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STEADY PROPERTIES OF ELEMENTS IN BANACH SPACES

Abstract. In this paper the most general statements concerning steady properties of elements in Banach spaces are stated. The paper provides a device for research of concrete steady properties of linear operators and linear boundary value problems. It is proved that each steady property is generated by some positively homogeneous and nonexpanding functional. The concept of the spectrum of an element concerning steady property is introduced and basic statements about the spectrum are formulated.

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Introduction

In the theory of linear boundary value problems for ordinary differential equations, partial differential equations, functional differential equations, and as well as in the theory of integral equations, to investigate some properties (solvability, Fredholm property, and the like), an approach is productively enough applied, the idea of which can be expressed as follows: a modelling equation (modelling boundary value problem) is considered and a property of this equation is established which is preserved under small perturbations. Application of the functional analysis methods in the theory of boundary value problems allowed to formulate the mentioned approach in terms of additive perturbations of linear operators [1].

The analysis of known steady properties of linear operators shows that they have a number of common properties. So there arose the idea of axiomatic definition of the steady property concept of the linear operator and study of abstract steady properties in the space of bounded linear operators. As it turned out, the main facts about steady properties admit natural and laconic treatment for elements of a Banach space.

In the proposed work the most general statements concerning steady properties of elements in Banach spaces are stated. This material is designed to give one an adequate background for studying steady properties of linear operators and linear boundary value problems. Both from the theoretical point of view a fundamental fact is that each steady property is generated by some positively homogeneous and nonexpanding functional. Such a functional, named the generating one, is a quite effective tool of studying a concrete steady property. So, for example, the study of surjectivity coefficient [2] has allowed to solve the problem of the Green operator with minimal norm. This problem for functional differential equations for the first time was considered in the work [3].

The basic material of the paper is divided into sections. Theorems are numbered within each section in the way that the first number corresponds to the number of a section. The text of the paper does not contain any references. The preliminary results used in the work can be found in [4,5].

0. Notation

Throughout this paper we will use the following notation and definitions.

0.1. Let $\mathbf{R}$ be the set of real numbers and $\mathbf{C}$ be the field of complex numbers. The field $\mathbf{K}$ of scalars is $\mathbf{R}$ or $\mathbf{C}$. Below all spaces under consideration are assumed linear and are denoted by the letters B, D, T, X, Y, Z. If not specially otherwise mentioned, under a space we mean a Banach space, i.e., a complete normed linear space.

Let E be a normed space and E_0 be a linear space such that $E_0 \subset E$. If we consider E_0 as a subspace of E, we mean that E_0 is endowed with the norm of E. In other cases the inclusion $E_0 \subset E$ is understood in algebraic sense, i.e., E_0 is a subset of E which has the linear structure.

Let E be a normed space with a norm $\|\cdot\|$. For a fixed element $a_0 \in E$ and a positive number r, the open ball $V(a_0, r)$ and the closed ball $U(a_0, r)$ centered at a_0 with radius r are defined by

$$V(a_0, r) = \{a \in E\colon \|a_0 - a\| < r\},$$
$$U(a_0, r) = \{a \in E\colon \|a_0 - a\| \leq r\}.$$

0.2. Let E_1 and E_2 be subspaces of a linear space E. We say that E is decomposed into the direct (algebraic) sum of subspaces E_1 and E_2 (and write $E = E_1 \dot{+} E_2$), if $E = E_1 + E_2$ and $E_1 \cap E_2 = \{\theta\}$. In case $E = E_1 \dot{+} E_2$ each element $a \in E$ is uniquely presented in the form $a = a_1 + a_2$, where $a_1 \in E_1$, $a_2 \in E_2$.

Let E be a Banach space, $E = E_1 \dot{+} E_2$ and E_1, E_2 be closed subspaces of E. Then we say that E is the direct (topological) sum of the closed subspaces E_1 and E_2 and write $E = E_1 \oplus E_2$.

0.3. Let E_1 and E_2 be linear spaces over a commonfield of scalars. The space $E_1 \times E_2$, named direct (algebraic) product of the spaces E_1 and E_2, is defined as the set of all ordered pairs (a_1, a_2), $a_1 \in E_1$, $a_2 \in E_2$, with component-wise operations of multiplication by a scalar and addition of elements.

If E_1 and E_2 are two Banach spaces with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively, the direct product $E_1 \times E_2$ becomes a Banach space after endowing it with the norm

$$\|(a_1, a_2)\|_p = \begin{cases} (\|a_1\|_1^p + \|a_2\|_2^p)^{1/p}, & 1 \leq p < \infty, \\ \max\{\|a_1\|_1, \|a_2\|_2\}, & p = \infty, \end{cases}$$

for a fixed p.

0.4. Let $\mathbf{R}^n$ be an n-dimensional real linear space of vectors $\alpha = \operatorname{col}(\alpha_1, \alpha_2, \dots, \alpha_n)$, $\alpha_i \in \mathbf{R}^1$, $i = \overline{1, n}$, with one of the following norms

$$|\alpha|_1 = \sum_{i=1}^{n} |\alpha_i|, \quad |\alpha|_2 = \max_{i=\overline{1,n}} |a_i|, \quad |\alpha|_3 = \left\{ \sum_{i=1}^{n} \alpha_i^2 \right\}^{1/2}.$$

We will specify a concrete norm only in case of need.

0.5. Denote by $C[a; b]$ the Banach space of all continuous functions $x\colon [a, b] \to \mathbf{R}^1$ with the norm defined by

$$\|x\|_C = \max_t |x(t)|.$$

1. Definition of Steady Properties

Let E be a Banach space over the field $\mathbf{K}$ of reals or of complexes. Further we denote elements of the space E by a, b, c etc., and elements of $\mathbf{K}$ by α, β, γ.

Definition 1.1. A set $R \subset E$ is said to be the *resolvent* of a steady property over E if the following axioms are fulfilled:

1) $\theta \notin R$,
2) if $\lambda \neq 0$, $a \in R$, then $\lambda a \in R$,
3) R is an open subset of E.

We say that the elements of the set R and only they possess the given steady property.

We will denote steady properties by the symbols μ, γ, ν etc. and, if necessary, use the appropriate index in the notation of the resolvent (for example, R_μ). Note that, as a rule, concrete steady properties can be defined by a phrase (term). So, for example, further it will be shown, that the invertibility is a steady property of invertible linear bounded operators. The steady properties over E with the resolvents $R = \varnothing$ and $R = E \setminus \{\theta\}$ are named trivial.

Axioms 1) and 2) of Definition 1.1 characterize a resolvent algebraically, and Axiom 3) does it topologically.

Let us consider a disjunctive splitting of the space E:

$$E = R \cup Q$$

produced by a resolvent R.

The set $Q = E \setminus R$ has the following properties:

4) If $\alpha \in Q$, then $\lambda\alpha \in Q$, for every $\lambda \in \mathbf{K}$,
5) Q is a closed set.

The set Q unites all elements of space E not having the given steady property. We can say that the elements of Q have unsteady property in a sense. If a steady property is defined over E by the resolvent, then each element of E either has that property or has not. On the other hand, if we fix some set $Q \subset E$ satisfying conditions 4) and 5), it generates the resolvent $R = E \setminus Q$. The fact that R satisfies the axioms 1–3) of Definition 1.1 can be checked directly.

Taking into account the above said, it is possible to speak about two ways of definition of steady properties. Namely:

1) Directly due to Definition 1.1;
2) Allocation of the set having properties 4) and 5).

2. Generating Functionals

In this section it will be shown that each steady property is generated by a positively homogeneous and nonexpanding functional.

Definition 2.1. A non-negative functional $\mu\colon E \to R^+$ satisfying the conditions

$$a1)\ \ \mu(\lambda a) = |\lambda|\mu(a), \quad \lambda \in K, \quad a \in E,$$
$$a2)\ \ \mu(a + b) \le \mu(a) + \mu(b), \quad a, b \in E,$$

is called *the generating functional.*

The condition a1) implies $\mu(a) \leq \|a\|$, $a \in E$, and the condition a2) is equivalent to each of the following inequalities:

$$\mu(a) - \|b\| \leq \mu(a+b),$$
$$|\mu(a) - \mu(b)| \leq \|a-b\|.$$

From the last inequality, it follows that any generating functional is continuous.

Theorem 2.1. *Let a functional* $\mu\colon E \to R^+$ *satisfy the conditions* a1) *and* a2). *Then the set*

$$R_\mu = \{a \in E\colon \mu(a) > 0\} \tag{1}$$

is the resolvent of a steady property.

Proof. It suffices to show that the set defined by the equality (1) satisfies the axioms 1)–3) of Definition 1.1.

Since

$$\mu(\theta) = \mu(0 \cdot a) = 0,$$

we have $\theta \notin R_\mu$. If $\lambda \neq 0$ and $a \in R_\mu$ (i.e., $\mu(a) > 0$), from the equality

$$\mu(\lambda a) = |\lambda| \mu(a)$$

it follows that $\lambda a \in R_\mu$. Thus the validity of the axioms 1) and 2) is proved. For testing the axiom 3), let us fix $a \in R_\mu$ and consider an ε-neighborhood $V = V(a, \varepsilon)$ of the element $a \in E$, where $0 < \varepsilon < \mu(a)$. For every $b \in V$, we have

$$\mu(b) = \mu(a - (a-b)) \geq \mu(a) - \|b-a\| \geq \mu(a) - \varepsilon > 0.$$

Thus each element belongs to the resolvent set along with some its neighborhood and R_μ is the resolvent of a steady property. □

In addition to two ways of definition of a steady property in a Banach space, Theorem 2.1 offers the third one: definition on the space E of a positive functional with the properties a1) and a2). The following theorem represents the inverse to Theorem 2.1 statement and says that each steady property is generated by some functional satisfying the conditions a1), a2).

Theorem 2.2. *Let* $R \subset E$ *be the resolvent of a steady property. Then there exists a functional* $\mu\colon E \to \mathbf{R}^+$ *satisfying the conditions* a1) *and* a2) *such that*

$$R = \{a \in E\colon \mu(a) > 0\}.$$

Proof. Restrict us to the case of a real Banach space E ($\mathbf{K} = \mathbf{R}$). Let R be the resolvent of a nontrivial steady property. Define $\mu\colon E \to \mathbf{R}^+$ by the equality

$$\mu(a) = \begin{cases} \sup\{r > 0\colon V(a,r) \subset R\}, & \text{if } a \in R, \\ 0, & \text{if } a \notin R, \end{cases}$$

and show that it satisfies the conditions a1) and a2). If $\lambda = 0$ or $a \notin R$, then $\lambda a \notin R$ and, consequently,

$$\mu(\lambda a) = |\lambda|\mu(a) = 0.$$

Let $\lambda \neq 0$ and $a \in R$. Since by definition $\mu(-a) = \mu(a)$, without less of generality we can suppose that $\lambda > 0$. The resolvent of a steady property has the property that $M \subset R$ is equivalent to the inclusion $\lambda M \subset R$ for every $\lambda > 0$. For any fixed $a \in E$, $\lambda > 0$ and $r > 0$, the equality

$$V(\lambda a, r) = \lambda V\left(a, \frac{r}{\lambda}\right)$$

takes place.

Therefore, for any $\lambda > 0$ and any element $a \in R$, we have

$$\begin{aligned}\mu(\lambda a) &= \sup\{r > 0\colon V(\lambda a, r) \subset R\} = \\ &= \lambda \sup\left\{\frac{r}{\lambda} > 0\colon V\left(a, \frac{r}{\lambda}\right) \subset R\right\} = \lambda\mu(a).\end{aligned}$$

This finishes testing of the condition a1) for the functional μ. Next let us test a condition equivalent to the condition a2):

$$\mu(a) - \|b\| \leq \mu(a-b), \quad a, b \in E. \tag{2}$$

If $\|b\| \leq \mu(a)$, then the validity of (2) is obvious. Let $a, b \in E$ be such that $\|b\| < \mu(a)$. For any $r_0 \in (\|b\|, \mu(a))$, prove the validity of the implication

$$z \in V(a-b, r_0 - \|b\|) \Rightarrow z \in V(a, r_0).$$

Let $z \in V(a-b, r_0 - \|b\|)$, i.e., $\|a-b-z\| < r_0 - \|b\|$. Then

$$\|a-z\| \leq \|a-b-z\| + \|b\| < r_0.$$

Hence $z \in V(a, r_0)$ and

$$V(a-b, r_0 - \|b\|) \subset V(a, r_0).$$

In view of this, the following estimate holds:

$$\begin{aligned}\mu(a) - \|b\| &= \sup\{r\colon V(a,r) \subset R\} - \|b\| = \sup\{r - \|b\|\colon V(a,r) \subset R\} \leq \\ &\leq \sup\{r - \|b\|\colon V(a-b, r-\|b\|) \subset R\} = \\ &= \sup\{r_1\colon V(a-b, r_1) \subset R\} = \mu(a-b).\end{aligned}$$

Thus the validity of the inequality (2), which is equivalent to the condition a2), is shown. □

The fact that each steady property is generated by a functional with the properties a1), a2) gives a convenient enough tool for research of steady properties. However, the answer to the question on uniqueness of generating functional turns out to be negative. Indeed, if the functional μ generates a steady property with the resolvent R_μ, a direct check shows that, for any $0 < \alpha < 1$, the functional

$$\mu_\alpha(a) = \alpha\mu(a)$$

also has the properties a1) and a2) and generates the steady property with the same resolvent R_μ. The inconvenience related to the last circumstance may be overcome as follows.

Denote by $M = M(R)$ the set of all functionals generating the steady property with the resolvent R and put

$$k(a) = \sup\{\mu(a)\colon \mu \in M\}. \tag{3}$$

Here the supremum is taken at every fixed $a \in E$.

Since for every fixed $a \in E$ and any $\mu \in M$ the inequality $\mu(a) \leq \|a\|$ holds, the functional $k(a)$ is finite-valued. Thus, the functional $k\colon E \to \mathbf{R}^+$ is determined correctly.

Theorem 2.3. *Let $R \subset E$ be the resolvent of a nontrivial steady property. Then the functional determined by the equality* (3) *has the properties* a1) *and* a2), *and*

$$R = \{a \in E\colon k(a) > 0\}. \tag{4}$$

Proof. The validity of the equality (4) is obvious. Let us check up the properties a1) and a2).

a1) For any $\lambda \in K$ and $a \in E$, we have

$$\begin{aligned} k(\lambda a) &= \sup\{\mu(\lambda a)\colon \mu \in M\} = \sup\{|\lambda|\mu(a)\colon \mu \in M\} = \\ &= |\lambda| \sup\{\mu(a)\colon \mu \in M\} = |\lambda| k(a). \end{aligned}$$

a2) Fix arbitrary $a, b \in E$ and consider the inequality

$$\mu(a+b) \leq \mu(a) + \|b\|.$$

In this inequality, applying sup with respect to all $\mu \in M$, due to properties of this operation we obtain

$$\begin{aligned} k(a+b) &= \sup\{\mu(a+b)\colon \mu \in M\} \leq \sup\{\mu(a) + \|b\|\colon \mu \in M\} = \\ &= \sup\{\mu(a)\colon \mu \in M\} + \|b\| = k(a) + \|b\|. \quad \square \end{aligned}$$

Definition 2.2. The generating functional determined by the equality (3) is called *canonical*.

The canonical functional not only identifies uniquely the steady property, but also gives the maximal quantitative characteristic of the degree of steadiness with respect to perturbations. By definition

$$\mu(a) \leq k(a), \quad \mu \in M, \quad a \in E. \tag{5}$$

The following statement shows that the canonical functional coincides with the functional, construction of which is used in the proof of Theorem 2.2.

Theorem 2.4. *For a steady property with the resolvent $R \subset E$, the canonical generating functional is defined by the equality*

$$k(a) = \begin{cases} \sup\{r > 0 \colon V(a,r) \subset R\}, & \text{if } a \in R, \\ 0, & \text{if } a \notin R. \end{cases}$$

Proof. Denote by μ_0 the functional in the right-hand side. The inequality $k(a) \leq \mu_0(a)$ is obvious, as μ_0 is one of the generating functionals. Prove the inverse inequality. As $k(a) = \mu_0(a)$ at $a \in q\ker$, assume that $a \in R$. We have

$$k(a+b) \geq k(a) - \|b\|.$$

From this inequality it follows that $(a+b) \in R$ for all $\|b\| \leq k(a)$. This means that

$$U(a, k(a)) \subset R.$$

This inclusion completes the proof of the statement.

In situations, when the exact value $k(a)$ cannot be found, any value $\mu(a)$ in the inequality (5) gives a lower bound. Such way of research of steady properties appears sufficiently effective.

The set $S = E \setminus R$, being the disjunctive complement of the resolvent, is defined by the equality

$$S = \{a \in E \colon k(a) = 0\}. \tag{6}$$

Therefore we introduce the following □

Definition 2.3. The set S determined by the equality (6) is called the *quasi-kernel* of the steady property with the resolvent $R = \{a \in E \colon k(a) > 0\}$ and is denoted by $q\ker k$.

3. Examples of Steady Properties

In this section some elementary examples of steady properties are given. Three various ways of defining steady properties are used. All three characteristics of the steady property (resolvent, quasi-kernel, generating functional) are discussed.

Example 1. Let X be a one-dimensional space and $x_0 \in X$ be a nonzero element of this space. Let the functional μ determine a steady property in

X such that x_0 belongs to the resolvent of the given steady property, i.e., $\mu(x_0) > 0$.

Since any element $x \in X$ admits the presentation $x = tx_0$, $t \in \mathbf{R}^1$, we have

$$\mu(x) = \mu(tx_0) = |t|\mu(x_0) > 0.$$

Hence

$$R_\mu = \{x \in X : x \neq \theta\} = x \setminus \{\theta\}$$

and the appropriate quasi-kernel consists of only the zero element

$$Q_\mu = \{\theta\}.$$

Thus on a one-dimensional space it is possible to define the unique non trivial steady property and its resolvent coincides with $X \setminus \{\theta\}$.

Example 2. Let us fix a δ from the interval $(0;1)$. Extend all elements of the space $C[0;1]$ on $[1;1+\delta]$ according to the equality

$$x(t) = -t + x(1) + 1, \quad t \in [1; 1+\delta].$$

Define the functional $\mu\colon C[0;1] \to \mathbf{R}^+$ by the equality

$$\mu(x) = \min_{t\in[0;1]} \frac{1}{\delta} \int_t^{t+\delta} |x(s)|\, ds.$$

This functional μ satisfies the axiom a1). Besides, for any $x, y \in C[0;1]$, the inequality

$$\int_t^{t+\delta} |x(s) + y(s)|\, ds \leq \int_t^{t+\delta} |x(s)|\, ds + \delta \max_t |y(t)|$$

holds. From here, the validity of the axiom a2) follows.

Hence the functional μ generates a steady property on the space $C[0;1]$.

Thus the resolvent of the given steady property is the set of continuous functions such that the plot of the function has no horizontal piece of length δ or greater.

If the plot of a function $x(\cdot)$ has a horizontal piece of length δ or greater, then such a function belongs to the quasi-kernel of the given steady property.

Example 3. Define the functional $\mu\colon C[0;1] \to \mathbf{R}^+$ by the equality

$$\mu(x) = \frac{1}{2} \max_t |x(t) - x(1-t)|.$$

The validity of the axiom a1) is obvious.

For any elements $x, y \in C[0;1]$, we have

$$\begin{aligned}\mu(x,y) &= \frac{1}{2} \max_t |x(t) + y(t) - x(1-t) - y(1-t)| \leq \\ &\leq \frac{1}{2} \max_t |x(t) - x(1-t)| + \frac{1}{2} \max_t |y(t) - y(1-t)| \leq\end{aligned}$$

$$\leq \mu(x) + \frac{1}{2} \cdot 2 \max_t |y(t)| = \mu(x) + \|y\|.$$

The validity of the axiom a2) is proved.

Thus the functional μ defines a steady property on the space $C[0;1]$. The quasi-kernel of the given steady property consists of the continuous functions $x(\cdot)$ such that $x(t) = x(1-t)$ for any $t \in [0;1]$. In other words, the generating functional takes the zero-value on all functions with plots having the straight line $t = \frac{1}{2}$ as an axis of symmetry.

Accordingly, all functions not having this property are elements of the resolvent of the given steady property.

Example 4. Let E and F be Banach spaces over a common field of scalars, and $B = B(E,F)$ be the Banach space of all bounded linear operators $A\colon E \to F$.

Denote by $R \subset B(E;F)$ the set of all invertible operators. The zero operator is not invertible. If an operator $A \in B(E;F)$ is invertible, then the operator λA, $(\lambda \neq 0)$ is also invertible. Besides, the operator $A + B$, where $B \in B(E,F)$ and $\|B\| < \frac{1}{\|A^{-1}\|}$, is invertible along with the operator A. Hence the set of invertible operators satisfies the axioms 2) and 3) and consequently it is the resolvent of a steady property.

Thus the invertibility is a steady property on the space of bounded linear operators.

4. Space of Steady Properties

Let E be a Banach space. We will consider the set of all resolvents of steady properties of elements of E including the trivial one i.e. open sets $R \subset E$ such that $\theta \notin R$ and $\lambda a \in R$ for $\lambda \neq 0$, $a \in R$. Denote

$$R(E) = \{R \subset E\colon R \text{ is the resolvent of a steady property}\}.$$

Definition 4.1. The set $R(E)$ is called the *space of steady properties* on E.

To each steady property with the resolvent R, let us put in correspondence the canonical generating functional $k\colon E \to \mathbf{R}^+$ with

$$R_k = R = \{a \in E\colon k(a) > 0\}.$$

Denote by E^p the set of all canonical generating functionals. By virtue of Theorems 2.1 and 2.2 there is a one-to-one correspondence between $R(E)$ and E^p. Therefore, sometimes instead of $R(E)$ it is convenient to consider E^p, examining the latter as a realization of the space of steady properties.

Define on the space of steady properties the operations of the product and the sum of steady properties. First, we note that, if R_1 and R_2 are resolvents, then both $R_1 \cup R_2$ and $R_1 \cap R_2$ also are resolvents.

Definition 4.2. The union (*the sum*) of two steady properties with the resolvents R_1 and R_2 is the steady property with the resolvent $R_1 \cup R_2$.

Definition 4.3. *The composition* (*the product*) of two steady properties with the resolvents R_1 and R_2 is the steady property with the resolvent $R_1 \cap R_2$.

The introduced operations are naturally extended to finite number of steady properties.

Theorem 4.1. *Let the steady properties with the resolvents R_1 and R_2 be generated by the functionals μ_1 and μ_2 respectively. Then*

1) *the sum is generated by the functional*

$$\mu(a) = \max\{\mu_1(a), \mu_2(a)\};$$

2) *the product is generated by the functional*

$$v(a) = \min\{\mu_1(a), \mu_2(a)\}.$$

Proof. The functional μ has the properties a1) and a2), which is checked directly. Further, we have $\mu_1(a) > 0$ or $\mu_2(a) > 0$, i.e.,

$$R_1 \cup R_2 = \{a \in E \colon \mu(a) > 0\}.$$

Thus the first statement of the theorem is proved. Similar arguments confirm the validity of the second statement. □

To denote the sum and the product of the steady properties with the resolvents

$$R_i = \{a \in E \colon \mu_i(a) > 0\}, \quad i = 1, 2,$$

we will write $\mu_1 + \mu_2$ and $\mu_1\mu_2$ respectively.

Let us emphasize that under conditions of Theorem 4.1 it is not supposed that the functionals μ_1 and μ_2 are canonical. It is possible to specify other functionals which generate the sum or the product of steady properties. So the sum $\mu_1 + \mu_2$ is generated also by the functional

$$\mu(a) = \frac{1}{2}\big(\mu_1(a) + \mu_2(a)\big).$$

The following statement may be checked directly making use of definition of the quasi-kernel.

Theorem 4.2. *There take place the following equalities:*

$$q\ker(\mu_1 + \mu_2) = (q\ker\mu_1) \cap (q\ker\mu_2),$$
$$q\ker(\mu_1\mu_2) = (q\ker\mu_1) \cup (q\ker\mu_2).$$

On the space of steady properties, we will define the relation of order according to the following definition. Thus $R(E)$ becomes a partially ordered space, since the existence of incomparable steady properties is obvious.

Definition 4.4. Let $R_1, R_2 \in R(E)$. We say that the steady property with the resolvent R_1 is a more strong steady property than the one with the resolvent R_2, if $R_1 \subset R_2$.

It is easily checked that if, for generating functionals, the inequality $\mu_1(a) \leq \mu_2(a)$, $a \in E$, holds, then there takes place the inclusion $R_{\mu_1} \subset R_{\mu_2}$. The converse is formulated as

Theorem 4.3. *Let $R_1, R_2 \in R(E)$ and μ_1, μ_2 be the corresponding canonical generating functionals. If $R_1 \subset R_2$, then*

$$\mu_1(a) \leq \mu_2(a), \quad a \in E. \tag{7}$$

Proof. Since $\mu_1(a) = 0$ for $a \in R_1$, it suffices to check the inequality only for $a \in R_1$. By virtue of the theorem on representation of the canonical functional, for every $a \in R_1$ we have

$$U(a, \mu_1(a)) \subset R_1.$$

Since $R_1 \subset R_2$, there take place

$$\mu_2(a) = \sup\{r > 0 \colon U(a,r) \subset R_2\} \geq \mu_1(a). \quad \square$$

The statement of Theorem 4.3 conditionally allows to write the inclusion $R_1 \subset R_2$ as the inequality $\mu_1 \leq \mu_2$ (the steady property μ_1 is a more strong steady property than μ_2), where $\mu_1, \mu_2 \in E^p$ are the generating functionals related to the resolvents R_1 and R_2 respectively.

Theorem 4.4. *For any pair $\mu_1, \mu_2 \in E^p$, the inequalities*

$$\mu_1\mu_2 \leq \mu_i \leq \mu_1 + \mu_2, \quad i = 1, 2,$$

hold.

Proof. As

$$R_1 \cap R_2 \subset R_i \subset R_1 \cup R_2, \quad i = 1, 2,$$

it is suffice to take advantage of Theorems 4.1 and 4.3. $\square$

5. Theorems on Steady Properties

The statements in this section have a property in common, namely, they allow construct new steady properties by known steady properties.

Let E and F be Banach spaces over a common field of scalars.

Theorem 5.1. *Let $R_0 \subset E$ be the resolvent of a steady property and $E_0 \subset E$ be a subspace. If R_0 and E_0 are disjunctive, then $R = R_0 + E_0$ is the resolvent of a steady property.*

Proof. We will show, that the set $R = R_0 + E_0$ satisfies all the resolvent axioms (see Definition 1.1). The set R contains every possible sums of the kind $a + b$, where $a \in R$ and $b \in E_0$. As R and E are disjunctive and $\theta \notin R_0$, the sum $a + b$, where $a \in R_0$, $b \in E_0$ are arbitrary elements of the correspoding sets, may not be equal to the zero element. So, $\theta \notin R$. Let $\lambda \neq 0$ and $z = (a + b) \in R$, $a \in R$, $b \in E_0$. Then the element λz can be represented if the form $\lambda z = \lambda a + \lambda b$, where $\lambda a \in R_0$ and $\lambda b \in E_0$. Hence $\lambda z \in R$. Thus the validity of first two axioms is shown. To complete the proof we note that R is an open set, as the direct sum of the open set R_0 and the subspace E_0. $\square$

In connection with the statement of Theorem 5.1 we will bring attention to the question on possibility of representation of any resolvent $R \subset E$ in the form

$$R = R_0 \dot{+} E_0,$$

where R_0 is a resolvent and $E_0 \subset E$ is a subspace.

If the subspace E_0 in the resolvent decomposition is nontrivial, i.e., $E_0 \neq \{\theta\}$, then the corresponding steady property is characterized by that the element $a \in R_0$ keeps its property under perturbations: $(a+b) \in R_0$, where b runs the space E_0.

Definition 5.1. The representation (7) is called *the decomposition of the resolvent* R, if E_0 is the maximal subspace. R_0 is then called *the main component* of the resolvent R.

Maximality of E_0 is understood in the following sense: for any representation

$$R = R_1 \dot{+} E_1,$$

where R_1 is a resolvent, E_1 is isomorphic to some subspace of E_0 (in particular, $E_1 \subset E_0$).

Definition 5.2. Two steady properties on E are called *similar*, if the main components of their resolvents coincide.

Remark. The analysis of the proof of Theorem 5.1 shows that the statement of the theorem remains true for the direct sum $R_0 \dot{+} R_1$ of disjunctive resolvents R_1 and R_2.

Theorem 5.2. *Let the functional μ generate a steady property on F with the resolvent R_μ and $A\colon E \to F$ be a nonzero bounded linear operator. Then the functional $\nu\colon E \to R^+$ determined by the equality*

$$\nu(a) = \|A\|^{-1}\mu(Aa),$$

generates on E the steady property with the resolvent

$$R_\nu = A^{-1}(R_\mu).$$

Proof. Positive homogeneity of the functional ν is obvious. Further, for any $a, b \in E$, we have

$$\nu(a+b) \leq \frac{1}{\|A\|}\big(\mu(Aa) + \|Ab\|\big) \leq \frac{1}{\|A\|}\mu(Aa) + \|b\| = \nu(a) + \|b\|.$$

Thus the functional ν generates a steady property on E.

Since

$$R_\nu = \{a \in E\colon \nu(a) > 0\} = \{a \in E\colon \mu(Aa) > 0\}$$

and $\mu(Aa) > 0$ only in the case $Aa \in R_\mu$, we have $R_\nu = A^{-1}(R_\mu)$. $\square$

Let us consider a corollary of Theorem 5.2 in the case where the space admits the representation in the form of the direct sum of subspaces.

Corollary. *Assume that*

1) $E = E_0 \oplus E_1$ *and* $P\colon E \to E$ *is a bounded linear projector on the subspace* E_1*;*

2) $\mu_0\colon E_1 \to R^+$ *is a generating functional with the resolvent* $R_1 \subset E_1$*.*

Then the functional

$$\mu(a) = \|P\|^{-1}\mu_0(Pa) \tag{8}$$

generates on E *a steady property with the resolvent* $R = E_0 \dot{+} R_1$*.*

Proof. The fact that $\mu(\cdot)$ is a generating functional follows from the Theorem 5.2. Therefore it suffices to check up the equality

$$P^{-1}(R_1) = E_0 \dot{+} R_1. \tag{9}$$

Since $R_1 \subset E_1$, it follows that $Pa_1 = a_1$ for every $a_1 \in R_1$ and $Pa_0 = \emptyset$ for every $a_0 \in E_0$. Hence

$$P^{-1}(R_1) = \{a \in E\colon Pa \in R_1\} = \{a_0 + a_1, a_0 \in E_0, Pa_1 \in R_1\} = E_0 \dot{+} R_1. \ \square$$

It should be noted that according to Definition 5.2 the steady property generated by the functional (8) is similar to the original one. On the other hand, it is possible to take advantage of this statement for construction of the generating functional of a steady property if the functional of a representative of the class of similar steady properties is known.

Let E_1 and E_2 be Banach spaces, $E = E_1 \times E_2$ with one of the norms of direct product. We will define projection operators $\Pi_i\colon E \to E_i$, $i = 1, 2$, by the equality

$$\Pi_1 z = a, \ \Pi_2 z = b, \quad z = (a, b).$$

Let us note that, at any choice of the norm on the space $E_1 \times E_2$, the inequalities

$$\|\Pi_1\| \leq 1, \quad i = 1, 2,$$

take place.

Theorem 5.3. *Let the functionals* μ_1 *and* μ_2 *generate on the spaces* E_1 *and* E_2 *steady properties with the resolvents* R_1 *and* R_2*, respectively. Then the functional* $\mu\colon E \to R^+$*,*

$$\mu(z) = \min\{\mu_1(\Pi_1 z), \mu_2(\Pi_2 z)\}, \tag{10}$$

generates on E *the steady property with the resolvent* $R_1 \times R_2$*.*

Proof. The fact that the functional μ determined by the equality (10) is a generating one can be checked directly. The inequality $\mu(z) > 0$ holds if and only if $\Pi_1 z \in R_1$ and $\Pi_2 z \in R_2$. Therefore the functional μ generates on $E = E_1 \times E_2$ a steady property with the resolvent $R_1 \times R_2$. $\square$

In view of the isomorphism of the spaces $E_1 \times E_2$ and $E_1 \oplus E_2$, the statement of Theorem 5.3 can be formulated for the case $E = E_1 \oplus E_2$.

Theorem 5.4. *Let the functionals μ_1 and μ_2 generate on the spaces E_1 and E_2 steady properties with the resolvents R_1 and R_2, respectively. Then the functional μ: $E = E_1 \oplus E_2 \to R^+$ defined by*

$$\mu(z) = \min\{\mu_1(z_1), \mu_2(z_2)\},$$

where $z = z_1 + z_2$, $z_i \in E_i$, $i = 1, 2$, generates on E a steady property with the resolvent $R_1 \dot{+} R_2$.

6. Continuation of the Steady Property by Parameter

This section consist of statements on conditions of the extension of the steady property by a parameter. The statement of a similar sort for the space $E = B(X, Y)$ of bounded linear operators is known in the literature as the Schauder theorem. Generalization of the Schauder theorem makes a basis of the method of continuation on parameter. The most simple case of dependence on a parameter is considered here. A confirmation of the importance of the general statements stated below is the use of them as applied to some concrete steady properties. We begin with the theorem, which is an analogue of the classical Schauder theorem.

Let the functionals $\mu_1(\cdot)$ and $\mu_2(\cdot)$ generate steady properties with the resolvents R_1 and R_2 respectively.

Put $\mu(a) = \min\{\mu_1(a), \mu_2(a)\}$.

Recall that the functional $\mu(\cdot)$ generates the steady property called the product of the original steady properties and has the resolvent $R = R_1 \cap R_2$.

Further for convenience the following notation is accepted

$$a(\lambda) = (1 - \lambda)a_0 + \lambda a_1,$$

where $a_0, a_1 \in E$ are fixed elements and $\lambda \in [0; 1]$ is a parameter.

Theorem 6.1. *Suppose that*

1) *$a_0 \in R$, $a_1 \in E$,*

2) *$\mu_1(a) \leq \mu_2(a)$ for all $a \in R$,*

3) *There is a constant $m > 0$ such that $\mu_1(a(\lambda)) \geq m$ for $\lambda \in [0; 1]$.*

Then $a(1) = a_1 \in R$.

Proof. The condition 3) implies that all the elements $a(\lambda) = (1-\lambda)a_0 + \lambda a_1$, $\lambda \in [0; 1]$, belong to R_1. Will show that the same segment belongs to the resolvent R_2. From this the required statement will follow, namely,

$$a_1 = a(1) \in R_1 \cap R_2 = R.$$

Put $\delta = \frac{m}{2}\|a_1 - a_0\|^{-1}$. For any $\lambda \in [0; \delta]$ the estimate

$$\mu_2(a(\lambda)) = \mu_2(a_0 + \lambda(a_1 - a_0)) \geq \mu_2(a_0) - |\lambda| \, \|a_1 - a_0\|$$

takes place.

Since $a_0 \in R$, by virtue of the condition 2) we have

$$\mu_1(a_0) \leq \mu_2(a_0).$$

Taking into account this inequality and the condition 3), we continue estimating:

$$\mu_2(a(\lambda)) \geq \mu_1(a_0) - \delta\|a_1 - a_0\|^{-1} \geq m - \frac{m}{2} = \frac{m}{2}.$$

Thus it is proved that for any $\lambda \in [0;\delta]$ the elements $a(\lambda)$ belong to R_2.

In case $\delta \geq 1$ this completes the proof of the theorem. If $\delta < 1$, we consider the segment $[\delta; 2\delta]$ for the parameter λ. Preliminary note the following. Since $a(\delta) \in R_1 \cap R_2 = R$, there takes place

$$\mu_2(a(\delta)) \geq \mu_1(a(\delta)) \geq m.$$

For $\lambda \in [\delta; 2\delta]$, we have

$$\begin{aligned} \mu_2(a(\lambda)) &= \mu_2(a(\delta) + (\lambda - \delta)(a_1 - a_0)) \geq \\ &\geq \mu_2(a(\delta)) - |\lambda - \delta| \, \|a_1 - a_0\| \geq m - \delta\|a_1 - a_0\| = \frac{m}{2}. \end{aligned}$$

Thus $a(\lambda) \in R_2$ for all $\lambda \in [\delta; 2\delta]$. If $2\delta \geq 1$, then the statement is proved. In case $2\delta < 1$ we use the above procedure several times until $n\delta$ becomes greater than 1. It is obvious that this will require a finite number of steps. □

Let us emphasize that the essential role in the proof of Theorem 6.1 is played by that the estimate $0 < m \leq \mu_1(a(\lambda))$ is uniform in $\lambda \in [0;1]$. The analysis of the proof shows that if the condition 1) is replaced by 1a): $a(\lambda) \in R$ for some $\lambda \in [0;1]$, $a_1 \in E$, then the statement of the theorem is still true. The same remark concerns the following theorem.

Theorem 6.2. *Let the functionals $\mu_1(\cdot)$ and $\mu_2(\cdot)$ generate steady properties corresponding to the resolvents R_1 and R_2. Suppose that*

1) *$a_0 \in R_1$, $a_1 \in E$;*
2) *$\mu_1(a) \leq \mu_2(a)$, $a \in E$;*
3) *There exists a constant $m > 0$ such that $\mu_2(a(\lambda)) \geq m$ for all $\lambda \in [0;1]$;*
4) *There exists a constant $0 < c < 1$ such that $c\mu_2(a) \leq \mu_1(a)$ for all $a \in R_1$.*

Then $a(1) = a_1 \in R_1$.

Proof. At first note that $a(1) = a_1 \in R_2$ and $R_1 \subset R_2$ by virtue of the conditions 2) and 3). Put $\delta = \frac{cm}{2}\|a_1 - a_0\|^{-1}$. For any $\lambda \in [0;\delta]$, we have

$$\mu_1(a(\lambda)) = \mu_1(a_0 + \lambda(a_1 - a_0)) \geq \mu_1(a_0) - |\lambda| \, \|a_1 - a_0\|.$$

From this inequality, in view of the condition 4), we obtain

$$\mu_1(a(\lambda)) \geq cm - \frac{cm}{2} = \frac{cm}{2}.$$

Thus $a(\lambda) \in R_1$ for all $\lambda \in [0;\delta]$. Further arguments are the same as in the proof of Theorem 6.1. $\square$

In conclusion we formulate a corollary of Theorem 6.2 which is of interest in comparison with the statement of Theorem 6.1.

Corollary. *Let the functionals $\mu_1(\cdot)$ and $\mu_2(\cdot)$ generate steady properties corresponding to the resolvents R_1 and R_2, and R is the resolvent of the steady property generated by the functional*

$$\mu(a) = \min\{\mu_1(a), \mu_2(a)\}.$$

Let the following conditions be fulfilled:
1) *$a_0 \in R$, $a_1 \in E$;*
2) *There is a constant $0 < c < 1$ such that $c\mu_1(a) \le \mu(a)$ for all $a \in R$;*
3) *There is a constant $m > 0$ such that $\mu_1(a(\lambda)) \ge m$ for all $\lambda \in [0;1]$.*
Then $a(1) = a_1 \in R$.

It is interesting that the statements of Theorem 6.1 and Corollary coincide under the following condition:

$$\mu_1(a) = \mu_2(a) \quad \text{for} \quad a \in R.$$

The Schauder theorem mentioned in the beginning of this section is characterized just by this situation.

7. The Spectrum of an Element with Respect to a Steady Property

Let E be a Banach space over the field $\mathbf{K}$, $R \subset E$ be the resolvent of a nontrivial steady property and μ be the corresponding generating functional.

Let us fix an element $a_0 \in R$ and let us denote

$$\mu_0 = \mu(a_0) > 0.$$

Definition 7.1. *The spectrum* of the element $a \in E$ with respect to the pair (a_0, μ) is the set

$$\sigma(a,\mu) = \{\lambda \in \mathbf{K}\colon \mu(\lambda a_0 - a) = 0\} = \{\lambda \in \mathbf{K}\colon \lambda a_0 - a \in q\ker\mu\}.$$

In cases where the pair (a_0, μ) is fixed, we will use the notation $\sigma(a)$ instead of $\sigma(a,\mu)$. If the complex Banach space E is a Banach algebra with the unit $e = a_0$ and the resolvent R unites invertible elements of E, then Definition 7.1 coincides with the classical definition of the spectrum of an element of the Banach algebra.

Theorem 7.1. *For any $a \in E$, the inclusion*

$$\sigma(a) \subset \{\lambda \in \mathbf{K}\colon \lambda_1 \le |\lambda| \le \lambda_2\} \tag{11}$$

holds, where $\lambda_1 = \mu(a)\|a_0\|^{-1}$, $\lambda_2 = \mu_0^{-1}\|a\|$.

Proof. Denote the set in the right-hand side of (11) by S. Show that, for any $\lambda \in \mathbf{K} \setminus S$, the inclusion $a(\lambda) = \lambda a_0 - a \in R$ takes place. Thus the inclusion $\sigma(a) \subset S$ will be established.

Let us assume that $|\lambda| < \lambda_1 = \mu(a)\|a_0\|^{-1}$. By virtue of the properties of the generating functional, we have

$$\mu(a(\lambda)) = \mu(\lambda a_0 - a) \geq \mu(a) - |\lambda|\,\|a_0\| > \mu(a) - \lambda_1\|a_0\| = \mu(a) - \mu(a) = 0.$$

Thus, for any $|\lambda| < \lambda_1$, we have $a(\lambda) \in R$.

For $|\lambda| > \lambda_2 = \mu_0^{-1}\|a\|$, we obtain $\mu(a(\lambda)) = \mu(\lambda a_0 - a) \geq |\lambda|\mu(a_0) - \|a\| > 0$. Hence $a(\lambda) \in R$. $\square$

If $\mathbf{K} = \mathbf{C}$, i.e., E is a complex space, the set S represents a circular ring with radiuses λ_1 and λ_2. Besides $\lambda_1 > 0$ only in the case where $a \in R$. If the element a does not possess the steady property, then by Theorem 7.1 the spectrum is a part of a circle of radius $\lambda_2 > 0$. The proved statement has an analogue in the classical spectral theory. This is true for the following statement as well.

Theorem 7.2. *The spectrum of any element is a compact set, i.e., a bounded and closed subset of* $\mathbf{K}$.

Proof. The boundedness of the spectrum follows from Theorem 7.1. Next we will prove that $\sigma(a)$ is closed for any fixed element $a \in E$. Define the operator $T\colon \mathbf{K} \to E$ by the equality

$$T\lambda = a(\lambda) = \lambda a_0 - a.$$

Since $\|T\lambda_1 - T\lambda_2\| = |\lambda_1 - \lambda_2|\,\|a_0\|$, the operator T is continuous. By definition of the spectrum

$$\sigma(a) = T^{-1}(q\ker\mu) = T^{-1}(E \setminus R).$$

As it was shown above, the quasi-kernel of the generating functional is closed. Hence the spectrum $\sigma(a)$ is a closed set, as an inverse image of the closed set under a continuous mapping T. This completes the proof of the theorem. $\square$

Let us remind an analogous statement for the case where $E = B(X, X)$ is a complex Banach space of bounded linear operators $L\colon X \to X$, $R \subset B(X, X)$ is the set of all continuously invertible operators (i.e., in this case the steady property is the invertibility of the operator) and $a_0 = I$ is the identity operator. As is known, the spectrum $\sigma(L)$ is a nonempty closed bounded set.

Note that in the case of an arbitrary nontrivial steady property the statement about nonemptiness of the spectrum does not take place in general.

Further we will need the following auxiliary statement.

Let $M \subset \mathbf{K}$ be a subset and $\varepsilon > 0$. Denote by $U(M, \varepsilon)$ the ε-neighborhood of M, i.e.,

$$U(M, \varepsilon) = \{\alpha \in \mathbf{K}\colon d(M, \alpha) < \varepsilon\},$$

where $d(M, \alpha) = \inf\{|\alpha - \lambda| : \lambda \in M\}$ is the distance of the element α from the set M.

Lemma. *Let μ be the generating functional of a steady property with the resolvent $R \subset E$. Then for every $\varepsilon > 0$ and any $a \in E$ the inequality*

$$0 < b(\varepsilon) \stackrel{def}{=} \inf\{\mu(\lambda a_0 - a) : \lambda \in \mathbf{K} \setminus U(\sigma(a), \varepsilon)\}$$

holds.

Proof. First of all, we note that the set $U(\sigma(a), \varepsilon)$ is open as an ε-neighborhood of the closed set $\sigma(a)$. Hence $\mathbf{K} \setminus U(\sigma(a), \varepsilon)$ is closed.

Let us consider the function $\varphi \colon \mathbf{K} \to R^+$ determined by the equality

$$\varphi(\lambda) = \mu(\lambda a_0 - a).$$

If $\varphi(\lambda) = 0$, then $\lambda a_0 - a \subset \sigma(a)$. Therefore the function $\varphi(\cdot)$ takes the zero value only at $\lambda \in \sigma(a)$. Hence $\phi(\lambda) > 0$ for any $\lambda \in \mathbf{K} \setminus U(\delta(a), \varepsilon)$. Besides, the function $\varphi(\cdot)$ is continuous. This follows from the inequality

$$\begin{aligned} |\varphi(\lambda_1) - \varphi(\lambda_2))| &= |\mu(\lambda_1 a_0 - a) - \mu(\lambda_2 a_0 - a)| \le \\ &\le \|(\lambda_1 a_0 - a) - (\lambda_2 a_0 - a)\| = |\lambda_1 - \lambda_2| \, \|a_0\|. \end{aligned}$$

Finally, for the function $\varphi(\cdot)$ the estimate $\varphi(\lambda) \ge |\lambda| \, \|a - 0\| - \|a\|$ takes place. Therefore $\varphi(\lambda) \to +\infty$ as $|\lambda| \to \infty$.

Thus the function $\varphi(\cdot)$ has the following properties:

a) $\varphi(\lambda) > 0$ for $\lambda \in \mathbf{K} \setminus U(\sigma(a), \varepsilon)$,

b) $\varphi(\cdot)$ is continuous and

c) $\varphi(\lambda) \to +\infty$ as $|\lambda| \to \infty$.

Now it suffices to notice that the function with the properties a), b) and c) has the strictly positive infimum on the closed set $\mathbf{K} \setminus U(\sigma(a), \varepsilon)$. $\square$

This lemma allows us to establish the following statement describing the property of upper semicontinuity of the spectrum with respect to a steady property.

Theorem 7.3. *For any $a \in E$ and $\varepsilon > 0$, there exists $\delta = \delta(a, \varepsilon) > 0$ such that under the condition $\|a - b\| < \delta$ the inclusion*

$$\sigma(b) \subset U(\sigma(a), \varepsilon) \tag{12}$$

takes place.

Proof. Let us fix an arbitrary element $a \in E$ and a number $\varepsilon > 0$. To prove the inclusion (12), it suffices to show the existence of $\delta > 0$ such that both $\|a - b\| < \delta$ and $\lambda \in \mathbf{K} \setminus U(\sigma(a), \varepsilon)$ imply $\mu(\lambda a_0 - b) > 0$, where μ is some generating functional of the given steady property. Let $b(\varepsilon) > 0$ be the number related to the given $\varepsilon > 0$, the existence of which is provided by the Lemma. We make choice of δ so that $0 < \delta < b(\varepsilon)$. Then

$$\mu(\lambda a_0 - b) = \mu(\lambda a_0 - a + a - b) \ge \mu(\lambda a_0 - a) - \|a - b\| \ge b(\varepsilon) - \delta > 0.$$

Hence it follows that $\lambda \notin \sigma(b)$. Due to the above remark, the theorem is proved. $\square$

Theorem 7.3 admits the following interpretation. Let $2^{\mathbf{K}}$ be the space of closed bounded subsets of $\mathbf{K}$. In situation when a concrete steady property μ is considered with the resolvent $R \subset E$ and an element $a_0 \in R$ is fixed, i.e., the pair (a_0, μ) is fixed, the spectrum defines the multi-valued mapping

$$F\colon E \to 2^{\mathbf{K}}, \quad Fa = \sigma(a),$$

which, to each element $a \in E$ puts into correspondence the spectrum $\sigma(a) \in 2^{\mathbf{K}}$. The theorem says that such a mapping is upper semicontinuous.

For convenience, in the following statement, a steady property and its generating functional are denoted by the same symbol.

Theorem 7.4. *Let μ_1 and μ_2 be steady properties on E with the resolvents R_1 and R_2, and $a_0 \in R_1 \cap R_2$. Then the following statements take place:*
1) $\sigma(a, \mu_1 + \mu_2) = \sigma(a, \mu_1) \cap \sigma(a, \mu_2)$;
2) $\sigma(a, \mu_1\mu_2) = \sigma(a, \mu_1) \cup \sigma(a, \mu_2)$;
3) if $\mu_1 \le \mu_2$, then $\sigma(a, \mu_1) \subset \sigma(a, \mu_2)$.

Proof. Prove the first statement of the theorem. Let the steady properties μ_1, μ_2 be generated by the functionals $\mu_1(\cdot)$ and $\mu_2(\cdot)$ respectively (not necessary canonical). Then the sum $\mu_1 + \mu_2$ is generated by the functional

$$\mu(a) = \max\{\mu_1(a), \mu_2(a)\}.$$

$$\begin{aligned}\sigma(a, \mu_1 + \mu_2) =& \{\lambda \in K\colon\ \max\{\mu_1(\lambda a_0 - a), \mu_2(\lambda a_0 - a)\} = 0\} = \\ =& \{\lambda \in K\colon \mu_1(\lambda a_0 - a) = 0\} \cap \{\lambda \in K\colon \mu_2(\lambda a_0 - a) = 0\} = \\ =& \sigma(a, \mu_1) \cap \sigma(a, \mu_2).\end{aligned}$$

The first statement of the theorem is proved. The second statement can be proved similarly. Now pass to the proof of the statement 3). Let $\mu_1(\cdot)$ and $\mu_2(\cdot)$ be the canonical generating functionals of the steady properties μ_1 and μ_2, respectively. Then by Theorem 4.3 the inequality

$$\mu_2(a) \le \mu_1(a), \quad a \in E,$$

holds. For any $\lambda \in \sigma(a, \mu_1)$, we have $0 = \mu_1(\lambda a_0 - a) \ge \mu_2(\lambda a_0 - a)$. This implies $\mu_2(\lambda a_0 - a) = 0$, i.e., $\lambda \in \sigma(a, \mu_2)$. $\square$

Let us note that the full description of the spectrum of an element in concrete situations is a quite difficult problem. However, to answer some questions, it suffices to know some bounds of the spectrum. Bounds of the spectrum of an arbitrary element are established according to Theorem 7.1. The unimprovable estimate of the spectrum is connected to the following notion.

Definition 7.2. By the *spectral radius* of the element $a \in E$ (with respect to the fixed pair (a_0, μ)) we mean a non-negative number $r(a)$ determined by the following equality $r(a) = \max\{|\lambda|\colon \lambda \in \sigma(a)\}$.

Thus the spectral radius of the element $a \in E$ coincides with the minimal radius of the closed ball in K with the center in zero which contains $\sigma(a)$, therewith

$$r(a) \leq \frac{\|a\|}{\mu(a_0)}, \tag{13}$$

where μ is some generating functional of the given steady property. The number in the right-hand side of the inequality (13) is an upper estimate of the spectral radius. This estimate becomes exact if the generating functional is canonical.

If the spectral radius, $r(a)$, is known or its upper estimate, r_0, is known, then it is possible to make the conclusion: if λ is such that $|\lambda| > r(a)$ (or $|\lambda| > r_0$), then the element $(\lambda a_0 - a)$ belongs to R, i.e., it possesses the given steady property. This conclusion usually is useful in applications.

For concrete steady properties it is possible to establish effective formulas for calculation of the spectral radius and its estimates which can be useful due to specificity of the examined steady property.

If a generating functional $\mu(\cdot)$ of a steady property is known, then

$$r(a) = \max\{|\lambda| \colon \mu(\lambda a_0 - a) = 0\}.$$

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