Mem. Differential Equations Math. Phys. 20(2000), 133–135

E. Bravyi

A NOTE ON THE FREDHOLM PROPERTY OF BOUNDARY VALUE PROBLEMS FOR LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

(Reported on October 11, 1999)

The following standard notation will be used:

W is the Banach space of absolutely continuous functions $x : [a,b] \to \mathbb{R}^1$ with the norm

$$\|x\|_W = |x(a)| + \int_a^b |\dot{x}(s)|\, ds;$$

C is the Banach space of continuous functions $x : [a,b] \to \mathbb{R}^1$ with the norm

$$\|x\|_C = \max_{t\in[a,b]} |x(t)|;$$

L is the Banach space of summable functions $z : [a,b] \to \mathbb{R}^1$ with the norm

$$\|z\|_L = \int_a^b |z(t)|\, dx;$$

I is the identical operator in an appropriate space.

1. Consider the general boundary value problem in the space W

$$\begin{aligned} &(\mathcal{L}x)(t) \overset{\text{def}}{=} \dot{x}(t) - (Tx)(t) = f(t), \quad t \in [a,b], \\ &\ell x = \alpha, \end{aligned} \tag{1}$$

where $T : W \to L$ is a linear bounded operator, $\ell : W \to R^1$ is a linear bounded functional.

We say that the boundary value problem (1) has the Fredholm property if the operator

$$\begin{pmatrix} \mathcal{L} \\ \ell \end{pmatrix} : W \to L \times R^1$$

has the Fredholm property, that is, it is a Noether operator with zero index.

In his paper [5] V. P. Maksimov proved that the problem (1) has Fredholm property if T is an U-bounded operator [6, p. 157] acting from C into L. In this case, by definition there exists a function $u \in L$ such that

$$|(Tx)(t)| \le u(t), \quad t \in [a,b],$$

for every $x \in C$ with $\|x\|_C \le 1$. Such an operator T acts from W into L completely continuously.

2000 *Mathematics Subject Classification.* 34K10.

Key words and phrases. Functional differential equation, boundary value problem, Fredholm property.

As we will show, the problem (1) has Fredholm property for $T : C \to L$ without any additional assumptions.

Theorem 1. *Let T be bounded as an operator from C into L . Then boundary value problem* (1) *has Fredholm property.*

To prove Theorem 1 we need two lemmas.

Leema 1. *Let T be a linear bounded operator from C to L. Then the operator T is weakly completely continuous.*

Proof. A linear bounded operator, acting from C into any weakly complete Banach space, is a weakly completely continuous operator [2, VI.7.6]. The space L is weakly complete [2, IV.8.6]. Thus, the operator T is weakly completely continuous. □

Leema 2. *Let $T : C \to L$, $S : L \to C$ be linear bounded operators. Then the operators $I - ST : C \to C$ and $I - TS : L \to L$ both have Fredholm property.*

Proof. By Lemma 1, it follows that T is weakly completely continuous. A product of a weakly completely continuous linear operator and a bounded linear operator is a weakly completely continuous operator [2, VI.4.5]. So we see that the operators $ST : C \to C$ and $TS : L \to L$ are weakly completely continuous. Therefore, the operators $(ST)^2 : C \to C$ $(TS)^2 : L \to L$ both are completely continuous. Indeed, a product of weakly completely continuous operators in the space C or in the space L is a completely continuous operator [2, VI.7.5, VI.8.13].

By Nikol'skiĭs theorem (see [3, p. 504]), since the squares of the operators ST and TS are, since the squares of the operators ST and TS are completely continuous, the operators $I - ST : C \to C$ and $I - TS : L \to L$ have Fredholm property. □

Proof of Theorem 1. The boundary value problem (1) has Fredholm property if and only if the operator $Q \operatorname{def} \mathrm{L}\Lambda : L \to L$, where $(\Lambda z)(t) = \int_a^t z(s)\,ds$, $t \in [a,b]$, has Fredholm property. This is shown in [1].

We have $Q = I - T\Lambda$.

The operator $\Lambda : L \to C$ is bounded. By Lemma 2 for $S = \Lambda$, it follows that Q has Fredholm property. □

2. Let us obtain criteria of Fredholm property for the singular boundary value problem

$$\begin{aligned} &(\mathrm{L}_1 x)(t) \operatorname{def} (t-a)(b-t)\ddot{x}(t) - (Tx)(t) = f(t), \quad t \in [a,b], \\ &\ell_1 x = \beta, \end{aligned} \tag{2}$$

where $T : W \to L$ is a linear bounded operator and $\ell_1 : W \to R^2$ is a linear bounded functional.

Consider the problem (2) in the space $\mathcal{D}$ of all functions $x : [a,b] \to R^1$ such that

1) x is absolutely continuous on $[a,b]$;

2) $\dot{x}$ is locally absolutely continuous on (a,b);

3) $\int_a^b (t-a)(b-t)|\ddot{x}(t)|\,dt < +\infty$.

We say that boundary value problem (2) has the Fredholm property if the operator

$$\begin{pmatrix} \mathcal{L}_1 \\ \ell \end{pmatrix} : \mathcal{D} \to L \times R^2$$

has the Fredholm property.

Theorem 2. *Let T be bounded as an operator from C into L. Then the boundary value problem* (2) *has Fredholm property.*

Proof. In the article [4] it was proved that the space $\mathcal{D}$ with the norm

$$\|x\|_{\mathcal{D}} = |x(a)| + |x(b)| + \int_a^b (t-a)(b-t)|\ddot{x}(t)|\,dt$$

is continuously embedded into the space W. Moreover, the space $\mathcal{D}$ is isomorphic to the direct product $L \times R^2$. The isomorphism $\mathcal{J} : L \times R^2 \to \mathcal{D}$ is defined by the equality:

$$\mathcal{J}\{z, \beta\} = \Lambda_1 z + Y\beta,$$

where $\Lambda_1 : L \to \mathcal{D}$, $Y : R^2 \to D$,

$$(\Lambda_1 z)(t) = -\int_a^t \frac{b-t}{b-s} z(s)\,ds - \int_t^b \frac{t-a}{s-a} z(s)\,ds, \quad t \in [a,b],$$

$$Y\beta = (t-a)\beta_1 + (b-t)\beta_2, \quad t \in [a,b].$$

The Fredholm property of the boundary value problem (2) is equivalent to the Fredholm property of the operator $Q_1 \operatorname{def} \mathrm{L}\Lambda_1 : L \to L$.

We have $Q_1 = I - T\Lambda_1$.

Since the operator $\Lambda_1 : L \to C$ is bounded, the assertion of the theorem follows from Lemma 2 for $S = \Lambda_1$. $\square$

Acknowledgement

Supported by Grants 96-15-96195, 99-01-01278 of the Russia Foundation for Basic Research and the Competition Centre of the Fundamental Natural Science, 1998.

References

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, Introduction to the theory of functional differential equations. (Russian) *Naula, Moscow,* 1991.

2. N. Dunford and J. T. Schwvartz, Linear operators. General theory. *Wiley-Interscience*, 1961.

3. L. V. Kantorovich and G. P. Akilov, Functional analysis. (Russian) *Nauka, Moscow,* 1977.

4. S. M. Labovskiĭ, On positive solutions of a two-point boundary value problem for a linear singular functional differential equation. *(Russian) Differentsial'nye Uravneniya* **24**(1988), No. 10, 1695–1704.

5. V. P. Maksimov, Noether property of the general for linear functional differential equation. (Russian) *Differentsial'nye Uravneniya* **10**(1974), No. 12, 2288–2291.

6. Functional analysis. (Russian) *Nauka, Moscow,* 1972.

Author's address:
Perm State Technical University,
Perm, Russia