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M. Benchohra, J. Henderson, and S. K. Ntouyas

EXISTENCE RESULTS FOR IMPULSIVE SEMILINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

Abstract. In this paper, the Schaefer fixed point theorem is used to investigate the existence of mild solutions for first and second order impulsive semilinear neutral functional differential equations in Banach spaces.

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1. Introduction

This paper is concerned with the existence of solutions, for initial value problems, for first and second order semilinear neutral functional differential equations with impulsive effects, in Banach spaces. More precisely, in Section 3, we consider first order impulsive semilinear neutral functional differential equations of the form

$$\begin{aligned}
&\frac{d}{dt}[y(t) - g(t, y_t)] = Ay(t) + f(t, y_t),\\
&t \in J = [0, b], \quad t \neq t_k, \quad k = 1, \dots, m,
\end{aligned} \tag{1.1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.2}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $T(t)$ in E, f, $g : J \times C([-r, 0], E) \longrightarrow E$ are given functions, $\phi \in C([-r, 0], E)$, $(0 < r < \infty)$, $0 = t_0 < t_1 < \dots < t_m < t_{m+1} = b$, $I_k \in C(E, E)$ $(k = 1, 2, \dots, m)$, are bounded functions, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^-)$ and $y(t_k^+)$ represent the left and right limits of $y(t)$ at $t = t_k$, respectively and E a real Banach space with norm $|\cdot|$.

For any continuous function y defined on $[-r, b] - \{t_1, \dots, t_m\}$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t.

In one sense, these first order results extend to the semilinear case some recent work by the authors [2], while in another sense, this work is also an extension to neutral functional differential equations of the results in [3] and [4].

In Section 4 we study second order impulsive semilinear neutral functional differential equations of the form

$$\begin{aligned}
&\frac{d}{dt}[y'(t) - g(t, y_t)] = Ay(t) + f(t, y_t),\\
&t \in J = [0, b], \quad t \neq t_k, \quad k = 1, \dots, m,
\end{aligned} \tag{1.4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.5}$$

$$\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{1.6}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \; y'(0) = \eta, \tag{1.7}$$

where A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$, of bounded linear operators in E, f, g, I_k, and ϕ are as in problem (1.1)–(1.3), $\overline{I}_k \in C(E, E)$ and $\eta \in E$.

For the second order results of this paper, they constitute extensions of [2] to the semilinear setting.

The study of impulsive differential equations arises as a useful mathematical machinery in the modeling of many processes and phenomena studied in physics, chemical technology, population dynamics, medicine, mechanics, biotechnology and economics. That is why, in recent years they are an object of investigations. We refer to the monographs of Bainov and Simeonov [1], Lakshmikantham et al [10], and Samoilenko and Perestyuk [12] and the references cited therein.

The results of this paper also generalize to the impulsive case other results on neutral semilinear functional differential equations in Banach spaces in the literature; see, for instance, the monographs of Erbe et al [5], Hale and Verduyn Lunel [8], Henderson [9], and the survey paper of Ntouyas [11]. Our approach here is based on a fixed point theorem due to Schaefer [13] (see also Smart [14]).

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

$C([-r,0],E)$ is the Banach space of all continuous functions from $[-r,0]$ into E with the norm

$$\|\phi\| = \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

By $C(J,E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_J := \sup\{|y(t)| : t \in J\}.$$

$B(E)$ is the Banach space of all linear bounded operator from E into E. A measurable function $y : J \longrightarrow E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral, see for instance, Yosida [17]).

$L^1(J,E)$ denotes the Banach space of functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt \quad \text{for all} \quad y \in L^1(J,E).$$

Our main results are based on the following:

Lemma 2.1 ([13], see also [14], p. 29). *Let X be a Banach space, $N : S \longrightarrow S$ be a completely continuous operator, and let*

$$\Phi(N) = \{y \in S : y = \lambda N(y) \quad \textit{for some} \quad 0 < \lambda < 1\}.$$

Then either $\Phi(N)$ is unbounded or N has a fixed point.

3. First Order Impulsive Neutral Functional Differential Equations

In order to define the concept of mild solution of (1.1)–(1.3) we shall consider the following space

$$\Omega = \{y : [-r,b] \longrightarrow E : y_k \in C(J_k, E), k = 0, \dots, m \text{ and there exist } y(t_k^-) \text{ and } y(t_k^+), \text{ with } y(t_k^-) = y(t_k),\ k = 1, \dots, m,\ y(t) = \phi(t),\ \forall t \in [-r, 0]\}$$

which is a Banach space with the norm

$$\|y\|_\Omega = \max\{\|y_k\|_{J_k}, k = 0, \dots, m\},$$

where y_k is the restriction of y to $J_k = [t_k, t_{k+1}]$, $k = 0, \dots, m$. So let us start by defining what we mean by a mild solution of problem (1.1)–(1.3).

Definition 3.1. A function $y \in C([-r,b], E)$ is said to be a mild solution of (1.1)–(1.3) if $y(t) = \phi(t)$ on $[-r, 0]$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, for each $0 \le t < b$ the function $AT(t-s)g(s, y_s)$, $s \in [0, t)$ is integrable and

$$y(t) = T(t)[\phi(0) - g(0,\phi)] + g(t, y_t) + \int_0^t AT(t-s)g(s, y_s)\, ds + \\ + \int_0^t T(t-s)f(s, y_s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J.$$

We are now in a position to state and prove our existence result for the problem (1.1)–(1.3). For the study of this problem we first list the following hypotheses:

(H1) A is the infinitesimal generator of a compact semigroup of bounded linear operators $T(t)$ in E such that

$$|T(t)| \le M_1, \text{ for some } M_1 \ge 1 \text{ and } |AT(t)| \le M_2,\ M_2 \ge 0,\ t \in J.$$

(H2) there exist constants $0 \le c_1 < 1$ and $c_2 \ge 0$ such that

$$|g(t,u)| \le c_1\|u\| + c_2, \quad t \in J,\ u \in C([-r,0], E);$$

(H3) there exist constants d_k such that $|I_k(y)| \le d_k$, $k = 1, \dots, m$ for each $y \in E$;

(H4) $|f(t,u)| \le p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C([-r,0], E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$\int_0^b \widehat{m}(s) ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)};$$

where

$$c = \frac{1}{1-c_1}\Big\{M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2 M_2 b + c_2 + \sum_{k=1}^m d_k\Big\};$$

and

$$\widehat{m}(t) = \frac{1}{1-c_1}\{M_2 c_1, M_1 p(t)\}.$$

(H5) the function g is completely continuous and for any bounded set $D \subset \Omega$ the set $\{t \to g(t, y_t) : y \in D\}$ is equicontinuous in Ω.

We need the following auxiliary result. Its proof is very simple, so we omit it.

Lemma 3.2. *$y \in \Omega$ is a mild solution of* (1.1)–(1.3) *if and only if $y \in \Omega$ is a solution of the impulsive integral equation*

$$y(t) = T(t)[\phi(0) - g(0,\phi)] + g(t, y_t) + \int_0^t AT(t-s)g(s, y_s)\, ds + \\ + \int_0^t T(t-s)f(s, y_s)\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J.$$

Theorem 3.3. *Assume that hypotheses* (H1)–(H5) *hold. Then the* IVP (1.1)–(1.3) *has at least one mild solution on* $[-r, b]$.

Proof. Transform the problem into a fixed point problem. Consider the operator, $N : \Omega \longrightarrow \Omega$ defined by:

$$N(y)=\begin{cases} \phi(t), & t\in[-r,0], \\ T(t)[\phi(0)-g(0,\phi)] + g(t,y_t) + \displaystyle\int_0^t AT(t-s)g(s,y_s)\ ds + & \\ + \displaystyle\int_0^t T(t-s)f(s,y_s))\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t\in J. \ \square \end{cases}$$

Remark 3.4. Clearly from Lemma 3.2 the fixed points of N are mild solutions to (1.1)–(1.3).

We shall show that N satisfies the assumptions of Lemma 2.1. Using (H5) it suffices to show that the operator $N_1 : \Omega \longrightarrow \Omega$ defined by:

$$N_1(y)=\begin{cases} \phi(t), & t\in[-r,0], \\ T(t)\phi(0) + \displaystyle\int_0^t AT(t-s)g(s,y_s)ds + & \\ + \displaystyle\int_0^t T(t-s)f(s,y_s))\, ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & t\in J. \ \square \end{cases}$$

is completely continuous. The proof will be given in several steps.

Step 1: *N_1 maps bounded sets into bounded sets in Ω.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $y \in B_q = \{y \in \Omega : \|y\|_\Omega \le q\}$ one has $\|N_1(y)\|_\Omega \le \ell$.

Let $y \in B_q$, then for each $t \in J$ we have

$$N_1(y)(t) = T(t)\phi(0) + \int_0^t AT(t-s)g(s,y_s)\,ds + \\ + \int_0^t T(t-s)f(s,y_s)\,ds + \sum_{0<t_k<t} I_k(y(t_k^-)).$$

By (H1)–(H4) we have for eact $t \in J$

$$\begin{aligned} |N_1(y)(t)| \leq\, & M_1\|\phi\| + M_2\int_0^t |g(s,y_s)|ds + \\ & + M_1\int_0^t |f(s,y_s)|ds + \sum_{0<t_k<t} |I_k(y(t_k^-))| \leq \\ & \leq M_1\|\phi\| + M_2 b(c_1 q + c_2) + M_1 \sup_{y\in[0,q]} \psi(y)\Big(\int_0^t p(s)ds\Big) + \\ & + \sum_{k=1}^m \sup\{|I_k(|y|)| : \|y\|_\Omega \leq q\}. \end{aligned}$$

Then we have

$$\begin{aligned} \|N_1(y)\|_\Omega \quad \leq \quad & M_1\|\phi\| + M_2 b(c_1 q + c_2) + M_1 \sup_{y\in[0,q]} \psi(y)\Big(\int_0^b p(s)ds\Big) + \\ & + \sum_{k=1}^m \sup\{|I_k(|y|)| : \|y\|_\Omega \leq q\} := \ell. \end{aligned}$$

Step 2: *N_1 maps bounded sets into equicontinuous sets of Ω.*

Let τ_1, $\tau_2 \in J$, $\tau_1 < \tau_2$ and $B_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$ be a bounded set of Ω. Let $y \in B_q$. Then

$$\begin{aligned} |N_1(y)(\tau_2) - N_1(y)(\tau_1)| \leq\, & |T(\tau_2) - T(\tau_1)|q + \\ & + \int_0^{\tau_2} |AT(\tau_2 - s) - T(\tau_1 - s)|(c_1 q + c_2)\,ds + \\ & + \int_{\tau_1}^{\tau_2} |AT(\tau_1)|(c_1 q| + c_2)\,ds + \\ & + \int_{\tau_1}^{\tau_2} |T(\tau_2 - s) - T(\tau_1 - s) M_1 \sup_{y\in[0,q]} \psi(y)\Big(\int_0^b p(s)ds\Big)ds + \\ & + \int_{\tau_1}^{\tau_2} |T(\tau_1 - s)| M_1 \sup_{y\in[0,q]} \psi(y)\Big(\int_0^b p(s)ds\Big)ds + \sum_{0<t_k<\tau_2-\tau_1} d_k. \end{aligned}$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ is obvious.

Step 3: $N : \Omega \longrightarrow \Omega$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in Ω. Then there is an integer q such that $\|y_n\|_\Omega \leq q$ for all $n \in \mathbb{N}$ and $\|y\|_\Omega \leq q$, so $y_n \in B_q$ and $y \in B_q$.

We have then by the dominated convergence theorem

$$\begin{aligned}\|N_1(y_n) - N_1(y)\|_\Omega \leq \sup_{t\in J}\Big[\int_0^t |AT(t-s)||g(s, y_{ns}) - g(s, y_s)|ds + \\ + \int_0^t |T(t-s)||f(s, y_{ns}) - f(s, y_s)|ds + \\ + \sum_{0<t_k<t} |I_k(y_n(t_k)) - I_k(y(t_k))|\Big] \longrightarrow 0.\end{aligned}$$

Thus N_1 is continuous.

As a consequence of Steps 1 to 3 and (H5) together with the Arzela-Ascoli theorem we can conclude that $N : \Omega \longrightarrow \Omega$ is completely continuous.

Step 4: *Now it remains to show that the set*

$$\Phi(N) := \{y \in \Omega : y = \lambda N(y), \quad \textit{for some} \quad 0 < \lambda < 1\}$$

is bounded.

Proof. Let $y \in \Phi(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$

$$\begin{aligned}y(t) = \lambda T(t)[\phi(0) - g(0, \phi)] + \lambda g(t, y_t) + \lambda \int_0^t AT(t-s)g(s, y_s)\, ds + \\ + \lambda \int_0^t T(t-s)f(s, y_s))ds + \lambda \sum_{0<t_k<t} I_k(y(t_k^-)), \quad t \in J.\end{aligned}$$

This implies by (H1)–(H4) that for each $t \in J$ we have

$$\begin{aligned}|y(t)| \leq M_1[\|\phi\| + c_1\|\phi\| + c_2] + c_1\|y_t\| + c_2 + \\ + M_2 \int_0^t c_1\|y_s\|ds + c_2M_2b + M_1 \int_0^t p(s)\psi(\|y_s\|)ds + \sum_{k=1}^m d_k.\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \leq s \leq t\}, \quad 0 \leq t \leq b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\begin{aligned}\mu(t) \leq M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_1\mu(t) + c_2M_2b + c_2 + \\ + M_2c_1 \int_0^t \mu(s)\, ds + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m d_k.\end{aligned}$$

Thus

$$\mu(t) \leq \frac{1}{1-c_1}\Big[M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2M_2b + c_2 + \\ + M_2c_1\int_0^t \mu(s)\,ds + \int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^m d_k\Big], \quad t \in J.$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = \frac{1}{1-c_1}\Big\{M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2M_2b + c_2 + \sum_{k=1}^m d_k\Big\},$$

$$\mu(t) \leq v(t), \quad t \in J,$$

and

$$v'(t) = \frac{1}{1-c_1}[M_2c_1\mu(t) + p(t)\psi(\mu(t))], \ \ t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq \frac{1}{1-c_1}[M_2c_1v(t) + p(t)\psi(v(t))] \leq \widehat{m}(t)[v(t) + \psi(v(t)], \ \ t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{d\tau}{\tau + \psi(\tau)} \leq \int_0^b \widehat{m}(s)ds < \int_{v(0)}^{\infty} \frac{d\tau}{\tau + \psi(\tau)}.$$

This inequality implies that there exists a constant K such that $v(t) \leq K, \ t \in J$, and hence $\mu(t) \leq K, \ t \in J$. Since for every $t \in [0,b]$, $\|y_t\| \leq \mu(t)$, we have

$$\|y\|_\Omega \leq K' = \max\{\|\phi\|, K\},$$

where K' depends only b and on the functions p and ψ. This shows that $\Phi(N)$ is bounded. Set $X := \Omega$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a mild solution of (1.1)–(1.3). □

4. Second Order Impulsive Neutral Functional Differential Equations

In this section we study the initial value problem (1.4)–(1.7) by using the theory of strongly continuous cosine and sine families.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operator in E),

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,

(iii) the map $t \longmapsto C(t)y$ is strongly continuous for each $y \in E$; The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y ds, \quad y \in E, \ t \in \mathbb{R}.$$

The infinitesimal generator $A : E \longrightarrow E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ay = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [7] and Fattorini [6], and to the papers of Travis and Webb [15], [16].

Definition 4.1. A function $y \in C([-r, b], E)$ is said to be a mild solution of (1.4)–(1.7) if $y(t) = \phi(t)$ on $[-r, 0]$, $y'(0) = \eta$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1, \dots, m$, $\Delta y'|_{t=t_k} = \overline{I}_k(y(t_k^-))$, $k = 1, \dots, m$, and

$$y(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s)g(s, y_s)\, ds$$
$$+ \int_0^t S(t-s)f(s, y_s)\, ds + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k))], \quad t \in J.$$

We need the following auxiliary result. Its proof is very simple, so we omit it.

Lemma 4.2. *$y \in \Omega$ is a mild solution of* (1.4)–(1.7), *if and only if $y \in \Omega$ is a solution of the impulsive integral equation*

$$y(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s)g(s, y_s)\, ds +$$
$$+ \int_0^t S(t-s)f(s, y_s)\, ds + \sum_{0<t_k<t} [I_k(y(t_k^-)) + (t - t_k)\overline{I}_k(y(t_k))], \quad t \in J.$$

Assume that:

(A1) A is the infinitesimal generator of a strongly continuous cosine family $C(t), t \in \mathbb{R}$ of bounded linear operators from E into itself.

(A2) There exists constants c_1 and c_2 such that

$$|f(t, u)| \leq c_1 \|u\| + c_2, \quad t \in J, \ u \in C(J_0, E);$$

(A3) There exist constants $d_k, \overline{d}_k$ such that $|I_k(y)| \leq d_k$, $|\overline{I}_k(y)| \leq \overline{d}_k$ $k = 1, \dots, m$ for each $y \in E$;

(A4) $|f(t,u)| \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in C(J_0, E)$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$\int_0^b \widehat{m}(s)ds < \int_c^\infty \frac{d\tau}{\tau + \psi(\tau)};$$

where

$$c = M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + Mc_2b + \sum_{k=1}^m [d_k + (b - t_k)\overline{d}_k],$$
$$M = \sup\{|C(t)| : t \in J\},$$

and

$$\widehat{m}(t) = \max\{Mc_1, Mp(t)\}.$$

(A5) The function g is completely continuous and for any bounded set $A \subseteq C(J_1, E)$ the set $\{t \longrightarrow g(t, y_t) : y \in A\}$ is equicontinuous in $C(J, E)$;

(A6) $C(t), t \in J$ is completely continuous.

Now, we are in a position to state and prove our main theorem in this section.

Theorem 4.3. *Assume that hypotheses* (A1)–(A6) *hold. Then the* IVP (1.4)–(1.7) *has at least one mild solution on* J_1.

Proof. Transform the problem into a fixed point problem. This time define an operator $N : \Omega \longrightarrow \Omega$ by:

$$N(y) := \begin{cases} \phi(t), & \text{if } t \in J_0 \\ C(t)\phi(0) + S(t)[\eta - g(0,\phi)] + & \\ + \displaystyle\int_0^t C(t-s)g(s, y_s)\,ds + & \\ + \displaystyle\int_0^t S(t-s)f(s, y_s)\,ds + & \\ + \displaystyle\sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y(t_k))], & \text{if } t \in J. \end{cases} \qquad \square$$

Remark 4.4. It is clear that the fixed points of N are mild solutions to (1.4)–(1.7).

We shall show that N satisfies the assumptions of Lemma 2.1. The proof will be given in several steps.

Step 1: *N maps bounded sets into bounded sets in Ω.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in N(y)$, $y \in B_q = \{y \in \Omega : \|y\|_\Omega \leq q\}$ one has $\|N(y)\|_\Omega \leq \ell$.

If $y \in B(q)$, then for each $t \in J$ we have

$$N(t) = C(t)\phi(0) + S(t)[\eta - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)\,ds + \\ + \int_0^t S(t-s)f(s,y_s)ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))].$$

By (A2)–(A4) we have for each $t \in J$

$$|N(y)(t)| \leq M\|\phi\| + bM(|\eta| + c_1 q + c_2) + Mb(c_1 q + c_2) + \\ + M \sup_{y\in[0,q]} \psi(y)\Big(\int_0^t p(s)ds\Big) + \sum_{k=1}^m [d_k + (b-t_k)\overline{d}_k].$$

Then we have

$$\|N(y)\|_\Omega \leq M\|\phi\| + bM(|\eta| + c_1 q + c_2) + Mb(c_1 q + c_2) + \\ + M \sup_{y\in[0,q]} \psi(y)\Big(\int_0^b p(s)ds\Big) + \sum_{k=1}^m [d_k + (b-t_k)\overline{d}_k] := \ell.$$

Step 2: *N maps bounded sets into equicontinuous sets of Ω.*

Let τ_1, $\tau_2 \in J$, $\tau_1 < \tau_2$ and B_q be a bounded set of Ω as in Step 2. Let $y \in B_q$. Then

$$h(t) = C(t)\phi(0) + S(t)[\eta - g(0,\phi)] + \int_0^t C(t-s)g(s,y_s)\,ds + \\ + \int_0^t S(t-s)v(s)ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t-t_k)\overline{I}_k(y(t_k))].$$

Thus

$$|h(\tau_2) - h(\tau_1)| \leq |C(\tau_2) - C(\tau_1)| + (|\eta| + c_1\|\phi\| + c_2)|S(\tau_2) - S(\tau_1)| + \\ + \Big|\int_0^{\tau_2} [C(\tau_2 - s) - C(\tau_1 - s)]g(s,y_s)ds\Big| + \Big|\int_{\tau_1}^{\tau_2} C(\tau_1 - s)g(s,y_s)ds\Big| + \\ + \Big|\int_0^{t_2} [S(\tau_2 - s) - S(\tau_1 - s)]f(s,y_s)\,ds\Big| + \Big|\int_{\tau_1}^{\tau_2} S(\tau_1 - s)f(s,y_s)\,ds\Big| + \\ + \sum_{0<t_k<\tau_2-\tau_1} [d_k + (b-t_k)\overline{d}_k] \leq \\ \leq |C(t_2) - C(\tau_1)| + (|\eta| + c_1 q + c_2)|S(\tau_2) - S(\tau_1)| + \\ + \int_0^{\tau_2} |C(\tau_2 - s) - C(\tau_1 - s)]|(c_1 q + c_2)ds + \\ + \int_{\tau_1}^{\tau_2} |C(\tau_1 - s)|(c_1 q + c_2)ds +$$

$$+\int_0^{\tau_2} |S(\tau_2 - s) - S(\tau_1 - s)| M \sup_{y\in[0,q]} \psi(y) \Big(\int_0^b p(s)ds \Big) \, ds +$$
$$+\int_{\tau_1}^{\tau_2} |S(t_1 - s)| M \sup_{y\in[0,q]} \psi(y) \Big(\int_0^b p(s)ds \Big) \, ds +$$
$$+ \sum_{0<t_k<\tau_2-\tau_1} [d_k + (b - t_k)\overline{d}_k].$$

As $\tau_2 \longrightarrow \tau_1$ the right-hand side of the above inequality tends to zero.

The equicontinuity for the cases $\tau_1 < \tau_2 \leq 0$ and $\tau_1 \leq 0 \leq \tau_2$ areobvious.

Step 3: $N : \Omega \to \Omega$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in Ω. Then there is an integer q such that $\|y_n\|_\Omega \leq q$ for all $n \in \mathbb{N}$ and $\|y\|_\Omega \leq q$, so $y_n \in B_q$ and $y \in B_q$.

We have then by the dominated convergence theorem

$$\|N(y_n) - N(y)\|_\Omega \leq \sup_{t\in J} \Big[\int_0^t |C(t-s)||g(s, y_{ns}) - g(s, y_s)| ds +$$
$$+ \int_0^t |S(t-s)||f(s, y_{ns}) - f(s, y_s)| ds +$$
$$+ \sum_{0<t_k<t} [|I_k(y_n(t_k)) - I_k(y(t_k))| +$$
$$+ |t - t_k||\overline{I}_k(y_n(t_k)) - \overline{I}_k(y(t_k))] \Big] \longrightarrow 0.$$

Thus N is continuous.

As a consequence of Steps 1 to 3 and (A6) together with the Arzela-Ascoli theorem we can conclude that $N : \Omega \longrightarrow \Omega$ is completely continuous.

Step 4: *The set*

$$\Phi(N) := \{y \in \Omega : y = \lambda N(y), \quad \textit{for some} \quad 0 < \lambda < 1\}$$

is bounded.

Proof. Let $y \in \Phi(N)$. Then $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus for each $t \in J$

$$y(t) = \lambda C(t)\phi(0) + \lambda S(t)[\eta - g(0, \phi)] + \lambda \int_0^t C(t-s) g(s, y_s) ds +$$
$$+\lambda \int_0^t S(t-s) f(s, y_s) ds + \lambda \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\overline{I}_k(y(t_k))], \; t \in J.$$

This implies by (A2)–(A4) that for each $t \in J$ we have

$$|y(t)| \leq M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + M \int_0^t (c_1\|y_s\| + c_2) ds +$$

$$+ M\int_0^t p(s)\psi(\|y_s\|)ds + \sum_{k=1}^{m}[d_k + (b-t_k)\overline{d}_k].$$

We consider the function μ defined by

$$\mu(t) = \sup\{|y(s)| : -r \le s \le t\}, \quad 0 \le t \le b.$$

Let $t^* \in [-r, t]$ be such that $\mu(t) = |y(t^*)|$. If $t^* \in J$, by the previous inequality we have for $t \in J$

$$\mu(t) \le M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + Mc_1\int_0^t \mu(s)ds + Mc_2 b +$$
$$+ M\int_0^t p(s)\psi(\mu(s))ds + \sum_{k=1}^{m}[d_k + (b-t_k)\overline{d}_k].$$

If $t^* \in J_0$ then $\mu(t) = \|\phi\|$ and the previous inequality holds.

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$c = v(0) = M\|\phi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + Mc_2 b + \sum_{k=1}^{m}[d_k + (b-t_k)\overline{d}_k],$$
$$\mu(t) \le v(t), \quad t \in J,$$

and

$$v'(t) = Mc_1\mu(t) + Mp(t)\psi(\mu(t)) \le$$
$$\le Mc_1 v(t) + Mp(t)\psi(v(t)) \le \widehat{m}(t)[v(t) + \psi(v(t))], \quad t \in J.$$

This implies for each $t \in J$ that

$$\int_{v(0)}^{v(t)} \frac{du}{u + \psi(u)} \le \int_0^b \widehat{m}(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \psi(u)}.$$

This inequality implies that there exists a constant L such that $v(t) \le L,\ t \in J$, and hence $\mu(t) \le L,\ t \in J$. Since for every $t \in J, \|y_t\| \le \mu(t)$, we have

$$\|y\|_\Omega \le L' = max\{\|\phi\|, L\},$$

where L' depends on b and on the functions p and ψ. This shows that Ω is bounded. Set $X := \Omega$. As a consequence of Lemma 2.1 we deduce that N has a fixed point which is a mild solution of (1.4)–(1.7). $\square$

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Authors' addresses:

M. Benchohra
Department of Mathematics
University of Sidi Bel Abbes
BP 89, 22000 Sidi Bel Abbes
Algeria

J. Henderson
Department of Mathematics
Auburn University
Auburn, Alabama 36849-5310
USA

S. Ntoyhas
Department of Mathematics
University of Ioannina
451 10 Ioannina
Greece