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ON UNIQUENESS OF SOLUTION OF THE WEIGHTED INITIAL VALUE PROBLEM FOR HIGHER ORDER EVOLUTION SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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In the present paper optimal, in a certain sense, sufficient conditions for uniqueness of solution of the weighted initial value problem

$$u^{(n)}(t) = f(u)(t), \tag{1}$$

$$\lim_{t\to a} \frac{u^{(k)}(t)}{h^{(k)}(t)} = 0 \quad (k = 0, \dots, n-1) \tag{2}$$

are given, where $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\mathrm{loc}}(]a,b];\mathbb{R}^m)$ is a continuous Volterra operator and $h : [a,b] \to [0,+\infty[$ is an $(n-1)$-times continuously differentiable function such that

$$h^{(k)}(a) = 0 \ \ (k = 0, \dots, n-2), \ \ h^{(n-1)}(t) > 0 \ \text{ for } \ a < t \le b.$$

A particular case of the equation (1) is the vector differential equation with delay

$$u^{(n)}(t) =$$

$$= f_0(t, u(\tau_{10}(t)), \dots, u^{(n-1)}(\tau_{1n-1}(t)), \dots, u(\tau_{l0}(t)), \dots, u^{(n-1)}(\tau_{ln-1}(t))), \tag{3}$$

where $f_0 :]a,b] \times \mathbb{R}^{lmn} \to \mathbb{R}^m$ satisfies the local Carathéodory conditions, and $\tau_{ik} : [a,b] \to [a,b]$ $(i = 1, \dots, l;\ k = 0, \dots, n-1)$ are measurable functions such that

$$\tau_{ik}(t) \le t \quad \text{for} \quad a \le t \le b \ \ (i = 1, \dots, l;\ k = 0, \dots, n-1).$$

Throughout the paper the use will be made of the following notation.

$\mathbb{R}^m$ is the space of m-dimensional column vectors $x = (x_i)_{i=1}^m$ with real components and the norm

$$\|x\| = \sum_{i=1}^m |x_i|.$$

$x \cdot y$ is the scalar product of the vectors x and $y \in \mathbb{R}^m$.

If $x = (x_i)_{i=1}^m \in \mathbb{R}^m$, then $\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^m$.

$C^{n-1}([a,b];\mathbb{R}^m)$ is the space of $(n-1)$-times continuously differentiable vector functions $x : [a,b] \to \mathbb{R}^m$ with the norm

$$\|x\|_{C^{n-1}} = \max\left\{ \sum_{k=1}^{n-1} \|x^{(k-1)}(t)\| : a \le t \le b \right\}.$$

$C_h^{n-1}([a,b];\mathbb{R}^m)$ is the set of $u \in C^{n-1}([a,b];\mathbb{R}^m)$ such that

$$\sup\left\{ \frac{\|u^{(k)}(t)\|}{h^{(k)}(t)} : a < t \le b \right\} < +\infty \ \ (k = 0, \dots, n-1).$$

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If $x :]a,b] \to \mathbb{R}^m$ is a bounded function and $a \le s < t \le b$, then

$$\nu(x)(s,t) = \sup\{\|x(\xi)\| : s < \xi < t\}.$$

$L_{\mathrm{loc}}(]a,b];\mathbb{R}^m)$ is the space of locally summable vector functions $x :]a,b] \to \mathbb{R}^m$ with the topology of convergence in the mean on each segment from $]a,b]$.

Definition 1. $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\mathrm{loc}}(]a,b];\mathbb{R}^m)$ is called a Volterra operator if the equality $f(x)(t) = f(y)(t)$ holds almost everywhere on $]a,t_0[$ for any $t_0 \in]a,b]$ and any vector functions x and $y \in C^{n-1}([a,b];\mathbb{R}^m)$ satisfying the condition $x(t) = y(t)$ for $a \le t \le t_0$.

Definition 2. We will say that the operator $f : C^{n-1}([a,b];\mathbb{R}^m) \to L_{\mathrm{loc}}(]a,b];\mathbb{R}^m)$ satisfies the local Carathéodory conditions if it is continuous and there exists a nondecreasing with respect to the second argument function $\gamma :]a,b] \times [0;+\infty[\to [0;+\infty[$ such that $\gamma(\cdot,\rho) \in L_{\mathrm{loc}}(]a,b];\mathbb{R})$ for any $\rho \in]0;+\infty[$ and the inequality

$$\|f(x)(t)\| \le \gamma(t, \|x\|_{C^{n-1}})$$

is fulfilled for any $x \in C^{n-1}([a,b];\mathbb{R}^n)$ almost everywhere on $]a,b[$.

Definition 3. A function $u : [a,b] \to \mathbb{R}^m$ is called a solution of the problem (1), (2) if:

(i) $u \in C^{n-1}([a,b];\mathbb{R}^m)$ and $u^{(n-1)}$ is absolutely continuous on each segment contained in $]a,b]$;

(ii) u satisfies (1) almost everywhere on $]a,b[$;

(iii) u satisfies initial conditions (2).

The following theorem is valid.

Theorem 1. *Let there exist summable functions $p_k : [a,b] \to [0,+\infty[$ $(k = 0,\dots,n-1)$ such that*

$$\limsup_{t\to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_a^t p_k(s)\,ds\right) < 1 \tag{4}$$

and for any $u_i \in C_h^{n-1}([a,b];\mathbb{R}^m)$ the inequality

$$\begin{aligned}\big(f(u_1)(t) - f(u_2)(t)\big)\,\mathrm{sgn}(u_1(t) - u_2(t)) \le \\ \le \sum_{k=0}^{n-1} p_k(t)\nu\left(\frac{u_1 - u_2}{h^{(k)}}\right)(a,t)\end{aligned} \tag{5}$$

be fulfilled almost everywhere on $]a,b[$. Then the problem (1), (2) has at least one solution.

From the above theorem and Theorem 2 from [3] follows

Theorem 2. *Let the conditions of Theorem 1 hold and*

$$\lim_{t\to a} \left(\frac{1}{h^{(n-1)}(t)} \int_a^t \|f(0)(s)\|ds\right) = 0. \tag{6}$$

Then the problem (1), (2) has one and only one solution.

Theorems 1 and 2 for the problem (3), (2) take the following form.

Corollary 1. *Let $\tau_{1n-1}(t) \equiv t$ and let there exist summable functions $p_{ik} : [a, b] \to [0, +\infty[$ $(i = 1, \dots, \ell; k = 0, \dots, n-1)$ such that*

$$\limsup_{t \to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \sum_{i=1}^{\ell} \int_a^t p_{ik}(s)\, ds \right) < 1 \tag{7}$$

and the inequality

$$\begin{aligned} &\Big(f_0(t, h(\tau_{10}(t))x_{10}, \dots, h^{(n-1)}(\tau_{\ell n-1}(t))x_{\ell n-1}) - \\ &- f_0(t, h(\tau_{10}(t))y_{10}, \dots, h^{(n-1)}(\tau_{\ell n-1}(t))y_{\ell n-1}) \Big) \operatorname{sgn}(x_{\ell n-1} - y_{\ell n-1}) \leq \\ &\leq \sum_{k=0}^{n-1} \sum_{i=1}^{\ell} p_{ik}(t) \|x_{ik} - y_{ik}\| \end{aligned}$$

is fulfilled for any $t \in]a, b[$, $x_{ik} \in \mathbb{R}^m$ and $y_{ik} \in \mathbb{R}^m$ $(i = 1, \dots, \ell; k = 0, \dots, n-1)$. Then the problem (3), (2) has at most one solution.

Corollary 2. *Let the conditions of Corollary 1 hold and*

$$\lim_{t \to a} \left(\frac{1}{h^{(n-1)}(t)} \int_a^t \|f_0(s, 0, \dots, 0)\|\, ds \right) = 0.$$

Then the problem (3), (2) has one and only one solution.

The above-formulated Theorems 1 and 2 and their corollaries generalize the results of [1] and make the results of §5 from monograph [2] more complete.

As an example, in the interval $]0, 1/2]$ we consider the boundary value problem

$$u^{(n)}(t) = \sum_{k=0}^{n-1} g_k(t) |u^{(k)}(t)| + g(t), \tag{8}$$

$$\lim_{t \to 0} \frac{u^{(k)}(t)}{t^{n-k}} = 0 \quad (k = 0, \dots, n-1), \tag{9}$$

where

$$\begin{aligned} g(t) = w^{(n)}(t), \quad w(t) = t^n \big/ |\ln t|, \\ g_k(t) = \ell_k w^{(n)}(t) \big/ w^{(k)}(t), \quad \ell_k \geq 0 \quad (k = 0, \dots, n-1). \end{aligned} \tag{10}$$

The problem (8), (9) is a particular case of the problem (1), (2), where $a = 0$, $b = 1/2$,

$$f(u)(t) = \sum_{k=0}^{n-1} g_k(t) |u^{(k)}(t)| + g(t)$$

and $h(t) = t^n$. Obviously, the operator f satisfies conditions (5) and (6), where

$$p_k(t) = \frac{n!}{(n-k)!} t^{n-k} g_k(t).$$

On the other hand,

$$\lim_{t \to 0} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_0^t p_k(s)\, ds \right) = \sum_{k=0}^{n-1} \ell_k.$$

According to Corollary 1, if

$$\sum_{k=0}^{n-1} \ell_k < 1,$$

then the problem (8), (9) has a unique solution.

Let us show that if

$$\sum_{k=0}^{n-1} \ell_k \geq 1, \tag{11}$$

then the problem (8), (9) has no solution. Assume the contrary that this problem has a solution u. Then with regard for (10) from (8) we obtain

$$u^{(k)}(t) \geq w^{(k)}(t) \ \ (k = 0, \ldots, n) \ \text{ for } \ 0 < t \leq 1/2.$$

Put

$$\rho_0 = \inf \left\{ \frac{u^{(n-1)}(t)}{w^{(n-1)}(t)} : \ 0 < t \leq 1/2 \right\}. \tag{12}$$

Then we have

$$u^{(k)}(t) \geq \rho_0 w^{(k)}(t) \ \text{ for } \ 0 < t \leq 1/2 \ \ (k = 0, \ldots, n-1).$$

If along with this we take into account (10) and (11), then from (8) we get

$$u^{(n)}(t) \geq (\rho_0 + 1) w^{(n)}(t) \ \text{ for } \ 0 < t \leq 1/2.$$

Thus

$$u^{(n-1)}(t) \geq (\rho_0 + 1) w^{(n-1)}(t) \ \text{ for } \ 0 < t \leq 1/2$$

which contradicts (12).

The above-constructed example shows that the condition (4) (the condition 7) in Theorems 1 and 2 (in Corollaries 1 and 2) cannot be replaced by the condition

$$\limsup_{t \to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \int_a^t p_k(s)\, ds \right) \leq 1$$

$$\left(\limsup_{t \to a} \left(\frac{1}{h^{(n-1)}(t)} \sum_{k=0}^{n-1} \sum_{i=1}^{\ell} \int_a^t p_{ik}(s)\, ds \right) \leq 1 \right).$$

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