a^{n-2}$*;*

(d) *operators* $U'_\omega : H_1'(s,p) \to H_2(s,p)$ *are invertible for any* $\omega \in S_a^{n-2}$*;*

(e) *the boundary value problem* (1.111), (1.112), *where*

$$(u_+, w) \in H_p^{s/a_n}(\overline{\mathbb{R}^1_+}, \mathbb{C}^N) \oplus \mathbb{C}^{m_-} \quad \left(B_{p,q}^{s/a_n}(\overline{\mathbb{R}^1_+}, \mathbb{C}^N) \oplus \mathbb{C}^{m_-}\right),$$

is uniquely solvable for any right-hand sides and any $\omega \in S_a^{n-2}$*;*

(f) *the Shapiro–Lopatinskiĭ condition is fulfilled.*

In any of points (a)–(d) *we can replace* $\{H_1'(s,p), H_2(s,p)\}$ *by* $\{B_1'(s,p,q), B_2(s,p,q)\}$, $\{H_1'(s^*,p^*), H_2(s^*,p^*)\}$ *or* $\{B_1'(s^*,p^*,q^*), B_2(s^*,p^*,q^*)\}$ *if for* s^* *and* p^* *the condition of type* (1.142) *is fulfilled. The same is true for point* (e).

For points (a)–(f) *to be fulfilled, it is necessary that the equality* (1.149) *take place.*

Remark. In the case under consideration the analogues of Remarks 1.26 and 1.27 are valid.

Remark. As it was noted above, the Shapiro–Lopatinskiĭ condition does not depend on the fact in what pair of spaces of type (1.143) we consider the operator U'. We need only that the inequalities (1.142) be fulfilled. At the same time the invertibility conditions for the operator (1.72) are the same only for those pairs $(s^*, p^*) \in \mathbb{R}\times]1,\infty[$ for which $s_n^* - 1/p^*$ belongs to the interval $]s_-, s_+[$ of the length not exceeding 1 (see (1.113)–(1.116)). Such a distinction is caused by the following fact. Assuming in §1.4 that $u_+ \in \widetilde{H}_p^{\overline{s}}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N)$ $(\widetilde{B}_{p,q}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N))$ we actually add to boundary value problem (1.61), (1.62) the boundary conditions $\frac{\partial^m u_+}{\partial x_n^m}(x', 0) = 0$, $m = 0, \dots, [s_n - 1/p]^-$ (see (1.17), Theorem 1.5 and Lemma 1.15) the number of which increases with the increase of s. In this section we assume that $u_+ \in H_p^{\overline{s}}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N)$ $(B_{p,q}^{\overline{s}}(\overline{\mathbb{R}_+^n}, \mathbb{C}^N))$ and there do not appear additional boundary conditions.

Remark. Theorem 1.37 admits evident generalization to the operators with discontinuous symbols as in §1.5 (see also Remarks 1.31 and 1.38).

Remark. In the given chapter we have considered matrix a-elliptic ΨDOs whose all elements are of the same order. By the use of order reduction operators (see (1.31) and Theorem 1.12), we can easily transfer all the results to a-elliptic in the Douglis-Nirenberg sense ΨDOs (the elements of such ΨDOs are, in general, of different orders).

The class of a-elliptic ΨDOs considered in the present chapter covers besides usual (isotropic) elliptic operators (corresponding to the case $a = (1, \dots, 1)$) parabolic operators. We complete this chapter by an example from the theory of parabolic partial differential equations (see also [80]).

The symbol of the operator $\frac{\partial}{\partial x_n} \pm \left(\sum\limits_{m=1}^{n-1} \frac{\partial^2}{\partial x_m^2} - 1\right)$ is equal to $-i(\xi_n \mp i\langle\xi'\rangle^2)$. It follows from the results obtained in the present chapter that the Cauchy problem

$$u\big|_{x_n=0} = \varphi, \quad \varphi \in B_{p,p}^{\sigma-2/p}(\mathbb{R}^{n-1}) \ \ \big(B_{p,q}^{\sigma-2/p}(\mathbb{R}^{n-1})\big) \tag{1.150}$$

for the heat equation

$$\begin{gathered} \frac{\partial u}{\partial x_n} - \Big(\sum_{m=1}^{n-1} \frac{\partial^2 u}{\partial x_m^2} - u\Big) = f, \\ f \in H_p^{\sigma-2,(\sigma-2)/2}(\overline{\mathbb{R}_+^n}) \ \ \big(B_{p,q}^{\sigma-2,(\sigma-2)/2}(\overline{\mathbb{R}_+^n})\big), \end{gathered} \tag{1.151}$$

has a unique solution

$$u \in H_p^{\sigma,\sigma/2}(\overline{\mathbb{R}_+^n}) \ \ \big(B_{p,q}^{\sigma,\sigma/2}(\overline{\mathbb{R}_+^n})\big) \tag{1.152}$$

for $\sigma > 2/p$ while for the backward heat equation

$$\frac{\partial u}{\partial x_n} + \Big(\sum_{m=1}^{n-1} \frac{\partial^2 u}{\partial x_m^2} - u \Big) = f \tag{1.153}$$

the initial conditions are superfluous. For any f such as in (1.151), equation (1.153) has a unique solution (1.152) for $\sigma > 2(1/p-1)$. (In (1.151), (1.152) in notation of functional spaces we write $\tau, \tau/2$ instead of $(\tau, \ldots, \tau, \tau/2)$).

It is clear that we can obtain analogous results for equations (and systems) which are significantly more general than (1.151), (1.153).

Chapter II

Ψ

§

For an arbitrary $m \in \mathbb{R}$ we denote by $S^m(\mathbb{R}^n \times \mathbb{R}^n)$ the set of all functions $A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for any multiindices $\alpha, \beta \in \mathbb{Z}_+^n = (\mathbb{Z}_+)^n$ the estimate

$$|\partial_x^\alpha \partial_\xi^\beta A(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \quad \forall (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n, \tag{2.1}$$

holds.

The class O_μ^∞, $\mu \in \mathbb{C}$, consists of functions $A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \backslash \mathbb{R}^n \times \{0\})$ homogeneous of order μ in the second argument:

$$A(x, t\xi) = t^\mu A(x,\xi), \ \ \forall t > 0, \ \ \forall x \in \mathbb{R}^n, \ \ \forall \xi \in \mathbb{R}^n \backslash \{0\}. \tag{2.2}$$

We shall say that A belongs to the class $\widehat{O}_\mu^\infty$, $\mu \in \mathbb{C}$, if $A \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and there is a function $A_0 \in O_\mu^\infty$ such that

$$A(x,\xi) - A_0(x,\xi)(1 - \chi(\xi)) \in S^{\operatorname{Re}\mu - \varepsilon}(\mathbb{R}^n \times \mathbb{R}^n),$$

where $\varepsilon > 0$, $\chi \in \mathcal{D}(\mathbb{R}^n)$ and $\chi(\xi) = 1$ for $|\xi| \le 1$.

We shall say that an operator $\mathcal{A} : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ belongs to the class $OP_0(\widehat{O}_\mu^\infty)$ if there are $A \in \widehat{O}_\mu^\infty$ and $\varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\mathcal{A} = \varphi A(x,D) \psi I$ where $A(x,D)$ is the ΨDO with the symbol $A(x,\xi)$:

$$\big(A(x,D)u\big)(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} A(x,\xi)(Fu)(\xi)\, d\xi, \tag{2.3}$$
$$\forall x \in \mathbb{R}^n, \quad \forall u \in S(\mathbb{R}^n).$$

It is well known that an operator of the class $OP_0(\widehat{O}_\mu^\infty)$ can be extended to a continuous one from $H_p^s(\mathbb{R}^n)$ $(B_{p,q}^s(\mathbb{R}^n))$ to $H_p^{s-\operatorname{Re}\mu}(\mathbb{R}^n)$ $(B_{p,q}^{s-\operatorname{Re}\mu}(\mathbb{R}^n))$, $\forall s \in \mathbb{R}$, $\forall p \in]1,\infty[$, $\forall q \in [1,\infty]$ (see [58] or [107, Ch. XI], [98]).

An operator $K : \mathcal{D}(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^n)$ belongs to the class $OPI(\gamma, r)$ if

$$(Kv)(x) = \sum_{m=1}^{m_0} C_m(x,D)\big((E_m(x',D')v)(x') \times \delta(x_n)\big), \ \forall v \in \mathcal{D}(\mathbb{R}^{n-1}), \tag{2.4}$$

where $C_m(x,D) \in OP_0(\widehat{O}_{\gamma_m}^\infty)$, $E_m(x',D') \in OP_0(\widehat{O}_{\gamma-\gamma_m}^\infty)$, $\gamma, \gamma_m \in \mathbb{C}$, $r \in \mathbb{R}$, $\operatorname{Re}\gamma_m \le r$, $m_0 \in \mathbb{N}$.

An operator $T : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^{n-1})$ belongs to the class $OPII(\gamma, r)$ if

$$(Tu)(x') = \sum_{m=1}^{m_0} E_m(x', D')\pi_0\big(B_m(x, D)u\big), \quad \forall u \in \mathcal{D}(\mathbb{R}^n), \tag{2.5}$$

where $B_m(x, D) \in OP_0(\widehat{O}^{\infty}_{\gamma_m})$, $E_m(x', D') \in OP_0(\widehat{O}^{\infty}_{\gamma-\gamma_m})$, $\gamma, \gamma_m \in \mathbb{C}$, $r \in \mathbb{R}$, $\operatorname{Re}\gamma_m \le r$, $m_0 \in \mathbb{N}$, $\pi_0 = \pi_0^0$ is the operator of restriction to the hyperplane $x_n = 0$ (see (1.21)).

Using Theorem 1.13, we can easily prove that an operator of the class $OPI(\gamma, r)$ admits extension to the bounded operator from $B^s_{p,p}(\mathbb{R}^{n-1})$ $(B^s_{p,q}(\mathbb{R}^{n-1}))$ to $H_p^{s-\operatorname{Re}\gamma-1+1/p}(\mathbb{R}^n)$ $(B_{p,q}^{s-\operatorname{Re}\gamma-1+1/p}(\mathbb{R}^n))$ for $r < \operatorname{Re}\gamma - s$. Similarly, using Theorem 1.5, it is not difficult to prove that an operator of the class $OPII(\gamma, r)$ admits extension to the bounded operator from $H^s_p(\mathbb{R}^n)$ $(B^s_{p,q}(\mathbb{R}^n))$ to $B_{p,p}^{s-\operatorname{Re}\gamma-1/p}(\mathbb{R}^{n-1})$ $(B_{p,q}^{s-\operatorname{Re}\gamma-1/p}(\mathbb{R}^{n-1}))$ for $r < s - 1/p$.

We shall say that an operator

$$U : \mathcal{D}(\mathbb{R}^n_+) \oplus \mathcal{D}(\mathbb{R}^{n-1}) \to \mathcal{D}'(\mathbb{R}^n_+) \oplus \mathcal{D}'(\mathbb{R}^{n-1})$$

belongs to the class

$$OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix},$$

if

$$U = \begin{pmatrix} \pi_+\mathcal{A} & \pi_+K \\ T & Q \end{pmatrix} : \begin{array}{c} \mathcal{D}(\mathbb{R}^n_+) \\ \oplus \\ \mathcal{D}(\mathbb{R}^{n-1}) \end{array} \to \begin{array}{c} \mathcal{D}'(\mathbb{R}^n_+) \\ \oplus \\ \mathcal{D}'(\mathbb{R}^{n-1}) \end{array}, \tag{2.6}$$

where $\mathcal{A} \in OP_0(\widehat{O}^{\infty}_{\mu})$, $K \in OPI(\gamma_1, r_1)$, $T \in OPII(\gamma_2, r_2)$, $Q \in OP_0(\widehat{O}^{\infty}_{\lambda})$, $\lambda = \gamma_1 + \gamma_2 - \mu + 1$.

From the above-said it follows that an operator of the class

$$OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}$$

admits extension to the bounded operator from

$$\widetilde{H}^s_p(\overline{\mathbb{R}^n_+}) \oplus B_{p,p}^{s-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1-1/p}(\mathbb{R}^{n-1})$$
$$\big(\widetilde{B}^s_{p,q}(\overline{\mathbb{R}^n_+}) \oplus B_{p,q}^{s-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1-1/p}(\mathbb{R}^{n-1})\big)$$

to

$$H_p^{s-\operatorname{Re}\mu}(\overline{\mathbb{R}^n_+}) \oplus B_{p,p}^{s-\operatorname{Re}\gamma_2-1/p}(\mathbb{R}^{n-1}) \ \ \big(B_{p,q}^{s-\operatorname{Re}\mu}(\overline{\mathbb{R}^n_+}) \oplus B_{p,q}^{s-\operatorname{Re}\gamma_2-1/p}(\mathbb{R}^{n-1})\big)$$

if $r_1 < \operatorname{Re}\mu - s - 1 + 1/p$, $r_2 < s - 1/p$, $(1 < p < \infty$, $1 \le q \le \infty)$.

We can transfer word by word all the above definitions and results to the matrix case. Below we shall make no distinctions between scalar and matrix ΨDOs (symbols).

§ Ψ

Let X be a smooth compact n-dimensional ($n \geq 2$) manifold with a boundary Y embedded in a compact closed smooth n-dimensional manifold M. (Throughout this chapter under the smoothness of a manifold or vector bundle will be meant, if not otherwise stated, its C^∞-smoothness). For instance, we may assume that $M = 2X$ is a duplicate of the manifold X obtained by pasting two copies of X along Y (see, e.g., [65, §8]).

Any smooth vector bundle E over X is regarded to be a restriction on X of a smooth vector bundle E_0 over M (see, e.g., [75, Ch. X, §4, Theorem 5]).

Spaces $H_p^s(E_0)$ and $B_{p,q}^s(E_0)$ of sections of bundle E_0 are defined in a standard way using partition of unity.

Introduce the notation:

$$\Omega = Int X = X \backslash Y, \quad H_p^s(E) = \{u\big|_\Omega : u \in H_p^s(E_0)\},$$
$$\widetilde{H}_p^s(E) = \{u \in H_p^s(E_0) : \operatorname{supp} u \subset X\}.$$

The notation $B_{p,q}^s(E)$, $\widetilde{B}_{p,q}^s(E)$ is treated analogously.

The restriction operator from M to Ω will be denoted by π_+ (cf. (1.30)).

Let E, F be smooth vector bundles over X and $\mathcal{I}, G$ over $Y = \partial X$. Consider an operator (cf. (2.6))

$$U = \begin{pmatrix} \pi_+\mathcal{A} & \pi_+K \\ T & Q \end{pmatrix} : \begin{matrix} \mathcal{D}(E|_\Omega) \\ \oplus \\ \mathcal{D}(\mathcal{I}) \end{matrix} \to \begin{matrix} \mathcal{D}'(F|_\Omega) \\ \oplus \\ \mathcal{D}'(G) \end{matrix} . \tag{2.7}$$

Assume W to be an open (generally speaking, nonconnected) subset of X, $W' = W \cap Y$ (W' may be empty), and bundles $E|_W$, $F|_W$, $\mathcal{I}|_{W'}$, $G|_{W'}$ to be trivial. We shall also assume any connected component W_0 of W intersecting the boundary to be diffeomorphic to $W_0' \times [0,1[$, where $W_0' = W_0 \cap Y$.

Denote by

$$\chi_{E} : E\big|_W \to V \times \mathbb{C}^k, \quad \chi_{F} : F\big|_W \to V \times \mathbb{C}^{k'},$$
$$\chi_{\mathcal{I}} : \mathcal{I}\big|_{W'} \to V' \times \mathbb{C}^j, \quad \chi_{G} : G\big|_{W'} \to V' \times \mathbb{C}^{j'}$$

trivialization of the corresponding bundles (V is a set open in $\overline{\mathbb{R}^n_+}$, $V' = V \cap \mathbb{R}^{n-1}$, k, k', j, j' are the fibre dimensions of the bundles).

The operator U_W defined by the commutative diagram

$$\begin{array}{ccc} \mathcal{D}(E|_\Omega)\oplus\mathcal{D}(\mathcal{I}) & \xrightarrow{U} & \mathcal{D}'(F|_\Omega)\oplus\mathcal{D}'(G) \\ \uparrow{\scriptstyle \chi_E^*\oplus\chi_{\mathcal{I}}^*} & & \downarrow{\scriptstyle (\chi_F)_*\oplus(\chi_G)_*} \\ \mathcal{D}(V\cap\mathbb{R}^n_+,\mathbb{C}^k)\oplus\mathcal{D}(V',\mathbb{C}^j) & \xrightarrow{U_W} & \mathcal{D}'(V\cap\mathbb{R}^n_+,\mathbb{C}^{k'})\oplus\mathcal{D}'(V',\mathbb{C}^{j'}) \end{array} \tag{2.8}$$

is called a local representation of U over W (with respect to the given trivialization).

Let $\varphi,\psi\in\mathcal{D}(V)$. The operator U_W induces the operator $\varphi U_W\psi I$: $\mathcal{D}(\mathbb{R}^n_+,\mathbb{C}^k)\oplus\mathcal{D}(\mathbb{R}^{n-1},\mathbb{C}^j)\to\mathcal{D}'(\mathbb{R}^n_+,\mathbb{C}^{k'})\oplus\mathcal{D}'(\mathbb{R}^{n-1},\mathbb{C}^{j'})$ (here the operator φI denotes multiplication of all components by φ or $\varphi|_{\mathbb{R}^{n-1}}$ respectively).

We shall say that an operator U of type (2.7) belongs to the class

$$OP\begin{pmatrix} \mu & \gamma_1,r_1 \\ \gamma_2,r_2 & \lambda \end{pmatrix}(E,F,\mathcal{I},G)$$

if for any open set $W\subset X$ possessing the above described properties and any functions $\varphi,\psi\in\mathcal{D}(V)$ the induced operator $\varphi U_W\psi\mathcal{I}$ belongs to the class $OP\begin{pmatrix} \mu & \gamma_1,r_1 \\ \gamma_2,r_2 & \lambda \end{pmatrix}$ (see Definition 2.7).

We shall give here another description of the class

$$OP\begin{pmatrix} \mu & \gamma_1,r_1 \\ \gamma_2,r_2 & \lambda \end{pmatrix}(E,F,\mathcal{I},G).$$

Let Π_0 be the bundle of densities over M, and Π_1 over Y (see, e.g., [49, v. 1, §§6.3, 6.4 and v.3, §18.1], [108, v. 2, §2.5]). Denote by Π the restriction of Π_0 to X. We can easily see that $\Pi' := \Pi\big|_Y = \Pi_0\big|_Y \cong \Pi_1$. This follows, for example, from the "collar" theorem (see [67, Theorem 5.9]). Therefore we shall identify Π' and Π_1.

Denote by F_0 an extension of bundle F on M and by F_0^* the bundle dual to F_0. Thus the transition matrices g_{ij} corresponding to F_0 are replaced by ${}^t g_{ij}^{-1}$ in the case of F_0^*.

Let $v\in\mathcal{D}'(F')$, where $F'=F|_Y$. Denote by $v\times\delta_Y$ distribution from $\mathcal{D}'(F_0)$ acting as follows:

$$\langle v\times\delta_Y,u\rangle=\langle v,\pi_0 u\rangle,\quad \forall u\in\mathcal{D}(F_0^*\otimes\Pi_0), \tag{2.9}$$

where π_0 is a restriction operator from M (or from X) to Y. Such a definition is correct. Indeed, M,Y are compact manifolds (without a boundary) and the restriction operator transforms $\mathcal{D}(F_0^*\otimes\Pi_0)$ to $\mathcal{D}((F')^*\otimes\Pi')$.

Similarly to Theorem 1.13 we can prove that if $v\in B^{s+1-1/p}_{p,p}(F')$ $\big(B^{s+1-1/p}_{p,q}(F')\big)$, then $v\times\delta_Y\in H^s_p(F_0)$ $\big(B^s_{p,q}(F_0)\big)$, provided $s<1/p-1$, $1<p<\infty$, $1\le q\le\infty$.

Assume

$$H_l \in OP(\widehat{O}^{\infty}_{\rho_l})(F_0, F_0), \quad Z_l \in OP(\widehat{O}^{\infty}_{\gamma_1-\rho_l})(\mathcal{I}, F'),$$
$$\mathcal{R}_m \in OP(\widehat{O}^{\infty}_{\gamma_2-\beta_m})(E', G), \quad L_m \in OP(\widehat{O}^{\infty}_{\beta_m})(E_0, E_0),$$
$$\mathcal{A} \in OP(\widehat{O}^{\infty}_{\mu})(E_0, F_0), \quad Q \in OP(\widehat{O}^{\infty}_{\lambda})(\mathcal{I}, G),$$

i.e., all they are ΨDOs acting in the section spaces of appropriate vector bundles which, in local coordinates, are written in terms of ΨDOs with symbols from the corresponding classes (see Definition 2.3). Here E_0 is an extension of the bundle E from X to M and E' is a restriction of E from X to Y; $\rho_l, \beta_m \in \mathbb{C}$, $\operatorname{Re}\rho_l \leq r_1$, $\operatorname{Re}\beta_m \leq r_2$, $\mu, \gamma_1, \gamma_2 \in \mathbb{C}$, $r_1, r_2 \in \mathbb{R}$, $\lambda = \gamma_1 + \gamma_2 - \mu + 1$, $l = 1, \ldots, l_0$, $m = 1, \ldots, m_0$, $l_0, m_0 \in \mathbb{N}$.

Consider an operator

$$U_0 = \begin{pmatrix} \pi_+ \mathcal{A} & \pi_+ \left(\sum_{l=1}^{l_0} H_l (Z_l(\cdot) \times \delta_Y) \right) \\ \sum_{m=1}^{m_0} \mathcal{R}_m \pi_0 L_m & Q \end{pmatrix} :$$
$$: \begin{array}{c} \mathcal{D}(E|_\Omega) \\ \oplus \\ \mathcal{D}(\mathcal{I}) \end{array} \to \begin{array}{c} \mathcal{D}'(F|_\Omega) \\ \oplus \\ \mathcal{D}'(G) \end{array}. \tag{2.10}$$

The space $\mathcal{D}(E|_\Omega)$ is considered to be embedded in $\mathcal{D}(E_0)$.

The set of operators of type (2.10) we denote by

$$OP' \begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix} (E, F, \mathcal{I}, G).$$

It is not difficult to see that

$$OP \begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix} (E, F, \mathcal{I}, G) = OP' \begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix} (E, F, \mathcal{I}, G). \tag{2.11}$$

In fact, to verify the embedding $OP' \subset OP$ let us take a sufficiently fine partition of unity $\sum_{i=1}^{i_0} \psi_i^2 = 1$ and insert it between operators H_l and Z_l, $\mathcal{R}_m$ and L_m. Upon localization the operators corresponding to a fixed value $i = 1, \ldots, i_0$ ($i_0 \in \mathbb{N}$), induce operators of the classes $OPI(\gamma_1, r_1)$ and $OPII(\gamma_2, r_2)$. Hence the operator U_0 induces that of the class

$$OP \begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}.$$

To verify the embedding $OP \subset OP'$ we shall again need sufficiently fine partition of unity $\sum_{i=1}^{i_0} \varphi_i = 1$. An operator of type (2.7) we represent as follows: $U = \sum\limits_{i,e=1}^{i_0} \varphi_i U \varphi_e I$. An operator $\varphi_i U \varphi_e I$ can be localized and then "removed" back on the manifold X in such a way that we obtain an operator of type (2.10). "Removing" is performed by means of the diagram of type (2.8). Thus we can represent the operator $\varphi_i U \varphi_e I$ in the form (2.10). Hence the operator U itself belongs to the class OP'.

In a standard way we prove the following statement.

An operator $U \in OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}(E, F, \mathcal{I}, G)$ *admits extension to the bounded operator*

$$\begin{gathered} U : H_1(s,p) = \widetilde{H}^s_p(E) \oplus B^{s-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1-1/p}_{p,p}(\mathcal{I}) \to \\ \to H_2(s,p) = H^{s-\operatorname{Re}\mu}_p(F) \oplus B^{s-\operatorname{Re}\gamma_2-1/p}_{p,p}(G) \\ \big(B_1(s,p,q) = \widetilde{B}^s_{p,q}(E) \oplus B^{s-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1-1/p}_{p,q}(\mathcal{I}) \to \\ \to B_2(s,p,q) = B^{s-\operatorname{Re}\mu}_{p,q}(F) \oplus B^{s-\operatorname{Re}\gamma_2-1/p}_{p,q}(G)\big) \end{gathered} \tag{2.12}$$

if $r_1 < \operatorname{Re}\mu - s - 1 + 1/p$, $r_2 < s - 1/p$, $1 < p < \infty$, $1 \le q \le \infty$.

The left upper corner of an operator U of type (2.7) contains a pseudodifferential operator $\mathcal{A}$. The principal homogeneous symbol of $\mathcal{A}$ (to which in local coordinates there corresponds A_0 from Definition 2.3) is a bundle morphism $\sigma_{\mathcal{A}} : \pi r^* E \to \pi r^* F$ (see, e.g., [82, 1.2.4.1]). Here $\pi r : T^*X \setminus \{0\} \to X$ is a canonic projection.

The morphism $\sigma_{\mathcal{A}}$ is said to be the principal interior symbol of the operator U and is denoted by $\sigma_\Omega(U)$.

The operator U is said to be elliptic if the operator $\mathcal{A}$ is elliptic, i.e. $\sigma_\Omega(U) = \sigma_{\mathcal{A}}$ is an isomorphism.

It is clear that for the operator U to be elliptic, it is necessary that the fibre dimensions of E and F be the same. We shall use the notations as in Chapter I: $k = k' = N$, $j = m_-$, $j' = m_+$.

Return now to the localization of the operator U of type (2.7). Let us take an open in W subset W_1 such that $\overline{W}_1 \subset W$. It maps on a set V_1. We shall assume the functions φ and ψ (see Definition 2.8) to be equal to identity in a neighbourhood of closure $\overline{V}_1 \subset V$. The operator $\varphi U_W \psi I$ belongs to the class $OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}$.

Take an arbitrary point $(x'_{(0)}, 0) \in V'_1 = V_1 \cap \mathbb{R}^{n-1}$. From the symbols of all ΨDOs composing the operator $\varphi U_W \psi I$ (see Definitions 2.4–2.7) we choose homogeneous principal parts (see Definitions 2.2–2.3). In these homogeneous symbols instead of arbitrary x and ξ (x' and ξ') we substitute $(x'_{(0)}, 0)$ and (ω, ξ_n), where $\omega \in S^{n-2} \subset \mathbb{R}^{n-1}$ ($x'_{(0)}$ and ω). For fixed $x'_{(0)}$ and ω we compose from the obtained symbols the operator on a semi-axis corresponding to the operator $\varphi U_W \psi I$ (see Definition 2.7 and also (1.111), (1.112)). Denote it by

$$\sigma_{W_1}(U)(x'_{(0)}, \omega) : \begin{matrix} \mathcal{D}(\mathbb{R}_+) \otimes \mathbb{C}^k \\ \oplus \\ \mathbb{C}^j \end{matrix} \to \begin{matrix} \mathcal{D}'(\mathbb{R}_+) \otimes \mathbb{C}^{k'} \\ \oplus \\ \mathbb{C}^{j'} \end{matrix}. \tag{2.13}$$

Let S^*Y be a cospherical bundle realized in cotangential bundle T^*Y by choosing a Riemannian metric on Y. The projection $\mathrm{pr} : S^*Y \to Y$ induces on S^*Y the bundles $\mathrm{pr}^* E'$, $\mathrm{pr}^* F'$, $\mathrm{pr}^* \mathcal{I}$, $\mathrm{pr}^* G$.

An easy checking shows (cf. [82, 2.3.3.1, Theorem 3]) that $\sigma_{W_1}(U)$ is a local representation of a bundle morphism

$$\sigma_Y(U) : \begin{matrix} \mathrm{pr}^* E' \otimes \mathcal{D}(\mathbb{R}_+) \\ \oplus \\ \mathrm{pr}^* \mathcal{I} \end{matrix} \to \begin{matrix} \mathrm{pr}^* F' \otimes \mathcal{D}'(\mathbb{R}_+) \\ \oplus \\ \mathrm{pr}^* G \end{matrix} . \tag{2.14}$$

The morphism $\sigma_Y(U)$ will be called a principal boundary symbol of the operator U.

Clearly the morphism $\sigma_Y(U)$ can be extended to the (continuous) morphism of bundles whose fibers are the corresponding Besov and Bessel-potential spaces (cf. Theorem 2.9 and (1.111), (1.112)).

We shall say that for an operator $U : H_1(s,p) \to H_2(s,p)$ $(B_1(s,p,q) \to B_2(s,p,q))$ of the class $OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}(E,F,\mathcal{I},G)$ (see Theorem 2.9) the Shapiro–Lopatinskiĭ condition is fulfilled if

$$\sigma_Y(U) : \begin{matrix} \mathrm{pr}^* E' \otimes \widetilde{H}^s_p(\overline{\mathbb{R}^1_+}) \\ \oplus \\ \mathrm{pr}^* \mathcal{I} \end{matrix} \to \begin{matrix} \mathrm{pr}^* F' \otimes \widetilde{H}^{s-\mathrm{Re}\,\mu}_p(\overline{\mathbb{R}^1_+}) \\ \oplus \\ \mathrm{pr}^* G \end{matrix} \tag{2.15}$$

$$\left(\sigma_Y(U) : \begin{matrix} \mathrm{pr}^* E' \otimes \widetilde{B}^s_{p,q}(\overline{\mathbb{R}^1_+}) \\ \oplus \\ \mathrm{pr}^* \mathcal{I} \end{matrix} \to \begin{matrix} \mathrm{pr}^* F' \otimes \widetilde{B}^{s-\mathrm{Re}\,\mu}_{p,q}(\overline{\mathbb{R}^1_+}) \\ \oplus \\ \mathrm{pr}^* G \end{matrix} \right)$$

is an isomorphism.

Note that the operator σ_{W_1} (see (2.13)) looks like the operator defined by the left parts of (1.111), (1.112). Hence due to the arguments following after Lemma 1.23, in Definition 2.13 we can restrict ourselves by consideration of the morphism $\sigma_Y(U)$ only in the scale of the Bessel-potential spaces.

§

Let $A(x,D) \in OP(S^m(\mathbb{R}^n \times \mathbb{R}^n))$, $m \in \mathbb{R}$ (see Definition 2.1 *and* (2.3)). *Then for any $x^0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there is a neighbourhood W_0 of the point x^0 such that for any $\varphi \in \mathcal{D}(W_0)$ the equality*

$$\varphi A(x,D) = \varphi\big(A(x^0,D) + A_{m-1}(x,D) + \mathcal{A}_\varepsilon\big)$$

is valid, where $A_{m-1}(x,D) \in OP(S^{m-1}(\mathbb{R}^n \times \mathbb{R}^n))$, $\mathcal{A}_\varepsilon$ is an operator which in a corresponding pair of (*Besov or Bessel-potential*) *spaces has a norm not exceeding ε.*

Proof. Let us take $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $\psi(x) = 1$ for $|x| \le 1$ and put

$$\psi_R(x) = \psi(x/R), \quad R > 0,$$
$$\psi_\rho^{x^0}(x) = \psi_\rho(x - x^0) = \psi((x - x^0)/\rho), \quad \rho > 0.$$

Consider operators

$$c(x, D) = \psi_\rho^{x^0}\big(A(x, D) - A(x^0, D)\big), \quad c_1(x, D) = c(x, D)I^{-m} \tag{2.16}$$

(see (1.10) for $a = (1, \dots, 1)$). We have

$$c_1(x, D) = c_1(x, D)(I - \psi_R(D)) + c_1(x, D)\psi_R(D) \equiv$$
$$\equiv c_2(x, D) + c_3(x, D). \tag{2.17}$$

It is easily seen (see (2.1)) that

$$c_3(x, D) \in OP\big(S^\sigma(\mathbb{R}^n \times \mathbb{R}^n)\big), \quad \forall \sigma \in \mathbb{R}, \tag{2.18}$$
$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n} |c_2(x, \xi)| \le const \cdot \rho \tag{2.19}$$

(here and up to the end of the proof "const" denotes values not depending on ρ). Moreover, $c_2(x, \xi) = 0$ if $|\xi| \le R$. Choose R large enough to attain the fulfillment of the inequality

$$\|c_2(x, D)|L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)\| \le const \cdot \rho \tag{2.20}$$

(see [49], v. 3, Theorem 18.1.15).

On the other hand

$$\|c_2(x, D)|L_r(\mathbb{R}^n) \to L_r(\mathbb{R}^n)\| \le \|\psi_\rho^{x^0} I|L_r(\mathbb{R}^n) \to L_r(\mathbb{R}^n)\| \times$$
$$\times\|(A(x, D) - A(x^0, D))I^{-m}(I - \psi_R(D))|L_r(\mathbb{R}^n) \to L_r(\mathbb{R}^n)\| \le$$
$$\le const, \quad \forall r \in]1, \infty[, \tag{2.21}$$

(see (2.16),(2.17)).

Apply Riesz–Torin convexity theorem (see, e.g., [109, Theorem 1.1.2.1]) or more general interpolation theorem 1.2 -c) (or -d)) to obtain

$$\|c_2(x, D)|L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| \le const \cdot \rho^{\frac{2}{\max\{p,p'\}} - \delta}, \ \ p' = \frac{p}{p-1}, \tag{2.22}$$

where $\delta > 0$ is an arbitrarily small number (and the constant depends on δ). Below we shall take $\delta = 1/\max\{p, p'\}$.

According to [49, v. 3, Theorem 18.1.8],

$$c(x, D) = I^{m-s}c_2(x, D)I^s + A_{m-1}(x, D), \ \ A^{m-1} \in S^{m-1}(\mathbb{R}^n \times \mathbb{R}^n) \tag{2.23}$$

(see (2.16)–(2.18)). For the operator

$$\mathcal{A}_\varepsilon = I^{m-s}c_2(x, D)I^s \tag{2.24}$$

we have

$$\|\mathcal{A}_\varepsilon | H_p^s(\mathbb{R}^n) \to H_p^{s-m}(\mathbb{R}^n)\| =$$
$$= \|c_2(x,D) | L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| \le const \cdot \rho^{1/max\{p,p'\}} \tag{2.25}$$

(see (1.11)).

Making $\rho > 0$ sufficiently small and taking as W_0 a neighbourhood whose diameter does not exceed ρ, we can see that the statement of the lemma is valid for Bessel-potential spaces since $\varphi\psi_\rho^{x^0} = \varphi$, $\forall\varphi \in \mathcal{D}(W_0)$ (see also (2.16), (2.23)–(2.25)). It remains for us to consider the case of the Besov spaces.

Take $\tau \in]0, \frac{1}{2\max\{p,p'\}}[$ and represent the operator $c(x,D)$ as follows:

$$c(x,D) = c_4(x,D) + A_{m-1}(x,D), \tag{2.26}$$

where

$$c_4(x,D) = I^{m-s+\tau} c_2(x,D) I^{s-\tau}, \quad A_{m-1} \in S^{m-1}(\mathbb{R}^n \times \mathbb{R}^n) \tag{2.27}$$

(see [49, v. 3, Theorem 18.1.8] and (2.23)). Note that in (2.23) and (2.26) $A_{m-1}(x,D)$ denotes different operators.

Due to (2.22) and (1.11)

$$\|c_4(x,D) | H_p^{s-\tau}(\mathbb{R}^n) \to H_p^{s-\tau-m}(\mathbb{R}^n)\| \le const \cdot \rho^{1/max\{p,p'\}}. \tag{2.28}$$

It is easily seen that

$$\|\psi_\rho^{x^0} I | L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| \le const,$$
$$\|\psi_\rho^{x^0} I | W_p^1(\mathbb{R}^n) \to W_p^1(\mathbb{R}^n)\| \le const \cdot \rho^{-1}.$$

Using the interpolation (see (1.13) and Theorem 1.2-c)) we obtain

$$\|\psi_\rho^{x^0} I | H_p^{2\tau}(\mathbb{R}^n) \to H_p^{2\tau}(\mathbb{R}^n)\| \le const \cdot \rho^{-2\tau}.$$

Therefore

$$\|c_2(x,D) | H_p^{2\tau}(\mathbb{R}^n) \to H_p^{2\tau}(\mathbb{R}^n)\| \le const \cdot \rho^{-2\tau}$$

(cf. (2.21)). Hence

$$\|c_4(x,D) | H_p^{s+\tau}(\mathbb{R}^n) \to H_p^{s+\tau-m}(\mathbb{R}^n)\| =$$
$$= \|c_2(x,D) | H_p^{2\tau}(\mathbb{R}^n) \to H_p^{2\tau}(\mathbb{R}^n)\| \le const \cdot \rho^{2\tau} \tag{2.29}$$

(cf. (2.27) and (1.11)).

Let us use interpolation once more (see Theorem 1.2-e)):

$$\|c_4(x,D) | B_{p,q}^s(\mathbb{R}^n) \to B_{p,q}^{s-m}(\mathbb{R}^n)\| \le$$
$$\le const \|c_4(x,D) | H_p^{s-\tau}(\mathbb{R}^n) \to H_p^{s-\tau-m}(\mathbb{R}^n)\|^{1/2} \times$$
$$\times \|c_4(x,D) | H_p^{s+\tau}(\mathbb{R}^n) \to H_p^{s+\tau}(\mathbb{R}^n)\|^{1/2} \le$$

$$\leq const \cdot \rho^{\frac{1}{2\max\{p,p'\}}} \cdot \rho^{-\tau} = const \cdot \rho^{\theta},$$

where $\theta > 0$ since $\tau < 1/(2\max\{p,p'\})$ (see (2.28), (2.29)).

Letting $\mathcal{A}_\varepsilon = c_4(x, D)$, we can accomplish the proof in exactly the same way as in the case of Besel-potential spaces. ■

Basing on the proven lemma we can use partition of unity, "freezing of coefficients", straightening of the boundary (see [37, §22] and [31, part II]) and then Theorem 1.24 to prove the following result.

Let for an elliptic operator (*see Definition* 2.11)

$$U \in OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}(E, F, \mathcal{I}, G)$$

the conditions of Theorem 2.9 *and the Shapiro–Lopatinskiĭ condition* (*see Definition* 2.13) *be fulfilled. Then the operator* $U : H_1(s,p) \to H_2(s,p)$ $(B_1(s,p,q) \to B_2(s,p,q))$ *is Noetherian.*

The proof is similar to that of [37, Theorem 22.1]. (Instead of [37, Theorem 19.4] we have to use Lemma 2.14 and instead of [37, Lemma 21.1 and Remark 21.1] we need [110, Remark 4.3.2.1]).

In the sequel we shall need some facts from the functional analysis.

Suppose that the Banach spaces Z_0, Z_1 are embedded continuously into a Hausdorff topological vector space Z. In this case a pair $\{Z_0, Z_1\}$ is called compatible.

It is not difficult to prove that for any compatible pair $\{Z_0, Z_1\}$ the spaces $Z_0 \cap Z_1$ and $Z_0 + Z_1 = \big\{f \in Z : f = f_0 + f_1,\ f_j \in Z_j, j = 0, 1\big\}$ are Banach ones with respect to the norms

$$\|f|Z_0 \cap Z_1\| = \max\big\{\|f|Z_0\|,\ \|f|Z_1\|\big\},$$
$$\|f|Z_0 + Z_1\| = \inf\big\{\|f_0|Z_0\| + \|f_1|Z_1\| \big| f = f_0 + f_1,\ f_j \in Z_j,\ j = 0, 1\big\}$$

and continuous embeddings

$$Z_0 \cap Z_1 \subset Z_j \subset Z_0 + Z_1, \ \ j = 0, 1, \tag{2.30}$$

hold (see, e.g., [109, Lemma 1.21]).

For brevity the use will be made of the following notation:

$$Z_{\min} = Z_0 \cap Z_1, \quad Z_{\max} = Z_0 + Z_1. \tag{2.31}$$

For any Banach spaces Z_0, Q_0 we shall denote by $\mathcal{L}(Z_0, Q_0)$ $(\mathrm{Com}(Z_0, Q_0))$ the set of all linear continuous (compact) operators acting from Z_0 to Q_0.

For any compatible pairs $\{Z_0, Z_1\}$, $\{Q_0, Q_1\}$ *the embeddings*

$$\mathcal{L}(Z_0, Q_0) \cap \mathcal{L}(Z_1, Q_1) \subset \mathcal{L}(Z_{\min}, Q_{\min}) \cap \mathcal{L}(Z_{\max}, Q_{\max}),$$
$$\mathrm{Com}(Z_0, Q_0) \cap \mathrm{Com}(Z_1, Q_1) \subset \mathrm{Com}(Z_{\min}, Q_{\min}) \cap \mathrm{Com}(Z_{\max}, Q_{\max}).$$

hold.

Proof. The first embedding follows directly from the definition of norms in spaces $Z_{\min}$ and $Q_{\min}$, $Z_{\max}$ and $Q_{\max}$. Let us prove the second embedding.

Take an arbitrary $T \in \mathrm{Com}(Z_0, Q_0) \cap \mathrm{Com}(Z_1, Q_1)$ and an arbitrary bounded sequence $\{x_n\}_{n\in\mathbb{N}}$ in $Z_{\min}$. From the compactness of $T : Z_0 \to Q_0$ follows the existence of a subsequence (we denote it again by $\{x_n\}_{n\in\mathbb{N}}$) such that $\{Tx_n\}_{n\in\mathbb{N}} \subset Q_0 \cap Q_1$ converges in Q_0. Using the compactness of $T : Z_1 \to Q_1$, we can choose from that subsequence a subsequence (which we again denote by $\{x_n\}_{n\in\mathbb{N}}$) such that $\{Tx_n\}_{n\in\mathbb{N}} \subset Q_0 \cap Q_1$ converges in Q_1. Then the sequence $\{Tx_n\}_{n\in\mathbb{N}}$ converges in $Q_{\min}$. Therefore $T \in \mathrm{Com}(Z_{\min}, Q_{\min})$.

Denote by S_0, S_1 and $S_{\max}$ unit balls in spaces Z_0, Z_1 and $Z_{\max}$, respectively. Clearly, $S_{\max} \subset S_0 + S_1$. Take an arbitrary $\varepsilon > 0$. From the compactness of $T : Z_j \to Q_j$, $j = 0, 1$, it follows that for the set $T(S_j)$ in the space Q_j there exists an $\varepsilon/2$-net $\{y_1^{(j)}, \ldots, y_m^{(j)}\}$:

$$\forall y^{(j)} \in T(S_j) \ \ \exists k \in \overline{1,m} : \|y^{(j)} - y_k^{(j)}|Q_j\| < \frac{\varepsilon}{2} \ \ (j = 0, 1).$$

It is evident that the set $\{y_k^{(0} + y_l^{(1)}\}_{k,l\in\overline{1,m}}$ is an ε-net in the space $Q_{\max}$ for $T(S_0) + T(S_1) = T(S_0 + S_1)$, and hence for $T(S_{\max})$, since $S_{\max} \subset S_0 + S_1$. Thus $T \in \mathrm{Com}(Z_{\max}, Q_{\max})$. ∎

For any Banach spaces Z_0, Q_0 we denote by $\Phi(Z_0, Q_0)$ the set of all Noetherian operators acting from Z_0 to Q_0.

[3] *Let $\{Z_0, Z_1\}$, $\{Q_0, Q_1\}$ be compatible pairs and the embedding $Z_{\min} \subset Z_{\max}$, $Q_{\min} \subset Q_{\max}$ be dense. If the operator $\mathcal{A} \in \Phi(Z_0, Q_0)$ $\cap\, \Phi(Z_1, Q_1)$ has a common regularizer $\mathcal{R} \in \mathcal{L}(Q_0, Z_0) \cap \mathcal{L}(Q_1, Z_1)$:*

$$\begin{aligned} &\mathcal{R}\mathcal{A} - I \in \mathrm{Com}(Z_0, Z_0) \cap \mathrm{Com}(Z_1, Z_1), \\ &\mathcal{A}\mathcal{R} - I \in \mathrm{Com}(Q_0, Q_0) \cap \mathrm{Com}(Q_1, Q_1), \end{aligned} \tag{2.32}$$

then

$$\mathrm{Ind}_{Z_{\min}\to Q_{\min}} \mathcal{A} = \mathrm{Ind}_{Z_j\to Q_j} \mathcal{A} = \mathrm{Ind}_{Z_{\max}\to Q_{\max}} \mathcal{A}, \ \ j = 0, 1, \tag{2.33}$$

any solution $f \in Z_{\max}$ of equation $\mathcal{A}f = g$, $g \in Q_j$, belongs to the space Z_j and, in particular,

$$\mathrm{Ker}_{Z_{\min}} \mathcal{A} = \mathrm{Ker}_{Z_j} \mathcal{A} = \mathrm{Ker}_{Z_{\max}} \mathcal{A}, \quad j = 0, 1. \tag{2.34}$$

Proof. From Lemma 2.16 and the conditions of the statement it follows that $\mathcal{R} \in \mathcal{L}(Q_{\min}, Z_{\min}) \cap \mathcal{L}(Q_{\max}, Z_{\max})$, $\mathcal{R}\mathcal{A} - I \in \mathrm{Com}(Z_{\min}, Z_{\min})$ $\cap \mathrm{Com}(Z_{\max}, Z_{\max})$, $\mathcal{A}\mathcal{R} - I \in \mathrm{Com}(Q_{\min}, Q_{\min}) \cap \mathrm{Com}(Q_{\max}, Q_{\max})$. Therefore $\mathcal{A} \in \Phi(Z_{\min}, Q_{\min}) \cap \Phi(Z_{\max}, Q_{\max})$ (see, e.g., [79, Ch. I, Theorem 4.3]), and the equality (2.33) makes sense. Note that each of relations (2.32) is a

[3] Similar statement can be found in [53]. The idea of the proof given here is borrowed from [76, 6.4].

consequence of the other since $\mathcal{A} \in \Phi(Z_0, Q_0) \cap \Phi(Z_1, Q_1)$ (see [79, Ch. I, Corollary 4.3]).

Let us show now that from density of the embedding $Q_{\min} \subset Q_{\max}$ follows that of $Q_{\min} \subset Q_j$, $j = 0, 1$. For the sake of definiteness we take $j = 0$. According to the condition for any $\varepsilon > 0$ and $g \in Q_0$ there exists $h \in Q_{\min}$ such that $\|g - h|Q_{\max}\| < \varepsilon$, i.e. $\exists g_0 \in Q_0$, $g_1 \in Q_1 : g - h = g_0 + g_1$,

$$\|g_0|Q_0\| + \|g_1|Q_1\| < \varepsilon;$$

$$g \in Q_0, \quad h \in Q_{\min} \subset Q_0 \Rightarrow g - h \in Q_0 \Rightarrow g_1 = (g - h) - g_0 \in Q_0 \Rightarrow$$
$$\Rightarrow g_1 \in Q_0 \cap Q_1 = Q_{\min} \Rightarrow g_1 + h \in Q_{\min}.$$

Moreover, $\|g - (g_1 + h)|Q_0\| = \|g_0|Q_0\| < \varepsilon$. Thus the embedding $Q_{\min} \subset Q_0$ is dense.

Density of embeddings $Q_{\min} \subset Q_j \subset Q_{\max}$, $j = 0, 1$, implies embeddings of conjugate spaces $Q^*_{\max} \subset Q^*_j \subset Q^*_{\min}$, $j = 0, 1$. This and embeddings $Z_{\min} \subset Z_j \subset Z_{\max}$, $j = 0, 1$, yield

$$\operatorname{Ker}_{Z_{\min}} \mathcal{A} \subset \operatorname{Ker}_{Z_j} \mathcal{A} \subset \operatorname{Ker}_{Z_{\max}} \mathcal{A}, \tag{2.35}$$

$$\operatorname{Ker}_{Q^*_{\max}} \mathcal{A}^* \subset \operatorname{Ker}_{Q^*_j} \mathcal{A}^* \subset \operatorname{Ker}_{Q^*_{\min}} \mathcal{A}^*. \tag{2.36}$$

We shall denote by $n_{\min}, n_j, n_{\max}$ kernel dimensions $\dim \operatorname{Ker} \mathcal{A}$ in the corresponding spaces and by $m_{\min}, m_j, m_{\max}$ cokernel dimensions $\dim \operatorname{Coker} \mathcal{A}$ $= \dim \operatorname{Ker} \mathcal{A}^*$ (see, e.g., [79, Ch.I, (3.1)]). Then from (2.35), (2.36) it follows that

$$n_{\min} \leq n_j \leq n_{\max}, \quad m_{\min} \geq m_j \geq m_{\max}, \quad j = 0, 1. \tag{2.37}$$

By the definition of the index we have

$$\operatorname{Ind}_{Z_{\min} \to Q_{\min}} \mathcal{A} \leq \operatorname{Ind}_{Z_j \to Q_j} \mathcal{A} \leq \operatorname{Ind}_{Z_{\max} \to Q_{\max}} \mathcal{A}. \tag{2.38}$$

Analogous inequalities for the regularizer $\mathcal{R}$ can be proved similarly. But $\operatorname{Ind} \mathcal{R} = -\operatorname{Ind} \mathcal{A}$ (see, e.g., [79, Ch. I, Theorems 3.6, 3.7]). Therefore the equalities take place in (2.38), hence we have proved (2.33).

In virtue of (2.37) the equalities in (2.38) may be achieved if and only if $n_{\min} = n_j = n_{\max}$, $\quad m_{\min} = m_j = m_{\max}$. Taking into account (2.35), we obtain (2.34). It is also obvious that the equalities take place in (2.36). Hence if the equation $\mathcal{A}f = g$, $g \in Q_j$, has a solution $f \in Z_{\max}$, then it is solvable in the space Z_j too (see [79, 1.2.4]). It remains to note that any two solutions of that equation in the space $Z_{\max}$ differ by an element from

$$\operatorname{Ker}_{Z_{\max}} \mathcal{A} = \operatorname{Ker}_{Z_j} \mathcal{A} \subset Z_j. \qquad \blacksquare$$

Return now to the operator U (see (2.7)). Its principal interior symbol $\sigma_\Omega(U) = \sigma_{\mathcal{A}}$ in local coordinates defines the matrix $\sigma^{(l)}_{\mathcal{A}}(x^{(l)}, \xi^{(l)})$ (l is a number of local coordinate system). Assume that we have to do with the boundary coordinate neighbourhood. Consider the matrix

$$\sigma^{(l)}(x'_{(l)}) = \left(\sigma^{(l)}_{\mathcal{A}}(x'_{(l)}, 0, 0, -1)\right)^{-1} \sigma^{(l)}_{\mathcal{A}}(x'_{(l)}, 0, 0, +1).$$

When we study boundary value problems in a half-space (see [37], [31] and Ch. I) eigenvalues $\lambda_1^{(l)}(x'_{(l)}), \dots, \lambda_N^{(l)}(x'_{(l)})$ of the matrix $\sigma^{(l)}(x'_{(l)})$ play an essential role. It is not difficult to see (cf. [37, §22] and [82, Theorem 2.3.3.1-3]) that these eigenvalues in fact do not depend on the choice of local coordinate system. Thus functions $\lambda_1(x'), \dots, \lambda_N(x')$ are defined on Y.

From the arguments following after Lemma 1.22 it follows that for the Shapiro–Lopatinskiĭ condition (see Definition 2.13) to be fulfilled, it is necessary the fulfillment of the condition

$$s - \frac{\operatorname{Re}\mu}{2} + \frac{1}{2\pi}\arg\lambda_m(x') - \frac{1}{p} \notin \mathbb{Z},\ \ m=1,\dots,N,\ \ \forall x' \in Y. \tag{2.39}$$

Remark. Consider, for example, the case of a scalar elliptic ΨDO. Then the continuous function $\lambda_1(x')$ is defined uniquely. If $\frac{\lambda_1(x')}{|\lambda_1(x')|}$ fills the entire unit circumference when $x' \in Y$ varies, then (2.39) obviously fails to be fulfilled for any s and p.

If for given elliptic ΨDO condition (2.39) is fulfilled for no s, then it is natural to consider for it boundary value problems in function spaces of piecewise-constant order of smoothness. This can be done as in [37, §25].

Consider the set

$$\mathbb{Z}(\mathcal{A}) = \left\{ \frac{\operatorname{Re}\mu}{2} - \frac{1}{2\pi}\arg\lambda_m(x') + \ell \,\middle|\, \ell \in \mathbb{Z},\ m=1,\dots,N,\ x' \in Y \right\}. \tag{2.40}$$

This set is closed. Really, by virtue of the compactness of Y we can see that the set $\{\lambda_m(x') | m = 1, \dots, N,\ x' \in Y\}$ of zeros of a polynomial whose coefficients depend continuously on $x' \in Y$ is compact. It remains to note that the function $\frac{1}{2\pi}\arg$ has an integer jump at a point of discontinuity.

We can rewrite (2.39) as follows:

$$s - 1/p \notin \mathbb{Z}(\mathcal{A}). \tag{2.41}$$

Suppose (2.41) is fulfilled and introduce the notation (cf. (1.113)–(1.115))

$$s_+ = \min\left\{ \operatorname{Re}\mu - r_1 - 1, t | t \in \mathbb{Z}(\mathcal{A}),\ t > s - 1/p \right\}, \tag{2.42}$$

$$s_- = \max\left\{ r_2, t | t \in \mathbb{Z}(\mathcal{A}),\ t < s - 1/p \right\} \tag{2.43}$$

($\min\{\cdots\}$ and $\max\{\cdots\}$ do exist since $\mathbb{Z}(\mathcal{A})$ is closed). Clearly $s_- < s - 1/p < s_+$ if the conditions of Theorem 2.9 are fulfilled.

For arbitrary $t_-, t_+ \in \mathbb{R}$, $t_- < t_+$, denote by $\sum(t_-, t_+)$ the union of all spaces $H_1(s,p)$, $s \in \mathbb{R}$, $p \in]1,\infty[$, $t_- < s - 1/p < t_+$. From the embedding theorem (see, for example, [109, Theorem 4.6.1] or Theorem 1.6) it follows that $\sum(t_-, t_+)$ is equal to the union of all spaces $B_1(s,p,q)$, $s \in \mathbb{R}$, $p \in]1,\infty[$, $q \in [1,\infty]$, $t_- < s - 1/p < t_+$.

Let for an elliptic operator $U : H_1(s,p) \to H_2(s,p)$ $(B_1(s,p,q) \to B_2(s,p,q))$ *of the class* $OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}(E,F,\mathcal{I},G)$

the conditions of Theorem 2.9 *and the Shapiro–Lopatinskiĭ condition be fulfilled. Then for any* $s^* \in \mathbb{R}$, $p^* \in]1,\infty[$, *satisfying*

$$s_- < s^* - 1/p^* < s_+, \tag{2.44}$$

and any $q, q^* \in [1,\infty]$ *the equalities*

$$\begin{aligned}\operatorname{Ind} U\big(H_1(s,p) \to H_2(s,p)\big) = \operatorname{Ind} U\big(B_1(s,p,q) \to B_2(s,p,q)\big) = \\ = \operatorname{Ind} U\big(H_1(s^*,p^*) \to H_2(s^*,p^*)\big) = \\ = \operatorname{Ind} U\big(B_1(s^*,p^*,q^*) \to B_2(s^*,p^*,q^*)\big)\end{aligned} \tag{2.45}$$

are valid. Moreover, if $g \in H_2(s^*,p^*)$ $(B_2(s^*,p^*,q^*))$, *then any solution* $f \in \sum(s_-,s_+)$ *of the equation*

$$Uf = g \tag{2.46}$$

(*if it exists*) *belongs to* $H_1(s^*,p^*)$ $(B_1(s^*,p^*,q^*))$. *If however*

$$g \in B^{t-\operatorname{Re}\mu}_{\infty,\infty}(F) \oplus B^{t-\operatorname{Re}\gamma_2}_{\infty,\infty}(G), \quad t \in]s_-,s_+[,$$

then

$$f \in \bigcap_{\tau<t} \big(\widetilde{B}^{\tau}_{\infty,\infty}(E) \oplus B^{\tau-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1}_{\infty,\infty}(\mathcal{I})\big).$$

Proof. Let us begin with the equality (2.45). By Theorems 2.15 and 1.24, all the operators contained in it are Noetherian. Analyzing the proofs of these theorems, we can see that the above-mentioned operators have a common regularizer (in the sense of (2.32)). It suffices for us to prove the equality

$$\operatorname{Ind} U\big(H_1(s,p) \to H_2(s,p)\big) = \operatorname{Ind} U\big(B_1(s^*,p^*,q^*) \to B_2(s^*,p^*,q^*)\big) \tag{2.47}$$

since s^*, p^* and q^* may in particular coincide with s, p and q, respectively.

If $q^* < \infty$, then the conditions of Lemma 2.17 are fulfilled (see, e.g., [109, Theorem 2.3.2] and [109, Remark 2.10.3.-1] or Lemma 1.8), and (2.47) follows from (2.33). If however $q^* = \infty$, then we have to apply Lemma 2.17 first to the pairs $\{H_1(s,p), B_1(s^*-\varepsilon,p^*,1)\}$, $\{H_2(s,p), B_2(s^*-\varepsilon,p^*,1)\}$ and then to the pairs $\{B_1(s^*-\varepsilon,p^*,1), B_1(s^*,p^*,\infty)\}$, $\{B_2(s^*-\varepsilon,p^*,1), B_2(s^*,p^*,\infty)\}$, where $\varepsilon > 0$ is a sufficiently small number (see [109, Theorem 2.3.2-(c)] or Theorem 1.6-a)).

The first assertion of the theorem concerning equation (2.46) is proved analogously since by the definition of $\sum(s_-,s_+)$ there are numbers $s^0 \in \mathbb{R}$ and $p^0 \in]1,\infty[$ for $f \in \sum(s_-,s_+)$ such that $f \in H_1(s^0,p^0)$ and $s_- < s^0 - 1/p^0 < s_+$.

Let us prove now the last statement of the theorem. Fix an arbitrary $\tau \in]s_-,t[$ and take $s^* \in]\tau,t[$, $p^* \in]1,\infty[$ such that $s^* - n/p^* \geq \tau$. Then $g \in B_2(s^*,p^*,\infty)$ (see [110, Theorem 3.3.1]) and according to already proven and embedding theorem (see [110, 3.3.1])

$$f \in B_1(s^*,p^*,\infty) \subset \widetilde{B}^{\tau}_{\infty,\infty}(E) \oplus B^{\tau-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1}_{\infty,\infty}(\mathcal{I}). \qquad \blacksquare$$

Remark. The above proven theorem enables us to reduce the investigation of the problem on the index of boundary value problems for elliptic ΨDOs in Besov and Bessel-potential spaces to its investigation in the case of spaces H_2^σ. For this it suffices to replace p by 2 and s by $s^* \in]s_- + 1/2, s_+ + 1/2[$ in the indices of the corresponding spaces.

Note that the index L_2-theory for a wide algebra of elliptic boundary value problems (without transmission property) has been constructed in [83] (see also [84]).

Remark. The second part of Theorem 2.19 is an assertion typical for the theory of elliptic boundary value problems that the increase of data smoothness implies that of the solution smoothness. (Recall incidentally that for $s > 0$ the space $B^s_{\infty,\infty}$ is a Hölder–Zygmund space by (1.18), (1.19)). In the case under consideration this however takes place within rather narrow interval $]s_-, s_+[$ (see (2.44)). In particular the solution of equation (2.46) may not belong to $\bigcup\limits_{\tau > s_+} \big(\widetilde{B}^\tau_{\infty,\infty}(E) \oplus B^{\tau - \mathrm{Re}\,\mu + \mathrm{Re}\,\gamma_1 + 1}_{\infty,\infty}(\mathcal{I})\big)$ even for infinitely smooth g. This follows from asymptotic properties of solutions of boundary value problems for elliptic ΨDOs in the neighbourhood of the boundary (see [37, §9], [85], [102], [103], [104]).

§

Ψ

Consider an (elliptic) operator

$$U \in OP \begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix} (E, F, \mathcal{I}, G). \tag{2.48}$$

For the Shapiro–Lopatinskiĭ conditions to be fulfilled, it is necessary that the mapping

$$\sigma_Y(\mathcal{A}) = P_1 \sigma_Y(U) I_1 : \mathrm{pr}^* E' \otimes \widetilde{H}^s_p(\overline{\mathbb{R}^1_+}) \to \mathrm{pr}^* F' \otimes H^{s - \mathrm{Re}\,\mu}_p(\overline{\mathbb{R}^1_+}) \tag{2.49}$$

generated by the morphism (2.15) be a family of Noetherian operators. Here

$$I_1 : \mathrm{pr}^* E' \otimes \widetilde{H}^s_p(\overline{\mathbb{R}^1_+}) \to \mathrm{pr}^* E' \otimes \widetilde{H}^s_p(\overline{\mathbb{R}^1_+}) \oplus \mathrm{pr}^* \mathcal{I} \tag{2.50}$$

is an embedding and

$$P_1 : \mathrm{pr}^* F' \otimes H^{s - \mathrm{Re}\,\mu}_p(\overline{\mathbb{R}^1_+}) \oplus \mathrm{pr}^* G \to \mathrm{pr}^* F' \otimes H^{s - \mathrm{Re}\,\mu}_p(\overline{\mathbb{R}^1_+}) \tag{2.51}$$

is a canonic projection. The expression "family of Noetherian operators" means that $\sigma_Y(\mathcal{A})$ defines a Noetherian operator for each fibre.

Noetherity of the family $\sigma_Y(\mathcal{A})$ is equivalent to the condition (2.39) (see [29], [30, §12]). We shall assume this condition to be fulfilled. Then the family $\sigma_Y(\mathcal{A})$ defines the index element $\mathrm{ind}_{S^*Y}\, \sigma_Y(\mathcal{A}) \in K(S^*Y)$ (see [82, 3.1.1.1], [37, §16]).

For an elliptic operator (2.48) *let the conditions of Theorem* 2.9 *and the Shapiro–Lopatinskiĭ condition be fulfilled. Then*

$$\operatorname{ind}_{S^*Y} \sigma_Y(\mathcal{A}) = [\operatorname{pr}^* G] - [\operatorname{pr}^* \mathcal{I}] \tag{2.52}$$

and hence

$$\operatorname{ind}_{S^*Y} \sigma_Y(\mathcal{A}) \in \operatorname{pr}^* K(Y) \tag{2.53}$$

($\operatorname{pr}^* K(Y)$ *is an inverse image of the group* $K(Y)$ *with respect to the canonic projection* $\operatorname{pr} : S^*Y \to Y$).

Proof. The proof is analogous to that of [82, Theorem 3.1.1.1–11]. The difference is that we cannot take the family of operators defined by $\sigma_{\mathcal{A}}^{-1}$ as a regularizer of $\sigma_Y(\mathcal{A})$. We should act as follows: consider the morphism $(\sigma_Y(U))^{-1}$ inverse to $\sigma_Y(U)$. The family $P_1(\sigma_Y(U))^{-1}I_1$ (see (2.50), (2.51)) will be the family of regularizers of $\sigma_Y(\mathcal{A})$. ■

Let $\mathcal{A} \in OP(\widehat{O}_\mu^\infty)(E_0, F_0)$ (*where* E_0 *and* F_0 *are extensions of bundles* E *and* F *from* X *to* M) *be an elliptic* ΨDO *satisfying the condition* (2.39), *and for the morphism* (2.49) *the condition* (2.53) *be fulfilled. Then there exist bundles* $\mathcal{I}, G$ *over* Y *and the operator* (2.48) *with* $\pi_+\mathcal{A}$ (*see* (2.7)) *in the left upper corner for which the Shapiro–Lopatinskiĭ condition is fulfilled. Moreover,* $r_1, r_2 \in \mathbb{R}$, $\gamma_1, \gamma_2 \in \mathbb{C}$ *are arbitrary numbers satisfying the conditions* $r_1 < \operatorname{Re}\mu - s - 1 + 1/p$, $r_2 < s - 1/p$.

Proof. It is not difficult to see that there exists a finite-dimensional subspace L of the space $S(\mathbb{R}^1)$ such that

$$\begin{aligned}\big(\operatorname{im} \sigma_Y(\mathcal{A})\big)_{(x',\xi')} + (\operatorname{pr}^* F')_{(x',\xi')} \otimes \pi_+ L = \\ = (\operatorname{pr}^* F')_{(x',\xi')} \otimes H_p^{s-\operatorname{Re}\mu}(\overline{\mathbb{R}^1_+}), \quad \forall (x',\xi') \in S^*Y\end{aligned} \tag{2.54}$$

(see [82, 3.1.1.1], [37, §16]).

Denote $l = \dim \pi_+ L$. Then the isomorphism $\operatorname{pr}^* F' \otimes \mathbb{C}^l \cong \operatorname{pr}^* F' \otimes \pi_+ L$ can be realized in terms of

$$I \otimes \pi_+ \sigma_Y(K)[\cdot\delta] : \operatorname{pr}^* F' \otimes \mathbb{C}^l \to \operatorname{pr}^* F' \otimes \pi_+ L$$

(see [37, §16]). Explain the notation: $\sigma_Y(K)$ is a matrix ΨDO on $\mathbb{R}^1$ acting with respect to the variable in the argument of δ-function. Components of this ΨDO have infinitely smooth rapidly decreasing symbols ("with constant coefficients").

We can easily verify that

$$\operatorname{ind}_{S^*Y} \sigma_Y(\mathcal{A}) = \big[\ker_{S^*Y} \big(\sigma_Y(\mathcal{A}), I \otimes \pi_+ \sigma_Y(K)\delta\big)\big] - [\operatorname{pr}^* F' \otimes \mathbb{C}^l] \tag{2.55}$$

(see [82, 1.1.3.4]).

Note that the assertion of [82, Lemma 3.1.1.2-1] will hold if instead of $\mathbb{C}^N$ we take $p^*\mathcal{I}_0$ (in notations of [82]) for a bundle $\mathcal{I}_0 \in \operatorname{Vect}(Y)$. Therefore it follows from (2.53), (2.55) that there exist smooth vector bundles $\mathcal{I}_1, G_0$

over Y for which $G_1 \oplus \mathrm{pr}^* \mathcal{I}_1 \cong \mathrm{pr}^* G_0$, where $G_1 = \ker_{S^*Y}(\sigma_Y(\mathcal{A}), I \otimes \pi_+\sigma_Y(K)\delta)$.

Let us take a zero morphism $O : \mathrm{pr}^* \mathcal{I}_1 \to \mathrm{pr}^* F' \otimes H_p^{s-\mathrm{Re}\,\mu}(\overline{\mathbb{R}^1_+})$ and consider an epimorphism

$$\mathrm{Epi} := \big(\sigma_Y(\mathcal{A}), I \otimes \pi_+\sigma_Y(K)\delta, O\big) : \big(\mathrm{pr}^* E' \otimes \widetilde{H}_p^s(\overline{\mathbb{R}^1_+})\big) \oplus \\ \oplus(\mathrm{pr}^* F' \otimes \mathbb{C}^l) \oplus \mathrm{pr}^* \mathcal{I}_1 \to \mathrm{pr}^* F' \otimes H_p^{s-\mathrm{Re}\,\mu}(\overline{\mathbb{R}^1_+}). \tag{2.56}$$

Clearly

$$\widetilde{G}_0 := \ker_{S^*Y} \mathrm{Epi} = G_1 \oplus \mathrm{pr}^* \mathcal{I}_1 \cong \mathrm{pr}^* G_0. \tag{2.57}$$

As usual, we denote by $(E')^*$, $(F')^*$, $\mathcal{I}_1^*$ the bundles dual respectively to $E', F', \mathcal{I}_1$ and by Epi^* the morphism dual to Epi.

We can prove (see [37, §16], [82, 3.1.1.1]) that there exist $f_j \in S(\mathbb{R}^1)$, $j = 1, \dots, m$, and smooth sections b_k, c_k, d_k, $k = 1, \dots r$, of bundles $\mathrm{pr}^*(E')^*$, $\mathrm{pr}^*(F')^* \otimes \mathbb{C}^l$, $\mathrm{pr}^* \mathcal{I}_1^*$, respectively, such that

$$(\mathrm{im}\,\mathrm{Epi}^*)_{(x',\xi')} + \mathcal{L}\{b_k \otimes \pi_+ f_j\}_{(x',\xi')} \oplus \mathcal{L}\{c_k\}_{(x',\xi')} \oplus \mathcal{L}\{d_k\}_{(x',\xi')} = \\ = \big(\mathrm{pr}^*(E')^* \otimes H_{p'}^{-s}(\overline{\mathbb{R}^1_+})\big)_{(x',\xi')} \oplus \big(\mathrm{pr}^*(F')^* \otimes \mathbb{C}^l\big)_{(x',\xi')} \oplus \\ \oplus(\mathrm{pr}^* \mathcal{I}_1^*)_{(x',\xi')}, \quad \forall (x',\xi') \in S^*Y \quad (p' = p/(p-1)), \tag{2.58}$$

where $\mathcal{L}\{\cdots\}$ denotes a linear span of the corresponding vector system.

It is clear that $b_{k_1} \otimes \pi_+ f_j = (b_{k_1} \otimes \pi_+ f_j) \oplus O \oplus O$, $c_{k_2} = O \oplus c_{k_2} \oplus O$ and $d_{k_3} = O \oplus O \oplus d_{k_3}$ $(j = 1, \dots, m,\ k_1, k_2, k_3 = 1, \dots, r)$ are sections of the bundle dual to $Z := \big(\mathrm{pr}^* E' \otimes \widetilde{H}_p^s(\overline{\mathbb{R}^1_+})\big) \oplus \big(\mathrm{pr}^* F' \otimes \mathbb{C}^l\big) \oplus \mathrm{pr}^* \mathcal{I}_1$. For these sections let us introduce a common notation φ_e, $e = 1, \dots, mr + r + r = r(m+2)$, and consider a bundle morphism

$$\Phi_1 = \big(\langle \cdot, \varphi_e \rangle\big) : Z \to \mathbb{C}^{r(m+2)},$$

where $\mathbb{C}^{r(m+2)}$ denotes a trivial bundle $S^*Y \times \mathbb{C}^{r(m+2)}$.

Taking into account that $(im\,\mathrm{Epi}^*)_{(x',\xi')}$ is an annihilator of the corresponding fibre of the bundle $\ker_{S^*Y} \mathrm{Epi}$, we obtain from (2.58) that Φ_1 is a monomorphism on $\widetilde{G}_0 = \ker_{S^*Y} \mathrm{Epi}$. Therefore $\mathrm{im}(\Phi_1|_{\widetilde{G}_0})$ is a smooth vector bundle (see, e.g., [66, Ch. I, §4, Theorem 1]) which is isomorphic to the bundle $\widetilde{G}_0$, and hence to $\mathrm{pr}^* G_0$ (see (2.57)). There exists subbundle $\mathcal{I}_2$ of a trivial bundle $\mathbb{C}^{r(m+2)}$ such that $\mathcal{I}_2 \oplus \mathrm{im}(\Phi_1|_{\widetilde{G}_0}) = \mathbb{C}^{r(m+2)}$ (see [66, Ch. I, §4, Theorem 1]). Denote by

$$\Phi_2 := \mathbb{C}^{r(m+2)} \to \mathrm{pr}^* G_0$$

the bundle morphism equal to zero on $\mathcal{I}_2$ and realizing an isomorphism

$$\Phi_2 : \mathrm{im}\,\big(\Phi_1|_{\widetilde{G}_0}\big) \to \mathrm{pr}^* G_0.$$

Thus we have obtained the isomorphism

$$\begin{pmatrix} (\sigma_Y(\mathcal{A}),\ I\otimes\pi_+\sigma_Y(K)\delta,\ O) \\ \\ \Phi_2\Phi_1 \end{pmatrix} \begin{array}{c} \mathrm{pr}^*\,E'\otimes\widetilde{H}^s_p(\overline{\mathbb{R}^1_+}) \\ \oplus \\ \mathrm{pr}^*\,F'\otimes\mathbb{C}^l \\ \oplus \\ \mathrm{pr}^*\,\mathcal{I}_1 \end{array} \longrightarrow$$

$$\longrightarrow \begin{array}{c} \mathrm{pr}^*\,F'\otimes H^{s-\operatorname{Re}\mu}_p(\overline{\mathbb{R}^1_+}) \\ \oplus \\ \mathrm{pr}^*\,G_0 \end{array}. \tag{2.59}$$

Note that we can consider the operator

$$\widetilde{H}^s_p(\overline{\mathbb{R}^1_+})\ni f\longmapsto\langle f,\pi_+f_j\rangle=\langle f,f_j\rangle\in\mathbb{C}$$

as an operator $\pi_0B_j:\widetilde{H}^s_p(\overline{\mathbb{R}^1_+})\to\mathbb{C}$, π_0 being the value at the point 0, and B_j is a ΨDO with infinitely smooth rapidly decreasing symbol "with constant coefficients" (see [37, §16]).

Denote $\mathcal{I}=(F'\otimes\mathbb{C}^l)\oplus\mathcal{I}_1$, $G=G_0$. It is easily seen that the morphism (2.59) looks like the isomorphism (2.15) and possesses the same properties.

Now the assertion of the theorem follows from the following general consideration. Let $c\in C^\infty(S^{n-2})$, $b\in S(\mathbb{R}^1)$. Then the symbol A,

$$A(\xi',\xi_n)=|\xi'|^\gamma c\Big(\frac{\xi'}{|\xi'|}\Big)b\Big(\frac{\xi_n}{|\xi'|}\Big),\quad \gamma\in\mathbb{C},$$

belongs to O^∞_γ. ■

Performing order reduction (see Theorem 1.12), we can reduce the investigation of $\sigma_Y(\mathcal{A})$ to that of a family

$$\sigma_Y(\mathcal{A}_0):\mathrm{pr}^*\,E'\otimes L_p(\mathbb{R}^1_+)\to\mathrm{pr}^*\,F'\otimes L_p(\mathbb{R}^1_+)$$

for which the condition (2.39) takes the form

$$\frac{1}{2\pi}\arg\lambda^0_m(x')-\frac{1}{p}\notin\mathbb{Z},\quad m=1,\dots,N,\ \ \forall x'\in Y. \tag{2.60}$$

Comparing the proofs of [37, Theorem 16.3] and [82, Theorem 3.2.1.2-1], we can easily get the result given below (see also Theorem 1.28).

Let $\mathcal{A}_0\in OP(\widehat{O}^\infty_0)(E_0,F_0)$ be an elliptic pseudodifferential operator satisfying condition (2.60)*. Then the following statements are equivalent:*

a) *there exist smooth vector bundles $\mathcal{I}$ and G over Y for which*

$$\operatorname{ind}_{S^*Y}\sigma_Y(\mathcal{A}_0)=[\mathrm{pr}^*\,G]-[\mathrm{pr}^*\,\mathcal{I}];$$

b) *there exist smooth vector bundles* $\mathcal{I}_0, G_0, L$ *and* P *over* Y *such that in the class of homogeneous zero order elliptic symbols satisfying condition* (2.60) *there is a homotopy* (*over a tubular neighbourhood of* $\partial X = Y$)*:*

$$\begin{pmatrix} \sigma_{\mathcal{A}_0}(x',0,\xi',\xi_n) & 0 \\ 0 & 1_{\mathrm{pr}^* L} \end{pmatrix} \simeq c \begin{pmatrix} \frac{\xi_n+i|\xi'|}{\xi_n-i|\xi'|} 1_{\mathrm{pr}^* G_0} & 0 & 0 \\ 0 & \frac{\xi_n-i|\xi'|}{\xi_n+i|\xi'|} 1_{\mathrm{pr}^* \mathcal{I}_0} & 0 \\ 0 & 0 & 1_{\mathrm{pr}^* P} \end{pmatrix},$$

where $[\mathrm{pr}^* G_0] - [\mathrm{pr}^* \mathcal{I}_0] = [\mathrm{pr}^* G] - [\mathrm{pr}^* \mathcal{I}]$, $c : E' \oplus L \to F' \oplus L$ *is an isomorphism,* $\mathrm{pr}^*(E' \oplus L) \cong \mathrm{pr}^*(G_0 \oplus \mathcal{I}_0 \oplus P)$.

Remark. In Theorems 2.22 - 2.24 the talk was, in general, about non-trivial bundles. If we restrict ourselves to the consideration only of trivial bundles, then we can get the results analogous to Theorems $16.2'$ and $16.3'$ from [37, §22]. These results do not follow from Theorems 2.22-2.24. Indeed, the bundles whose existence is established in Theorems 2.23 and 2.24 are not a priori trivial even if E and F are trivial and

$$\mathrm{ind}_{S^*Y}\, \sigma_Y(\mathcal{A}) = \mathrm{sgn}\, m[S^*Y \times \mathbb{C}^{|m|}], \quad m \in \mathbb{Z}.$$

Remark. In this section we could restrict ourselves to the consideration of spaces H_2^σ (see [83]). To this end it suffices to replace p by 2 and s by $s^* \in]s_- + 1/2, s_+ + 1/2[$ (see Remark 2.20).

§

For an elliptic operator (2.48) let the conditions of Theorem 2.9 and the Shapiro–Lopatinskiĭ condition be fulfilled for some $s \in \mathbb{R}$ and $p \in]1,\infty[$. Then Theorem 2.19 permits to obtain automatically the results on regularity of solutions of boundary value problem (2.46). In particular these results can be obtained directly from the L_2-theorems on the Noetherity.

Usually we act as follows. First we establish the fulfillment of Shapiro–Lopatinskiĭ conditions for a pair $(s,p) \in \mathbb{R}\times]1,\infty[$. This is not an easy procedure because we have to factorize matrices. In practice the following argument is very helpful. We may know that the boundary value problem is Noetherian for some s and p and as a rule, $p = 2$. In some cases we can specify this by the methods of the theory of Hilbert spaces (variational methods, Lax–Milgram theorem, coercive estimates, Gårding inequality, etc.). On the other hand, it is established in [31] that Shapiro–Lopatinskiĭ condition is not only sufficient but also necessary for singular integral operators to be Noetherian in spaces H_p^σ. This result can be easily transferred to ΨDOs of non-zero order on manifolds with boundary. The necessity of Shapiro–Lopatinskiĭ condition is proved in [83] for a wide algebra of elliptic boundary value problems in H_2^σ spaces.

After the Shapiro–Lopatinskiĭ condition is established for some s and p, we have to cover $s - 1/p$ by an interval which is supplementary to a closed set $\mathbb{Z}(\mathcal{A})$ (see (2.40)). Intersection of this interval with $]r_2, \mathrm{Re}\,\mu - r_1 - 1[$

is, in fact, the interval $]s_-, s_+[$ from Theorem 2.19. Note only that in order to construct the set $\mathbb{Z}(\mathcal{A})$ we must find eigenvalues of some matrix. This procedure is more easy than factorization of a matrix function.

The results of the present chapter can be transferred to elliptic in the Douglis–Nirenberg sense pseudodifferential operators. Boundary value problems for such ΨDOs are reduced to those considered in the present chapter with the help of order reduction operators (see [45], and [21, §2.7]). One can act in a more simple way first passing to the boundary value problem for elliptic in the Douglis-Nirenberg sense ΨDO (with "frozen coefficients") in a half-space and then performing order reduction (see Theorem 1.12 and Remark 1.41).

We have considered above the case of infinitely smooth manifolds and symbols. In practice finite smoothness is frequently quite enough. It must only ensure straightening of the boundary and "freezing of coefficients" (cf. [31] and §3.5).

In the case when pseudodifferential operators possess the transmission property, the L.Boutet de Monvel method (see [20]) allows us to obtain results about boundary value problems in Besov-Triebel-Lizorkin spaces and, in particular, in Hölder spaces (see , [38], [45], [82, 3.1.1.4] and also J. Johnsen's papers indicated in the footnote on page 43). Using the results of section 1.6 and applying the methods of this chapter, enable us investigate boundary value problems on manifolds for elliptic ΨDOs with the transmission property in Besov and Bessel-potential spaces. Such an approach apparently has the right to exist since in its realization the restriction $\mu \in \mathbb{Z}$ which is necessary for the Boutet de Monvel method to be applicable can be neglected.

Chapter III

Ψ

§

The case of two-dimensional manifolds ($n = 2$) which will be considered in the present chapter is a particular one from the viewpoint of the theory of boundary value problems for elliptic (pseudo-)differential operators. Indeed, if $n \geq 3$, then, as it has been noted at the end of §1.3, the index $\varkappa(\omega)$, $\omega \in S^{n-2}$, of an elliptic symbol (scalar or matrix) is constant. If however $n = 2$, then the sphere $S^{n-2} = S^0 = \{-1, 1\}$ is disconnected, and it may happen that $\varkappa(-1) \neq \varkappa(1)$. Note that this is not a pathology. For example, for a classical operator such as $\overline{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\,\frac{\partial}{\partial x_2}\right)$ we have $\varkappa(-1) = -1$, $\varkappa(1) = 0$. If $\varkappa(-1) \neq \varkappa(1)$, then no boundary value problem of type (1.61), (1.62) is uniquely solvable since condition (1.110) is necessarily violated (see Theorem 1.24). Indeed, this condition expresses the difference between the number of coboundary (potential) and boundary operators in a uniquely solvable boundary value problem in a half-space, i.e. the value not depending on $\omega = \pm 1$, by $\varkappa(\omega)$ and values also independent of $\omega = \pm 1$.

If for an elliptic differential operator $\varkappa(-1) \neq \varkappa(1)$, then this operator is not proper elliptic. In the theory of boundary value problems for partial differential operators difficulties connected with improper elliptic operators are well known (see, e.g., [2, Part I, Ch. I, §1]).

In the present chapter we consider boundary value problems for elliptic ΨDOs in the case when $\varkappa(-1)$ does not, in general, equal $\varkappa(1)$. Frequently we do not formulate final results on the problems but only show how they can be reduced to the boundary value problems investigated in the previous chapters. When formulating boundary value problems we try to modify problems of type (2.46) ((1.61), (1.62)) as little as possible, achieving nevertheless Noetherity (the unique solvability). Hereat only the boundary conditions rather than the equation involving an elliptic pseudodifferential operator are subjected to the modification.

We consider two types of boundary value problems. The problems with complex conjugation ((3.7)–(3.9), (3.29)) belong to the first type. These problems are analogues of the Hilbert problem for analytic functions (see [40], [70]). Boundary value problems containing analytic projectors $P_{\pm} = \frac{1}{2}\big(I \pm S\big)$ where S is a singular integral operator with the Cauchy kernel ((3.18)–(3.21), (3.33)), belong to the second type. In a sense they are similar to the problem of linear conjugation for analytic functions. Really, the problem of linear conjugation of the type $G\Phi^{+} + \Phi^{-} = g$ (see [40], [70]) can be easily reduced to the problem $P_{+}(G\Phi^{+}) = P_{+}g$. The connection here is the same as between paired operators and the Wiener–Hopf operators (see, e.g., [42, Ch. V, Theorem 1.1]).

§

Let L, L' be linear spaces over the field $\mathbb{C}$. A mapping $V_0 : L \to L'$ is said to be antilinear if $V_0(\lambda\varphi + \mu\psi) = \overline{\lambda} V_0\varphi + \overline{\mu} V_0\psi$, $\forall \lambda, \mu \in \mathbb{C}$, $\forall \varphi, \psi \in L$. A mapping $V : L \to L$ is said to be involution if $V^2 = I$.

Suppose that L, L_j, L'_j, $j = 1, 2$, are linear spaces over the field $\mathbb{C}$, $V : L \to L$ is an antilinear involution, $V_j : L_j \to L'_j$, $V_j^{-1} : L'_j \to L_j$, $j = 1, 2$, are reciprocal antilinear operators, $\mathcal{A} : L_1 \to L_2$, $B : L'_1 \to L_2$ and $C : L_1 \to L$ are linear operators. Introduce the notation

$$\mathcal{A}_* = V_2 \mathcal{A} V_1^{-1}, \;\; B_* = V_2 B V_1, \;\; C_* = V C V_1^{-1}, \;\; (I \pm V)L = \big\{(I \pm V)\psi | \psi \in L\big\}.$$

It is clear that $(I \pm V)L$ is a linear space over $\mathbb{R}$ (not over $\mathbb{C}$).

Consider the operators

$$U_\pm = \begin{pmatrix} \mathcal{A} \pm BV_1 \\ (I \pm V)C \end{pmatrix} : L_1 \longrightarrow \begin{matrix} L_2 \\ \oplus \\ (I \pm V)L \end{matrix}, \tag{3.1}$$

$$U_* = \begin{pmatrix} \mathcal{A} & B \\ B_* & \mathcal{A}_* \\ C & C_* \end{pmatrix} : \begin{matrix} L_1 \\ \oplus \\ L'_1 \end{matrix} \longrightarrow \begin{matrix} L_2 \\ \oplus \\ L'_2 \\ \oplus \\ L \end{matrix}. \tag{3.2}$$

Obviously the operators $U_\pm$ are $\mathbb{R}$-linear and the operator U_* is $\mathbb{C}$-linear.

$$\dim_{\mathbb{R}} \operatorname{Ker} U_\pm = \dim_{\mathbb{C}} \operatorname{Ker} U_*, \;\; \dim_{\mathbb{R}} \operatorname{Coker} U_\pm = \dim_{\mathbb{C}} \operatorname{Coker} U_*$$

(*infinite values being admitted*).

Proof. Multiplication by i is an automorphism of spaces L_j, L'_j as well as an isomorphism $(I + V)L \to (I - V)L$. It is easily seen that $U_- = iU_+ i^{-1}$. Therefore

$$\dim_{\mathbb{R}} \operatorname{Ker} U_- = \dim_{\mathbb{R}} \operatorname{Ker} U_+, \;\; \dim_{\mathbb{R}} \operatorname{Coker} U_- = \dim_{\mathbb{R}} \operatorname{Coker} U_+. \tag{3.3}$$

Let us introduce the following subspaces (with respect to the field $\mathbb{R}$) of the space $L_j \oplus L'_j$: $L_j^\pm = \big\{(\psi_j, \pm V_j\psi_j) | \psi_j \in L_j\big\}$. It is not difficult to see that

$$L_j^+ \oplus L_j^- = L_j \oplus L'_j. \tag{3.4}$$

Indeed,

$$L_j^+ \cap L_j^- = \{0\},$$

$$(\psi_j, \psi'_j) = \frac{1}{2}(\psi_j + V_j^{-1}\psi'_j, V_j\psi_j + \psi'_j) + \frac{1}{2}(\psi_j - V_j^{-1}\psi'_j, -V_j\psi_j + \psi'_j) =$$

$$= \frac{1}{2}\big(\psi_j + V_j^{-1}\psi'_j, V_j(\psi_j + V_j^{-1}\psi'_j)\big) + \frac{1}{2}\big(\psi_j - V_j^{-1}\psi'_j, -V_j(\psi_j - V_j^{-1}\psi'_j)\big) \in$$

$$\in L_j^+ \oplus L_j^-, \quad \forall(\psi_j, \psi_j') \in L_j \oplus L_j' \ \ (j = 1, 2).$$

Moreover,

$$(I+V)L \oplus (I-V)L = L. \tag{3.5}$$

Only the fact that $(I+V)L \cap (I-V)L = \{0\}$ needs to be proved. Let $(I+V)\chi_1 = (I-V)\chi_2$ for some $\chi_1, \chi_2 \in L$. When both sides of the equality are affected by the operator V, we obtain $(I+V)\chi_1 = -(I-V)\chi_2$. Hence $(I+V)\chi_1 = \pm(I-V)\chi_2 = 0$.

It is evident that the mappings

$$\mathcal{I}_1^{\pm} = \begin{pmatrix} I \\ \pm V_1 \end{pmatrix} : L_1 \longrightarrow L_1^{\pm},$$

$$\mathcal{I}_2^{\pm} = \begin{pmatrix} I & 0 \\ \pm V_2 & 0 \\ 0 & I \end{pmatrix} : \begin{matrix} L_2 \\ \oplus \\ (I \pm V)L \end{matrix} \longrightarrow \begin{matrix} L_2^{\pm} \\ \oplus \\ (I \pm V)L \end{matrix}$$

are the isomorphisms and $U_* \mathcal{I}_1^{\pm} = \mathcal{I}_2^{\pm} U_{\pm}$. Therefore

$$\dim_{\mathbb{R}} \operatorname{Ker} \big(U_* : L_1^{\pm} \to L_2^{\pm} \oplus (I \pm V)L\big) = \dim_{\mathbb{R}} \operatorname{Ker} U_{\pm},$$
$$\dim_{\mathbb{R}} \operatorname{Coker} \big(U_* : L_1^{\pm} \to L_2^{\pm} \oplus (I \pm V)L\big) = \dim_{\mathbb{R}} \operatorname{Coker} U_{\pm}.$$

Taking into account (3.3)–(3.5), we get

$$\begin{aligned} 2\dim_{\mathbb{C}} \operatorname{Ker} U_* = \dim_{\mathbb{R}} \operatorname{Ker} U_* = \dim_{\mathbb{R}} \operatorname{Ker} \big(U_* : L_1^+ \to L_2^+ \oplus (I+V)L\big) + \\ + \dim_{\mathbb{R}} \operatorname{Ker} \big(U_* : L_1^- \to L_2^- \oplus (I-V)L\big) = \dim_{\mathbb{R}} \operatorname{Ker} U_+ + \\ + \dim_{\mathbb{R}} \operatorname{Ker} U_- = 2\dim_{\mathbb{R}} \operatorname{Ker} U_{\pm} \end{aligned}$$

and analogously $\dim_{\mathbb{C}} \operatorname{Coker} U_* = \dim_{\mathbb{R}} \operatorname{Coker} U_{\pm}$. ■

Let V be the operator of complex conjugation and $A(x,D)$ be a pseudo-differential operator with a symbol $A(x,\xi)$ (see (1.28) and (2.3)). It is easily seen that the equality

$$A_*(x,D) := VA(x,D)V = \overline{A}(x,-D) = F^{-1}\overline{A(x,-\xi)}F$$

holds.

Suppose $A \in (O^{a,\mu}_{b,[n/2]+3})^{N\times N}$ to be an a-elliptic symbol (see §1.3). Assume $A_*(\xi) = \overline{A(-\xi)}$. By virtue of Lemma 1.19 (see (1.57)) we have for A_*

$$\begin{aligned} A_{*\omega}(\xi) = const(\xi_n - i|\xi'|_a^{a_n})^{\overline{\mu}/2a_n} \overline{A^-_{-\omega}(-\xi)} \times \\ \times \overline{\mathcal{D}(-\omega,-\xi)}\,\overline{A^+_{-\omega}(-\xi)}(\xi_n + i|\xi'|_a^{a_n})^{\overline{\mu}/2a_n}. \end{aligned}$$

(Here $\overline{\mu}$ denotes a number complex conjugate to $\mu \in \mathbb{C}$ not the vector of type (1.2). The notation $\overline{\delta}_k$ is understood analogously). On the diagonal of $\overline{\mathcal{D}(-\omega,-\xi)}$ there are elements

$$\left(\frac{\xi_n - i|\xi'|_a^{a_n}}{\xi_n + i|\xi'|_a^{a_n}}\right)^{\varkappa_k(-\omega)+\overline{\delta}_k}.$$

Note that $\overline{A^-_{-\omega}(-\xi)}$ $(\overline{A^+_{-\omega}(-\xi)})$ and its inverse matrix admit bounded analytic continuation with respect to ξ_n the lower (upper) complex half-plane.

For the order of a-homogeneity of A_* to be equal to the order of a-homogeneity of A, it is necessary and sufficient that $\mu \in \mathbb{R}$. Below when studying boundary value problems with complex conjugation on a half-plane (see §3.3), we shall always assume this condition to be fulfilled. This does not restrict the generality since the general case can be easily reduced to the case $\mu \in \mathbb{R}$ by means of order reduction operators (see Theorem 1.12 and Remark 1.41).

Thus let $\mu \in \mathbb{R}$. It is easy to see that an a-elliptic symbol

$$\begin{pmatrix} A(\xi) & 0 \\ 0 & A_*(\xi) \end{pmatrix} \tag{3.6}$$

has constant index $\widetilde{\varkappa}(\omega) = const$, $\omega = \pm 1$, even in two-dimensional case $(n = 2)$. Indeed, $\widetilde{\varkappa}(\omega) = \sum\limits_{k=1}^{N} \varkappa_k(\omega) + \sum\limits_{k=1}^{N} \varkappa_k(-\omega) = \varkappa(-1) + \varkappa(1)$ where $\varkappa(\omega)$ is an index of A (see lemma 1.19). Hence for the pseudodifferential operator with the symbol (3.6) the results of Chapter I are valid irrespective of the fact whether condition (1.58) is fulfilled for A or not. (The same is true for the scalar symbols $A \in O^{a,\mu}_{b,[n/2]+2}$).

§ a Ψ

1^0. Let $A \in (O^{a,\mu}_{b,4})^{N\times N}$ be an a-elliptic symbol, $\mu \in \mathbb{R}$. In our case $n = 2$, $b = b_1 > 0$, $[n/2] + 3 = 4$.

Consider a boundary value problem

$$\pi_+ \widehat{A}(D) u_+ + \sum_{k=1}^{m_-} \pi_+ \widehat{C}_k(D)\big(w_k(x_1) \times \delta(x_2)\big) = f(x), \tag{3.7}$$

$$\begin{aligned} \pi_0 \widehat{B}_j(D) u_+ + \pi_0 \widehat{B}'_j(D)\overline{u}_+ + \sum_{k=1}^{m_-} \big(\widehat{E}_{jk}(D_1) w_k(x_1) + \\ + \widehat{E}'_{jk}(D_1)\overline{w_k(x_1)}\big) = g_j(x_1), \quad 1 \le j \le m_0, \end{aligned} \tag{3.8}$$

$$\operatorname{Re}\Big(\pi_0 \widehat{B}_j(D) u_+ + \sum_{k=1}^{m_-} \widehat{E}_{jk}(D_1) w_k(x_1)\Big) = g_j(x_1), \ \ m_0+1 \le j \le m_1, \tag{3.9}$$

where $\widehat{B}_j, \widehat{B}'_j, \widehat{C}_k, \widehat{E}_{jk}, \widehat{E}'_{jk}, f, g_j, u_+, w_k$ satisfy the same conditions as in §1.4 with the only exception that g_j are real-valued functions for $m_0 + 1 \le j \le m_1$. To the system (3.7)–(3.9) there corresponds a boundary value problem

$$\pi_+ \widehat{A}(D) u_+^{(1)} + \sum_{k=1}^{m_-} \pi_+ \widehat{C}_k(D)\big(w_k^{(1)}(x_1) \times \delta(x_2)\big) = f^{(1)}(x), \tag{3.10}$$

$$\pi_+\widehat{A}_*(D)u_+^{(2)}+\sum_{k=1}^{m_-}\pi_+\widehat{C}_{k*}(D)\big(w_k^{(2)}(x_1)\times\delta(x_2)\big)=f^{(2)}(x), \quad (3.11)$$

$$\pi_0\widehat{B}_j(D)u_+^{(1)}+\pi_0\widehat{B}_j'(D)u_+^{(2)}+\sum_{k=1}^{m_-}\big(\widehat{E}_{jk}(D_1)w_k^{(1)}(x_1)+$$
$$+\widehat{E}_{jk}'(D_1)w_k^{(2)}(x_1)\big)=g_j^{(1)}(x_1),\ \ 1\le j\le m_0, \quad (3.12)$$

$$\pi_0\widehat{B}_{j*}'(D)u_+^{(1)}+\pi_0\widehat{B}_{j*}(D)u_+^{(2)}+\sum_{k=1}^{m_-}\big(\widehat{E}_{jk*}'(D_1)w_k^{(1)}(x_1)+$$
$$+\widehat{E}_{jk*}(D_1)w_k^{(2)}(x_1)\big)=g_j^{(2)}(x_1),\ \ 1\le j\le m_0, \quad (3.13)$$

$$\pi_0\widehat{B}_j(D)u_+^{(1)}+\pi_0\widehat{B}_{j*}(D)u_+^{(2)}+\sum_{k=1}^{m_-}\big(\widehat{E}_{jk}(D_1)w_k^{(1)}(x_1)+$$
$$+\widehat{E}_{jk*}(D_1)w_k^{(2)}(x_1)\big)=g_j^{(0)}(x_1),\ \ m_0+1\le j\le m_1. \quad (3.14)$$

The relation between the boundary value problems (3.7)–(3.9) and (3.10)–(3.14) is the same as between the operators (3.1) and (3.2). Note that the system (3.10)–(3.14) belongs to the class of boundary value problems (1.61), (1.62) (see the end of the previous section), and hence Theorem 1.24 is valid for it.

From Lemma 3.1 we have the following statement.

The unique solvability (*in the corresponding function spaces*) *for any right-hand sides of the system* (3.7)–(3.9) *is equivalent to that of the system* (3.10)–(3.14) *for any right-hand sides.*

For an a-elliptic symbol $A\in(O_{b,4}^{a,\mu})^{N\times N}$ *let the condition* (1.95) *be fulfilled. Then there exists a boundary value problem of type* (3.7)–(3.9) *which is uniquely solvable* (*in the corresponding function spaces*) *for any right-hand sides. Moreover, equations* (3.8) *can be assumed to be absent in it.*

Proof. First we construct symbols C_k such that the system

$$\pi_+A(\omega,D_2)u_+^{(1)}(x_2)+\sum_{k=1}^{m_-}w_k^{(1)}\pi_+C_k(\omega,D_2)\delta(x_2)=f^{(1)}(x_2) \quad (3.15)$$

has a solution $\big(u_+^{(1)},w_1^{(1)},\dots,w_{m_-}^{(1)}\big)\in\widetilde{H}_p^{s/a_2}(\overline{\mathbb{R}_+^1},\mathbb{C}^N)\oplus\mathbb{C}^{m_-}$ for any $f^{(1)}\in H_p^{(s-\mu)/a_2}(\overline{\mathbb{R}_+^1},\mathbb{C}^N)$ (see [37, §16, point 2], as well as the proof of Theorem 2.23). It is clear that the system

$$\pi_+A_*(\omega,D_2)u_+^{(2)}(x_2)+\sum_{k=1}^{m_-}w_k^{(2)}\pi_+C_{k*}(\omega,D_2)\delta(x_2)=f^{(2)}(x_2) \quad (3.16)$$

has a solution for any right-hand sides.

For $\omega = \pm 1$ the left-hand sides of (3.15), (3.16) determine a surjective Noetherian operator (see [37, §§12, 16] and §§1.4, 3.2). Its kernel is a direct sum of kernels of the operators defined separately by the left-hand sides of (3.15) and (3.16).

It follows from the arguments at the end of §3.2 that the kernel of the operator which corresponds to (3.16) for $\omega = \pm 1$ consists of functions which are complex conjugate to the functions composing the kernel of the operator corresponding to (3.15) for $\omega = \mp 1$.

Let $\varphi_1, \dots, \varphi_{n_+}$, $(\overline{\varphi}_1, \dots, \overline{\varphi}_{n_+})$ be a kernel basis of the operator corresponding to (3.15) (to (3.16)) for $\omega = +1$ (for $\omega = -1$) and $\psi_1, \dots, \psi_{n_-}$, $(\overline{\psi}_1, \dots, \overline{\psi}_{n_-})$ for $\omega = -1$ (for $\omega = +1$). We construct on $\widetilde{H}_p^{s/a_2}(\overline{\mathbb{R}^1_+}, \mathbb{C}^N) \oplus \mathbb{C}^{m_-}$ linear functionals $\theta_1, \dots, \theta_{n_+}$ and $\chi_1, \dots, \chi_{n_-}$ satisfying the conditions

$$\langle \theta_j, \varphi_l \rangle = \langle \overline{\theta}_j, \overline{\varphi}_l \rangle = \delta_j^l, \quad \langle \chi_j, \psi_l \rangle = \langle \overline{\chi}_j, \overline{\psi}_l \rangle = \delta_j^l,$$

where δ_j^l is the Kronecker symbol: $\delta_j^l = 0$ for $l \neq j$, $\delta_l^l = 1$.

Due to the duality theorem (see, e.g., [109, 2.6.1, 2.10.5]), $\theta_1, \dots, \theta_{n_+}$, $\chi_1 \dots, \chi_{n_-}$ can be assumed to be the elements of $H_{p'}^{-s/a_2}(\mathbb{R}^1, \mathbb{C}^N) \oplus \mathbb{C}^{m_-}$, $p' = p/(p-1)$.

Add to (3.15), (3.16) the following boundary conditions:

$$\begin{gathered}
\frac{1+\operatorname{sgn}\omega}{2} \langle \theta_j, (u_+^{(1)}, w_1^{(1)}, \dots, w_{m_-}^{(1)}) \rangle + \\
+ \frac{1+\operatorname{sgn}(-\omega)}{2} \langle \overline{\theta}_j, (u_+^{(2)}, w_1^{(2)}, \dots, w_{m_-}^{(2)}) \rangle = g_j^{(0)}, \quad j = 1, \dots, n_+, \\
\frac{1-\operatorname{sgn}\omega}{2} \langle \chi_{j-n_+}, (u_+^{(1)}, w_1^{(1)}, \dots, w_{m_-}^{(1)}) \rangle + \\
+ \frac{1-\operatorname{sgn}(-\omega)}{2} \langle \overline{\chi}_{j-n_+}, (u_+^{(2)}, w_1^{(2)}, \dots, w_{m_-}^{(2)}) \rangle = g_j^{(0)}, \\
j = n_+ + 1, \dots, n_+ + n_-.
\end{gathered} \tag{3.17}$$

From the above-said it follows that the boundary value problem (3.15)–(3.17) is uniquely solvable for any right-hand sides when $\omega = \pm 1$.

Further reasonings are rather standard. Elements of $H_{p'}^{-s/a_2}(\mathbb{R}^1)$ are approximated by functions from $S(\mathbb{R}^1)$. We obtain a uniquely solvable for any right-hand sides boundary value problem of type (3.15)–(3.17) in which instead of θ_j, χ_j there appear elements from $S(\mathbb{R}^1, \mathbb{C}^N) \oplus \mathbb{C}^{m_-}$. The functionals corresponding to the latter ones are interpreted by means of ΨDOs with symbols from appropriate classes (see [37, §16, point 2] as well as the end of the proof of Theorem 2.23). The proof is accomplished by applying Theorems 1.24 and 3.2. ∎

2^0. Let $A \in (O_{b,4}^{a,\mu})^{N \times N}$ be an a-elliptic symbol, $\mu \in \mathbb{C}$.

Consider a boundary value problem

$$\pi_+\widehat{A}(D)u_+ + \sum_{k=1}^{m_-}\pi_+\widehat{C}_k(D)\big(w_k(x_1)\times\delta(x_2)\big) = f(x), \tag{3.18}$$

$$\pi_0\widehat{B}_j(D)u_+ + \sum_{k=1}^{m_-}\widehat{E}_{jk}(D_1)w_k(x_1) = g_j(x_1), \quad 1\le j\le m_+, \tag{3.19}$$

$$P_-\Big(\pi_0\widehat{B}_j(D)u_+ + \sum_{k=1}^{m_-}\widehat{E}_{jk}(D_1)w_k(x_1)\Big) = g_j(x_1), \tag{3.20}$$
$$m_+ + 1\le j\le m_+ + m_1,$$

$$P_+\Big(\pi_0\widehat{B}_j(D)u_+ + \sum_{k=1}^{m_-}\widehat{E}_{jk}(D_1)w_k(x_1)\Big) = g_j(x_1), \tag{3.21}$$
$$m_+ + m_1 + 1\le j\le m_+ + m_1 + m_2,$$

where $P_\pm = \frac{1}{2}(I\pm S_{\mathbb{R}})$ are analytic projectors,

$$(S_{\mathbb{R}}\varphi)(t) = \frac{1}{\pi i}\int_{\mathbb{R}}\frac{\varphi(\tau)}{\tau - t}\,d\tau, \quad t\in\mathbb{R}, \tag{3.22}$$

and the integral is understood in the sense of Cauchy principal value (see also [37, §5]); $\widehat{B}_j, \widehat{C}_k, \widehat{E}_{jk}, f, g_j, u_+, w_k$ satisfy the same conditions as in §1.4 with the only exception that

$$g_j\in P_-B_{p,p}^{\overline{r(j)}}(\mathbb{R}^1)\ \ \big(P_-B_{p,q}^{\overline{r(j)}}(\mathbb{R}^1)\big)\ \text{ for }\ m_+ + 1\le j\le m_+ + m_1,$$
$$g_j\in P_+B_{p,p}^{\overline{r(j)}}(\mathbb{R}^1)\ \ \big(P_+B_{p,q}^{\overline{r(j)}}(\mathbb{R}^1)\big)\ \text{ for }\ m_+ + m_1 + 1\le j\le m_+ + m_1 + m_2,$$
$$\overline{r^{(j)}} = \frac{1}{a_1}(s - \operatorname{Re}\beta_j - a_2/p)\ \ \text{(cf. (1.70))}.$$

To the system (3.18)–(3.21) there correspond two boundary value problems

$$\pi_+\widehat{A}_{+1}(D)u_+^{(1)} + \sum_{k=1}^{m_-}\pi_+\widehat{C}_{k,+1}(D)\big(w_k^{(1)}(x_1)\times\delta(x_2)\big) = f^{(1)}(x), \tag{3.23}$$

$$\pi_0\widehat{B}_{j,+1}(D)u_+^{(1)} + \sum_{k=1}^{m_-}\widehat{E}_{jk,+1}(D_1)w_k^{(1)}(x_1) = g_j^{(1)}(x_1), \tag{3.24}$$
$$1\le j\le m_+ + m_1,$$

and

$$\pi_+\widehat{A}_{-1}(D)u_+^{(2)} + \sum_{k=1}^{m_-}\pi_+\widehat{C}_{k,-1}(D)\big(w_k^{(2)}(x_1)\times\delta(x_2)\big) = f^{(2)}(x), \tag{3.25}$$

$$\pi_0 \widehat{B}_{j,-1}(D)u_+^{(2)} + \sum_{k=1}^{m_-} \widehat{E}_{jk,-1}(D_1)w_k^{(2)}(x_1) = g_j^{(2)}(x_1), \tag{3.26}$$
$$1 \le j \le m_+ \quad \text{or} \quad m_+ + m_1 + 1 \le j \le m_+ + m_1 + m_2,$$

where as usual

$$\widehat{A}_\omega(\xi) = \widehat{A}(\omega|\xi_1|, \xi_2) = A(\omega\langle\xi_1\rangle, \xi_2), \quad \omega = \pm 1, \tag{3.27}$$

(see (1.60), (1.73) as well as (1.7), (1.8)), and the notations $\widehat{C}_{k,\omega}$, $\widehat{B}_{j,\omega}$, $\widehat{E}_{jk,\omega}$ have analogous meaning.

Note that the left-hand sides of systems (3.23), (3.24) and (3.25), (3.26) define operators of type (1.74) and for them the corresponding assertions from Theorem 1.24 are valid.

Similarly to Theorem 1.29 we obtain the following result.

The unique solvability (*in the corresponding function spaces*) *of the system* (3.18)–(3.21) *for any right-hand sides is equivalent to that of the boundary value problems* (3.23), (3.24) *and* (3.25), (3.26) *for any right-hand sides.*

For an a-elliptic symbol $A \in (O_{b,4}^{a,\mu})^{N\times N}$ *let the condition* (1.95) *be fulfilled. Then there exists a boundary value problem* (3.18)–(3.21) *which is uniquely solvable for any right-hand sides* (*in the corresponding function spaces*). *Moreover, equations* (3.19) *can be assumed to be absent in it.*

Proof. It follows from the above theorem that it suffices to construct boundary value problems of type (3.23), (3.24) and (3.25), (3.26) which are uniquely solvable for any right-hand sides. This is not difficult to perform taking m_- sufficiently large and choosing m_1, m_2 since $\widehat{A}_{\pm 1}(\xi)$ does not depend on $\operatorname{sgn}\xi_1$ (cf. [37, §16, point 2]). ■

In the scalar case we can slightly weaken the restriction imposed on the smoothness of the symbol A. All the results of this section are valid for the symbols $A \in O_{b,3}^{a,\mu}$ (see§1.3).

§ Ψ

1^0. Let X be a C^∞-smooth compact two-dimensional manifold with a boundary Y embedded in C^∞-smooth compact closed two-dimensional manifold M and let E, F be C^∞-smooth complex vector bundles over X and $\mathcal{I}, G_1$ over $Y = \partial X$. Consider also a real C^∞-smooth vector bundle G_2 over Y. Denote by $G_3 = cG_2$ its complexification (see, e.g., [66, Ch.I, §4]) and by $\operatorname{Re} : G_3 \to G_2$ the corresponding projection.

Let $\overline{E}$ ($\overline{F}$) be the bundle complex conjugate to E (F). Thus transition matrices g_{ij} corresponding to the bundle E (F) are replaced by complex conjugate matrices $\overline{g}_{ij}$ in the case of $\overline{E}$ ($\overline{F}$) (see [66, Ch. I, §4]). We shall denote the antilinear morphism of complex conjugation $E \to \overline{E}$ ($F \to \overline{F}$) by V (for all bundles). The same letter will denote the corresponding mapping of K-groups (see [8, §2.1]) and an antilinear operator of complex conjugation acting on sections of the corresponding bundles.

Note that the bundle H admits an antilinear involution $V : H \to H$ (i.e. $\overline{H} \cong H$) if and only if it is a complexification of a real bundle (see [66, Ch. I, §4, Proposition 2]). In particular, for the above-introduced bundle G_3 we can assume that $\mathrm{Re} = \frac{1}{2}(I + V)$.

Consider an operator

$$U = \begin{pmatrix} \pi_+\mathcal{A} & \pi_+K \\ T_1 & Q_1 \\ T_2 & Q_2 \end{pmatrix} : \begin{matrix} \mathcal{D}(E|_\Omega) \\ \oplus \\ \mathcal{D}(\mathcal{I}) \end{matrix} \to \begin{matrix} \mathcal{D}'(F|_\Omega) \\ \oplus \\ \mathcal{D}'(G_1) \\ \oplus \\ \mathcal{D}'(G_3) \end{matrix}, \tag{3.28}$$

($\Omega = \mathrm{Int}\, X = X \backslash Y$) belonging to the class

$$OP \begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix} (E, F, \mathcal{I}, G_1 \oplus G_3)$$

(see Definition 2.8). Assume T_1' and Q_1' to be operators of the same type as T_1 and Q_1, respectively, with the only difference that we have to replace bundles E and $\mathcal{I}$ by their complex conjugates $\overline{E}$ and $\overline{\mathcal{I}}$.

Using the operators U, T_1', Q_1' we construct the following operators

$$U_{\mathrm{Re}} = \begin{pmatrix} \pi_+\mathcal{A} & \pi_+K \\ T_1 + T_1' \circ V & Q_1 + Q_1' \circ V \\ \mathrm{Re} \circ T_2 & \mathrm{Re} \circ Q_2 \end{pmatrix} : \begin{matrix} \mathcal{D}(E|_\Omega) \\ \oplus \\ \mathcal{D}(\mathcal{I}) \end{matrix} \to \begin{matrix} \mathcal{D}'(F|_\Omega) \\ \oplus \\ \mathcal{D}'(G_1) \\ \oplus \\ \mathcal{D}'(G_2) \end{matrix}, \tag{3.29}$$

$$U_* = \begin{pmatrix} \pi_+\mathcal{A} & \pi_+K & 0 & 0 \\ 0 & 0 & \pi_+\mathcal{A}_* & \pi_+K_* \\ T_1 & Q_1 & T_1' & Q_1' \\ T_{1*}' & Q_{1*}' & T_{1*} & Q_{1*} \\ T_2 & Q_2 & T_{2*} & Q_{2*} \end{pmatrix} : \begin{matrix} \mathcal{D}(E|_\Omega) \\ \oplus \\ \mathcal{D}(\mathcal{I}) \\ \oplus \\ \mathcal{D}(\overline{E}|_\Omega) \\ \oplus \\ \mathcal{D}(\overline{\mathcal{I}}) \end{matrix} \longrightarrow \begin{matrix} \mathcal{D}'(F|_\Omega) \\ \oplus \\ \mathcal{D}'(\overline{F}|_\Omega) \\ \oplus \\ \mathcal{D}'(G_1) \\ \oplus \\ \mathcal{D}'(\overline{G}_1) \\ \oplus \\ \mathcal{D}'(G_3) \end{matrix}, \tag{3.30}$$

($\mathcal{A}_* = V\mathcal{A}V$, etc.) with the same correspondence as between (3.1) and (3.2) ((3.7)–(3.9) and (3.10)–(3.14)).

Operator (3.30) is almost of the same type as operator (2.7), (2.48). The difference is that the order of homogeneity of $\mathcal{A}_*$ is equal to $\overline{\mu}$ rather than to μ (and similarly for K_*, T_{1*}', Q_{1*}', T_{1*}, Q_{1*}, T_{2*}, Q_{2*}). But this however is not of principal importance. Indeed, after reducing the investigation of

U_* to that of an operator on a half-space, the latter can be reduced to an operator of type (1.72) as it has been noted in §3.2 (see Theorem 1.12 and Remark 1.41). As it was noted in §2.5, we can also apply order reduction operators directly to U_* (see [45] as well as [21, §2.7]). Thus, the problem on the Noetherity of the operator U_* in the corresponding function spaces can be solved by the methods from previous chapters.

From Lemma 3.1 we obtain the following result.

Operators U_{Re} and U_* (of type (3.29) and (3.30)) are simultaneously either Noetherian or not (in the corresponding function spaces), and the following equality

$$\mathrm{Ind}_{\mathbb{R}}\, U_{\mathrm{Re}} = \mathrm{Ind}_{\mathbb{C}}\, U_*. \tag{3.31}$$

holds.

Remark. The above theorem allows us to reduce the problem on the existence of the Noetherian boundary value problem of type (3.29) for a given elliptic pseudodifferential operator $\mathcal{A}$ to the problem on the existence of the Noetherian boundary value problem of type (3.30) (in the corresponding Besov and Bessel-potential spaces) for $B = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A}_* \end{pmatrix}$. Combining the methods of the proof of Theorems 2.22, 2.23 and 3.3 (and taking into account the special type of the boundary value problem (3.30)) enables us to obtain for B the analogues of Theorems 2.22 and 2.23. In our case $S^{n-2} = \{\pm 1\}$, $\mathrm{pr}^* G$ for any bundle G over Y is in fact two copies of G. From the proof of Theorem 3.3 it follows that the condition $\mathrm{ind}_{S^*Y}\, \sigma_Y(B) \in \mathrm{pr}^*\, K(Y)$ is equivalent to

$$\mathrm{ind}_Y\, \sigma_Y(B)(\omega) = V\big(\mathrm{ind}_Y\, \sigma_Y(B)(\omega)\big), \quad \omega = \pm 1. \tag{3.32}$$

2^0. Let E, F be C^∞-smooth complex vector bundles over a C^∞-smooth compact two-dimensional manifold X and $\mathcal{I}$, G_1, G_2, G_3 over $Y = \partial X$.

The boundary $Y = \partial X$ is a C^∞-smooth compact closed (i.e. $\partial Y = \varnothing$) one-dimensional manifold (generally speaking, disconnected). We can choose a positive direction on Y. As boundary local coordinate diffeomorphisms of the manifold X we shall consider only those mappings into the upper half-plane which transfer the positive direction chosen on Y in a positive direction of the axis of abscissae. Clearly, these diffeomorphisms induce an atlas on Y.

Denote by $\widetilde{P}_\pm$ a pseudodifferential operator acting on the manifold Y whose principal homogeneous symbol in local coordinates is equal to $\frac{1 \mp \mathrm{sgn}\, \xi_1}{2}$ (cf. [108, v. I, Ch. I, Theorem 5.3]). Introduce the notation $G = G_1 \oplus G_2 \oplus G_3$ and take an operator $U \in OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}(E, F, \mathcal{I}, G)$ for which the conditions of Theorem 2.9 are assumed to be fulfilled. Then the linear operator

$$\begin{gathered}
U_c=(I\oplus I\oplus \widetilde{P}_-\oplus \widetilde{P}_+)U : H_1^c(s,p)=H_1(s,p)= \\
=\widetilde{H}_p^s(E)\oplus B_{p,p}^{s-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1-1/p}(\mathcal{I})\to H_2^c(s,p)= \\
=H_p^{s-\operatorname{Re}\mu}(F)\oplus B_{p,p}^{s-\operatorname{Re}\gamma_2-1/p}(G_1)\oplus \widetilde{P}_-B_{p,p}^{s-\operatorname{Re}\gamma_2-1/p}(G_2)\oplus \\
\oplus\widetilde{P}_+B_{p,p}^{s-\operatorname{Re}\gamma_2-1/p}(G_3)\subset H_2(s,p) \qquad (3.33)\\
\big(B_1^c(s,p,q)=B_1(s,p,q)=\widetilde{B}_{p,q}^s(E)\oplus B_{p,q}^{s-\operatorname{Re}\mu+\operatorname{Re}\gamma_1+1-1/p}(\mathcal{I})\to \\
\to B_2^c(s,p,q)=B_{p,q}^{s-\operatorname{Re}\mu}(F)\oplus B_{p,q}^{s-\operatorname{Re}\gamma_2-1/p}(G_1)\oplus \\
\oplus\widetilde{P}_-B_{p,q}^{s-\operatorname{Re}\gamma_2-1/p}(G_2)\oplus \widetilde{P}_+B_{p,q}^{s-\operatorname{Re}\gamma_2-1/p}(G_3)\subset B_2(s,p,q)\big)
\end{gathered}$$

is bounded.

The principal boundary symbol

$$\sigma_Y(U):\big(\operatorname{pr}^*E'\otimes\mathcal{D}(\mathbb{R}_+)\big)\oplus\operatorname{pr}^*\mathcal{I}\to\big(\operatorname{pr}^*F'\otimes\mathcal{D}'(\mathbb{R}_+)\big)\oplus\operatorname{pr}^*G_1\oplus\operatorname{pr}^*G_2\oplus\operatorname{pr}^*G_3$$

of the operator U (see (2.14)) defines two morphisms

$$\begin{aligned}
\sigma_{Y,+1}(U)&=\big(E'\otimes\mathcal{D}(\mathbb{R}_+)\big)\oplus\mathcal{I}\to\big(F'\otimes\mathcal{D}'(\mathbb{R}_+)\big)\oplus G_1\oplus G_2,\\
\sigma_{Y,-1}(U)&=\big(E'\otimes\mathcal{D}(\mathbb{R}_+)\big)\oplus\mathcal{I}\to\big(F'\otimes\mathcal{D}'(\mathbb{R}_+)\big)\oplus G_1\oplus G_3
\end{aligned}\qquad(3.34)$$

which correspond to the values $\omega=\pm1$. (Recall that in the case under consideration $S^{n-2}=\{\pm1\}$ and for any bundle H over Y the bundle pr^*H represents, in fact, two copies of H).

We shall say that the Shapiro–Lopatinskiĭ condition is fulfilled for operator (3.33) if

$$\begin{aligned}
\sigma_{Y,+1}(U)&=\big(E'\otimes\widetilde{H}_p^s(\overline{\mathbb{R}_+^1})\big)\oplus\mathcal{I}\to\big(F'\otimes H_p^{s-\operatorname{Re}\mu}(\overline{\mathbb{R}_+^1})\big)\oplus G_1\oplus G_2,\\
\sigma_{Y,-1}(U)&=\big(E'\otimes\widetilde{H}_p^s(\overline{\mathbb{R}_+^1})\big)\oplus\mathcal{I}\to\big(F'\otimes H_p^{s-\operatorname{Re}\mu}(\overline{\mathbb{R}_+^1})\big)\oplus G_1\oplus G_3
\end{aligned}\qquad(3.35)$$

are isomorphisms.

Note that we can investigate $\sigma_{Y,\pm1}(U)$ by the methods of Chapter I.

Operators $\widetilde{P}_\pm$ may not be normally solvable, i.e. their images may be unclosed. In this case the normed spaces

$$\begin{gathered}
\widetilde{P}_-B_{p,p}^{s-\operatorname{Re}\gamma_2-1/p}(G_2),\quad \widetilde{P}_+B_{p,p}^{s-\operatorname{Re}\gamma_2-1/p}(G_3)\\
\big(\widetilde{P}_-B_{p,q}^{s-\operatorname{Re}\gamma_2-1/p}(G_2),\quad \widetilde{P}_+B_{p,q}^{s-\operatorname{Re}\gamma_2-1/p}(G_3)\big)
\end{gathered}$$

are incomplete and it is more convenient for us to consider U_c as an operator acting from $H_1^c(s,p)=H_1(s,p)$ to $H_2(s,p)$ and from $B_1^c(s,p,q)=B_1(s,p,q)$ to $B_2(s,p,q)$ (see (2.12) where we take $G=G_1\oplus G_2\oplus G_3$). In the case when operators $\widetilde{P}_\pm$ are normally solvable, the space $H_2^c(s,p)$ $(B_2^c(s,p,q))$ is a Banach one, and there is no need for (3.33) to be changed.

Analogously to Theorem 2.15 we can prove the following statement (see also Theorems 3.4 and 1.24).

Let $U \in OP\begin{pmatrix} \mu & \gamma_1, r_1 \\ \gamma_2, r_2 & \lambda \end{pmatrix}(E, F, \mathcal{I}, G)$ *be an elliptic operator,* $r_1 < \operatorname{Re}\mu - s - 1 + 1/p$, $r_2 < s - 1/p$, $1 < p < \infty$, $1 \le q \le \infty$, *and the Shapiro–Lopatinskiĭ condition be fulfilled for* U_c. *Then there exists an operator*

$$\mathcal{R} : H_2(s,p) \to H_1^c(s,p) \ \big(B_2(s,p,q) \to B_1^c(s,p,q)\big)$$

such that the operators

$$\mathcal{R}U_c - I : H_1^c(s,p) \to H_1^c(s,p) \ \big(B_1^c(s,p,q) \to B_1^c(s,p,q)\big),$$
$$U_c\mathcal{R} - (I \oplus I \oplus \widetilde{P}_- \oplus \widetilde{P}_+) : H_2(s,p) \to H_2(s,p) \ \big(B_2(s,p,q) \to B_2(s,p,q)\big)$$

are compact. If moreover the operators $\widetilde{P}_\pm$ *are normally solvable, then the operator* (3.33) *is Noetherian.*

Note one circumstance which is important for applications. In the next subsection we shall show that if X is embedded in $\mathbb{R}^2$, then under certain conditions we may take as operators $\widetilde{P}_\pm$ the analytic projectors $P_\pm$ (see (3.36), (3.37)). The equality $S_Y^2 = I$ is fulfilled for the operator S_Y (see [40, 7.3] or [70, §32]), therefore the operators $P_\pm$ are projectors, i.e. they satisfy the condition $P_\pm^2 = P_\pm$. On the other hand, projectors are normally solvable operators (see, e.g., [43, Ch. II, §4] or [79, 1.2]), hence in the case under consideration the use can be made of the last assertion of Theorem 3.9. In §3.5 we shall do this without additional comments.

From the proof of Theorem 2.23, using Theorem 3.9, we can easily obtain that for any elliptic operator $\mathcal{A} \in OP(\overset{\circ}{O}{}_\mu^\infty)(E_0, F_0)$ (where E_0 and F_0 are extensions of bundles E and F from X to M) satisfying the condition (2.39), there exists a boundary value problem of type (3.33) for which the Shapiro–Lopatisnkiĭ condition is fulfilled.

We can easily check that the analogues of Theorems 2.19 are valid for boundary value problems of type (3.29) and (3.33). These problems can be considered in the function spaces of piecewise-constant order of smoothness analogously to §25, [37]. Note finally that by the above methods we can consider boundary value problems containing complex conjugation and the operators $\widetilde{P}_\pm$ simultaneously.

3^0. Let Ω be a bounded open finitely connected domain in $\mathbb{C}$ with a boundary $Y = \partial\Omega$, $X = \Omega \cup Y$. On the components of the curve Y we choose the orientation such that when moving in positive direction the domain Ω remains on the left. Our aim is to show that when studying the Noetherity of boundary value problems for elliptic ΨDOs we can under certain restrictions on the smoothness of Y consider instead of operators $\widetilde{P}_\pm$

the analytic projectors

$$P_{\pm} = \frac{1}{2}(I \pm S_Y), \tag{3.36}$$

where

$$(S_Y\varphi)(t) = \frac{1}{\pi i}\int_Y \frac{\varphi(\tau)}{\tau - t}\,d\tau, \quad t \in Y \tag{3.37}$$

(cf. (1.136), (3.22)).

Introduce the following notation

$$l[s] = \begin{cases} s & \text{if } s \in \mathbb{N}, \\ \max\{1, s+\varepsilon\} & \text{if } s > 0,\ s \notin \mathbb{N}, \\ |s| + 1 & \text{if } s \in \mathbb{Z}\backslash\mathbb{N}, \\ |s| + 1 + \varepsilon & \text{if } s < 0,\ s \notin \mathbb{Z}, \end{cases} \tag{3.38}$$

$$l(s) = \begin{cases} \max\{1, s+\varepsilon\} & \text{if } s > 0, \\ |s| + 1 + \varepsilon & \text{if } s \le 0, \end{cases} \tag{3.39}$$

where $\varepsilon > 0$ is an arbitrarily small number.

It is well known that diffeomorphisms of the class $C^{l[s]}$ ($C^{l(s)}$) preserve spaces H_p^s ($B_{p,q}^s$), $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$ (see, e.g., [52, Theorem 3], and [12, Lemma 21.2]). Indeed, for the spaces $H_p^s = W_p^s$, $s \in \mathbb{N}$, this can be proved by direct calculation of derivatives (see [12, Lemma 21.9]) and for $H_p^0 = L_p$ (see (1.11)) this is obvious. For the spaces H_p^s, $s \in]0,1[$, the statement can be obtained by interpolation (see [109, 2.4.2] or Theorem 1.2-c)). Using for the space H_p^s, $s = k + \delta$, $k \in \mathbb{N}$, $\delta \in]0,1[$, the equivalent norm

$$\|f|H_p^s\|^* = \sum_{|\alpha| \le k} \|\partial^\alpha f|H_p^\delta\| \tag{3.40}$$

(see [110, 2.3.8]), by theorems on pointwise multipliers (see [110, Corollary 2.8.2]) and the already proven facts we can see that for the spaces H_p^s, $s > 0$, the assertion is valid. For the spaces H_p^s, $s < 0$, the assertion follows from the already proven, from the duality theorem (see [109, 2.6.1] or Theorem 1.1) and theorems on pointwise multipliers. In the case of $B_{p,q}^s$ spaces it suffices to apply interpolation (see [109, 2.4.2] or Theorem 1.2-e)). (Note that more precise results are valid for the Nikol'skiĭ spaces $B_{p,\infty}^s$, $s > 0$. Due to [12, Lemma 21.2], the diffeomorphism of the class C^l preserves these spaces if $l \ge \max\{1, s\}$ for noninteger s and $l > s$ for integer s).

Thus to define correctly the spaces $H_p^s(Y)$ and $B_{p,q}^s(Y)$, it suffices to assume that Y belongs to the class C^l, where $l \ge l[s]$ in the case of Bessel-potential spaces and $l \ge l(s)$ in the case of Besov spaces. Below in considering function spaces on Y we shall always assume these conditions to be fulfilled.

Let the curve Y belong to the class C^1. Then the operator S_Y (see (3.37)) is bounded in L_p, $1 < p < \infty$ (see [22], [25], [27], [68], [69], [36]). Using the equality for the derivatives

$$(S_Y\varphi)^{(m)} = S_Y\varphi^{(m)}, \quad m \in \mathbb{N} \tag{3.41}$$

(see [40, 4.4]), we can easily get that the operator S_Y is bounded in the space $W_p^1(Y) = H_p^1(Y)$. By means of interpolation (see [109, 2.4.2] or Theorem 1.2-c)) we see that S_Y is bounded in $H_p^s(Y)$, $0 \le s \le 1$. Using equivalent norm (3.40) for the spaces $H_p^s(Y)$, $s = k + \delta$, $k \in \mathbb{N}$, $\delta \in [0,1]$, due to equality (3.41) and the already proven we obtain that the operator S_Y is bounded in $H_p^s(Y)$ for $s \ge 0$. Using now the transposition formula

$$\int_Y \varphi S_Y \psi = -\int_Y \psi S_Y \varphi$$

(see, e.g., [40, 7.1]) and the duality theorem (see [109, 2.6.1] or Theorem 1.1), it is not difficult to prove that S_Y is bounded in $H_p^s(Y)$ for $s < 0$. The boundedness of S_Y in $B_{p,q}^s(Y)$ follows from the already proven and from the interpolation theorem (see [109, 2.4.2] or Theorem 1.2-e)).

Taking into account that $l[s], l(s) \ge 1$, $\forall s \in \mathbb{R}$, we obtain from the above-said that the operator S_Y is bounded in the space $H_p^s(Y)$ $(B_{p,q}^s(Y))$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$, if the curve Y belongs to the class $C^{l[s]}$ $(C^{l(s)})$.

Let $W \subset X$ be a coordinate neighbourhood (generally speaking, disconnected), $W \cap Y \ne \varnothing$. It is diffeomorphic to an open in $\overline{\mathbb{R}^2_+}$ set V. Denote by $z : V \to W$ the corresponding diffeomorphism of the class $C^{l[s]}$ $(C^{l(s)})$. Take arbitrary functions $\varphi, \psi \in \mathcal{D}(W)$ and consider the operator $\varphi S_Y \psi I$.

$$\begin{aligned}
\big((\varphi S_Y \psi I)f\big)(z(t)) &= \frac{\varphi(z(t))}{\pi i}\int_Y \frac{\psi(z)f(z)}{z - z(t)}\,dz = \\
&= \frac{\varphi(z(t))}{\pi i}\int_{\mathbb{R}} \frac{\psi(z(\tau))f(z(\tau))}{z(\tau) - z(t)}\,dz(\tau) = \\
&= \frac{\varphi(z(t))}{\pi i}\int_{\mathbb{R}} \frac{z'(\tau)}{z(\tau) - z(t)}\,\psi(z(\tau))f(z(\tau))\,dz = \\
&= \big[\big((\varphi\circ z)S_{\mathbb{R}}(\psi\circ z)I\big)(f\circ z)\big](t) + \frac{\varphi(z(t))}{\pi i}\int_{\mathbb{R}}\Big(\frac{z'(\tau)}{z(\tau) - z(t)} - \\
&\quad - \frac{1}{\tau - t}\Big)\psi(z(\tau))f(z(\tau))\,dz, \quad t \in \mathbb{R}\cap V.
\end{aligned} \tag{3.42}$$

The last operator in (3.42) is compact in $L_p(\mathbb{R})$, $1 < p < \infty$, since $l[s], l(s) \ge 1$ (see [46]). Using the interpolation theorems (see [109, 2.4.2] or Theorem 1.2) and the boundedness of the operators S_Y and $S_{\mathbb{R}}$ in the corresponding function spaces, by the well-known method (see [57, Ch.I, Theorem 4.1]), it is not difficult to prove that this operator is compact in the spaces $H_p^s(\mathbb{R})$, $s \in \mathbb{R}\backslash\mathbb{Z}$, $1 < p < \infty$, and $B_{p,q}^s(\mathbb{R})$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$. Indeed, in the spaces $H_p^s(\mathbb{R}^n)$, $B_{p,q}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$,

$1 \leq q < \infty$, there exists a common Schauder basis (see [28, Ch. IV, §3]). Such a basis is composed, for example, by wavelets (see [60], [99], [39], [11]). To prove that the operator under consideration is compact in $H_p^s(\mathbb{R})$, $s \in \mathbb{Z}\backslash\{0\}$, we shall have to raise the smoothness of the curve Y and assume it to belong to the class $C^{l[s]+\varepsilon'} = C^{l(s)}$ (see (3.38), (3.39)). In this case there takes place the previous proof based on the interpolation.

Thus if the curve Y belongs to the class $C^{l(s)}$, then the operator S_Y in local coordinates differs from $S_{\mathbb{R}}$ by an operator compact in H_p^s and $B_{p,q}^s$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, and what is more, in the case of the space $H_p^0 = L_p$ it suffices to require of the curve Y to be C^1-smooth. Note that in proving this fact, the use is made of the boundedness of the multiplication operator by $\varphi \circ z \quad (\psi \circ z)$ in the corresponding function space (see [110, Corollary 2.8.2]).

4^0. We have considered above the boundary value problems for elliptic ΨDOs not possessing, in general, the transmission property. Using §1.6, we can transfer the results of this chapter to the boundary value problems for ΨDOs with the transmission property. We shall not formulate the corresponding theorems but in the next section we consider instead the examples of the boundary value problems for elliptic differential equations.

§

Let Ω be a bounded open finitely connected domain in $\mathbb{C}$ with a boundary $Y = \partial\Omega$ of the class C^1, $X = \Omega \cup Y$. Components of the curve Y are oriented so that in moving in positive direction the domain Ω remains on the left. (Components of Y are simple closed curves).

We shall use the following standard notation:

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big), \quad \frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \quad z = z + iy \in \mathbb{C}, \quad P_{\pm} = \frac{1}{2}(I \pm S_Y)$$

(see (3.36), (3.37)).

1^0. Consider the system of the theory of generalized analytic vectors (see [19]):

$$\frac{\partial u}{\partial \overline{z}} + Q(z)\frac{\partial u}{\partial z} + A(z)u + B(z)\overline{u} = f(z), \quad z \in \Omega, \tag{3.43}$$

where Q is a triangular $N \times N$-matrix whose diagonal elements satisfy the condition

$$|q_{jj}(z)| \leq q_0 < 1, \;\; j = 1, \ldots, N, \;\; \forall z \in X = \Omega \cup Y, \tag{3.44}$$

which ensures ellipticity; $f \in H_p^{s-1}(X, \mathbb{C}^N)$ $(B_{p,q}^{s-1}(X, \mathbb{C}^N))$ is a given vector function and $u \in H_p^s(X, \mathbb{C}^N)$ $(B_{p,q}^s(X, \mathbb{C}^N))$ is an unknown vector function, $1 < p < \infty$, $1 \leq q \leq \infty$.

For the system (3.43) let us pose two boundary value problems:

$$\operatorname{Re}\Big(\sum_{k=0}^{m} A_k(t)\,\frac{\partial^m u}{\partial z^k \partial \overline{z}^{m-k}}\Big|_Y + P_{m-1}u\big|_Y\Big) = \varphi(t), \quad t \in Y, \tag{3.45}$$

or

$$P_+\Big(\sum_{k=0}^{m} A_k(t)\,\frac{\partial^m u}{\partial z^k \partial \overline{z}^{m-k}}\Big|_Y + P_{m-1}u\big|_Y\Big) = \psi(t), \quad t \in Y, \tag{3.46}$$

where P_{m-1} is a differential operator of order not higher than $m-1$,

$$\varphi \in B^{s-m-1/p}_{p,p}(Y, \mathbb{R}^N)\ \ \big(B^{s-m-1/p}_{p,q}(Y, \mathbb{R}^N)\big),$$
$$\psi \in P_+B^{s-m-1/p}_{p,p}(Y, \mathbb{C}^N)\ \ \big(P_+B^{s-m-1/p}_{p,q}(Y, \mathbb{C}^N)\big)$$

are given vector functions.

Remark. The spaces $P_+B^{\sigma}_{p,q}(Y, \mathbb{C}^N)$ are the analogues of Smirnov classes $E_p(\Omega)$ (see, e.g., [78]). Indeed, if the curve Y belongs to the class C^1, then boundary values of the functions from $E_p(\Omega)$, $1 < p < \infty$, form the space $P_+L_p(Y)$ (see [47], [36]).

For the boundary conditions (3.45), (3.46) to have sense, we assume $m < s - 1/p$. In particular $s > 1/p$.

The conditions which the coefficients Q, A, B, A_k and the curve Y should satisfy will be formulated below. We shall start with the curve Y.

Smoothness of the curve Y must ensure the possibility to straighten the boundary. From the arguments given in point 3^0, §3.4, it follows that the coordinate diffeomorphism of the class C^l preserves the spaces taking part in formulation of boundary value problems (3.43), (3.45) and (3.43), (3.46) if $l \geq l[s]$ and $l \geq l(s)$ in the case of the spaces $H^s_p(X, \mathbb{C}^N)$ and $B^s_{p,q}(X, \mathbb{C}^N)$, respectively. To work with the operator P_+ it suffices for the curve Y to belong to the class C^l, $l \geq l[s]$ ($l \geq l(s)$). Indeed, according to (3.38), (3.39), $l(s-m-1/p) \leq l(s)$, $l(s-m-1/p) \leq l[s]$ (see point 3^0, §3.4).

Thus, we shall assume the curve Y to belong to the class C^l, where $l \geq l[s]$ in the case of Bessel-potential spaces and $l \geq l(s)$ in the case of Besov spaces.

In investigating the Noetherity of boundary value problems the restrictions imposed on the coefficients Q, A, B, A_k and those of the operator P_{m-1} must ensure the possibility to "freeze" the coefficients in leading terms and to discard lowest terms (i.e. lowest terms must generate compact operators in the corresponding function spaces). Many such possibilities are available (see [63, 2.2.9, 2.3.1, 2.3.3], [18, Ch. I, §6], [110, Remark 4.3.2-1] and §3.6 below). We shall restrict ourselves to the cases which allow us to investigate boundary value problems (3.43), (3.45) and (3.43), (3.46) under the restrictions on the coefficients as in the classical monograph [113] as well as in the works [18], [1], [6], [50].

It follows from [63, Theorem 2.2.9] and [110, Remark 4.3.2-1] that if

$$a \in H_p^{s-1}(X), \quad 1 < p < \infty, \ \ s \geq 1, \ \ s > 2/p, \tag{3.47}$$

then multiplication by a is a compact operator from $H_p^s(X)$ to $H_p^{s-1}(X)$.

Analogously, using the results from [18, Ch. I, §6], we can prove that if either

$$a \in B_{p,q}^{s-1}(X), \ \ 1 < p < \infty, \ \ s > 1, \ \ s > 2/p, \ \ 1 \leq q \leq \infty, \tag{3.48}$$

or

$$a \in B_{p,q}^{s-1}(X), \quad 1 < p < 2, \ \ s = 2/p > 1, \ \ q = 1, \tag{3.49}$$

then multiplication by a is a compact operator from $B_{p,q}^s(X)$ to $B_{p,q}^{s-1}(X)$. (In the case (3.49) we have first to approximate a by smooth functions and then to apply [110, Remark 4.3.2-1]).

As above we can prove that if coefficients of the operator P_{m-1} belong to the space $B_{p,p}^{s-m-1/p}(Y, \mathbb{C}^{N\times N})$ $(B_{p,q}^{s-m-1/p}(Y, \mathbb{C}^{N\times N}))$ and $m < s - 1/p$, then in investigating the Noetherity of boundary value problems (3.43), (3.45) and (3.43), (3.46) this operator can be neglected.

Let one of the conditions

$$\begin{gathered} A_k \in B_{p,q}^{s-m-1/p}(Y, \mathbb{C}^{N\times N}), \\ 1 < p < \infty, \ \ 1 \leq q \leq \infty, \ \ s - m - 1/p > 1/p, \end{gathered} \tag{3.50}$$

$$\begin{gathered} A_k \in B_{p,q}^{s-m-1/p}(Y, \mathbb{C}^{N\times N}), \\ 1 < p < \infty, \ \ q = 1, \ \ s - m - 1/p = 1/p \end{gathered} \tag{3.51}$$

be fulfilled. Then multiplication by A_k is a bounded operator in $B_{p,q}^{s-m-1/p}(Y, \mathbb{C}^N)$ (see [18, Ch. I, §6]). From the embedding theorems (see, e.g., [109, 4.6.1]) it follows that $A_k \in C(Y, \mathbb{C}^{N\times N})$. Therefore we can "freeze" the coefficients. If $q < \infty$, then to prove this it suffices to approximate A_k by a smooth matrix function (see [109, 2.3.2]) and then to use Lemma 2.14. In general case the possibility of "freezing" the coefficients follows from the results of §3.6.

In the case of Bessel-potential spaces we shall assume that either

$$Q \in H_{p_1}^{s-1}(X, \mathbb{C}^{N\times N}), \ \ 1 < p \leq p_1 < \infty, \ \ p_1 > 2/(s-1), \ \ s > 1, \tag{3.52}$$

or

$$Q \in C(X, \mathbb{C}^{N\times N}), \quad \text{if} \ \ s = 1, \tag{3.53}$$

while in the case of Besov spaces either

$$\begin{gathered} Q \in B_{p_1,q}^{s-1}(X, \mathbb{C}^{N\times N}), \ \ 1 < p \leq p_1 < \infty, \\ p_1 > 2/(s-1), \ \ 1 \leq q \leq \infty, \ \ s > 1, \end{gathered} \tag{3.54}$$

or

$$Q \in B^{s-1}_{p_1,1}(X, \mathbb{C}^{N\times N}), \quad 1 < p < p_1 < \infty,$$
$$s = 1 + 2/p_1, \quad 1 \le q \le \frac{2p}{2 - p(s-1)}, \tag{3.55}$$

or

$$Q \in B^{s-1}_{p,q}(X, \mathbb{C}^{N\times N}), \quad 1 < p < \infty, \quad q = 1, \quad s = 1 + 2/p. \tag{3.56}$$

As above we can prove that if one of the conditions (3.52), (3.53) or, respectively, one of the conditions (3.54)–(3.56) is fulfilled, then we can "freeze" the coefficients.

Suppose u to be a solution of equation (3.43). Differentiating (3.43) $(m-1)$ times with respect to z and then differentiating m obtained equalities (including (3.43)) with respect to $\overline{z}$ as many times as required enable us to express all the derivatives of type $\frac{\partial^m u}{\partial z^k \partial \overline{z}^{m-k}}$, $k = 1, \dots, m-1$, by $\frac{\partial^m u}{\partial z^m}$ and by derivatives of the lowest order. Substitute the obtained expressions in (3.45), (3.46) to get

$$\operatorname{Re}\Big(C(t)\frac{\partial^m u}{\partial z^m}\Big|_Y + \widetilde{P}_{m-1}u\big|_Y\Big) = \widetilde{\varphi}(t), \quad t \in Y, \tag{3.57}$$

$$P_+\Big(C(t)\frac{\partial^m u}{\partial z^m}\Big|_Y + \widetilde{P}_{m-1}u\big|_Y\Big) = \widetilde{\psi}(t), \quad t \in Y, \tag{3.58}$$

where

$$C = \sum_{k=0}^{m} (-1)^{m-k} A_k Q^{m-k} \tag{3.59}$$

and $\widetilde{P}_{m-1}$, $\widetilde{\varphi}$, $\widetilde{\psi}$ possess the same properties as P_{m-1}, φ, ψ in (3.45), (3.46). (We suppose that one of conditions (3.52), (3.53) or (3.54)–(3.56) as well as corresponding condition (3.50), (3.51) are fulfilled and the elements of the matrices A and B satisfy (3.47) or (3.48), (3.49)).

Thus the boundary value problems (3.43), (3.45) and (3.43), (3.46) are equivalent to the problems (3.43), (3.57) and (3.43), (3.58), respectively.

It is not difficult to investigate Noetherity of boundary value problems (3.43), (3.57) and (3.43), (3.58). Indeed, the symbol of the operator $\partial/\partial\overline{z} + Q(z)\,\partial/\partial z$ in local coordinates is a triangular matrix which, together with its inverse, extends analytically with respect to ξ_2 into upper or lower half-plane depending on $\operatorname{sgn}\xi_1$. Therefore there is no need to look for factorization. Using the above obtained results (see §§1.6, 2.3, 3.4), we can prove (see also [37, §11]) that for boundary value problems (3.43), (3.57) and (3.43), (3.58) to be Noetherian, it is sufficient that the condition

$$\det C(t) \ne 0, \quad \forall t \in Y, \tag{3.60}$$

be fullfilled. Moreover, we do not use the equality (3.59). We need it to return to boundary value problems (3.43), (3.45) and (3.43), (3.46).

Our task now is to calculate the indices of boundary value problems (3.43), (3.57) and (3.43), (3.58). Neglecting in (3.43) the lowest terms we perform homotopy of the matrix $Q(z)$ to zero one: $Q_\tau(z) = (1-\tau)Q(z)$, $\tau \in [0,1]$, not changing the matrix $C(t)$. The Noetherity and the index of the corresponding boundary value problems here will be invariant. Therefore when calculating the index of boundary value problems (3.43), (3.57) and (3.43), (3.58), we may assume $Q \equiv 0$, $A \equiv 0$, $B \equiv 0$.

Let $\zeta : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism of the class $C^{l[s]}$ ($C^{l(s)}$) which is close to the identical one in $C^{l[s]}$-norm ($C^{l(s)}$-norm) and maps Ω onto the domain Ω_1 with a C^∞-smooth boundary Y_1. It is not difficult to prove the existence of such a diffeomorphism by means of "collar" theorem (see, e.g., [67, Theorem 5.9]) and of smoothing theorems (see the proof of [67, Theorem 4.8]). Using ζ, we can reduce boundary value problems under consideration to those of type (3.43), (3.45) and (3.43), (3.46) in the domain Ω_1 (see [110, Corollary 2.8.2] as well as point 3^0, §3.4). In their turn they are reduced to the boundary value problems of type (3.43), (3.57) and (3.43), (3.58), the determinant indices on the components of Y_1 for the corresponding matrix C_1 (see (3.59)) being equal to those of the matrix C on the components of Y (if ζ is close enough to the identical diffeomorphism). Neglect the lowest terms and perform the homotopy as above to arrive at the boundary value problems

$$\frac{\partial u}{\partial \bar z} = f_1(z), \quad z \in \Omega_1, \tag{3.61}$$

$$\operatorname{Re}\Big(C_1(t)\,\frac{\partial^m u}{\partial z^m}\Big|_{Y_1}\Big) = \varphi_1(t), \quad t \in Y_1, \tag{3.62}$$

$$P_+\Big(C_1(t)\,\frac{\partial^m u}{\partial z^m}\Big|_{Y_1}\Big) = \psi_1(t), \quad t \in Y_1, \tag{3.63}$$

where $f_1, \varphi_1, \psi_1, C_1$ have the same properties as in (3.43), (3.57), (3.58), (3.60).

Let the curve Y (Y_1) consist of simple closed contours $Y^{(0)}, Y^{(1)}, \dots, Y^{(n)}$ ($Y_1^{(0)}, Y_1^{(1)}, \dots, Y_1^{(n)}$) and moreover, let the contours $Y^{(1)}, \dots, Y^{(n)}$ ($Y_1^{(1)}, \dots, Y_1^{(n)}$) be interior to $Y^{(0)}$ ($Y_1^{(0)}$). Introduce the notation (see (3.59))

$$\varkappa_j = \frac{1}{2\pi}\big[\arg\det C_1(t)\big]_{Y_1^{(j)}} =$$
$$= \frac{1}{2\pi}\Big[\arg\det\Big(\sum_{k=0}^{m}(-1)^{m-k}A_k(t)Q^{m-k}(t)\Big)\Big]_{Y^{(j)}}, \tag{3.64}$$

$$\varkappa = \sum_{j=0}^{n}\varkappa_j. \tag{3.65}$$

In the class of non-degenerate matrices we perform homotopy of the matrix $C_1(t)$ to $\mathcal{D}(t) = \operatorname{diag}[d_e(t)]_{e=1}^N$, where $d_e = 1$ for $e > 1$, $d_1(t) =$

$(t-t_0)^{\varkappa}(t-t_1)^{-\varkappa_1}\cdots(t-t_n)^{-\varkappa_n}$, the points t_j, $j\ge 1$, being interior to $Y_1^{(j)}$, $t_0\in\Omega_1$ (see [70, Appendix VI]).

Thus we have obtained the boundary conditions

$$\operatorname{Re}\Big(\mathcal{D}(t)\,\frac{\partial^m u}{\partial z^m}\Big|_{Y_1}\Big)=\varphi_1(t),\quad t\in Y_1, \tag{3.66}$$

$$P_+\Big(\mathcal{D}(t)\,\frac{\partial^m u}{\partial z^m}\Big|_{Y_1}\Big)=\psi_1(t),\quad t\in Y_1. \tag{3.67}$$

Since $\mathcal{D}$ and Y_1 are C^∞-smooth, the index of the boundary value problem (3.61), (3.66), ((3.61), (3.67)) is the same in all the above considered spaces (see Lemma 2.17, the proof of Theorem 2.19 as well as [82, Theorems 3.1.1.1-5, 3.1.1.4-3]). Take, for example, the spaces $H_2^s(X_1,\mathbb{C}^N)=B_{2,2}^s(X_1,\mathbb{C}^N)$, $s=m+2$ $(X_1=\Omega_1\cup Y_1)$ (see [109, 2.3.2]).

The equation (3.61) is solvable for any right-hand side $f_1\in H_2^{s-1}(X_1,\mathbb{C}^N)$. Indeed, let us take an extension $F_1\in H_2^{s-1}(\mathbb{R}^2,\mathbb{C}^N)$ of the function f_1 onto $\mathbb{R}^2$ (see [109, 4.2.2, 4.2.3]). We may assume F_1 to have a compact support. Consider the function

$$u_0(z)=TF_1(z)=-\frac{1}{\pi}\iint_{\mathbb{R}^2}\frac{F_1(\zeta)\,d\xi\,d\eta}{\zeta-z},\quad \zeta=\xi+i\eta.$$

From the boundedness of Calderon–Zygmund–Mikhlin singular integral operators in Sobolev spaces (see, e.g., [64, Ch. XI, Theorem 9.1]) and the properties of weakly singular operators (see, e.g., [56, Theorem 8.1]) it follows that the function $u=u_0|_{\Omega_1}$ belongs to $H_2^s(X_1,\mathbb{C}^N)$ (see also [64, Ch. X, Theorem 7.1 and Ch. XI, Theorem 11.1]). Moreover, $\frac{\partial u}{\partial\bar z}=f_1$ (see [113, Ch. I, (5.8)]).

Thus, it suffices to calculate the index of the following problem: find a holomorphic vector of the class $H_2^s(X_1,\mathbb{C}^N)$ satisfying boundary condition (3.66) (respectively (3.67)). This problem is divided into N boundary value problems for analytic functions and the unknown index is equal to the sum of indices of scalar problems.

Consider boundary value problems for the analytic function $v\in H_2^s(X_1)$:

$$\operatorname{Re}\Big(d_1(t)\,\frac{\partial^m v}{\partial z^m}\Big|_{Y_1}\Big)=\chi_1(t),\quad t\in Y_1, \tag{3.68}$$

or

$$P_+\Big(d_1(t)\,\frac{\partial^m v}{\partial z^m}\Big|_{Y_1}\Big)=\rho_1(t),\quad t\in Y_1, \tag{3.69}$$

where $d_1(t)=(t-t_0)^{\varkappa}(t-t_1)^{-\varkappa_1}\cdots(t-t_n)^{-\varkappa_n}$, $\chi_1\in H_2^{s-m-1/2}(Y_1,\mathbb{R})$, $\rho_1\in P_+H_2^{s-m-1/2}(Y_1)$.

Introduce the analytic function $w=\frac{\partial^m v}{\partial z^m}\in H_2^{s-m}(X_1)$ for which we have boundary value problems

$$\operatorname{Re}\big(d_1(t)w(t)\big)=\chi_1(t),\quad t\in Y_1, \tag{3.70}$$

$$P_+\big(d_1(t)w(t)\big) = \rho_1(t), \quad t \in Y_1. \tag{3.71}$$

The index of the first problem with respect to the field $\mathbb{R}$ is equal to $-2\varkappa - n + 1$, while the index of the second problem with respect to the field $\mathbb{C}$ is equal to $-\varkappa$ (see [40, §§16, 37], [70, §§34–37] and the proof of [18, Ch. I, §8, Lemma 1.2]).

It is easily seen that a function holomorphic in Ω_1 and continuous in $X_1 = \Omega_1 \cup Y_1$ has its primitive if and only if its integrals along $Y_1^{(j)}$, $j = 1, \dots, n$, are equal to zero (see, e.g., [59, Ch. I, §4, point 13]). Moreover, the primitive is determined to within a constant. Taking into account this remark and the connection between w and v, we obtain that the $\mathbb{R}$-index of the boundary value problem (3.68) is equal to $-2\varkappa - (2m+1)(n-1)$ and the $\mathbb{C}$-index of the problem (3.69) is equal to $-\varkappa - m(n-1)$.

The remaining scalar boundary value problems are investigated in a more easy way for, instead of d_1 there we have $d_e = 1$, $e > 1$.

Thus, the $\mathbb{R}$-index of the boundary value problem (3.61), (3.66) is equal to $-2\varkappa - (2m+1)N(n-1)$ and the $\mathbb{C}$-index of the boundary value problem (3.61), (3.67) is equal to $-\varkappa - mN(n-1)$.

Summing up all the above-said, we obtain the following assertion. (Similar results concerning the problem (3.43), (3.45) are contained in the dissertation [50]. Boundary conditions more general than (3.46) for generalized analytic vectors have been considered in [62, Ch. III, §5] and in the works mentioned in the references therein. See also [41] and references therein).

Let Ω be a bounded open $(n+1)$-connected domain in $\mathbb{C}$ with a boundary $Y = \partial\Omega$ belonging to the class C^l, where $l \geq l[s]$ $(l \geq l(s))$, $X = \Omega \cup Y$; the triangular matrix Q satisfy condition (3.44) *as well as one of conditions* (3.52), (3.53) ((3.54)–(3.56))*; elements of the matrices A and B satisfy* (3.47) (*one of conditions* (3.48), (3.49))*; A_k satisfies* (3.50) *with $q = p$* (*one of conditions* (3.50), (3.51))*; coefficients of the operator P_{m-1} belong to $B^{s-m-1/p}_{p,p}(Y, \mathbb{C}^{N\times N})$ $(B^{s-m-1/p}_{p,q}(Y, \mathbb{C}^{N\times N})$. If the condition*

$$\det\Big(\sum_{k=0}^{m}(-1)^{m-k}A_k(t)Q^{m-k}(t)\Big) \neq 0, \quad \forall t \in Y, \tag{3.72}$$

is fulfilled, then the linear with respect to the field $\mathbb{R}$ operator defined by boundary value problem (3.43), (3.45) *is Noetherian from $H^s_p(X, \mathbb{C}^N)$ $(B^s_{p,q}(X, \mathbb{C}^N))$ to*

$$H^{s-1}_p(X, \mathbb{C}^N) \oplus B^{s-m-1/p}_{p,p}(Y, \mathbb{R}^N) \ \ \big(B^{s-1}_{p,q}(X, \mathbb{C}^N) \oplus B^{s-m-1/p}_{p,q}(Y, \mathbb{R}^N)\big)$$

and the $\mathbb{R}$-linear operator defined by boundary value problem (3.43), (3.46) *is Noetherian from $H^s_p(X, \mathbb{C}^N)$ $(B^s_{p,q}(X, \mathbb{C}^N))$ to*

$$H^{s-1}_p(X, \mathbb{C}^N) \oplus P_+ B^{s-m-1/p}_{p,p}(Y, \mathbb{C}^N)$$
$$\big(B^{s-1}_{p,q}(X, \mathbb{C}^N) \oplus P_+ B^{s-m-1/p}_{p,q}(Y, \mathbb{C}^N)\big).$$

The index of the first operator with respect to the field $\mathbb{R}$ is equal to $-2\varkappa-(2m+1)N(n-1)$, while the $\mathbb{R}$-index of the second operator is equal to $-2\varkappa-2mN(n-1)$, where

$$\varkappa=\frac{1}{2\pi}\Big[\arg\det\Big(\sum_{k=0}^{m}(-1)^{m-k}A_k(t)Q^{m-k}(t)\Big)\Big]_Y. \qquad (3.73)$$

Recall that the numbers $l[s]$ and $l(s)$ in Theorem 3.11 are defined by formulas (3.38) and (3.39), respectively.

2^0. In the domain Ω consider the equation

$$\frac{\partial^{m+n}z}{\partial z^m\partial\overline{z}^n}=f(z), \quad m,n\in\mathbb{Z}_+, \;\; m\le n. \qquad (3.74)$$

In case $n=m$ it turns into a polyharmonic equation

$$\Delta^m u=f(z)$$

for which there exist Noetherian boundary value problems of type

$$\sum_{k=0}^{m_r}a_{rk}(t)\,\frac{\partial^{m_r}u}{\partial\nu^k\partial s^{m_r-k}}\Big|_Y=\varphi_r(t), \;\; t\in Y, \;\; r=1,\dots,m, \qquad (3.75)$$

where $\partial/\partial\nu$ is the derivative with respect to the interior normal and $\partial/\partial s$ is the derivative with respect to the tangent directed positively. If however $n\ne m$, then for the operator in the left-hand side of (3.74) we have $\varkappa(-1)\ne\varkappa(1)$ in any local coordinate system. Hence the Noetherian boundary value problem of type $B_ru\big|_Y=\varphi_r$, $r=1,\dots,N$, where B_r are $\mathbb{C}$-linear differential operators does not exist for it (see §3.1 as well as [82, Theorem 3.1.1.1-7]). In particular, this is true for Bitsadze equation (see [15], [16, Ch. IV, §9], [17, Ch. II, §1, point 1^0]) $\frac{\partial^2 u}{\partial\overline{z}^2}=f(z)$. (Note, by the way, that solutions of the homogeneous equation (3.74) for $m=0$ are called polyanalytic functions. The survey [10] is devoted to the theory of such functions).

Thus we add to (3.75) the following boundary conditions:

$$\mathrm{Re}\Big(\sum_{k=0}^{m_r}a_{rk}(t)\frac{\partial^{m_r}u}{\partial\nu^k\partial s^{m_r-k}}\Big|_Y\Big)=\varphi_r(t), \; t\in Y, \; r=m+1,\dots,n, \qquad (3.76)$$

or

$$P_+\Big(\sum_{k=0}^{m_r}a_{rk}(t)\,\frac{\partial^{m_r}u}{\partial\nu^k\partial s^{m_r-k}}\Big|_Y\Big)=\psi_r(t), \;\; t\in Y, \;\; r=m+1,\dots,n, \qquad (3.77)$$

and assume that

$$f\in H_p^{s-m-n}(X)\;\;\big(B_{p,q}^{s-m-n}(X)\big),$$
$$u\in H_p^{s}(X)\;\;\big(B_{p,q}^{s}(X)\big),$$

$$
\begin{aligned}
&\varphi_r \in B^{s-m_r-1/p}_{p,p}(Y) \ \ \big(B^{s-m_r-1/p}_{p,q}(Y)\big) \ \text{ for } \ r=1,\dots,m,\\
&\varphi_r \in B^{s-m_r-1/p}_{p,p}(Y,\mathbb{R}) \ \big(B^{s-m_r-1/p}_{p,q}(Y,\mathbb{R})\big) \ \text{for } r=m+1,\dots,n,\\
&\psi_r \in P_+B^{s-m_r-1/p}_{p,p}(Y) \ \big(P_+B^{s-m_r-1/p}_{p,q}(Y)\big) \ \text{for } r=m+1,\dots,n,\\
&a_{rk} \in B^{s-m_r-1/p}_{p,q}(Y), \ \ s>m_r+2/p, \ \ 1<p<\infty, \ \ 1\le q\le\infty,
\end{aligned} \tag{3.78}
$$

or

$$
a_{rk} \in B^{s-m_r-1/p}_{p,q}(Y), \ \ s \ge m_r+2/p, \ \ 1<p<\infty, \ \ q=1. \tag{3.79}
$$

Remark. The lowest terms in equation (3.74) and in boundary conditions (3.75)–(3.77) are absent. Under appropriate restrictions these terms generate compact operators which can be neglected as in point 1^0 (this exactly has been done). If however we consider boundary value problems with the loss of smoothness, then the corresponding operators are unbounded, and the lowest terms cease to be subordinate and may essentially influence the character of solvability of the boundary value problems (see, e.g., [88], [16, Ch. IV, §10]).

Assume the boundary $Y=\partial\Omega$ to belong to the class C^l, where $l \ge \max\{l[s],\ l[s-m-n]\}$ in the case of Bessel-potential spaces and $l \ge \max\{l(s),\ l(s-m-n)\}$ in the case of Besov spaces. At every point $t\in Y$ let us introduce the local coordinate system with the abscissae axis directed positively along the tangent and with the ordinate axis directed along the inner normal. The operator from (3.74) in this system has the symbol $(-1)^m 2^{-m-n} e^{i(n-m)\theta}(\xi_2+i\xi_1)^m(\xi_2-i\xi_1)^n$ where $\theta=\theta(t)$ is the angle between positive directions of the tangent to the curve Y at the point t and of the axis Ox.

Introduce the following notation:

$$
\begin{aligned}
A_r(t,\xi_1,\xi_2) &= \sum_{k=0}^{m_r} a_{rk}(t)\xi_2^k\xi_1^{m_r-k},\\
\overline{A}_r(t,\xi_1,\xi_2) &= \sum_{k=0}^{m_r} \overline{a_{rk}(t)}\xi_2^k\xi_1^{m_r-k}, \quad r=1,\dots,n.
\end{aligned} \tag{3.80}
$$

By Taylor formula we have

$$
\begin{aligned}
\frac{\xi_2^j A_r(t,1,\xi_2)}{(\xi_2+i)^m} &= \frac{Q^{m-2}_{r,j}(t,\xi_2)}{(\xi_2+i)^m} + \frac{\frac{\partial^{m-1}(\xi_2^j A_r)}{\partial\xi_2^{m-1}}(t,1,-i)}{\xi_2+i} + \\
&\quad + Q^{m_r+j-m}_{r,j,1}(t,\xi_2),\\
\frac{\xi_2^j A_r(t,-1,\xi_2)}{(\xi_2+i)^n} &= \frac{Q^{n-2}_{r,j}(t,\xi_2)}{(\xi_2+i)^n} + \frac{\frac{\partial^{n-1}(\xi_2^j A_r)}{\partial\xi_2^{n-1}}(t,-1,-i)}{\xi_2+i} + \\
&\quad + Q^{m_r+j-n}_{r,j,-1}(t,\xi_2),
\end{aligned}
$$

$$\frac{\xi_2^j\overline{A}_r(t,1,\xi_2)}{(\xi_2+i)^n} = \frac{G_{r,j}^{n-2}(t,\xi_2)}{(\xi_2+i)^n} + \frac{\frac{\partial^{n-1}(\xi_2^j\overline{A}_r)}{\partial\xi_2^{n-1}}(t,1,-i)}{\xi_2+i} + \\ + G_{r,j,1}^{m_r+j-n}(t,\xi_2),$$

$$\frac{\xi_2^j\overline{A}_r(t,-1,\xi_2)}{(\xi_2+i)^m} = \frac{G_{r,j}^{m-2}(t,\xi_2)}{(\xi_2+i)^m} + \frac{\frac{\partial^{m-1}(\xi_2^j\overline{A}_r)}{\partial\xi_2^{m-1}}(t,-1,-i)}{\xi_2+i} + \\ + G_{r,j,-1}^{m_r+j-m}(t,\xi_2), \quad j\in\mathbb{Z}_+,$$

where

$$Q_{r,j}^{m-2}(t,\xi_2),\ Q_{r,j,1}^{m_r+j-m}(t,\xi_2),\ Q_{r,j}^{n-2}(t,\xi_2),\ Q_{r,j,-1}^{m_r+j-n}(t,\xi_2),$$
$$G_{r,j}^{n-2}(t,\xi_2),\ G_{r,j,1}^{m_r+j-n}(t,\xi_2),\ G_{r,j}^{m-2}(t,\xi_2),\ G_{r,j,-1}^{m_r+j-m}(t,\xi_2)$$

are polynomials with respect to ξ_2 with superscripts indicating polynomial degrees.

Using the results from §§1.6, 3.3, 3.4 (see also [37, Examples 11.1 and 11.2]) we can prove that for boundary value problem (3.74)–(3.76) to be Noetherian, it is sufficient the invertibility of the matrix

$$\|Z_{\lambda\tau}(t,\omega)\|_{\lambda,\tau=1}^{m+n},, \quad \omega=\pm 1,\ \ t\in Y,$$

where

$$Z_{\lambda\tau}(t,1) = \begin{cases} \frac{\partial^{m-1}(\xi_2^{\tau-1}A_r)}{\partial\xi_2^{m-1}}\,(t,1,-i), & \tau=1,\dots,m,\ r=\lambda=1,\dots,m \\ \qquad\qquad\qquad\text{or} & r=\lambda-m,\ \lambda=2m+1,\dots,m+n, \\ \frac{\partial^{n-1}(\xi_2^{j-1}\overline{A}_r)}{\partial\xi_2^{n-1}}\,(t,1,-i), & j=\tau-m,\ \tau=m+1,\dots,m+n, \\ & r=\lambda-m,\ \lambda=m+1,\dots,m+n, \\ 0, & \lambda=1,\dots,m,\ \tau=m+1,\dots,m+n \\ \qquad\qquad\qquad\text{or} & \lambda=m+1,\dots,2m,\ \tau=1,\dots,m, \end{cases}$$

$$Z_{\lambda\tau}(t,-1) = \begin{cases} \frac{\partial^{n-1}(\xi_2^{\tau-1}A_r)}{\partial\xi_2^{n-1}}\,(t,-1,-i), & \tau=1,\dots,n,\ r=\lambda=1,\dots,m \\ \qquad\qquad\qquad\text{or} & r=\lambda-m,\ \lambda=2m+1,\dots,m+n, \\ \frac{\partial^{m-1}(\xi_2^{j-1}\overline{A}_r)}{\partial\xi_2^{m-1}}\,(t,-1,-i), & j=\tau-n,\ \tau=n+1,\dots,m+n, \\ & r=\lambda-m,\ \lambda=m+1,\dots,m+n, \\ 0, & \lambda=1,\dots,m,\ \tau=n+1,\dots,m+n \\ \qquad\qquad\qquad\text{or} & \lambda=m+1,\dots,2m,\ \tau=1,\dots,n. \end{cases}$$

The matrix $\|Z_{\lambda\tau}(t,1)\|$ is non-degenerate if and only if the matrices

$$\left\|\frac{\partial^{m-1}(\xi_2^{j-1}A_r)}{\partial\xi_2^{m-1}}\,(t,1,-i)\right\|_{r,j=1}^{m}, \tag{3.81}$$

$$\left\| \frac{\partial^{n-1}(\xi_2^{j-1}\overline{A}_r)}{\partial \xi_2^{n-1}} (t, 1, -i) \right\|_{r,j=1}^{n} \tag{3.82}$$

are non-degenerate. Analogously, the matrix $\|Z_{\lambda\tau}(t,-1)\|$ is non-degenerate if and only if the matrices

$$\left\| \frac{\partial^{n-1}(\xi_2^{j-1}A_r)}{\partial \xi_2^{n-1}} (t, -1, -i) \right\|_{r,j=1}^{n}, \tag{3.83}$$

$$\left\| \frac{\partial^{m-1}(\xi_2^{j-1}\overline{A}_r)}{\partial \xi_2^{m-1}} (t, -1, -i) \right\|_{r,j=1}^{m} \tag{3.84}$$

are non-degenerate. From (3.80) it follows that invertibility of matrix (3.81) ((3.82)) is equivalent to that of matrix (3.84) ((3.83)).

Thus for boundary value problem (3.74)–(3.76) to be Noetherian, it is sufficient that matrices (3.81), (3.83) be non-degenerate for any $t \in Y$.

In a similar way, owing to the results from §§1.6, 3.3, 3.4 (see also [37, Examples 11.1 and 11.2]), we can prove that the invertibility of matrices (3.81), (3.83) for any $t \in Y$ is also sufficient for boundary value problem (3.74), (3.75), (3.77) to be Noetherian. Investigation of this problem is easier than that of the previous problem for, this time there do not appear matrices (3.82), (3.84).

Direct calculation shows that the condition

$$\begin{aligned} &\det \left\| \frac{\partial^{m-1}(\xi_2^{j-1}A_r)}{\partial \xi_2^{m-1}} (t, 1, -i) \right\|_{r,j=1}^{m} \neq 0, \quad \forall t \in Y, \\ &\det \left\| \frac{\partial^{n-1}(\xi_2^{j-1}A_r)}{\partial \xi_2^{n-1}} (t, -1, -i) \right\|_{r,j=1}^{n} \neq 0, \quad \forall t \in Y, \end{aligned} \tag{3.85}$$

is equivalent to

$$\begin{aligned} &\det \left\| \sum_{k=m-j}^{m_r} a_{rk}(t) \frac{(k+j-1)!}{(k+j-m)!} (-i)^k \right\|_{r,j=1}^{m} \neq 0, \quad \forall t \in Y, \\ &\det \left\| \sum_{k=n-j}^{m_r} a_{rk}(t) \frac{(k+j-1)!}{(k+j-n)!} i^k \right\|_{r,j=1}^{n} \neq 0, \quad \forall t \in Y. \end{aligned} \tag{3.86}$$

We can slightly simplify (3.85). Really, due to the Leibniz formula

$$\begin{aligned} \frac{\partial^{m-1}}{\partial \xi_2^{m-1}} (\xi_2^{j-1}A_r) &= \sum_{e=0}^{m-1} \binom{m-1}{e} \frac{\partial^e \xi_2^{j-1}}{\partial \xi_2^e} \, \frac{\partial^{m-1-e}A_r}{\partial \xi_2^{m-1-e}} = \\ &= \sum_{e=0}^{j-1} \binom{m-1}{e} \frac{(j-1)!}{(j-1-e)!} \, \xi_2^{j-1-e} \, \frac{\partial^{m-1-e}A_r}{\partial \xi_2^{m-1-e}}. \end{aligned} \tag{3.87}$$

Use (3.87) and perform $n-1$ steps to transform the matrix (3.81). At the j-th step in the matrix resulting from the previous step we subtract from the columns with numbers $\lambda = j+1, \dots, m$ the column with the number j multiplied by $\binom{\lambda-1}{j-1}\xi_2^{\lambda-j}$ $(\xi_2 = -i)$. Finally we shall get the matrix

$$\left\| \frac{(m-1)!}{(m-j)!} \frac{\partial^{m-j} A_r}{\partial \xi_2^{m-j}} (t, 1, -i) \right\|_{r,j=1}^{m}.$$

The matrix (3.83) may be treated analogously. It is clear that upon transformations the determinants remain unchanged. Therefore (3.85) is equivalent to the condition

$$\begin{aligned} &\det \left\| \frac{\partial^{m-j} A_r}{\partial \xi_2^{m-j}} (t, 1, -i) \right\|_{r,j=1}^{m} \neq 0, \quad \forall t \in Y, \\ &\det \left\| \frac{\partial^{n-j} A_r}{\partial \xi_2^{n-j}} (t, -1, -i) \right\|_{r,j=1}^{n} \neq 0, \quad \forall t \in Y, \end{aligned} \tag{3.88}$$

that is to the condition

$$\begin{aligned} &\det \left\| \sum_{k=m-j}^{m_r} a_{rk}(t) \frac{k!}{(k+j-m)!} (-i)^k \right\|_{r,j=1}^{m} \neq 0, \quad \forall t \in Y, \\ &\det \left\| \sum_{k=n-j}^{m_r} a_{rk}(t) \frac{k!}{(k+j-n)!} i^k \right\|_{r,j=1}^{n} \neq 0, \quad \forall t \in Y. \end{aligned} \tag{3.89}$$

Let Ω be a bounded open finitely connected domain in $\mathbb{C}$ with a boundary $Y = \partial\Omega$ belonging to the class C^l, where $l \geq l[s]$ $(l \geq l(s))$, $X = \Omega \cup Y$; let a_{rk} satisfy (3.78) *with $q = p$ (one of conditions* (3.78), (3.79)) *and equivalent conditions* (3.85), (3.86), (3.88), (3.89) *be fulfilled. Then boundary value problem* (3.74)–(3.76) *defines a Noetherian operator from $H_p^s(X)$ $(B_{p,q}^s(X))$ to*

$$H_p^{s-m-n}(X) \oplus \bigoplus_{r=1}^{m} B_{p,p}^{s-m_r-1/p}(Y) \oplus \bigoplus_{r=m+1}^{n} B_{p,p}^{s-m_r-1/p}(Y, \mathbb{R})$$
$$\left(B_{p,q}^{s-m-n}(X) \oplus \bigoplus_{r=1}^{m} B_{p,q}^{s-m_r-1/p}(Y) \oplus \bigoplus_{r=m+1}^{n} B_{p,q}^{s-m_r-1/p}(Y, \mathbb{R})\right)$$

while boundary value problem (3.74), (3.75), (3.77) *defines that from $H_p^s(X)$ $(B_{p,q}^s(X))$ to*

$$H_p^{s-m-n}(X) \oplus \bigoplus_{r=1}^{m} B_{p,p}^{s-m_r-1/p}(Y) \oplus \bigoplus_{r=m+1}^{n} P_+ B_{p,p}^{s-m_r-1/p}(Y)$$
$$\left(B_{p,q}^{s-m-n}(X) \oplus \bigoplus_{r=1}^{m} B_{p,q}^{s-m_r-1/p}(Y) \oplus \bigoplus_{r=m+1}^{n} P_+ B_{p,q}^{s-m_r-1/p}(Y)\right).$$

Remark. Particular cases of the problem (3.74)–(3.76) have been considered in detail by N. E. Tovmasyan's pupils (see [122], [123] and [5]). They determined the indices of the corresponding boundary value problems. In many cases the number of linearly independent solutions of homogeneous problems has been found, and what is more, sometimes even explicit formulas for solutions have been obtained.

The results of this section can be generalized to the case of equations on the Riemann surfaces. When investigating the problem of the Noetherity there appear no additional difficulties, although calculation of the index requires special consideration (see [13], [14]).

§

Let us take arbitrary Banach spaces E_1, E_2 and a continuous linear operator $\mathcal{A} : E_1 \to E_2$. The value

$$|||\mathcal{A}||| = \inf \left\{ \|\mathcal{A} + K\| : K \text{ is compact from } E_1 \text{ to } E_2 \right\}$$

is called an essential norm of the operator $\mathcal{A}$.

Essential norms of pointwise multipliers are of great importance when we "freeze" coefficients in partial differential equations. In §3.5 the coefficients were "frozen" as follows: first we approximated the coefficients by smooth functions and then applied Lemma 2.14. This method does not do for $q = \infty$ since C^∞ is not dense in $B^\sigma_{p,\infty}$. The estimates of essential norms of pointwise multipliers in Besov spaces ensure the possibility of "freezing" coefficients in this case. These estimates have been obtained for any $q \in [1, \infty]$, although we need them only for $q = \infty$. The proof of the above-mentioned estimates is based on the idea of using Kuratovski measure of noncompactness borrowed by us from the paper [77] in which operators in Hölder spaces have been considered.

Let E be a Banach space. By definition the Kuratovski measure of noncompactness $\alpha(\Omega)$ of the set $\Omega \subset E$ is an infinum of $d > 0$ such that Ω admits a finite covering by the sets whose diameters are less than d.

For a continuous linear operator $\mathcal{A} : E_1 \to E_2$ the Kuratovski measure of noncompactness is defined by the equality $\|\mathcal{A}\|^{(\alpha)} = \frac{1}{2}\,\alpha(\mathcal{A}S)$ where S is the unit sphere in E_1.

It is not difficult to see that $\|\mathcal{A}\|^{(\alpha)} \le |||\mathcal{A}|||$.

For a wide class of Banach spaces the values $|||\cdot|||$ and $\|\cdot\|^{(\alpha)}$ turn out to be equivalent.

We shall say that the Banach space E possesses the property of bounded approximation if for any given elements $x_1, \ldots, x_n \in E$ and any given $\varepsilon > 0$ there exists a finite-dimensional linear operator $T : E \to E$ such that $\|x_j - Tx_j\| \le \varepsilon$ for $j = 1, \ldots, n$, $\|T\| \le M < \infty$, where M depends on E only.

It is easy to see that if E_2 possesses the property of bounded approximation, then for any continuous linear operator $\mathcal{A} : E_1 \to E_2$ the inequality

$$|||\mathcal{A}||| \le C\|\mathcal{A}\|^{(\alpha)} \tag{3.90}$$

is valid, where $C = 2(M+1)$ depends on E_2 only. Indeed, let us divide $\mathcal{A}S$ into a finite number of sets of diameter less than $\alpha(\mathcal{A}S) + \varepsilon = 2\|\mathcal{A}\|^{(\alpha)} + \varepsilon$,

$\varepsilon > 0$. In every set let us choose one point y_j, $j = 1, \dots, m$, and take a finite-dimensional operator $T : E_2 \to E_2$ such that $\|T\| \le M$, $\|y_j - Ty_j\| \le \varepsilon$, $j = 1, \dots, m$.

We have

$$|||\mathcal{A}||| \le \|\mathcal{A} - T\mathcal{A}\| = \sup_{x \in S} \|(I - T)\mathcal{A}x\| \le$$
$$\le \sup_{x \in S} \left(\|(I - T)(\mathcal{A}x - y_j(x))\| + \|(I - T)y_j(x)\| \right),$$

where $y_j(x)$ denote one of the points y_j whose distance from $\mathcal{A}x$ is less than $\alpha(\mathcal{A}S) + \varepsilon$.

Thus $|||\mathcal{A}||| \le (M + 1)(2\|\mathcal{A}\|^{(\alpha)} + \varepsilon) + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get (3.90).

It is well known that

$$|||\mathcal{A}^*||| \le |||\mathcal{A}|||, \quad \|\mathcal{A}\|^{(\alpha)} \le 2\|\mathcal{A}^*\|^{(\alpha)} \tag{3.91}$$

(see [4, 2.5.1, 2.5.7]). Therefore if E_2 possesses the property of bounded approximation, then from (3.90), (3.91) we obtain

$$|||\mathcal{A}^*||| \le C_1 \|\mathcal{A}^*\|^{(\alpha)}, \tag{3.92}$$

where $C_1 = 4(M + 1)$ depends on E_2 only. (According to [7] the following improvement of (3.91) is valid: $\|\mathcal{A}^*\|^{(\alpha)} = \|\mathcal{A}\|^{(\alpha)}$. Hence in (3.92) we may take $C_1 = C = 2(M + 1)$.)

Let $1 \le p, q \le \infty$, $s > n/p$, $\varphi \in B^s_{p,q}(\mathbb{R}^n)$ have a compact support. Then for φI, the operator of multiplication by the function φ, acting in the space $B^s_{p,q}(\mathbb{R}^n)$, the inequality

$$\|\varphi I\|^{(\alpha)} \le const \|\varphi | C(\mathbb{R})\| = const \sup_{x \in \mathbb{R}^n} |\varphi(x)| < +\infty \tag{3.93}$$

is valid with a constant independent of φ.

Proof. By induction we can easily prove the equality

$$\left(\Delta_h^N(gf)\right)(x) = \sum_{k=0}^{N} \binom{N}{k} (\Delta_h^{N-k} g)(x + kh)(\Delta_h^k f)(x), \quad \forall x, h \in \mathbb{R}^n \tag{3.94}$$

(multiple differences Δ_h^l have been defined in §1.1 before the formula (1.14)). For $N = 2l$, $l \in \mathbb{N}$, from (3.94) it follows that

$$\left|\left(\Delta_h^{2l}(gh)\right)(x)\right| \le \sum_{k=0}^{l} 2^k \binom{2l}{k} \|f | C(\mathbb{R}^n)\| \cdot |(\Delta_h^{2l-k} g)(x + kh)| +$$
$$+ \sum_{k=l+1}^{2l} 2^{2l-k} \binom{2l}{k} \|g | C(\mathbb{R}^n)\| \cdot |(\Delta_h^k f)(x)|, \quad \forall x, h \in \mathbb{R}^n. \tag{3.95}$$

Take $N = 2l$, where $l > s$. By (3.95) and the equivalent norm (1.15) (where $m = 0$) we get

$$\|gf|B^s_{p,q}(\mathbb{R}^n)\| \le const\big(\|g|B^s_{p,q}(\mathbb{R}^n)\| \cdot \|f|C(\mathbb{R}^n)\| + \\ + \|g|C(\mathbb{R}^n)\| \cdot \|f|B^s_{p,q}(\mathbb{R}^n)\|\big) \tag{3.96}$$

with a constant not depending on g and f.

Due to the condition $s > n/p$ the space $B^s_{p,q}(\mathbb{R}^n)$ is continuously embedded in the Hölder space $C^\tau(\mathbb{R}^n)$, where $\tau < s - n/p$ (see, e.g., [109, Theorems 2.8.1 and 2.3.2-(c)]).

Let us fix an arbitrary function $\psi \in \mathcal{D}(\mathbb{R}^n)$ equal to unity in some neighbourhood of $\operatorname{supp}\varphi$. The norm of the operator ψI in the space $B^s_{p,q}(\mathbb{R}^n)$ is majorized by the C^λ-norm of function ψ for $\lambda > |s|$ (see [110, Corollary 2.8.2]). Therefore we can majorize it by the value not depending on φ.

From the above arguments and Arzela–Ascoli theorem (see, e.g., [87, Appendix A5]) it follows that the set

$$S_\psi = \big\{\psi f | f \in B^s_{p,q}(\mathbb{R}^n),\ \|f|B^s_{p,q}(\mathbb{R}^n)\| = 1\big\}$$

is precompact in $C(\mathbb{R})$. Hence for any $\varepsilon > 0$ there is a partition $S_\psi = \bigcup_{j=1}^{r} S_j$ such that $\operatorname{diam}_{C(\mathbb{R}^n)} S_j < \varepsilon$, $j = 1, \dots, r$.

If f_1, f_2 belong to the unit sphere of the space $B^s_{p,q}(\mathbb{R}^n)$ and $\psi f_1, \psi f_2 \in S_j$, then according to (3.96) we have

$$\|\varphi(f_1 - f_2)|B^s_{p,q}(\mathbb{R}^n)\| = \|\varphi(\psi f_1 - \psi f_2)|B^s_{p,q}(\mathbb{R}^n)\| \le \\ \le const\big(\|\varphi|C(\mathbb{R}^n)\| + \varepsilon\|\varphi|B^s_{p,q}(\mathbb{R}^n)\|\big) \tag{3.97}$$

with a constant depending on n, p, q and s only.

From (3.97) we can easily get that

$$\|\varphi I\|^{(\alpha)} \le const\big(\|\varphi|C(\mathbb{R}^n)\| + \varepsilon\|\varphi|B^s_{p,q}(\mathbb{R}^n)\|\big).$$

Since $\varepsilon > 0$ is arbitrary, (3.93) holds. ■

Using the Banach–Steinhaus theorem (see, e.g., [87, 2.6]), we can easily see that a Banach space in which there exists a sequence of continuous finite-dimensional linear operators strongly convergent to the unit operator, possesses the property of bounded approximation. In particular any Banach space in which there exists Shauder basis possesses the property of bounded approximation (see [28, Ch. IV, §3]). The wavelets mentioned in point 3^0, §3.4, form Shauder basis in the space $B^s_{p,q}(\mathbb{R}^n)$, $1 < p < \infty$, $1 \le q < \infty$, $s \in \mathbb{R}$ (the other Shauder bases are referred in [110, 2.5.5]). Therefore the above-mentioned space possesses the property of bounded approximation. We shall take advantage of this fact in proving the following statement.

Let $1 < p < \infty$, $1 \leq q \leq \infty$, $s > n/p$, $\varphi \in B^s_{p,q}(\mathbb{R}^n)$ have a compact support. Then for the operator φI acting in the space $B^s_{p,q}(\mathbb{R}^n)$ the inequality

$$|||\varphi I||| \leq const\|\varphi|C(\mathbb{R}^n)\| < +\infty \tag{3.98}$$

is valid, where the constant does not depend on φ.

Proof. In case $q < \infty$, (3.98) follows from (3.90) and (3.93). Suppose $q = \infty$. The space $B^s_{p,\infty}(\mathbb{R}^n)$ is conjugate to the space $B^{-s}_{p',1}(\mathbb{R}^n)$, $p' = p/(p-1)$, (see [109, 2.6.1] or Theorem 1.1) possessing the property of bounded approximation.

Operator of multiplication by φ is defined on the set $\mathcal{D}(\mathbb{R}^n)$ (which is dense in $B^{-s}_{p',1}(\mathbb{R}^n)$) since $s - n/p > 0 > -s - n/p'$ (see [109, 4.6.2]). Let us prove that this operator can be extended to the operator continuous in $B^{-s}_{p',1}(\mathbb{R}^n)$. Really, due to Hahn–Banach theorem and duality theorem (see [109, 2.6.1] or Theorem 1.1) for any $\psi \in \mathcal{D}(\mathbb{R}^n)$ there exists $g \in B^s_{p,\infty}(\mathbb{R}^n)$ such that

$$\|g|B^s_{p,\infty}(\mathbb{R}^n)\| = 1, \quad \langle g, \varphi\psi\rangle = \|\varphi\psi|B^{-s}_{p',1}(\mathbb{R}^n)\|.$$

Thus

$$\|\varphi\psi|B^{-s}_{p',1}(\mathbb{R}^n)\| = \langle g, \varphi\psi\rangle = \langle \varphi g, \psi\rangle \leq \|\varphi g|B^s_{p,\infty}(\mathbb{R}^n)\| \times$$
$$\times \|\psi|B^{-s}_{p',1}(\mathbb{R}^n)\| \leq const\|\varphi|B^s_{p,\infty}(\mathbb{R}^n)\| \cdot \|\psi|B^{-s}_{p',1}(\mathbb{R}^n)\| < +\infty$$

with a constant depending on n, p and s only (see (3.96)). Therefore we can extend φI by continuity to the operator bounded in $B^{-s}_{p',1}(\mathbb{R}^n)$. The operator of multiplication by φ in the space $B^s_{p,\infty}(\mathbb{R}^n)$ is its conjugate. For the latter the inequality (3.98) is a consequence of inequalities (3.92), (3.93). ■

Note that if $q < \infty$, then the requirement for $\operatorname{supp}\varphi$ to be compact in Lemma 3.15 and Theorem 3.16 is superfluous. Indeed, we can approximate φ by the functions with compact supports. In the case $q = \infty$ we lose this possibility. But in the present section this case is exactly one which is basic.

If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then the inequality (3.98) holds for any $s \in \mathbb{R}$, $p \in]1, +\infty[$, $q \in [1, +\infty]$. The proof of this fact is reduced to Theorem 3.16 by means of the order reduction operators

$$I^\sigma = F^{-1}\langle\xi\rangle^\sigma F \tag{3.99}$$

(cf. (1.10)). Indeed, let us take $r > 0$ and $\psi \in \mathcal{D}(\mathbb{R}^n)$ such that $s + r > n/p$, $\psi\varphi = \varphi$. The pseudodifferential operator $(\varphi I - I^r\varphi I^{-r})$ is of order -1 (see, e.g., [49, v. 3, Theorem 18.1.8]). Consequently, the operator $\psi(\varphi I - I^r\varphi I^{-r})$ is compact in $B^s_{p,q}(\mathbb{R}^n)$ (see [110, Remark 4.3.2-1] as well as [58] or [107, Ch. XI], [98]). Taking into account that $I^{\pm r}$ realize isomorphisms of the corresponding spaces (see Theorem 1.3), from Theorem 3.16 we obtain

$$|||\varphi I|B^s_{p,q}(\mathbb{R}^n)||| = |||\psi\varphi I|B^s_{p,q}(\mathbb{R}^n)||| =$$

$$= |||\psi I^r \varphi I^{-r} + \psi(\varphi I - I^r \varphi I^{-r}) | B^s_{p,q}(\mathbb{R}^n)||| =$$
$$= |||\psi I^r \varphi I^{-r} | B^s_{p,q}(\mathbb{R}^n)||| \leq const|||I^r \varphi I^{-r} | B^s_{p,q}(\mathbb{R}^n)||| \leq$$
$$\leq const|||\varphi I | B^{s+r}_{p,q}(\mathbb{R}^n)||| \leq const\|\varphi | C(\mathbb{R}^n)\| = const \sup_{x \in \mathbb{R}^n} |\varphi(x)|.$$

If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, then for the operator φI acting in the space $H^s_p(\mathbb{R}^n)$, we can easily reduce the proof of the inequality (3.98) with the help of operators (3.99) to the case of space $L_p(\mathbb{R}^n)$ in which this inequality is obvious (see [31, p. 204]). Then (3.98) is transferred to those pointwise multipliers φ which can be approximated by the functions from $\mathcal{D}(\mathbb{R}^n)$.

Combining the methods of the proof of Lemma 3.15, Theorem 3.16 and reasoning from [18, Ch. I, §6], we easily get the following result.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain satisfying the cone condition (see, e.g., [109, 4.2.3]*), $1 < p_1, p_2 < \infty$, $1 \leq q_1, q_2, \leq \infty$, $0 < s_1 \leq s_2 < \infty$, $q = \max\{q_1, q_2\}$, $s_2 > n/p_2$; $p = p_1$ if $s_1 \leq n/p_1$ and $p = \min\{p_1, p_3\}$ if $s_1 > n/p_1$, where p_3 is determined by the equality $s_1 = s_2 - n/p_2 + n/p_3$. If $\varphi \in B^{s_2}_{p_2,q_2}(\overline{\Omega})$, then for the operator φI acting from $B^{s_1}_{p_1,q_1}(\overline{\Omega})$ in $B^{s_1}_{p,q}(\overline{\Omega})$ the inequality*

$$|||\varphi I||| \leq const\|\varphi | C(\overline{\Omega})\|$$

is valid, where the constant does not depend on φ.

Note that more precise results on essential norms of pointwise multipliers in Sobolev–Slobodeckiĭ spaces $W^s_p(\mathbb{R}^n)$, $s > 0$, have been obtained in [63, Chapter IV].

REFERENCES

1. I. A. Abitbekov and N. K. Bliev, On solvability of generalized Riemann–Hilbert problem for multi-connected domain in fractional spaces. (Russian) *Izv. Akad. Nauk Kazakh. SSR, Ser. Fys.-Mat.* (1983), 1–4.

2. S. Agmon, A. Douglis and L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I, II. *Comm. Pure Appl. Math.* (1959), 623–727; (1964), 35–92.

3. M. S. Agranovich, Elliptic singular integro-differential operators. (Russian) *Uspekhi Mat. Nauk* (1965), No. 5, 3–120.

4. R. R. Akhmerov, M. I. Kamenskiĭ, A. S. Potapov, A. E. Rodkina and B. N. Sadovskiĭ, Measures of noncompactness and densifying operators. (Russian) *Nauka, Novosibirsk,* 1986.

5. V. V. Asatryan, Boundary value problems for proper and improper elliptic equations. (Russian) *Dissertation. Erevan,* 1992.

6. A. A. Askarov, On one problem with oblique derivative in fractional spaces. In: "Boundary value problems for differential equations and their application to mechanics and technique". (Russian) *Nauka, Alma-Ata,* 1983, 13–19.

7. K. Astala, On measures of noncompactness and ideal variations in Banach spaces. *Ann. Acad. Sci. Fenn., Ser. AI Math. Dissertationes,* (1980), 1–42.

8. M. F. Atiyah, K-theory. *Harvard Univ. Cambridge, Mass.,* 1965.

9. P. S. Avetisyan, Estimates of resolvent kernel of semi-elliptic operator. (Russian) *Dissertation. Erevan,* 1992.

10. M. B. Balk, Polyanalytic functions and their generalizations. *Itogi Nauki i Tekhniki, VINITI* (1991), 187–246; (*English transl.: Encyclopaedia of Math. Sci.* , *Springer-Verlag, Berlin.*)

11. M. Z. Berkolaĭko and I. Ya. Novikov, Wavelet bases in differentiable function spaces of anisotropic smoothness. (Russian) *Dokl. Akad. Nauk* (1992), No. 4, 615–618.

12. O. V. Besov, V. P. Il'in, S. M. Nikol'skiĭ, Integral representations of functions and embedding theorems. (Russian) *Nauka, Moscow,* 1975.

13. I. A. Bikchantaev, Differential problem of conjugation on a compact Riemann surface. (Russian) *Izv. Vysh Uchebn. Zaved. Mat.* (1985), 3–12.

14. I. A. Bikchantaev, Differential boundary value problems for first order elliptic systems on Riemann surfaces. (Russian) *Differentsial'nye Uravneniya* (1987), No. 10, 1725–1736.

15. A. V. Bitsadze, On the uniqueness of solution of Dirichlet problem for elliptic partial differential equations. (Russian) *Uspekhi Mat. Nauk* (1948), No. 6(28), 211–212.

16. A. V. Bitsadze, Boundary value problems for second order elliptic equations. (Russian) *Nauka, Moscow,* 1966.

17. A. V. Bitsadze, Some classes of partial differential equations. (Russian) *Nauka, Moscow,* 1981.

18. N. K. Bliev, Generalized analytic functions in fractional spaces. (Russian) *Nauka, Alma-Ata,* 1985.

19. B. Bojarski, Theory of the generalized analytic vector. (Russian) *Ann. Polon. Math.* (1966), 281–320.

20. L. Boutet de Monvel, Boundary problems for pseudo-differential operators. *Acta Math.* (1971), 11–51.

21. A. V. Brenner and E. M. Shargorodsky, Boundary value problems for elliptic pseudodifferential operators. *In: Encyclopaedia of Math. Sci. , Springer-Verlag, Berlin (to appear).*

22. A. P. Calderon, Cauchy integrals on Lipschitz curves and related operators. *Proc. Nat. Acad. Sci. U.S.A.* (1977), No. 4, 1324–1327.

23. O. O. Chkadua, The nonclassical boundary-contact problems of elasticity for homogeneous anisotropic media. *Math. Nachr.* (1995), 49–64.

24. O. O. Chkadua, The boundary-contact problem of elasticity for homogeneous anisotropic media with a contact on some part of the boundaries. *Georgian Math. J.* (1994), No. 2, 111–122.

25. R. R. Coifman, A. McIntosh et Y. Meyer, L'intégrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes. *Ann. of Math.* (1982), No. 2, 361–388.

26. M. Costabel and E. P. Stephan, An improved boundary element Galerkin method for three-dimensional crack problems. *Integral Equations Operator Theory* (1987), 467–504.

27. G. David, Opérateurs intégraux singuliers sur certaines courbes du plan complexe. *Ann. Sci. École Norm. Sup.* (1984), No. 1, 157–189.

28. M. M. Day, Normed linear spaces. *Springer-Verlag, Berlin–Göttingen–Heidelberg,* 1958.

29. R. V. Duduchava, Wiener-Hopf integral operators with discontinuous symbols. (Russian) *Dokl. Akad. Nauk SSSR* (1973), No. 2, 277–280.

30. R. V. Duduchava, Integral equations with fixed singularities. *Teubner, Leipzig,* 1979.

31. R. Duduchava, On multidimensional singular integral operators. I, II. *J. Operator Theory* (1984), 41–76, 199–214.

32. R. Duduchava, D. Natroshvili and E. Shargorodsky, Boundary value problems of the mathematical theory of cracks. *Proc. I. Vekua Inst. Appl. Math., Tbilisi State Univ.* (1990), 68–83.

33. R. Duduchava and R. Schneider, The algebra of non-classical singular integral operators on halfspace. *Integral Equations Operator Theory* (1987), 531–553.

34. A. S. Dynin, Multidimensional elliptic boundary value problems with one unknown function. (Russian) *Dokl. Akad. Nauk SSSR* (1961), No. 2, 285–287.

35. A. S. Dynin, On the theory of pseudo-differential operators on a manifold with boundary. (Russian) *Dokl. Akad. Nauk SSSR* (1969),

No. 2, 251–253.

36. E. M. Dynkin, Methods of the theory of singular integrals. I, II. (Russian) *Itogi Nauki i Tekhniki, VINITI* (1987), 197–292; (1989), 105–198; (*English transl.: Encyclopeadia of Math. Sci.* , *Springer-Verlag, Berlin*).

37. G. I. Eskin, Boundary value problems for elliptic pseudodifferential equations. *Transl. Math. Monographs* , *Providence, Rhode Island,* 1981.

38. J. Franke, Besov-Triebel-Lizorkin spaces and boundary value problems. *Semin. Anal.* 1984/85, *Inst. Math., Berlin,* 1985, 89–104.

39. M. Friezier, B. Jawerth and G. Weiss, Littlewood-Paley theory and the study of functional spaces. *CBMS Regional Conf. Ser. in Math.* (1991).

40. F. D. Gakhov, Boundary value problems. *Pergamon Press, Oxford,* 1966.

41. R. P. Gilbert and J. L. Buchanan, First order elliptic systems. *Academic Press, New York and London,* 1983.

42. I. Gohberg and I. A. Feldman, Convolution equations and projection methods for their solution. *Transl. Math. Monographs* , *Providence, Rhode Island,* 1974.

43. I. Gohberg and N. Ya. Krupnik, Einführung in die Theorie der Eindimensionalen Singulären Integraloperatoren. *Birkhäuser Verlag, Basel–Boston–Stuttgart,* 1979.

44. R. V. Goldshteĭn and V. M. Entov, Qualitative methods in solid mechanics. (Russian) *Nauka, Moscow,* 1989.

45. G. Grubb, Pseudo-differential boundary problems in L_p spaces. *Comm. Partial Differential Equations* (1990), No. 3, 289–340.

46. S. M. Grudskiĭ, On the compactness of one integral operator. (Russian) *Rostov-on-Don, Deposited in VINITI* 18.11.80, No. 4856.

47. V. P. Khavin, Boundary properties of the Cauchy type integral and harmonically conjugate functions in domains with rectifiable boundary. (Russian) *Mat. Sb.* (1965), No. 4, 499–517.

48. L. Hörmander, Estimates for translation invariant operators in L^p-spaces. *Acta Math.* (1960), 93–140.

49. L. Hörmander, The analysis of linear partial differential operators. Vol. I-IV, *Springer-Verlag, Berlin-Heidelberg-New York-Tokyo,* 1983, 1985.

50. Zh. Idirisov, Some general boundary value problems for elliptic equations on a plane in fractional spaces. (Russian) *Dissertation. Alma-Ata,* 1988.

51. L. Jentsch and D. Natroshvili, Non-classical mixed interface problems for anisotropic bodies. *Math. Nachr.* (*to appear*).

52. G. A. Kalyabin, Theorems on extension, multipliers and diffeomorphisms for generalized Sobolev-Liouville classes in domains with Lipschitz boundary. (Russian) *Trudy Mat. Inst. Steklov* (1985), 173–186.

53. R. V. Kapanadze, On some properties of singular operators in normed spaces. (Russian) *Trudy Tbiliss. Univ. Mat. Mekh. Astronom.* (1968),

17–26.
54. G. A. Karapetyan, Solution of semi-elliptic equations in a half-space. (Russian) *Trudy Mat. Inst. Steklov* (1984), 119–138.
55. P. Koosis, Introduction to H_p spaces. *Cambridge Univ. Press, Cambridge,* 1980.
56. M. A. Krasnoselskiĭ, P. P. Zabreiko, E. I. Pustylnik and P. E. Sobolevskiĭ, Integral operators in spaces of summable functions. (Russian) *Nauka, Moscow,* 1966.
57. S. G. Kreĭn, Yu. I. Petunin and E. M. Semënov, Interpolation of linear operators. (Russian) *Nauka, Moscow,* 1978.
58. H. Kumano-go and M. Nagase, L^p-theory of pseudo-differential operators. *Proc. Japan Acad. Ser. A Mat. Sci.* (1970), No. 2, 138–142.
59. M. A. Lavrentyev and B. V. Shabat, Methods of the theory of functions of a complex variable. (Russian) *Nauka, Moscow,* 1987.
60. P. C. Lemarié et Y. Meyer, Ondelettes et bases Hilbertiennes. *Rev. Mat. Ibero-Amer.* (1986), 1–18.
61. G. S. Litvinchuk, Boundary value problems and singular integral equations with shift. (Russian) *Nauka, Moscow,* 1977.
62. G. F. Mandzhavidze, Boundary value problems of conjugation with a shift for analytic and generalized analytic functions. (Russian) *I. Vekua Inst. Appl. Math., Tbilisi Univ. Press, Tbilisi,* 1990.
63. V. G. Maz'ya and T. O. Shaposhnikova, Multipliers in spaces of differentiable functions. (Russian) *Izd. Leningr. Univ., Leningrad,* 1986.
64. S. G. Mikhlin and S. Prössdorf, Singular Integral Operators. *Springer-Verlag, Berlin-Heidelberg-New York-Tokyo,* 1986.
65. J. W. Milnor, Topology from the differentiable viewpoint. *Univ. Press Virginia, Charlottesville,* 1965.
66. A.S. Mishchenko, Vector bundles and their applications. (Russian) *Nauka, Moscow,* 1984.
67. R. Munkres, Elementary Differential Topology. *Ann. Math. Studies* 54, *Princeton, New Jersey, Princeton University Press,* 1966.
68. T. Murai, Boundedness of singular integral operators of Calderon type. *Proc. Japan Acad. Ser. A Mat. Sci.* (1983), No. 8, 364–367.
69. T. Murai, A real variable method for the Cauchy transform and analytic capacity. *Springer-Verlag, Berlin-Heidelberg-New York-Tokyo,* 1988.
70. N. I. Muskhelishvili, Singular integral equations. (Russian) *Nauka, Moscow,* 1968.
71. D. Natroshvili, Mixed interface problems for anisotropic elastic bodies. *Georgian Math. J.* (*to appear*).
72. D. G. Natroshvili, O. O. Chkadua and E. M. Shargorodsky, Mixed problems for homogeneous anisotropic elastic media. (Russian) *Proc. I. Vekua Inst. Appl. Math., Tbilisi State Univ.* (1990), 133–181.
73. D. G. Natroshvili and E.M. Shargorodsky, Three-dimensional problems of dynamics for elastic bodies with cuts. (Russian) *Bull. Acad. Sci. Georgia* (*Soobshch. Akad. Nauk Gruzii*) (1990), No. 2, 281–284.

74. S. M. Nikol'skiĭ, Approximation of Functions of Several Variables and Imbedding Theorems. (Russian) *Nauka, Moscow,* 1977.

75. R. S. Palais, Seminar on the Atiyah-Singer index theorem. *Princeton Univ. Press, Princeton, New Jersey,* 1965.

76. V. S. Pilidi, Bisingular operators and operators of close to them classes. (Russian) *Dissertation. Rostov-on-Don,* 1990.

77. R. Pöltz, Operators of local type in the spaces of Hölder functions. (Russian) *Semin. Anal. Operat. Equat. Numer. Anal.,* 1986/87, *Inst. Math., Berlin,* 1987, 107–122.

78. I. I. Privalov, Boundary properties of analytic functions. (Russian) *GITTL, Moscow,* 1950.

79. S. Prössdorf, Einige Klassen Singulärer Gleichungen. *Akademie-Verlag, Berlin,* 1974.

80. V. S. Rabinovich, Quasielliptic pseudo-differential operators and Cauchy problem for parabolic equations. (Russian) *Dokl. Akad. Nauk SSSR* (1971), No. 5, 1055–1058.

81. V. S. Rabinovich, Pseudo-differential operators on one class of non-compact manifolds. (Russian) *Mat. Sb.* (1972), 46–60.

82. S. Rempel and B.-W. Schulze, Index theory of elliptic boundary problems. *Akademie-Verlag, Berlin,* 1982.

83. S. Rempel and B.-W. Schulze, Parametrices and boundary symbolic calculus for elliptic boundary problems without the transmission property. *Math. Nachr.* (1982), 45–149.

84. S. Rempel and B.-W. Schulze, A theory of pseudo-differential boundary value problems with discontinuous conditions. I. II. *Ann. Global Anal. Geom.* (1984), No. 2, 163–251; No. 3, 289–384.

85. S. Rempel and B.-W. Schulze, Complete Mellin symbols and the conormal asymptotics in boundary value problems. *Proc. Journées "Eqs. Dér. Part.," St. Jean de Monts* (1984).

86. S. Rempel and B.-W. Schulze, Mixed boundary value problems for Lamé's system in three dimensions. *Math. Nachr.* (1984), 265–290.

87. W. Rudin, Functional analysis. *Mc Graw-Hill Book Company, New York,* 1973.

88. R. S. Saks, Boundary value problems for elliptic systems of differential equations. (Russian) *Novosibirsk State Univ., Novosibirsk,* 1975.

89. H.-J. Schmeisser and H. Triebel, Topics in Fourier Analysis and Function Spaces. *Akad. Verl. Geest and Portig K.-G., Leipzig,* 1987.

90. E. Shamir, Une propriété des espaces $H^{s,p}$. *C. R. Acad. Sci. Paris. Sér. A-B,* (1962), 448–449.

91. E. Shamir, A remark on Mikhlin-Hörmander multipliers theorem. *J. Math. Anal. Appl.* (1966), 104–107.

92. E. Shamir, Elliptic systems of singular integral operators. I: The half-space case. *Trans. Amer. Math. Soc.* (1967), 107–124.

93. E. Shamir, Regularization of mixed second-order elliptic problems. *Israel J. Math.* (1968), 150–168.

94. E. Shargorodsky, Boundary value problems for elliptic pseudodifferential operators: the half-space case. (Russian) *Proc. Razmadze Math. Inst. Georgian Acad. Sci.* (*Trudy Tbilisi Mat. Inst. Razmadze Akad. Nauk. Gruzii*) 44–80 *(in print).*

95. E. Shargorodsky, Boundary value problems for elliptic pseudodifferential operators on manifolds. (Russian) *Proc. Razmadze Math. Inst. Georgian Acad. Sci.* (*Trudy Tbilisi Mat. Inst. Razmadze Akad. Nauk. Gruzii*) 108–132 *(in print).*

96. E. Shargorodsky, Boundary value problems for elliptic pseudodifferential operators: the case of two-dimensional manifolds. (Russian) *Proc. Razmadze Math. Inst. Georgian Acad. Sci.* (*Trudy Tbilisi Mat. Inst. Razmadze Akad. Nauk. Gruzii*) , 29–69 *(in print).*

97. E. Shargorodsky, On the continuity of generalized solutions of mixed problems for Lamé's system. (Russian) *Bull. Acad. Sci. Georgia* (*Soobshch. Akad. Nauk Gruzii*) (1989), No. 3, 493–496.

98. E. Shargorodsky, Some remarks on the boundedness of pseudodifferential operators. *Math. Nachr.* (*to appear*).

99. W. Sickel, Spline representations of functions in Besov-Triebel-Lizorkin spaces on $\mathbb{R}^n$. *Forum Math.* (1990), 451–475.

100. I. B. Simonenko, A new general method for the investigation of linear operator equations of singular integral equations type. I, II. (Russian) *Izv. Akad. Nauk SSSR, Ser. Mat.* (1965), 567–586, 757–782.

101. E. M. Stein and G. Weiss, Introduction to Fourier Analysis on Euclidean Spaces. *Princeton Univ. Press, Princeton, New Jersey,* 1971.

102. E. P. Stephan, Boundary integral equations for mixed boundary value problems, screen and transmission problems in $\mathbb{R}^3$. *Habilitationsschrift. Darmstadt,* 1984.

103. E. P. Stephan, A boundary integral equation method for three-dimensional crack problems in elasticity. *Math. Methods Appl. Sci.* (1986), 609–623.

104. E. P. Stephan, Boundary integral equations for mixed boundary value problems in $\mathbb{R}^3$. *Math. Nachr.* (1987), 21–53.

105. B. Stöckert and H. Triebel, Decomposition methods for functional spaces of $B^s_{p,q}$ type and $F^s_{p,q}$ type. *Math. Nachr.* (1979), 247–267.

106. R. S. Strichartz, Multipliers on fractional Sobolev spaces. *J. Math. Mech.* (1967), 1031–1060.

107. M. E. Taylor, Pseudodifferential Operators. *Princeton Univ. Press, Princeton, New Jersey,* 1981.

108. F. Trèves, Introduction to Pseudodifferential and Fourier Integral Operators. Vol. I-II. *Plenum Press, New York and London,* 1982.

109. H. Triebel, Interpolation Theory, Functional Spaces, Differential Operators. *VEB Deut. Verl. Wissen., Berlin,* 1978.

110. H. Triebel, Theory of Function Spaces. *Birkhäuser Verlag, Basel–Boston–Stuttgart,* 1983.

111. H. Triebel, A priori estimates and boundary value problems for semi-elliptic differential equations: A model case. *Comm. Partial Differential Equations* (1983), 1621–1664.

112. S. V. Uspenskiĭ, G. V. Demidenko, V. G. Perepëlkin, Embedding theorems and applications to differential equations. (Russian) *Nauka, Novosibirsk,* 1984.

113. I. N. Vekua, Verallgemeinerte analytische Functionen. *Akademie-Verlag, Berlin,* 1963.

114. M. I. Vishik and G. I. Eskin, Boundary value problems for general singular equations in a bounded domain. (Russian) *Dokl. Akad. Nauk SSSR* (1964), No. 1, 24–27.

115. M. I. Vishik and G. I. Eskin, Convolution equations in a bounded domain. (Russian) *Uspekhi Mat. Nauk* (1965), No. 3(123), 89–152.

116. M. I. Vishik and G. I. Eskin, Convolution equations in a bounded domain in spaces with weighted norms. (Russian) *Mat. Sb.* (1966), No. 1, 65–110.

117. M. I. Vishik and G. I. Eskin, Normally solvable problems for elliptic systems of convolution equations. (Russian) *Mat. Sb.* (1967), No. 3, 326–356.

118. M. I. Vishik and G. I. Eskin, Convolution equations of variable order. (Russian) *Trudy Moskov. Mat. Obshch.* (1967), 25–50.

119. M. I. Vishik and G. I. Eskin, Elliptic convolution equations in a bounded domain and their applications. (Russian) *Uspekhi Mat. Nauk* (1967), No. 1, 15–76.

120. M. I.Vishik and G. I. Eskin, Sobolev-Slobodeckiĭ's spaces of variable order and their applications to mixed boundary value problems. (Russian) *Sibirsk. Mat. Zh.* (1968), No. 5, 973–997.

121. W. L. Wendland and J. Zhu, The boundary element method for three-dimensional Stokes flows exterior to an open surface. *Math. Comput. Modelling* (1991), No. 6, 19–41.

122. A. A. Zakaryan, Boundary value problems for improper elliptic equations. (Russian) *Dissertation. Erevan,* 1992.

123. M. A. Ziroyan, Boundary value problems for improper elliptic equations of higher order. (Russian) *Dissertation. Erevan,* 1991.

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Author's address:
Department of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 380043
Republic of Georgia