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ASYMPTOTIC REPRESENTATIONS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF n-th ORDER

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Consider the differential equation

$$y^{(n)} = f\left(t, y, y', \dots, y^{(n-1)}\right), \tag{1}$$

where $f : [\alpha, \omega[\times D \longrightarrow R$ is continuous function, $-\infty < \alpha < \omega \leq +\infty$, $D = \{(y_1, \dots, y_n) \in R^n : 0 < |y_i| < +\infty, \ i = 1, \dots, n\}$.

For this equation in the second and third chapters of the monography of I.T.Kiguradze and T.A.Chanturija [1] at some estimations on function f are obtained: at $\omega = +\infty$ conditions of existence of solutions with a degree asymptotics $y(t) \sim t^{i-1}$ $(i = 1, \dots, n)$, and also estimations for Kneser's and fast-growing solutions; at $\omega < +\infty$ - estimations for singular solutions of the first and second kind.

In the present paper theorems of exact asymptotic formulas are reduced for those solutions y the equations (1), each of which is defined on some interval $[t_0, \omega[\subset [\alpha, \omega[$ and satisfies to conditions

$$1) \quad y^{(n-1)}(t) \neq 0 \quad \text{for} \quad t \in [t_0, \omega[;$$

$$2) \quad \lim_{t \uparrow \omega} y^{(k-1)}(t) = \begin{cases} \text{or} & 0, \\ \text{or} & \pm\infty \end{cases} \qquad (k = 1, \dots, n).$$

At an establishment of these theorems the ideas included in works [2-5] are used, devoted to the equations with nonlinearities of Emden - Fowler type.

Let's assume

$$\pi_\omega(t) = \begin{cases} t, & \text{if} \quad \omega = +\infty \\ t - \omega, & \text{if} \quad \omega < +\infty \end{cases}, \quad \Lambda_{n-1} = \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1, \pm\infty\right\},$$

also we will enter set $\Omega_{o\delta} = [\alpha_o, \omega[\times D_\delta$, where

$$\alpha_o \in [\alpha, \omega[, \quad D_\delta = \{(z_1, \dots, z_n) \in R^n : \ | z_i | \leq \delta < 1, \ i = 1, \dots, n\}.$$

All basic outcomes for the equation (1) are obtained in terms of existence some continuously or twice continuously differentiable function $\psi : [\alpha, \omega[\longrightarrow R \setminus \{0\}$, possessing those or other properties.

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For their formulation we will need the following notations:

$$\varphi_{k1}(t) = \frac{\psi(t)\left[(\lambda_{n-1}^{o}-1)\pi_\omega(t)\right]^{n-k}}{\prod\limits_{i=k}^{n-1} a_{0i}}, \qquad (k=1,\dots,n),$$

where $\quad a_{0k} = (n-k)\lambda_{n-1}^0 - (n-k-1), \qquad \lambda_{n-1}^0 \notin \Lambda_{n-1};$

$$\varphi_{k2}(t) = \frac{\psi(t)[\pi_\omega(t)]^{n-k}}{(n-k)!}, \qquad (k=1,\dots,n);$$

$$\varphi_{k3}(t) = \psi(t)\left(\frac{\psi(t)}{\psi'(t)}\right)^{n-k}, \qquad (k=1,\dots,n);$$

$$\varphi_{k3+i}(t) = \frac{\psi(t)[\pi_\omega(t)]^{i-k}}{(i-k)!}, \qquad (k=1,\dots,i),$$

$$\varphi_{k3+i}(t) = \frac{(-1)^{k-i-1}(k-i-1)!\,\psi'(t)}{[\pi_\omega(t)]^{k-i-1}}, \qquad (k=i+1,\dots,n),$$

$i=1,\dots,n-1,$

and also the following conditions (A_j) $(j=1,\dots,n+2)$:

(A_j) $(j\in\{1,2,3\})$. On some set $\Omega_{o\delta}$ the relation takes place

$$\frac{f\left(t,\varphi_{1j}(t)[1+z_1],\dots,\varphi_{nj}(t)[1+z_n]\right)}{\psi'(t)} = b_{0j}(t) + \sum_{k=1}^{n} b_{kj}(t)z_k + Z_j(t,z_1,\dots,z_n), \quad (2_j)$$

where functions $b_{kj}:[\alpha_o,\omega[\longrightarrow R$ $(k=0,1,\dots,n)$ - are continuous and have properties

$$\lim_{t\uparrow\omega} b_{0j}(t)=1, \qquad \lim_{t\uparrow\omega} b_{kj}(t) = b_{kj}^0 = \text{const} \quad (k=1,\dots,n), \qquad (3_j)$$

and function $Z_j:\Omega_{o\delta}\longrightarrow R$ is continuous and such, that

$$\frac{Z_j(t,z_1,\dots,z_n)}{\sum\limits_{k=1}^{n}|z_k|} \longrightarrow 0 \quad \text{for} \quad \sum_{k=1}^{n}|z_k| \longrightarrow 0 \quad \text{uniformly on} \quad t\in[\alpha_o,\omega[. \qquad (4_j)$$

(A_{3+i}) $(i\in\{1,\dots,n-1\})$. On some set $\Omega_{o\delta}$ the relation takes place

$$\frac{(-1)^{n-i}[\pi_\omega(t)]^{n-i} f\left(t,\varphi_{13+i}(t)[1+z_1],\dots,\varphi_{n3+i}(t)[1+z_n]\right)}{(n-i)!\,\psi'(t)} =$$
$$= b_{03+i}(t) + \sum_{k=1}^{n} b_{k3+i}(t)z_k + Z_{3+i}(t,z_1,\dots,z_n),$$

where functions $b_{k3+i}:[\alpha_o,\omega[\to R$ $(k=0,1,\dots,n)$ and $Z_{3+i}:\Omega_{o\delta}\to R$– - are continuous and such, that conditions (3_{3+i}) and (4_{3+i}) are observed.

Theorem 1. *Let there is continuously differentiable function* $\psi:[\alpha,\omega[\longrightarrow R\setminus\{0\}$ *such, that*

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)\psi'(t)}{\psi(t)} = \frac{1}{\lambda_{n-1}^0-1}, \quad \lambda_{n-1}^0\notin\Lambda_{n-1}$$

and the condition (A_1) *is observed. Then, if the algebraic equation*

$$\sum_{k=1}^{n} b_{k1}^{0} \prod_{i=k}^{n-1} a_{0i} \prod_{j=1}^{k-1} (a_{0j} + \rho) = (1+\rho) \prod_{j=1}^{n-1} (a_{0j} + \rho) \tag{5}$$

does not have roots with zero real part, the differential equation (1) *has at least one solution satisfyng asymptotic representations*

$$y^{(k-1)}(t) = \varphi_{k1}(t)[1+o(1)], \qquad (k=1,\dots,n) \qquad at \quad t \uparrow \omega.$$

Remark 1. The equation (5) obviously has no roots with a zero real part, if

$$\sum_{k=1}^{n} b_{k1}^{0} \neq 1 \quad \text{and} \quad \sum_{k=1}^{n-1} |b_{k1}^{0}| \leq |b_{n1}^{0} - 1|.$$

Theorem 2. *Let there is continuously differentiable function* $\psi : [\alpha, \omega[\longrightarrow R \setminus \{0\}$ *such, that*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega(t)\psi'(t)}{\psi(t)} = 0, \qquad \lim_{t \uparrow \omega} \psi(t) = \left\{ \begin{array}{ll} or & 0, \\ or & \pm\infty \end{array} \right.$$

and the condition (A_2) *is observed. Then, if* $\sum\limits_{k=1}^{n} b_{k2}^{0} \neq 0$*, at the differential equation* (1) *there is at least one solution satisfyng asymptotic representations*

$$y^{(k-1)}(t) = \varphi_{k2}(t)[1+o(1)], \qquad (k=1,\dots,n) \qquad at \quad t \uparrow \omega.$$

Theorem 3. *Let there is twice continuously differentiable function* $\psi : [\alpha, \omega[\longrightarrow R \setminus \{0\}$ *such, that*

$$\lim_{t \uparrow \omega} \frac{\psi''(t)\psi(t)}{[\psi'(t)]^2} = 1$$

and the condition (A_3) *is observed. Then, if the algebraic equation*

$$\sum_{k=1}^{n} b_{0k} (1+\rho)^{k-1} = (1+\rho)^n \tag{6}$$

as no roots with a zero real part, the differential equation (1) *has at least one solution satisfing asymptotic representations*

$$y^{(k-1)}(t) = \varphi_{k3}(t)[1+o(1)], \qquad (k=1,\dots,n) \qquad at \quad t \uparrow \omega.$$

Remark 2. The equation (6) obviously has no roots with a zero real part, if

$$\sum_{k=1}^{n} b_{k2}^{0} \neq 1 \quad \text{and} \quad \sum_{k=1}^{n-1} |b_{k2}^{0}| \leq |b_{n2}^{0} - 1|.$$

Theorem 4. *Let there is twice continuously differentiable function* $\psi : [\alpha, \omega[\longrightarrow R \setminus \{0\}$ *such, that*

$$\lim_{t \uparrow \omega} \frac{\pi_\omega \psi''(t)}{\psi'(t)} = -1, \qquad \lim_{t \uparrow \omega} \psi(t) = \left\{ \begin{array}{ll} or & 0, \\ or & \pm\infty \end{array} \right.$$

and the condition (A_{3+i}) *is observed at some* $i \in \{1, \dots, n-1\}$*. Then, if* $\sum\limits_{k=i+1}^{n} b^0_{k3+i} \neq 1$ *and the algebraic equation*

$$\sum_{k=i+1}^{n} \frac{b^0_{k3+i}}{(k-i-1)!} \prod_{j=i+1}^{k-1} (j-i+\rho) = \frac{(n-i+\rho)}{(n-i)!} \prod_{j=i+1}^{n-1} (j-i+\rho) \tag{7}$$

has no roots with a zero real part, the differential equation (1) *has at least one solution satisfyng asymptotic representations*

$$y^{(k-1)}(t) = \varphi_{k3+i}(t)[1+o(1)], \qquad (k=1,\dots,n) \qquad at \quad t \uparrow \omega.$$

Remark 3. The equation (7) obviously has no roots with a zero real part, if

$$\sum_{k=i+1}^{n} b^0_{k3+i} \neq 1 \quad \text{and} \quad \sum_{k=i+1}^{n-1} |b^0_{k3+i}| \leq |b^0_{n3+i} - 1|.$$

Remark 4. To find out to what extend theorems 1-4 sapplement each other, it is necessary to pay attention to a principal term φ_{nj} $(j \in \{1, \dots, n+2\})$ established asymptotic of $n-1$ a derivative of a solution y of the differential equation (1).

It is easy to notice, taking into account conditions of the appropriate theorems, that

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)\varphi'_{n1}(t)}{\varphi_{n1}(t)} = \frac{1}{\lambda^0_{n-1}-1}, \qquad \lambda^0_{n-1} \notin \left\{0, \frac{1}{2}, \frac{2}{3}, \dots, \frac{n-2}{n-1}, 1, \pm\infty\right\};$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)\varphi'_{n2}(t)}{\varphi_{n2}(t)} = 0, \qquad (\lambda^0_{n-1} = \pm\infty);$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)\varphi'_{n3}(t)}{\varphi_{n3}(t)} = \pm\infty, \qquad (\lambda^0_{n-1} = 1);$$

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)\varphi'_{n3+i}(t)}{\varphi_{n3+i}(t)} = i-n \quad \left(\lambda^0_{n-1} = \frac{n-i-1}{n-i}\right), \quad i = 1, \dots, n-1.$$

Moreover, it is possible to show, that each of these limits is equal $\lim\limits_{t\uparrow\omega} \dfrac{\pi_\omega(t)y^{(n)}(t)}{y^{(n-1)}(t)}$.

Therefore, in case of existence (final or equal $\pm\infty$) a $\lim\limits_{t\uparrow\omega} \dfrac{\pi_\omega(t)y^{(n)}(t)}{y^{(n-1)}(t)}$ all possible situations are enveloped.

Let's show now on the example of the differential equation

$$y^{(n)} = p(t)|y|^{\sigma_0}|y'|^{\sigma_1} \cdots |y^{(n-1)}|^{\sigma_{n-1}} \operatorname{sign} y, \tag{8}$$

where σ_j $(j = 0, 1, \dots, n-1)$– real constants and $p : [\alpha, \omega[\longrightarrow R \setminus \{0\}$– continuous function, how effectively theorems 1-4 work.

In case of the theorem 1, the left part of representation (2_1) from a condition (A_1) becomes

$$\frac{f\left(t, \varphi_{11}(t)[1+z_1], \dots, \varphi_{n1}(t)[1+z_n]\right)}{\psi'(t)} =$$

$$= \frac{\alpha_0 p(t)|\psi(t)|^{1-\gamma_0}|(\lambda^0_{n-1}-1)\pi_\omega(t)|^{\mu_n}}{\psi'(t)} \prod_{j=1}^{n} |1+z_j|^{\sigma_{j-1}},$$

where

$$\alpha_0 = \operatorname{sign}\,[\psi(t)[(\lambda_{n-1}^0 - 1)\pi_\omega(t)]^{n-1},$$

$$\gamma_0 = 1 - \sum_{j=0}^{n-1} \sigma_j, \qquad \mu_n = \sum_{j=0}^{n-2} \sigma_j (n-j-1).$$

From here it is clear, that the condition (A_1) will be hold, if

$$\lim_{t\uparrow\omega} \frac{\alpha_0 p(t)|\psi(t)|^{1-\gamma_0}|(\lambda_{n-1}^0 - 1)\pi_\omega(t)|^{\mu_n}}{\psi'(t)} = 1.$$

In this connection, let's search function ψ, aspiring at $t \uparrow \omega$ either to zero, or to $\pm\infty$, from the differential equation of the first order

$$\psi' = \alpha_0 p(t)|\psi|^{1-\gamma_0}|(\lambda_{n-1}^0 - 1)\pi_\omega(t)|^{\mu_n}.$$

From here we discover, that

$$|\psi(t)|^{\gamma_0} = \gamma_0|\lambda_{n-1}^0 - 1|^{\mu_n} J_n(t)\operatorname{sign}[(\lambda_{n-1}^0 - 1)\pi_\omega(t)]^{n-1},$$

where

$$J_n(t) = \int_{A_n}^{t} p(\tau)|\pi_\omega(\tau)|^{\mu_n}\, d\tau, \qquad A \in \{\omega; \alpha\}.$$

Hence, the inequality

$$\gamma_0[(\lambda_{n-1}^0 - 1)\pi_\omega(t)]^{n-1} J_n(t) > 0 \qquad \text{at} \quad t \in [\alpha, \omega[\tag{9}$$

should be fulfilled and thus we will have

$$\psi(t) = \pm \left|\gamma_0|\lambda_{n-1}^0 - 1|^{\mu_n} J_n(t)\right|^{\frac{1}{\gamma_0}}.$$

Due to the first of conditions of the theorem 1, this function should have property also

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t)\psi'(t)}{\psi(t)} = \frac{1}{\lambda_{n-1}^0 - 1}, \qquad (\lambda_{n-1}^0 \notin \Lambda_{n-1}),$$

i.e., the condition

$$\lim_{t\uparrow\omega} \frac{|\pi_\omega(t)|^{\mu_n+1} J_n'(t)}{J_n(t)} = \frac{1}{\lambda_{n-1}^0 - 1}, \qquad (\lambda_{n-1}^0 \notin \Lambda_{n-1}). \tag{10}$$

should be satisfied. Thus, from the theorem 1 we have

Corollary 1. *If $\gamma_0 \neq 0$, conditions* (9), (10) *are observed and the algebraic equation*

$$\sum_{k=1}^{n} \sigma_{k-1} \prod_{i=k}^{n-1} a_{0i} \prod_{j=1}^{k-1} (a_{0j} + \rho) = (1+\rho) \prod_{j=1}^{n-1} (a_{0j} + \rho)$$

has no roots with a zero real part, the differential equation (1) *has the solutions, satisfyng asymptotic representations*

$$y^{(k-1)}(t) = \pm \left|\gamma_0|\lambda_{n-1}^0 - 1|^{\mu_n} J_n(t)\right|^{\frac{1}{\gamma_o}} [(\lambda_{n-1} - 1)\pi_\omega(t)]^{n-k}[1 + o(1)],$$

$$(k = 1, \dots, n) \qquad \textit{at} \quad t \uparrow \omega.$$

Let's remark, that the conditions indicated in a corollary (9) and (10) are necessary for existence of the equation (8) solutions satisfying a condition

$$\lim_{t\uparrow\omega} \frac{\pi_\omega(t) y^{(n)}(t)}{y^{(n-1)}(t)} = \frac{1}{\lambda_{n-1}^0 - 1}, \qquad \lambda_{n-1}^0 \notin \Lambda_{n-1}.$$

Acknowledgment

The appropriate corollaries may be similarly obtained from theorems 2-4.

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