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THE EXISTENCE THEOREM FOR ONE CLASS OF OPTIMAL PROBLEMS IN BANACH SPACE

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The optimal problem with a quasilinear differential equation in Banach spaace is considered, where the linear part of the right-hand side contains unbounded operators. In the class of piecewise uniformly Lipschitz mappings, with values in a compact subset of finite-dimensional space, the existence of the optimal control is proved. As an illustration of the obtained result, the existence of the optimal control for an optimal problem containing an equation of hyperbolic type is established.

1. Statement of the Problem. The Existence Theorem

Let $J = [a, b]$ be a finite interval, X and X_0 be Banach spaces, X_0 be densely and continuously embedded in X. $B(X_0, X)$ denotes the Banach space of linear bounded operators from X_0 to X, $B(X) \equiv B(X, X)$; for each t, $A(t) : D(A(t)) \to X$, $D(A(t)) \subset X$, is a linear unbounded operator and the family $\{A(t)\}_{t\in J}$ satisfies the following assumptions:

(i) $X_0 \subset D(A(t))$, $A(t) \in B(X_0, X)$, $\forall t \in J$, and $t \mapsto A(t)$ is a norm-continuous mapping from $[a, b)$ to $B(X_0, X)$;

(ii) $A(t)$ generates a semigroup of class c_0 on X, $\forall t \in J$ (see[1]), and there are constants M, β such that

$$|\sum_{j=k}^{1}(\lambda I - A(t_j))^{-1}|_{B(X)} \le M(\lambda-\beta)^{-k}, \ \ \lambda > \beta,$$

for any finite family $\{t_j\}$ with $a \le t_1 \le \cdots \le t_k \le b$, $k = 1, 2, \cdots$;

(iii) There is a family $\{S(t)\}$ of isomorfisms of X_0 in X such that $t \mapsto S(t)x$ is cotinuously differentiable for each $x \in X_0$ and

$$S(t)A(t)S(t)^{-1} = A(t) + B(t), \ \ B(t) \in B(X),$$

where $B(t)$ is strongly continuous in X.

Further, let $U \subset \mathbb{R}^r$ be a compact set, $f : J \times X_0 \times U \to X_0$ be a continuous mapping and there exist $k \ge 0$, such that for every $t \in J$ and $u \in U$

$$|f(t, x_1, u) - f(t, x_2, u)| \le k|x_1 - x_2|_{X_0}, \ \ \forall x_1, x_2 \in X_0.$$

Moreover, as in [2], let $\Omega = \Omega(m, l)$ be the set of piecewise continuous functions $u(\cdot) : J \to U$ having the property: for each $u(\cdot) \in \Omega$ there exists a partition $a = \xi_0 < \cdots \xi_m = b$ such that on every (ξ_i, ξ_{i+1}) the restriction of $u(\cdot)$ satisfies the Lipschitz condition:

$$|u(t') - u(t'')| \le L|t' - t''|, \ \ \forall t', t'' \in (\xi_i, \xi_{i+1}), \ \ i = 0, \ldots, m - 1,$$

besides, the constants m and L do not depend on $u(\cdot) \in \Omega$; let $q^i : J^2 \times X_0^2 \to \mathbb{R}^1$, $i = 0, 1, \ldots, l$, be continuous functions; K be a compact set in X_0.

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Consider the problem:

$$\dot{x}(t) = A(t)x(t) + f(t, x(t), u(t)), \quad t \in [t_0, t_1] \subset J, \quad u(\cdot) \in \Omega, \tag{1}$$

$$x(t_0) = x_0 \in K, \tag{2}$$

$$q^i(t_0, t_1, x_0, x(t_1)) = 0, \quad i = 1, \ldots, l, \tag{3}$$

$$q^0(t_0, t_1, x_0, x(t_1)) \to \min. \tag{4}$$

Definition 1. A continuous function $x(t) = x(t, z) \in X_0, \ t \in [t_0, t_1]$, is said to be *a solution corresponding to an element*

$$z = ((t_0, t_1, x_0, u(\cdot)) \in J^2 \times K \times \Omega, \quad t_0 \leq t_1,$$

if it in X satisfies (1) at the points of continuity of $u(\cdot)$ and satisfies the initial condition (2).

Definition 2. The element $z \in J^2 \times K \times \Omega$ is said to be *admissible*, if the corresponding solution $x(t, z)$ satisfies (3).

The set of admissible elements will be denoted by Δ.

Definition 3. The element $\tilde{z} = ((\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{u}(\cdot)) \in \Delta$ is said to be *optimal*, if

$$\tilde{I} = I(\tilde{z}) = \inf_{z \in \Delta} I(z),$$

where

$$I(z) = q^0(t_0, t_1, x_0, x(t_1)), \quad x(t) = x(t, z).$$

Theorem 1. *if $\Delta \neq \emptyset$, then there exists an optinal element.*

Remark 1. If $\{A(t)\}$ is a family of bounded operators, then we can take $X_0 = X$ and the conclusions of Theorem 1 are valid without assumptions (ii), (iii), which easily follow from continuity of mapping $A(t)$.

2. Auxiliary Lemmas

First, the family $\{A(t)\}$ satisfies conditions of Theorem 6.1 of [3], so the following lemma is valid. Note that we denote the exponent of the generator A by e^A as in [4].

Lemma 1. *Let the family $\{A(t)\}_{t \in J}$ of unbounded operators satisfy the conditions* (i), (ii), (iii). *Then there exists a unique family of operators $T(t, s) \in B(X)$ – solutions of the homogeneous equation, defined for $a \leq s \leq t \leq b$ with the following properties.*

h1) *$T(t, s)$ is strongly continuous (X) in s, t,*

$$T(s, s) = I_X \quad \textit{and} \quad |T(t, s)|_{B(X)} \leq Me^{\beta(t-s)}, \quad a \leq s \leq t \leq b;$$

h2) $T(t, r) = T(t, s)T(s, r), \quad r \leq s \leq t;$

h3) *$T(t, s)X_0 \subset X_0$, $T(t, s)$ is strongly continuous (X_0) jointly in s, t and there exist constants $\tilde{M} \geq 1$, $\tilde{\beta} \geq 0$ such that*

$$|T(t, s)|_{B(X_0)} \leq \tilde{M}e^{\tilde{\beta}(t-s)}, \quad a \leq s \leq t \leq b;$$

h4) *for every $x \in X_0$ and $a \leq s \leq t \leq b$:*

$$\frac{\partial}{\partial t}[T(t, s)x] = A(t)T(t, s)x, \quad \frac{\partial}{\partial s}[T(t, s)x] = -T(t, s)A(s)x.$$

Lemma 2. *Let* $[t_0, t_1] = J_0 \subset J$; $x_0 \in K$; $g : J_0 \to X_0$ *is a continuous mapping and there exist* $k_1 \geq 0$ *such that*

$$|g(t, x_1) - g(t, x_2)|_{X_0} \leq |x_1 - x_2|_{X_0}, \quad \forall x_1, x_2 \in X_0, \quad \forall t \in J_0.$$

Then there exists a unique continuous function $x(\cdot) : J_0 \to X_0$, *satisfying in* X *the following Cauchy problem:*

$$\begin{cases} \dot{x}(t) = A(t)x(t) + g(t, x(t)), & t \in J_0, \\ x(t_0) = x_0. \end{cases} \tag{5}$$

The proof of Lemma 2 is a standard one and is based on the fact that the solution satisfies also

$$x(t) = T(t, t_0)x_0 + \int_{t_0}^{t} T(t, s)g(s, x(s))ds, \quad t \in J_0, \tag{6}$$

so we omit it.

Lemma 3. *To each element* $z = (t_0, t_1, x_0, u(\cdot)) \in J^2 \times K \times \Omega$, $t_0 \leq t_1$, *there corresponds a unique function* $x(t) = x(t, z)$, *which continuously maps* $[t_0, b]$ *into* X_0, *at the points of continuity of* $u(\cdot)$ *satisfies the equation* (1) *in* X *and the initial condition* (2), *and satisfies* (6) *on* $[t_0, b]$; *there exist a constsnt* $\gamma \geq 0$ *such that* $|x(t)|_{X_0} \leq \gamma$, $t_0 \leq t \leq b$, *and* γ *does not depend on* z.

Proof. For an arbitrary $z = (t_0, t_1, x_0, u(\cdot)) \in J^2 \times K \times \Omega$, $t_0 \leq t_1$, by definition of $u(\cdot)$ there exists a partition $t_0 = \xi_0 \cdots < \xi_j = b$, $j \leq m$, such that on every (ξ_i, ξ_{i+1}), $i = 0, \ldots, j-1$, $u(\cdot)$ satisfies Lipschitz condition with a constant L. Obviously, for every $t \in [t_0, b]$ there exist $u(t+)$, $u(t-)$.

Let $J_0 = [t_0, b]$, $g(t, x) = f(t, x, u(t))$, $u(\xi_1) = u(\xi_1-)$. Making use of Lemma 2 we construct a unique continuous mapping $x(\cdot) : [t_0, \xi_1] \to X_0$, satisfying (1) in X, (2) and (6) on $[t_0, \xi_1]$.

Further, let $J_0 = [\xi_1, \xi_2]$, g be as above, $u(\xi_1) = u(\xi_1+)$, $u(\xi_2) = u(\xi_2-)$, a new initial moment be $t = \xi_1$ and a new initial value be $\xi(t_1)$. Use Lemma 2 and the obtained solution denote by $x(\cdot)$, again. Now, $x(\cdot) : [t_0, \xi_2] \to X_0$ is continuous, differentiable in X on $[t_0, \xi_2]$ except for $t = \xi_1$, on $[t_0, \xi_1]$ we have (6) and

$$x(t) = T(t, \xi_1)x(\xi_1) + \int_{\xi_1}^{t} T(t, s)f(s, x(s), u(s))ds, \quad t \in [\xi_1, \xi_2]. \tag{7}$$

Substituting in (7) the value of $x(\xi_1)$ calculated by (6), we see that (6) takes place on $[t_0, \xi_2]$.

Continuing this process, after j steps we obtain the unknown $x(\cdot)$, which has the demanded properties.

Finally, using the Gronwall's lemma and (6) we can estimate $|x(t) - T(t, t_0)x_0|_{X_0}$, so there exists γ, such that $|x(t, z)|_{X_0} \leq \gamma$, $t_0 \leq t \leq b$, and γ does not depend on z. $\square$

3. The Proof of Theorem 1

There exists a sequence

$$z_j = (t_0^j, t_1^j, x_0^j, u_j(\cdot)) \in \Delta, \;\; j = 1, 2, \dots$$

such that

$$I(z_j) \to \tilde{I}, \;\; t_0^j \to \tilde{t}_0, \;\; t_1^j \to \tilde{t}_1, \;\; x_0^j \to \tilde{x}_0 \quad \text{when} \quad i \to \infty, \tag{8}$$

and there exists $\tilde{u}(\cdot) \in \Omega$, such that $\lim_{j\to\infty} u_j(t) = \tilde{u}(t)$ everywhere on J, except a finite set of points (see [2], Lemma 2).

By virtue of Lemma 3, there exists a continuous function $\tilde{x}(\cdot) : [\tilde{t}_0, b] \to X_0$, which corresponds to the element $\tilde{z} = (\tilde{t}_0, \tilde{t}_1, \tilde{x}_0, \tilde{u}(\cdot))$. Consider the nontrivial case $\tilde{t}_0 < \tilde{t}_1$. There exists $\gamma_1 \geq 0$, such that

$$|f(t, x, u)|_{X_0} \leq \gamma_1, \quad \forall t \in J, \;\; \forall u \in U, \;\; |x|_{X_0} \leq \gamma,$$

where γ is from Lemma 3.

With regard for (8), without loss of generality we can suppose $t_1^j > \tilde{t}_0 + \eta_0, \;\; \forall i \in \mathbb{N}$, for some $\eta_0 > 0$.

Taking into consideration the continuity of the mappings q^i, (8), Lemma 3 and

$$\tilde{x}(\tilde{t}_1) - x_j(t_1^j) = \big(\tilde{x}(\tilde{t}_1) - \tilde{x}(t_1^j)\big) + \big(\tilde{x}(t_1^j) - x_j(t_1^j)\big),$$

in order to prove Theorem 1 is sufficient to show

$$\lim_{j\to\infty} (x_j(t_1^j) - \tilde{x}(t_1^j)) = 0 \quad \text{in} \quad X_0. \tag{9}$$

Denote $c_1 = \tilde{M} e^{\tilde{\beta}(b-a)}$, $\;\tilde{f}(s) = f(s, \tilde{x}(s), \tilde{u}(s))$, $\;f_j(s) = f(s, x_j(s), u_j(s))$.

Let us take arbitrarily $\varepsilon > 0$, and choose $\eta \in (0, \eta_o]$ such that $\eta 3 c_1 \gamma_1 e^{c_1 k} < \varepsilon/3$. By virtue of (8), there exists $j_0 \in \mathbb{N}$ such that $|\tilde{t}_0 - t_0^j| < \eta$ when $j \geq j_0$. Thus, taking into consideration (8) and the Gronwall Lemma, we get:

$$|\tilde{x}(t) - x_j(t)|_{X_0} \leq |T(t, \tilde{t}_0)\tilde{x}_0 - T(t, t_0^j)x_0^j|_{X_0} e^{c_1 k(b-a)} + \frac{\varepsilon}{3} +$$
$$+ c_1 e^{c_1 k(b-a)} \int\limits_{\tilde{t}_0+\eta}^{b} |\tilde{f}(s) - f(s, \tilde{x}(s), u_j(s))|_{X_0} ds,$$

$\forall t \in [\tilde{t}_0 + \eta_0, b], \;\; j > j_0$.

By virtue of Lemma 1 and the Lebesgue theorem on passage to limit in integrals, there exists $j_1 \in \mathbb{N}$ such that $j_1 > j_0$ and from $\big(j > j_1, \;\; t \in [\tilde{t}_0 + \eta_0, b]\big)$ it follows $|\tilde{x}(t) - x_j(t)|_{X_0} < \varepsilon$. Due to the arbitrariness of ε, $x_j(\cdot)$ uniformly converges on $[\tilde{t}_0 + \eta_0, b]$ to $\tilde{x}(\cdot)$ in X_0. Thus (11) is valid.

4. Application to Optimal Problem with Hyperbolic Symmetric Object to be Controlled

Consider the optimal control problem

$$\frac{\partial y}{\partial t} + \sum_{j=1}^{n} a_j(\xi,t)\frac{\partial y}{\partial \xi_j} + b(\xi,t,u(t))y = c(\xi,t,u(t)), \tag{10}$$

$$\xi \in \mathbb{R}^n, \quad t \in [t_0,t_1] \subset [0,\eta], \quad u(\cdot) \in \Omega,$$

$$y(\xi,t_0) = y_0(\xi), \quad y_0(\cdot) \in K \subset H^1(\mathbb{R}^n), \tag{11}$$

$$\int\limits_{\mathbb{R}^n} |y_0(\xi) - \varphi_0(\xi)|^2 d\xi + \int\limits_{\mathbb{R}^n} |y(\xi,t_1) - \varphi_1(\xi)|^2 d\xi \to \min. \tag{12}$$

Here $y = (y_1,\dots,y_N)$ is an N-vector of unknown functions of $\xi,\ t$, $a_j(\xi,t)$ are Hermitian symmetric $N \times N$ matrix functions, $b(\xi,t,u)$ is an $N \times N$ matrix function, $c(\xi,t,u) \in \mathbb{C}^N$, Ω is determined above, $\varphi_0(\cdot),\ \varphi_1(\cdot) \in L^2(\mathbb{R}^n)$, K is a compact subset in H^1 (the Sobolev space).

We assume

(I) The maps $t \mapsto a_j(\cdot,t)$ are continuous on $[0,\eta]$ to $C^1(\mathbb{R}^n)$, $j = 1,\dots,n$;
(II) $(t,u) \mapsto b(\cdot,t,u)$ is continuous on $[0,\eta] \times U$ to $C^1(\mathbb{R}^n)$;
(III) $(t,u) \mapsto c(\cdot,t,u)$ is continuous on $[0,\eta] \times U$ to $H^1(\mathbb{R}^n)$.

Here $C^1(\mathbb{R}^n)$ denotes the set of all $N \times N$ matrix-valued functions g such that g and $\frac{\partial g}{\partial \xi_j}$ are continuous and bounded on $\mathbb{R}^n$. This is a Banach space with the corresponding supremum norm.

Denote: $X = L^2(\mathbb{R}^n)$, $X_0 = H^1(\mathbb{R}^n)$, $J = [0,\eta]$, $y_0(\cdot) = x_0 \in K$; for every $t \in J$, $A(t) : D(A(t)) \to X$ is a linear unbounded operator in X formally given by (see [3]):

$$A(t)x = -\sum_{j=1}^{n} a_j(\xi,t)\frac{\partial x}{\partial \xi_j};$$

for every $x_1,\ x_2 \in X_0$ we have

$$q^0(t_0,t_1,x_1,x_2) =$$
$$= \int\limits_{\mathbb{R}^n} |x_1(\xi) - \varphi_0(\xi)|^2 d\xi + \int\limits_{\mathbb{R}^n} |x_2(\xi) - \varphi_1(\xi)|^2 d\xi,$$

and for $(t,\varphi,u) \in J \times X_0 \times U$ we define $f(t,\varphi,u) \in X_0$ as follows:

$$f(t,\varphi,u)(\xi) = c(\xi,t,u) - b(\xi,t,u)\varphi(\xi).$$

Now we can rewrite (10)–(12) in the Banach space X in the following form:

$$\dot{x}(t) = A(t)x(t) + f(t,x(t),u(t)), \quad t \in [t_0,t_1] \subset J, \quad u(\cdot) \in \Omega, \tag{13}$$

$$x(t_0) = x_0 \in K, \tag{14}$$

$$q^0(t_0,t_1,x_0,x(t_1)) \to \min. \tag{15}$$

Lemma 1 is applicable to $\{A(\cdot)\}$ (see [3]), and a simple verification shows that Theorem 1 is applicable to (13)–(15). Thus, there exists an optimal element $(\tilde{t}_0,\tilde{t}_1,\tilde{x}_0,\tilde{u}(\cdot))$ whose correspondig solution $\tilde{x}(\xi,t)$, $t \in [\tilde{t}_0,\tilde{t}_1]$, has the following properties: $t \mapsto \tilde{x}(\cdot,t)$ is continuous in $H^1(\mathbb{R}^n)$, in $L^2(\mathbb{R}^n)$ it satisfies the equations (15) and (12) at the points of continuity of $u(\cdot)$, and $\tilde{x}(\cdot,\tilde{t}_0) = \tilde{x}_0$ holds.

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