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ON A BOUNDARY VALUE PROBLEM WITH A WEIGHTED CONDITION AT INFINITY FOR EVEN ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Consider the nonlinear ordinary differential equation

$$u^{(n)}+\sum_{k=1}^{n-1}p_k(t)u^{(k)}=f(t,u,u',\dots,u^{(n-1)}) \tag{1}$$

on the infinite interval $[a,+\infty[$, where $n\geq 2$ is an even number, $1<a<+\infty$, each of the functions $p_k:[a,+\infty[\to\mathbb{R}$ for $k\in\{1,\dots,n-1\}$ is locally absolutely continuous together with its derivatives up to the order $k-1$ inclusive (i.e., the functions $p_k^{(i)}$ $(i=0,\dots,k-1)$ are absolutely continuous on any finite segment contained in $[a,+\infty[)$, and the function $f:[a,+\infty[\times\mathbb{R}^n\to\mathbb{R}$ satisfies the local Carathéodory conditions.

Let $\nu<-1$, $n_0=\frac{n}{2}$. Consider the problem on the existence of a solution of the equation (1) satisfying the boundary conditions

$$\begin{gathered}u^{(i)}(a)=\varphi_i(u(a),u'(a),\dots,u^{(n-1)}(a)) \quad (i=0,\dots,n_0-1),\\ \int_a^{+\infty}t^{\nu}|u^{(j)}(t)|^2dt<+\infty \quad (j=0,\dots n_0),\end{gathered} \tag{2}$$

where the functions $\varphi_i:\mathbb{R}^n\to\mathbb{R}$ $(i=0,\dots,n_0-1)$ are continuous and satisfy the condition

$$\sum_{i=0}^{n_0-1}\Big|\varphi_i(x_0,x_1,\dots,x_{n-1})\Big|\leq c_1\Big(1+\sum_{i=0}^{n_0-1}|x_i|\Big)^{-\vartheta} \tag{3}$$

on $\mathbb{R}^n$, where $c>0$ and $\vartheta\in[0,1]$.

In the case where $p_k(t)\equiv 0$, problems of such type were investigated by I. Kiguradze [1]. The author has recently studied the case where $\nu=0$ (see [5]). Our interest to the problem (1), (2) is two-fold. First, to supplement the results of [1] and generalize those of [5] which correspond to the case of an even n. Second, to supplement in certain cases some results appearing in the qualitative theory.

Below the use will be made of the following notation:

$\mathbb{R}$ is the set of real numbers;

$\mathbb{R}^n$ the n-dimensional Euclidean space;

μ_i^k $(i=1,2,\dots;\ k=2i,\ 2i+1,\dots)$ are real constants defined by the recurrence relation

$$\mu_0^{i+1}=1/2,\ \ \mu_i^{2i}=1,\ \ \mu_{i+1}^{k}=\mu_{i+1}^{k-1}+\mu_i^{k-2}\ \ (i=0,1,\dots;\ k=2i+3,\dots).$$

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Everywhere below it will be assumed that the function f satisfies the condition

$$\Big|f(t,x_0,x_1,\dots,x_{n-1})\Big| \le h(t,|x_0|,|x_1|,\dots,|x_{n_0-1}|) \tag{4}$$

on $[a,+\infty[\times\mathbb{R}^n$, where the function $h:[a,+\infty[\times\mathbb{R}_+^{n_0}\to\mathbb{R}_+$ is locally summable in the first argument, nondecreasing in the last n_0 arguments, and for any $\rho_0>0$ satisfies the condition

$$\limsup_{\substack{t\to a\\ \rho\to+\infty}} \frac{1}{\rho^2}\Big|\int_a^t h(\tau,\rho_0,\rho,\dots,\rho)\,d\tau\Big|^{1-\vartheta} < +\infty. \tag{5}$$

Theorem 1. *Let the inequality*

$$(-1)^{n_0-1}f(t,x_0,x_1,\dots,x_{n-1})\operatorname{sgn}x_0 \ge -\sum_{i=0}^{n_0-1}\alpha_i(t)|x_i| - \alpha(t)$$

hold on $[a,+\infty[\times\mathbb{R}^n$, where the function $\alpha_0:[a,+\infty[\to\mathbb{R}$ and $\alpha_i:[a,+\infty[\to\mathbb{R}_+$ $(i=1,\dots,n_0-1)$ are locally summable, and $\alpha:[a,+\infty[\to\mathbb{R}$ is measurable and satisfies the condition

$$\int_a^{+\infty} t^\nu\alpha^2(t)dt < +\infty.$$

Moreover, let there exist the constants $\gamma_i\ge 0$ $(i=1,\dots,n_0-1)$, $\delta>0$ and $\eta>\max\{[\nu(\nu-1)]^{n_0-1},\,2^{(n_0-1)(n_0-2)}\}$ satisfying

$$1-\sum_{i=1}^{n_0-1}\gamma_i\prod_{k=i}^{n_0-1}\eta^{\frac1k}\ge\delta$$

and such that the inequalities

$$t^{-\nu}\sum_{k=2i}^{n}(-1)^{n+k-i-1}\mu_i^k\Big[t^\nu p_k(t)\Big]^{(k-2i)}+\alpha_i(t)\le\gamma_i\ \ (i=1,\dots,n_0-1),$$

$$t^{-\nu}\sum_{k=1}^{n}(-1)^{n_0+k}\mu_0^k\Big[t^\nu p_k(t)\Big]^{(k)}-\sum_{i=0}^{n_0-1}\alpha_i(t)\ge\sum_{i=1}^{n_0-1}\gamma_i\Big(\sum_{k=i}^{n_0-1}2^{k-1}\eta^{1+\frac12+\dots+\frac1k}\Big)+\delta$$

hold on $[a,+\infty[$, where $p_n(t)\equiv 0$. Then the problem (1), (2) *is solvable.*

Corollary 1. *Let the inequality $(-1)^{n_0-1}f(t,x_0,x_1,\dots,x_{n-1})\operatorname{sgn}x_0\ge\gamma(t)|x_0|^\lambda$ be fulfilled on $[a,+\infty[\times\mathbb{R}^n$, where $\lambda>1$ and the function $\gamma:[a,+\infty[\to]0,+\infty[$ is measurable and satisfies the condition*

$$\int_a^{+\infty} t^\nu[\gamma(t)]^{-\frac{2}{\lambda-1}}dt<+\infty.$$

Moreover, let there exist a constant $r>0$ such that the inequalities

$$t^{-\nu}\sum_{k=2i}^{n}(-1)^{n_0+k-i-1}\mu_i^k\Big[t^\nu p_k(t)\Big]^{(k-2i)}<r\ \ (i=1,\dots,n_0-1),$$

$$t^{-\nu}\sum_{k=1}^{n}(-1)^{n_0+k}\mu_0^k\Big[t^\nu p_k(t)\Big]^{(k)}>-r$$

hold on $[a, +\infty[$, *where* $p_n(t) \equiv 1$. *Then the problem* (1), (2) *is solvable.*

From these results and also from the existence of a so-called proper solution of (1) (i.e., a nontrivial solution of (1) defined in some neighbourhood at infinity) we obtain its asymptotic behaviour.

Theorem 2. *Assume that all the hypotheses of Theorem* 1 (*Corollary* 1) *hold. Then for arbitrary continuous functions* $\varphi_i (i = 0, \dots, n_0 - 1)$ *satisfying* (3) *on* $\mathbb{R}^n$, *where* $c > 0$ *and* $\vartheta \in [0, 1]$, *there exists at least one proper solution of the equation* (1) *satisfying the initial conditions*

$$u^{(i)}(a) = \varphi_i(u(a), u'(a), \dots, u^{(n-1)}(a)) \quad (i = 0, \dots, n_0 - 1), \tag{6}$$

and possessing the following asymptotic property:

$$\lim_{t \to +\infty} t^{\frac{\mu}{2}} |u^{(i)}(t)| = 0 \quad (i = 0, \dots, n_0 - 1). \tag{7}$$

Corollary 2. *Assume that all the hypotheses of Corollary* 1, *except that of the restriction* (5), *are satisfied. Then the equation* (1) *has an* n_0*-parametric family of proper solutions possessing the asymptotic property* (7).

These results generalize those obtained in [5] and complement the results of [1] concerning the case of even n.

On the other hand, Theorem 2 provides us with sufficient conditions on the existence of proper oscillatory (i.e., having a sequence of zeros converging at infinity) solutions of the equation

$$u^{(n)} + u^{(n-2)} = f(t, u, u', \dots, u^{(n-1)}) \tag{1_1}$$

which appear from qualitative theory (see Corollary 1.1[2], p. 208). Therefore, the result below fills in a certain way the gap in [4] (see Theorem 2, p. 39).

Corollary 3. *Let* $n = 2n_0 \geq 4$, *and along with* (4) *let the inequality*

$$\gamma t^{\mu} |x_0|^{\lambda} \leq (-1)^{n_0 - 1} f(t, x_0, x_1, \dots, x_{n-1}) \operatorname{sgn} x_0$$

hold on $[a, +\infty[\times \mathbb{R}^n$, *where* $\lambda > 1$, $\mu \geq 2 - n$, $\gamma > 0$, *the function* $h : [a, +\infty[\times \mathbb{R}_+^{n_0} \to \mathbb{R}_+$ *is locally summable in the first argument, nondecreasing in the last* n_0 *arguments and for any* $\rho_0 > 0$ *satisfies* (5), *where* $c \geq 0$ *and* $\vartheta \in [0, 1]$. *Then for arbitrary continuous functions* $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 0, \dots, n_0 - 1)$ *satisfying* (3) *on* $\mathbb{R}^n$ *there exists at least one proper solution of the problem* (1_1), (6) *such that*

$$\lim_{t \to +\infty} t^{\frac{\mu}{\lambda - 1} - \frac{1+\varepsilon}{2}} |u^{(i)}(t)| = 0 \quad (i = 0, \dots, n_0 - 1).$$

Moreover, in the case n_0 *is even, every proper solution is oscillatory.*

The last result is new even for the Emden-Fowler type equation

$$u^{(n)} + u^{(n-2)} = p(t) |u|^{\lambda} \operatorname{sgn} u \tag{1_2}$$

and gives an answer to some open problems of the oscillation theory. For example, as early as in 1992, I. Kiguradze [2] (see Corollary 1. 6) proved that if $n \geq 4$ is even, $\lambda > 1$ and $p : [0, +\infty[\to] - \infty, 0]$ is locally summable, then the condition

$$\int_0^{+\infty} t^{n-3} p(t) dt = -\infty$$

is necessary and sufficient for every proper solution of (1_2) to be oscillatory. However, the question on the existence of at least one proper solution of (1_2) remained open. Clearly, Corollary 3 implies

Corollary 4. *Let $n = 2n_0$, n_0 be even, and let the inequality $p(t) \leq -\gamma t^{2-n}$ on $[a, +\infty[$, where $\gamma > 0$. Then the equation (1_2) has an n_0-parametric family of proper oscillatory solutions.*

Finally, we consider the generalized Emden-Fowler equation

$$u^{(n)} + \sum_{k=0}^{n-1} p_k(t) u^{(k)} = -\delta(t)|u|^{\lambda} \operatorname{sgn} u, \tag{8}$$

where $n \geq 2$, $\lambda > 1$ and $\delta : [a, +\infty[\to [0, +\infty[$ is measurable. From the above reasoning we answer the question on the existence of proper solutions of (8). Moreover, using a result of T. Chanturia (see Theorem 1.9 in [3], p. 50), we obtain the following sufficient conditions for the existence of proper oscillatory solutions of (8).

Corollary 5. *Let $n = 2n_0$ and $n_0 > 1$ be even. Assume that all hypotheses of Corollary 1 are satisfied. Moreover, let the functions $p_k : [a, +\infty[\to \mathbb{R}$ $(k = 0, \dots, n-1)$ be summable and the condition*

$$\liminf_{t \to +\infty} p_0(t) > 0$$

be fulfilled. Then for arbitrary continuous functions $\varphi_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 0, \dots, n_0 - 1)$ satisfying (3) *on $\mathbb{R}^n$, there exists at least one proper oscillatory solution of* (8) *satisfying the initial condition* (6) *and possessing the asymptotic property* (7).

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