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ON NONINCREASING SINGULAR SOLUTIONS OF THE EMDEN–FOWLER EQUATION

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Consider the Emden–Fowler equation

$$u^{(n)} = (-1)^n p(t)|u|^\lambda sign\, u, \quad p(t) \ge 0, \quad \lambda \in (0,1), \quad n \ge 2, \tag{1}$$

with the coefficient p locally Lebesgue integrable on (a,b) and differing from zero on a set of positive measure in any left neighborhood of b.

A solution $u : [a,b) \to (0,+\infty)$ of the equation (1) is said to be a Kneser singular solution of the first kind, if

$$(-1)^i u^{(i)}(t) > 0 \quad (i = 0, \dots, n-1) \quad \text{for} \quad t \in (a,b), \quad \lim_{t\to b} u(t) = 0. \tag{2}$$

The problem (1), (2) is studied in [1, 2]. There, in particular, there are obtained the sufficient condition

$$J(a,b) < +\infty, \quad where \quad J(s,t) \equiv \int_s^t p(\tau)(b-\tau)^{(n-1)\lambda} d\tau \tag{3}$$

for solvability of the problem and an associated upper asymptotic estimate of its solutions

$$|u^{(n-1)}(t)| \le [(1-\lambda)J(t,b)/(n-1)!]^{1/(1-\lambda)}. \tag{4}$$

The conversion of (1), (2) to a similar problem on proper nonoscillatory Kneser solutions vanishing at infinity, which is more studied [1, 3], allows one to get necessary conditions of solvability and lower asymptotic estimates of solutions. In the same manner it is easy to conclude on the basis of [4, 5], that (3) is not a necessary condition.

In the present report we use a direct approach to the problem under consideration, which helps us to simplify proofs and to find out specific properties of solutions. Here new necessary conditions of solvability of the problem (1), (2) and lower asymptotic estimates of its solutions are obtained. Here also the necessity of the condition (3) is established in certain cases.

Begin with a simple Lemma which is basic in the subsequent discussion.

Lemma 1. *Let $u(t)$ be a solution of the problem* (1), (2). *Then the auxiliary functions*

$$v_i(t) = |u^{(i)}(t)|(b-t)^{i-n+1} \quad (i = 0, \dots, n-1)$$

increase on (a,b) and tend to zero as $t \to b$.

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Proof. Let u be a solution of the problem (1), (2). Then for all $i = 0, \dots, n-1$ the equalities $\dot{v}_i(t) = (b-t)^{i-n} R_i(t)$ hold, where $R_i(t) = (n-i-1)|u^{(i)}(t)| - (b-t)|u^{(i+1)}(t)|$ are continuous on (a,b) functions satisfying there $R_i(b) = 0$, $R_i'(t) = -R_{i+1}(t)$ $(i = 0, \dots, n-2)$, $R_{n-1}(t) = -(b-t)|u^{(n)}(t)| \leq 0$. Thus in view of the representations

$$R_i(t) = \int_t^b (\tau - t)^{n-i-2} R_{n-1}(\tau) d\tau/(n-i-2)! \quad (i = 0, \dots, n-2)$$

we see that $R_i(t) < 0$ on (a,b) for all i. □

Corollary 1. *Every solution of the problem* (1), (2) *satisfies the inequalities*

$$|u^{(i+1)}(t)|(b-t) > (n-i-1)|u^{(i)}(t)| \quad (i = 0, \dots, n-2), \quad t \in (a,b). \tag{6}$$

The proof of the next Lemma is a matter of easy calculation.

Lemma 2. *Let u be a solution of the problem* (1), (2) *and let $\varphi(t) > 0$ be a nondecreasing function on (a,b). Then for all $t \in (a,b)$*

$$-\frac{\dot{v}_{\varphi,n-1}(t)}{v_{\varphi,n-1}(t)} = \frac{p(t)(b-t)^{(n-1)\lambda} v_{\varphi,0}^{\lambda}(t)}{\varphi^{1-\lambda}(t) v_{\varphi,n-1}(t)} + \frac{\dot{\varphi}(t)}{\varphi(t)},$$

$$\frac{\dot{v}_{\varphi,i}(t)}{v_{\varphi,i+1}(t)} \leq -\frac{\varphi_M(t)}{n-i-1},$$

where $v_{\varphi,i}(t) = v_i(t)/\varphi(t)$, $\varphi_M(t) = \min\{(b-t)^{-1}, \dot{\varphi}(t)/(M\varphi(t))\}$.

The main result of the report contains the following

Theorem 1. *Let u be a solution of the problem* (1), (2) *and let $\varphi : (a,b) \to (0, +\infty)$ be any nondecreasing function. Then for any numbers $\nu \in [0,1)$, $\mu \in ((1-\nu)/n, (1-\nu)/n_1)$, $M > 0$ and $\sigma > 0$ the equality*

$$\lim_{t \to b} F_{\nu,\mu,\sigma,M}(\varphi)(t) = 0 \tag{7}$$

is fulfilled and the upper estimate

$$|u^{(n-1)}(t)| > \gamma [F_{\nu,\mu,\sigma,M}(\varphi)(t)]^{1/((1-\lambda)\mu)} \tag{8}$$

is true, where

$$F_{\nu,\mu,\sigma,M}(\varphi)(t) = \varphi^{\sigma}(t) \int_t^b \frac{(p(\tau)(b-\tau)^{(n-1)\lambda})^{\mu}}{\varphi^{\sigma}(\tau)\varphi_M^{\mu+\nu-1}(\tau)} \left(\frac{\dot{\varphi}(\tau)}{\varphi(\tau)}\right)^{\nu} d\tau,$$

$n_1 = 1 + (n-1)\lambda$ and $\gamma > 0$ depends only on n, λ, μ, ν.

Proof. In accordance with the given values of μ and ν define the numbers μ_i by the equalities $\mu_n = \mu$, $\mu_{n+1} = \nu$,

$$\mu_i = \begin{cases} \mu - (n-i)\varepsilon_1, & \varepsilon_1 = \dfrac{2(\nu + n\mu - 1)}{n(n-1)}, \quad \mu \in \left(\dfrac{1-\nu}{n}, \dfrac{2(1-\nu)}{n+n_1}\right) \\ \lambda\mu + i\varepsilon_1, & \varepsilon_1 = \dfrac{2(1-\nu-n_1\mu)}{n(n-1)}, \quad \mu \in \left(\dfrac{2(1-\nu)}{n+n_1}, \dfrac{1-\nu}{n_1}\right) \end{cases}$$

for $i = 1, \dots, n$. It is clear that these numbers satisfy the conditions

$$\mu_{i+1} - \mu_i \geq \varepsilon_1 > 0 \quad (i = 1, \dots, n-1), \quad \mu_1 - \lambda\mu_n \geq \varepsilon_1, \quad \sum_{i=1}^{n+1} \mu_i = 1. \tag{9}$$

Assume u is a solution of the problem (1), (2) and let $\varphi(t) > 0$ be a nondecreasing function on (a,b). Then by Lemma 2 and the well known inequality

$$\sum_{i=1}^{n} \beta_i x_i \geq \prod_{i=1}^{n} x_i^{\beta_i}, \quad x_i > 0, \quad \beta_i \geq 0, \quad \sum_{i=1}^{n} \beta_i = 1$$

we obtain for the derivative of the function $\omega_\varphi(t) = \prod\limits_{i=0}^{n-1} v_{\varphi,i}(t)$

$$\frac{|\dot\omega_\varphi(t)|}{\omega_\varphi(t)} = \sum_{i=0}^{n-1} \frac{|\dot v_{\varphi,i}(t)|}{v_{\varphi,i}(t)} > \frac{p(t)(b-t)^{(n-1)\lambda} v_{\varphi,0}^{\lambda}(t)}{\varphi^{1-\lambda}(t) v_{\varphi,n-1}(t)} + \varphi_M(t) \sum_{i=0}^{n-2} \frac{v_{\varphi,i+1}(t)}{v_{\varphi,i}(t)} +$$

$$+ \frac{\dot\varphi(t)}{\varphi(t)} \geq \gamma \Big(\frac{p(t)(b-t)^{(n-1)\lambda} v_{\varphi,0}^{\lambda}(t)}{\varphi^{1-\lambda}(t) v_{\varphi,n-1}(t)} \Big)^{\mu_n} \prod_{i=0}^{n-2} \Big(\frac{v_{\varphi,i+1}(t)}{v_{\varphi,i}(t)} \varphi_M(t) \Big)^{\mu_{i+1}} \times$$

$$\times \Big(\frac{\dot\varphi(t)}{\varphi(t)} \Big)^{\nu} = \gamma \Big(\frac{\dot\varphi(t)}{\varphi(t)} \Big)^{\nu} \frac{((p(t)(b-t)^{(n-1)\lambda})^{\mu} v_{\varphi,0}^{\lambda\mu-\mu_1}(t)}{\varphi^{1-\lambda}(t)(\varphi_M(t))^{\mu+\nu-1}} \prod_{i=1}^{n-1} v_{\varphi,i}^{\mu_i-\mu_{i+1}}(t).$$

Multiplying the above inequality by $\omega_\varphi^{\varepsilon}(t)\varphi^{-\delta}(t)$ with positive $\varepsilon < \min\{\varepsilon_1, \sigma/n\}$, $\delta = \sigma - n\varepsilon$ and integrating the result, we get

$$(\gamma\varepsilon)^{-1} w_\varphi^{\varepsilon}(t)\varphi^{-\delta}(t) \geq \gamma^{-1} \int_t^b \dot\omega_\varphi(s)\omega_\varphi^{-1+\varepsilon}(s)\varphi^{-\delta}(s)ds \geq$$

$$\geq \int_t^b \frac{(p(x)(b-x)^{(n-1)\lambda}\varphi^{\lambda-1})^{\mu}}{\varphi^{\delta}(x)\varphi_M^{\mu+\nu-1}(x)} \Big(\frac{\dot\varphi(x)}{\varphi(x)} \Big)^{\nu} v_{\varphi,0}^{\lambda\mu_n-\mu_1+\varepsilon}(x) \prod_{i=1}^{n-1} v_{\varphi,i}^{\mu_i-\mu_{i+1}+\varepsilon}(x)dx,$$

where γ depends on n, λ, μ, ν. From here by Lemma 1, (6) and (9) we get

$$v_{n-1}^{(1-\lambda)\mu}(t) \geq v_0^{\mu_1-\lambda\mu_n}(t) \prod_{i=1}^{n-1} v_i^{\mu_{i+1}-\mu_i}(t) \geq F_{\nu,\mu,\sigma,M}(\varphi)(t)$$

which completes the proof. □

By means of this theorem it is easy to find the case where the sufficient condition (3) turns out to be a necessary one as well.

Theorem 2. *Let the equation* (1) *with the coefficient* p *satisfying*

$$p(t)(b-t)^{n_1} < c(J(a,t)+1), \quad c > 0, \tag{10}$$

on (a,b) *have a solution of the type* (2). *Then the condition* (3) *must hold and for all* $\mu \in (0, n_1^{-1})$ *the estimate*

$$|u^{(n-1)}(t)| > \gamma J^{1/(\mu(1-\lambda))}(t,b), \quad \textit{where} \quad \gamma = \gamma(n,\lambda,\mu), \tag{11}$$

is fulfilled.

If along with (3)

$$p(t)(b-t)^{n_1} < cJ(t,b) \quad \textit{for} \quad t \in (a,b), \quad c > 0, \tag{12}$$

holds, then there takes place the two-sided estimate

$$0 < \gamma_1 < |u^{n-1}(t)| J^{1/(\lambda-1)}(t,b) < \gamma_2. \tag{13}$$

Here γ_1 and γ_2 depend only on n, λ, μ.

Proof. Assume that u is a solution of the problem (1), (2), and let (10) hold. Then, putting $\varphi(t) = J(a,t)+1$, we have for $\sigma > \mu$

$$F_{\nu,\mu,\sigma,c}(\varphi)(t) = \varphi^{\sigma}(t)\int_t^b \dot\varphi(x)\varphi^{\mu-\sigma-1}(x)dx = \varphi^{\mu}(t)/(\sigma-\mu)$$

which contradicts the conclusion of Theorem 1. Consequently, (3) must be fulfilled and $\varphi(t)$ is bounded. Owing to this fact we obtain $F_{\nu,\mu,\sigma,c}(\varphi)(t) > \gamma J(t,b)$ on (a,b), which by Theorem 1 yields (11).

To prove (13) we can use (4) and (8) with $\varphi(t) = 1/J(t,b)$ because

$$F_{\nu,\mu,\sigma,c}(\varphi)(t) = J^{-\sigma}(t,b)\int_t^b p(x)(b-x)^{(n-1)\lambda}J^{\mu+\sigma-1}(x,b)dx = \gamma J^{\mu}(t,b).$$

The theorem is proved. □

Corollary 2. *The problem (1), (2) with the function p satisfying $0 < c_1 < p(t)(b-t)^{(n-1)\lambda}\ln_k^{\sigma}(1/(b-t))l_k(1/(b-t)) < c_2$, $k \geq 0$ on (a,b) has a solution if and only if $\sigma > 0$ and every its solution admits the estimate $0 < \gamma_1 < |u^{(n-1)}(t)|\ln_k^{\sigma/(1-\lambda)}(1/(b-t)) < \gamma_2$, where $\ln_0 t = t$, $\ln_{j+1} t = \ln(\ln_j t)$, $l_j(t) = \prod_{i=0}^{j}\ln_i t$ and γ_1, γ_2 depend only on n, λ, σ.*

In the general case where (3) is not assumed to be fulfilled we will introduce into consideration the nonnegative functions $p_{f*}(t) = \min\{p(t),\ f(t)(b-t)^{n_1}\}$, $p_f^*(t) = p(t) - p_{f*}(t)$ and the integrals

$$J_{f*}(s,t) = \int_s^t \frac{p_{f*}(x)(b-x)^{(n-1)\lambda}}{f(x)}dx, \quad J_f^*(s,t) = \int_s^t \frac{p_f^*(x)(b-x)^{(n-1)\lambda}}{f(x)}dx,$$

where $f(x)$ is an arbitrary nondecreasing positive function.

Theorem 3. *If the equation (1) has a solution u of the type (2), then for an arbitrary nondecreasing positive function f and all $\mu \in (0, 1/n_1)$ there hold $J_{f*}(a,b) < +\infty$, $\lim_{t\to b} f^{\mu}(t)J_{f,*}(t,b) = 0$ and in some neighborhood of b the estimate*

$$|u^{(n-1)}(t)| > \gamma(f(t)J_{f*}^{1/\mu}(t,b))^{1/(1-\lambda)}$$

is true.

If in addition $p_{f*}(t)(b-t)^{n_1} < cf(t)J_{f*}(t,b)$ for $t \in (a,b)$, then $|u^{(n-1)}(t)| > \gamma(f(t)J_{f*}(t,b))^{1/(1-\lambda)}$. The constant γ depends only n, λ, μ.

Corollary 3. *If the problem (1), (2) is solvable with p satisfying*

$$J_f(t,b) = \int_t^b \frac{p(x)(b-x)^{(n-1)\lambda}}{f(x)}dx = +\infty \quad \text{for} \quad t < b,$$

where $f(x) > 0$ is an arbitrary nondecreasing function, then

$$\overline{\lim_{t\to b}}\, p_f^*(t)(b-t)^{(n-1)\lambda}/f(t) = +\infty, \quad J_f^*(t,b) = +\infty.$$

Theorem 4. *If the equation* (1) *with the function* p *satisfying the condition*

$$p_{f*}(t)(b-t)^{n_1} > cf(t)J_{f*}(t,b) \quad for \quad t \in (a,b),$$

where $f(x)$ *increases on* (a,b), *has a solution* u *of the type* (2), *then the estimate*

$$|u^{(n-1)}(t)| > \gamma\Big(f(t)J_{f*}^{(1-\nu)/\mu}(t,b)\Big)^{1/(1-\lambda)},$$

holds, where γ *depends only on* n, λ, μ.

Theorem 5. *If the problem* (1), (2) *with* p *satisfying* $p_f^*(t)(b-t)^{n_1} > cf(t)J_f^*(a,t)$ *on* (a,b), *where* $f(x)$ *increases on* (a,b), *has a solution* u *of the type* (2), *then*

$$|u^{(n-1)}(t)| > \gamma\Big(f(t)J_{f*}(a,t)\Big(\int\limits_t^b x^{-1} sgn p(x)dx\Big)^{1/\mu}\Big)^{1/(1-\lambda)},$$

where $\gamma = \gamma(n,\lambda,\mu) > 0$.

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