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ON A NONLINEAR TWO-POINT BOUNDARY VALUE PROBLEM FOR HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS

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The present paper deals with the problem of existence of solution of the n-th order nonlinear nonautonomous ordinary differential equation

$$u^{(n)}+\sum_{k=1}^{n-1}p_k(t)u^{(k)}=f(t,u,u',\dots,u^{(n-1)}) \tag{1}$$

satisfying the nonlinear two-point boundary conditions

$$\begin{aligned} u^{(i)}(a)&=\varphi_{1i}(u(a),u'(a),\dots,u^{(n-1)}(a)) \quad (i=0,\dots,n_0-1),\\ u^{(j)}(b)&=\varphi_{2j}(u(b),u'(b),\dots,u^{(n-1)}(b)) \quad (j=0,\dots,n-n_0-1), \end{aligned} \tag{2}$$

where $n\geq 2$, $0<a<b<+\infty$, n_0 is the entire part of $\frac{n}{2}$, each of the functions $p_k:[a,b]\to\mathbb{R}$ for $k\in\{1,\dots,n-1\}$ is absolutely continuous together with its derivatives up to the order $k-1$ inclusive, the function $f:[a,b]\times\mathbb{R}^n\to\mathbb{R}$ satisfies the local Carathéodory conditions and the functions $\varphi_{1i}:\mathbb{R}^n\to\mathbb{R}$ $(i=0,\dots,n_0-1)$ and $\varphi_{2j}:\mathbb{R}^n\to\mathbb{R}$ are continuous and satisfy the conditions

$$\begin{aligned} \sum_{i=0}^{n_0-1}\Big|\varphi_{1i}(x_0,x_1,\dots,x_{n-1})\Big| &\leq c_1\Big(1+\sum_{k=n_0}^{n-1}|x_k|\Big)^{-\vartheta_1},\\ \sum_{j=0}^{n-n_0-1}\Big|\varphi_{2j}(x_0,x_1,\dots,x_{n-1})\Big| &\leq c_2\Big(1+\sum_{k=n-n_0}^{n-1}|x_k|\Big)^{-\vartheta_2} \end{aligned} \tag{3}$$

on $\mathbb{R}^n$, where $c_i\geq 0$ and $\vartheta_i\in[0,1]$ $(i=1,2)$.

The above-given theorems on the existence and uniqueness of a solution of the problem (1), (2) supplement the results of the works [1–3, 6].

Everywhere in the sequel we will assume that $\mathbb{R}=]-\infty,+\infty[$, $\mathbb{R}_+=[0,+\infty[$ and $\mathbb{R}^n$ is an n dimensional real Euclidean space.

Following [4], by $\mu_i^k (k=1,2,\dots;\ k=2i,2i+1,\dots)$ we denote real constants defined by the recurrence relation

$$\mu_0^{i+1}=1/2,\ \ \mu_i^{2i}=1,\ \ \mu_{i+1}^k=\mu_{i+1}^{k-1}+\mu_i^{k-2}\ \ (i=0,1,\dots;\ k=2i+3,\dots).$$

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Below in Theorems 1 and 2 it will be assumed that the function f satisfies the inequalities

$$(-1)^{n-n_0-1} f(t, x_0, x_1, \dots, x_{n-1}) s_0 \geq -\sum_{i=0}^{n_0-1} \alpha_i(t)|x_i|, -\alpha(t), \tag{4}$$

$$\Big| f(t, x_0, x_1, \dots, x_{n-1}) \Big| \leq h(t, |x_0|, |x_1|, \dots, |x_{n_0-1}|) \tag{5}$$

on $[a,b] \times \mathbb{R}^n$, where the functions $\alpha_0 : [a,b] \to \mathbb{R}$ and $\alpha_i : [a,b] \to \mathbb{R}_+$ $(i = 1, \dots, n_0 - 1)$ are summable and $\alpha : [a,b] \to \mathbb{R}$ is square summable, while the function $h : [a,b] \times \mathbb{R}_+^{n_0} \to \mathbb{R}_+$ is summable in the first argument, nondecreasing in the last n_0 arguments and for any $\rho_0 > 0$ satisfies the conditions

$$\limsup_{\substack{t \to a_k \\ \rho \to +\infty}} \frac{1}{\rho^2} \Big(\int_{a_k}^{t} h(\tau, \rho_0, \rho, \dots, \rho)\, d\tau \Big)^{1-\lambda_k} < +\infty \quad (k = 1, 2), \tag{6}$$

where $\lambda_k \in [0,1]$, $a_1 = a$ and $a_2 = b$.

Theorem 1. *Let there exist constants $\gamma_i \leq 0$ $(i = 1, \dots, n_0 - 1)$, $\eta > 0$, and $\delta > 0$ such that*

$$\mu_{n_0}^n - \sum_{i=1}^{n-1} \frac{i\gamma_i}{n_0} \eta^{i-n_0} \delta \tag{7}$$

and the inequalities

$$\begin{gathered}
\sum_{k=2i}^{n-1} (-1)^{n-n_0+k-i-1} \mu_i^k \Big[t^{n-2n_0} p_k(t) \Big]^{(k-2i)} + \\
+ t^{n-2n_0} \alpha_i(t) \leq \gamma_i \quad (i = 1, \dots, n_0 - 1), \\
\sum_{k=1}^{n-1} (-1)^{n-n_0+k} \mu_0^k \Big[t^{n-2n_0} |p(t)\Big]^{(k)} - t^{n-2n_0} \sum_{i=0}^{n_0-1} \alpha_i(t) \geq \\
\geq \sum_{i=1}^{n_0-1} \frac{(n_0 - i)\gamma_i}{n_0} \eta^i + \delta
\end{gathered} \tag{8}$$

hold on $[a,b]$. Then the problem (1), (2) *is solvable.*

In the case where n is odd, we complement Theorem 1 with

Theorem 2. *Let $n = 2n_0 + 1$ and there exist a constant $\delta > 0$ such that the inequalities*

$$\begin{gathered}
p_{n-1}(t) \leq -\delta, \quad \sum_{k=2i}^{n-1} (-1)^{n-n_0+k-i} \mu_i^k \Big[p_k(t) \Big]^{(k-2i)} \leq 0 \quad (i = 1, \dots, n_0 - 1), \\
\sum_{k=1}^{n-1} (-1)^{n_0+k-1} \mu_0^k \Big[p_k(t) \Big]^{(k)} - \sum_{i=0}^{n_0-1} \alpha_i(t) \geq \delta
\end{gathered} \tag{9}$$

hold on $[a,b]$. Then the problem (1), (2) *is solvable.*

In the special case where

$$f(t,x_0,x_1,\dots,x_{n-1}) \equiv f(t,x_0,x_1,\dots,x_{n_0-1}),$$
$$\varphi_{1i}(x_0,x_1,\dots,x_{n-1}) \equiv 0 \quad (i=0,\dots,n_0-1),$$
$$\varphi_{2j}(x_0,x_1,\dots,x_{n-1}) \equiv 0 \quad (j=0,\dots,n-n_0-1),$$

i.e., when the problem (1), (2) is of the form

$$u^{(n)} + \sum_{k=1}^{n-1} p_k(t)u^{(k)} = f(t,u,u',\dots,u^{(n_0-1)}), \tag{1_0}$$

$$u^{(i)}(a)=0 \ \ (i=0,\dots,n_0-1), \ \ u^{(j)}(b)=0 \ \ (j=0,\dots,n-n_0-1), \tag{2_0}$$

we have the following results on the unique solvability of the problem (1_0), (2_0).

Theorem 3. *Let the inequality*

$$(-1)^{n-n_0-1}\Big[f(t,x_0,x_1,\dots,x_{n_0-1}) - f(t,y_0,y_1,\dots,y_{n_0-1})\Big] \geq - \\ - \sum_{i=0}^{n_0-1} \alpha_i(t)|x_i-y_i| \tag{10}$$

hold on $[a,b]\times\mathbb{R}^{n_0}$*, where the functions* $\alpha_0[a,b]\to\mathbb{R}$ *and* $\alpha_i:[a,b]\to\mathbb{R}_+$ $(i=1,\dots,n_0-1)$ *are summable. Moreover, assume that the condition*

$$\int_a^b \big|f(t,0,0,\dots,0)\big|^2 dt < +\infty \tag{11}$$

is fulfilled and there exist constants $\gamma_i \geq 0$ $(i=1,\dots,n_0-1)$, $\eta>0$ *and* $\delta>0$ *satisfying* (7) *and such that the inequalities* (8) *hold on* $[a,b]$*. Then the problem* (1_0), (2_0) *is uniquely solvable.*

Theorem 4. *Let* $n=2n_0+1$*, the inequality* (10) *hold on* $[a,b]\times\mathbb{R}^{n_0}$ *and the condition* (11) *be fulfilled. Moreover, assume that there exists a constant* $\delta>0$ *such that the inequalities* (3) *are satisfied on* $[a,b]$*. Then the problem* (1_0), (2_0) *is uniquely solvable.*

For proving the above-formulated theorems the use was made of the method of a priori estimates and the principle of a priori boundedness proven in [5].

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