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# VECTOR OPTIMIZATION TOOLS IN ASYMPTOTIC THEORY OF TOTAL DIFFERENTIAL EQUATIONS

**Abstract.** We apply the basic concepts of vector optimization theory to certain problems arising in the asymptotic theory of total differential equations. This approach enables us to establish interrelations between characteristic functionals and characteristic exponents, which are the main asymptotic characteristics of solutions of these equations. We are also enabled to construct a set of proper characteristic functionals with a number of useful properties. Using these results, we study the structure of characteristic sets, the properties of weakly regular linear total differential equations, and the behavior of characteristic functionals of equations under exponentially decreasing perturbations.

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**Key words and phrases**: Total differential equation, characteristic exponent, characteristic functional, proper efficiency.

**რეზიუმე.** ოპტიმიზაციის ვექტორული თეორიის ძირითად ცნებებს ჩვენ ვიყენებთ ტოტალური დიფერენციალური განტოლებების ასიმპტოტური თეორიის ზოგიერთი ამოცანის შესასწავლად. ეს მიდგომა საშუალებას გვაძლევს დავადგინოთ ურთიერთკავშირები მახასიათებელ ფუნქციონალებსა და მახასიათებელ მაჩვენებლებს შორის, რომლებიც ამ განტოლებების მთავარ ასიმპტოტურ მახასიათებლებს წარმოადგენს. ჩვენ აგრეთვე შესაძლებლობა გვეძლევა ავაგოთ საკუთრივი მახასიათებელი ფუნქციონალები მთელი რიგით სასარგებლო თვისებებისა. ამ შედეგების გამოყენებით ჩვენ შევისწავლით მახასიათებელი სიმრავლეების სტრუქტურას, სუსტად რეგულარული წრფივი ტოტალური დიფერენციალური განტოლებების თვისებებს, აგრეთვე განტოლებების მახასიათებელი ფუნქციონალების ყოფაქცევას ექსპონენციალურად კლებადი შეშფოთებებისას.

## 1. Introduction

Let $E$ and $F$ be real Banach spaces, $U$ be a connected open subset of $E$, and $L(E, F)$ be the Banach space of all linear continuous mappings from $E$ to $F$. We supply $L(E, F)$ with the operator norm. The set of all invertible elements of $L(F, F)$ is denoted by $GL(F)$. In this paper, we consider linear total differential equations (TDE) of the form

$$y'h = A(x)hy, \quad y \in F, \quad x \in U, \quad h \in E, \tag{1.1}$$

where the derivative $y'$ is the Frechét derivative with respect to $x \in U$ and the coefficient $A : U \to L(E, L(F, F))$ is continuous and bounded on $U$. We put $M := \sup_{x \in U} \|A(x)\|$. If $E$ is finite-dimensional, the equation (1.1) can be rewritten as follows

$$\begin{gathered} dy = A_1(x)ydx_1 + \ldots + A_m(x)ydx_m, \\ y \in F, \quad x \in U \subset E = \mathbb{R}^m, \end{gathered} \tag{1.2}$$

where the coefficients $A_i : U \to L(F, F)$ are continuous and bounded. Note that $h \in E$ in (1.1) corresponds to $dx_1, \ldots, dx_m$ in (1.2). In what follows, we suppose that all TDEs under consideration are completely integrable (see Section 2 below).

The regular way to obtain a linear total differential equation (1.1) is to linearize some nonlinear TDE

$$y' = f(x, y), \quad y \in D \subset F, \quad x \in U, \tag{1.3}$$

where $f : U \times D \to L(E, F)$ should be at least continuous. Since total differential equations are very similar to ordinary differential equations in their basic properties, some analogue of the Lyapunov exponents theory is therefore possible (and needed).

Foundations of this new theory were laid by E. I. Grudo in 1974. In his paper [18], he defined the key notion of characteristic vector (functional, in fact) and investigated basic properties of these objects in the case $\dim E < +\infty$. These results were completely published in [19] and [20].

In the subsequent years, the leading role in the asymptotic theory of TDEs was played by Byelorussian mathematicians. This theory was developed by I. V. Gaĭshun, E. I. Grudo, and M. V. Kozhero, as well as N. E. Bolshakov, P. T. Lasyĭ, L. F. Yanchuk, P. P. Potapenko, and others. By virtue of their efforts, the theory of characteristic functionals and exponents of TDEs was created as a very non-trivial generalization of the Lyapunov characteristic exponents theory.

Now the following asymptotic characteristics are mainly used for solutions of TDEs: strong exponents [39], (weak) characteristic exponents [39; 15, p. 115], and characteristic functionals (vectors) [18; 20; 15, p. 108; 11, p. 82]. Each of these notions is a straightforward generalization of the classical Lyapunov exponent and coincides with it when $E = \mathbb{R}$.

Strong exponents and weak characteristic exponents were introduced by M. V. Kozhero in [39]. Characteristic vectors were defined by E. I. Grudo for $E = \mathbb{R}^n$ and $K = \mathbb{R}^n_+$. Characteristic functionals are a generalization of characteristic vectors to an arbitrary Banach space $E$ with an appropriate closed convex cone $K$. This notion was introduced and investigated by I. V. Gaĭshun in [10]. He also obtained a number of fundamental results in the field including the development of Floquet theory for TDEs with periodic coefficients in infinite dimensional spaces [7, 8, 9, 13, 14]; construction of the Lyapunov classification of linear TDEs and profound investigation of regular and reducible TDEs [1, 10, 11, 12, 16, 15]; a proof of the theorem on stability by regular linear approximation in an infinite-dimensional space [16]; establishing relations between geometric properties of the ordering cone and uniqueness of characteristic functionals [16, 15].

Some of these issues were considered by E. I. Grudo [18–23] in the case $E = \mathbb{R}^2$. Different variants of the theorem on stability by an irregular linear approximation were proved by P. T. Lasyĭ [48], N. E. Bolshakov and P. P. Potapenko [5]. E. I. Grudo [21] and P. G. Lasyĭ [47] estimated the range of characteristic functionals of TDE under small perturbations. Certain problems closely related to the asymptotic theory of TDEs were also treated by D. A. Bože [3], I. P. Karkliņš [38], A. D. Myshkis [4, 60], A. I. Perov [62, 63], L. E. Reiziņš [66] and others.

This stage of investigations has ended at the beginning of 90th. The obtained results are summarized by I. V. Gaĭshun in the monographs [11] and [15], where the general and asymptotic theories of TDEs are systematically presented.

Recently N. A. Izobov together with his co-workers A. S. Platonov and E. N. Krupchik started systematic investigation of upper and lower characteristic and power characteristic sets of TDEs, see [28, 29, 30, 33, 34, 35, 36, 44]. Some results on central characteristic vectors were also obtained by P. P. Potapenko [64].

A brief inquiry into the matter shows that a good deal of difficulties arising in the asymptotic theory of TDEs is due to the complicated nature of asymptotic characteristics used in that theory. The aim of this paper is to demonstrate that some concepts of vector optimization theory can be fruitfully applied to studying characteristic functionals and exponents of solutions to TDEs.

The paper is organized in the following way. In Section 2 we recall some preliminary notions and results. In Section 3 we consider the relation between characteristic functionals and characteristic exponents of solutions to TDEs. In Section 4 the notion of proper characteristic functional is discussed. The paper is ended by conclusions.

## 2. Preliminaries

Here we recall and discuss some notions and results from the general theory of TDEs as well as from convex analysis and vector optimization theory, which will be necessary in the sequel. Detailed presentation of most of these issues can be found in [17, 42, 43, 65, 67, 37, 61], and also in [11, 15].

**Completely integrable total differential equations.** The system

$$\frac{\partial y}{\partial x_i} = A_i(x)y, \quad i = 1, \dots, m, \tag{2.1}$$

equivalent to (1.2), consists of $nm$ equations in $n$ unknown scalar functions $y_i$ being the components of $y$. This means that (2.1) is overdetermined and, therefore, (1.2) must satisfy some strongly restrictive conditions in order to have a rich collection of solutions. In the general case of (1.1), the same conclusion is the more so valid.

Let $B_F(0,R) := \{y \in F : \|y\| < R\}$ with some $R > 0$. Consider the nonlinear equation

$$y'h = f(x,y)h, \quad x \in U, \quad y \in F, \quad h \in E, \tag{2.2}$$

with a continuous right hand $f : U \times B_F(0,R) \to L(E,F)$ such that $f(x,0) = 0$ for all $x \in U$.

**Definition 2.1.** The equation (2.2) is said to be completely integrable at $(x_0, y_0) \in U \times B_F(0,R)$ if there exists a solution $y$ of the Cauchy problem for (2.2) with initial data $y(x_0) = y_0$ defined and bounded in some neighborhood of $x_0$. The equation (2.2) is said to be completely integrable on $G \subset U \times B_F(0,R)$ if it is completely integrable at each $x_0 \in G$.

The common way to prove complete integrability of (1.1) is to use infinitesimal sufficient conditions due to Frobenius, Perov and some others (see [15, p. 160; 24, p. 357; 62]).

Suppose that

$$f(x,y)h = Q(x)hy + \varphi(x,y),$$

where $Q : U \to L(E, L(F,F))$ is continuous and $\varphi$ satisfies the condition $r(x,y) := \|\varphi(x,y)\|/\|y\| \to 0$ as $y \to 0$ for each given $x \in U$. Then the equation

$$y'h = P(x)yh, \quad x \in U, \quad y \in F, \quad h \in E, \tag{2.3}$$

is a linear approximation of (2.2) along the trivial solution $y = 0$.

In order to use (2.3) within the standard scheme of the stability theory, we heve to be sure that (2.3) is completely integrable. If $f$ is $C^2$, then the required assertion follows from [3]. Some less restrictive conditions were obtained in [58, 57].

**Theorem 2.1** ([58, 57]). *If* (i) (2.2) *is completely integrable on* $U \times B_F(0,R)$, (ii) $f$ *is* $C^1(U)$, *and* (iii) $r$ *is finally bounded on* $U$ *uniformly in* $y \in B_F(0,R)$, *i.e., any* $x_0 \in U$ *has a neighborhood* $V$ *such that* $r$ *is bounded on* $V \times B_F(0,R)$, *then* (2.3) *is completely integrable on* $U \times F$.

**Vector optimization and convex analysis.** Let $X$ be a Banach space and $X^*$ be its topological dual. A set $K \subset X$ is said to be a cone if

$K \supset tK$ for each $t > 0$. In this subsection we denote cones by $K$ and an arbitrary subsets of $X$ by $Q$. The conical hull of arbitrary set $Q \subset X$ or the cone generated by $Q$ is the set $\operatorname{cone} Q := \{tx\colon x \in Q,\, t \geq 0\}$. Note that $0 \in \operatorname{cone} Q$ by definition. The convex hull of $Q$ is the least (with respect to inclusion) set $\operatorname{conv} Q$ containing $Q$. We use the notation $\overline{\operatorname{conv}}\, Q := \operatorname{cl} \operatorname{conv} Q$. If $Q$ is a cone, then $\operatorname{cl} Q$, $\operatorname{conv} Q$, and $\overline{\operatorname{conv}}\, Q$ are cones too.

For a mapping $f : Q \to \mathbb{R}$, the epigraph of $f$ is the set $\operatorname{epi} f := \{(x, s) \in Q \times \mathbb{R} : s \geq f(x)\}$. A mapping $f : K \to \mathbb{R}$ is said to be positively homogeneous if $f(tx) = tf(x)$ for each $x \in K$ and $t > 0$ or, equivalently, if $\operatorname{epi} f$ is a cone in $X \times \mathbb{R}$. A mapping $f : Q \to \mathbb{R}$ is convex iff $\operatorname{epi} f$ is convex. In what follows, we suppose that all convex functions are defined everywhere on $X$ and, therefore, we put $f(x) = +\infty$ for $x \notin Q$. The domain of a convex mapping $f$ is the projection of $\operatorname{epi} f$ onto $X$, i.e. $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$.

The closed convex hull of an arbitrary mapping $f : Q \to \mathbb{R}$ is the mapping $\overline{\operatorname{conv}}\, f : X \to \mathbb{R}$ defined by the condition $\operatorname{epi} \overline{\operatorname{conv}}\, f = \overline{\operatorname{conv}} \operatorname{epi} f$. According to the above, we assume $\overline{\operatorname{conv}}\, f(x) = +\infty$ for $x \notin \operatorname{cl} Q$.

A continuous linear functional $\mu \in X^*$ is said to be a subgradient of a convex function $f : X \to \mathbb{R}$ at $x \in X$ if $f(z) \geq f(x) + \mu(z - x)$ for all $z \in X$. The subdifferential of $f$ at $x \in X$ is the set $\partial f(x)$ containing all the subgradients of $f$ at $x$.

If the mapping $f : K \to \mathbb{R}$ is convex and positively homogeneous, then $\mu \in \partial f(x_0)$ for some $x_0 \in K$ iff $\mu x \leq f(x)$ for all $x \in K$ and $\mu x_0 = f(x_0)$.

Suppose that there exists an affine mapping majorizing the mapping $f$, i.e. $f(x) \leq \mu(x - x_0)$ for each $x \in X$ with some $\mu \in X^*$ and $x_0 \in X$. Then the set

$$\partial^{\geqslant} f(x) = -\partial(\overline{\operatorname{conv}}(-f))(x)$$

if defined for any $x \in \operatorname{dom} f$. We will refer to the set $\partial^{\geqslant} f(x)$ as the Penot superdifferential of $f$ at $x$. It should be stressed that the Penot superdifferential $\partial^{\geq} f$ considered in [26] does not coincide with $\partial^{\geqslant} f(0)$ in general. However, we have $\partial^{\geq} f(0) = \partial^{\geqslant} f(0)$ for any positively homogeneous $f$.

A cone $K$ is convex iff $K + K \subset K$. A convex cone $K$ is said to be pointed if $K \cap (-K) = \{0\}$.

For any given pointed convex cone $K$, we can define the following binary relation: $x \preceq y$ iff $y - x \in K$. It can be easily seen that $\preceq$ is a partial order.

A point $x \in Q \subset X$ is called maximal in $Q$ with respect to $K$ if $(x + K) \cap Q = \{x\}$. The set of all such points of $Q$ is denoted by $\operatorname{Max}(Q|K)$.

Suppose that $X$ is partially ordered by a closed convex pointed cone $K$. A continuous linear functional $f \in X^*$ is called positive if $f(x) \geq 0$ for any $x \in K$, and it is called strictly positive if $f(x) > 0$ for any $x \in K \setminus \{0\}$. The set of all positive elements $f \in X^*$ is called the dual cone of $K$. We denote it by $K^+$. The set of all strictly positive elements $f \in X^*$ is denoted by $K^{+i}$. A continuous linear functional $f \in X^*$ is called uniformly positive if there exists a positive number $c_f$ such that $f(x) \geq c_f \|x\|$ for any $x \in K$.

For any $K$ its dual cone is closed and convex. If $K$ is solid, then $K^+$ is pointed and, therefore, $X^*$ is partially ordered by $K^+$.

By $\operatorname{Pos}(Q|\Lambda)$ we denote the set of $x \in Q$ such that $f(x) = \max_{y \in Q} f(y)$ for some $f \in \Lambda \subset X^*$. If the set $Q$ is convex, then by linear scalarization we have $\operatorname{Pos}(Q|K^+ \setminus \{0\}) \supset \operatorname{Max}(Q|K)$ when $\operatorname{Int} K^+ \neq \varnothing$ and $\operatorname{Max}(Q|K) \supset \operatorname{Pos}(Q|K^{+i})$ when $K^{+i}$ is not empty. The elements of $\operatorname{Pos}(Q|K^{+i})$ are called positive proper efficient (maximal) elements of $Q$.

A convex set $B \subset X$ is said to be a base of some convex cone $K$ if $0 \notin \operatorname{cl} B$ and $K = \operatorname{cone} B$. The cone $K$ has a base iff the set $K^{+i}$ is not empty. If the cone $K$ has a closed bounded base, then $K$ is closed and pointed.

The following statements are equivalent:

(i) the cone $K$ has a bounded base;

(ii) there exists a uniformly positive functional on $K$.

Note that $\{x \in K\colon f(x) = 1\}$ is a bounded base of $K$ for any uniformly positive $f \in X^*$. In a finite-dimensional $X$, the conditions (i) and (ii) are valid for each pointed closed convex cone $K$.

## 3. Relations between Asymptotic Characteristics of Solutions

In this section, we study relations between two main notions used in the asymptotic theory of TDEs and derive some consequences from those relations.

**Characteristic exponents and functionals of mappings.** Let $E$ be a Banach space partially ordered by a closed convex pointed cone $K$ with a bounded base. By $\mathfrak{F}$ we denote the filter on $K$ generated by the sets $K \setminus B$, where $B$ is an arbitrary bounded subset of $E$. Further we suppose that the domain $U$ contains some element $D \in \mathfrak{F}$.

Take any $f : U \to \mathbb{R}$ such that $f(x) \geq 0$ for all $x$ from some $D_f \in \mathfrak{F}$, $D_f \subset U$. The following definitions are basic in our considerations.

**Definition 3.1** ([39; 15, p. 115])**.** The (weak) characteristic exponent of $f$ is the function $\chi[y] : K \setminus \{0\} \to \mathbb{R}$ defined by

$$\chi[f](x) := \overline{\lim_{t \to +\infty}} \frac{1}{t\|x\|} \ln f(tx).$$

**Definition 3.2** ([18; 20; 15, p. 108; 11, p. 82])**.** A functional $\lambda \in E^*$ is said to be a characteristic functional of $f$ if

$$\limsup_{\mathfrak{F}} \|x\|^{-1}(\lambda x + \ln f(x)) = 0$$

and

$$\limsup_{\mathfrak{F}} \|x\|^{-1}(\lambda x + \mu x + \ln f(x)) > 0$$

for all $\mu \in K^+$, $\mu \neq 0$.

The set of all characteristic functionals is called the characteristic set of $f$. We denote it by $\mathcal{M}[f]$. Both characteristic exponents and functionals are straightforward generalizations of Lyapunov exponents and coincide with them when $E$ is finite-dimensional.

If $\ln f$ is Lipschitzian, i.e., satisfies the condition

$$|\ln f(x) - \ln f(y)| \le L\|x - y\| \tag{3.1}$$

for all $x, y \in D_f$, then it follows from [15, pp. 111, 116] that $f$ has the continuous and bounded on $K \setminus \{0\}$ (weak) characteristic exponent and there exists at least one characteristic functional of $f$.

It should be noted that the function $\chi[f]$ has some anomalous properties. First of all, $\chi[f]$ depends on the norm in $E$, specifically, replacing the norm $\|\cdot\|$ by another norm $\|\cdot\|_1$ in $E$ we get $\chi_1[f] = \|x\|\chi[f]/\|x\|_1$. Furthermore, the function $\chi[f]$ is Lipschitzian on $E\setminus\{0\}$, but we can not define it at $x = 0$ preserving continuity if $\chi_1[f]$ is not a constant. To avoid these problems, the following definition was introduced in [52].

**Definition 3.3.** The modified characteristic exponent of $f$ is the function $\psi[f] : K \to \mathbb{R}$ such that $\psi[f](0) = 0$ and

$$\psi[f](x) := \varlimsup_{t\to+\infty} \frac{1}{t} \ln f(tx) \tag{3.2}$$

for $x \ne 0$.

It should be noted that $\psi[f](x) = \|x\|\chi[f](x)$ for $x \ne 0$ and, therefore, $\psi[f](x)$ does not depend on the norm in $E$. Useful properties of $\psi[f](x)$ are given in the following statement.

**Lemma 3.1** ([52]). *If $f$ satisfies the condition* (3.1)*, then $\psi[f]$ is positively homogeneous and satisfies the Lipschitz condition $|\psi(x) - \psi(y)| \le L\|x - y\|$ everywhere on $K$.*

*Proof.* For all $s > 0, x \ne 0$ we have

$$\psi[f](sx) = \varlimsup_{t\to+\infty} \frac{1}{t} \ln f(tsx) = \varlimsup_{t\to+\infty} \frac{s}{t} \ln f(tx) =$$
$$= s \varlimsup_{t\to+\infty} \frac{1}{t} \ln f(tx) = s\psi[f](x).$$

Since $\psi[f](0) = 0$, we can assume that the function $\psi[f]$ is positively homogeneous.

Since $D_f \in \mathfrak{F}$, the set $K\setminus D_f$ is bounded and $tx \in D_f$ for any $x \in K\setminus\{0\}$ and $t > 0$ sufficiently large. If $x, y \in K \setminus \{0\}$, then

$$|\psi[f](x) - \psi[f](y)| = |\varlimsup_{t\to\infty} t^{-1} \ln f(tx) - \varlimsup_{t\to\infty} t^{-1} \ln f(ty)| \le$$
$$\le \varlimsup_{t\to\infty} t^{-1} |\ln f(tx) - \ln f(ty)| \le \varlimsup_{t\to\infty} t^{-1} L\|tx - ty\| = L\|x - y\|.$$

If $x \in K \setminus \{0\}$ and $y = 0$, then

$$|\psi[f](x) - \psi[f](y)| = |\varlimsup_{t\to\infty} t^{-1} \ln f(tx)| =$$
$$= |\varlimsup_{t\to\infty} t^{-1} \ln f(tx) f(sx)^{-1}| \le \varlimsup_{t\to\infty} t^{-1} L\|tx - sx\| = L\|x\|$$

with $s > 0$ large enough. Finally, for $x = y = 0$ the required assertion is obvious. $\square$

Let us denote

$$g(f, \mu, x) := \|x\|^{-1}(\mu x + \ln f(x)),\ G(f, \mu) := \limsup_{\mathfrak{F}} g(f, \mu, x),$$

and put $\mathcal{E}(f) := \{\mu \in E^* : G(f, \mu) \leq 0\}$. Thus, $\mathcal{E}(f)$ is the set of all $\mu \in E^*$ such that

$$\limsup_{\mathfrak{F}} \|x\|^{-1}(\ln f(x) + \mu x) \leq 0,$$

and $\mathcal{M}[f]$ is the set of all $\lambda \in E^*$ such that $G(f, \lambda) = 0$ and $G(f, \lambda + \mu) > 0$ for each $\mu \in K^+ \setminus \{0\}$.

**Lemma 3.2.** *The set $\mathcal{E}[f]$ is convex.*

*Proof.* Take any $\mu_0, \mu_1 \in \mathcal{E}[f]$. Then for each $\mu_s = s\mu_1 + (1-s)\mu_0$ with $s \in ]0, 1[$, we have $g(f, \mu_s, x) = sg(f, \mu_1, x) + (1-s)g(f, \mu_0, x)$ and $G(f, \mu_s) \leq sG(f, \mu_1) + (1-s)G(f, \mu_0) \leq 0$. Thus, $\mu_s \in \mathcal{E}[f]$. $\square$

Since $K$ is solid, $K^+$ is pointed. Hence, the space $E^*$ is partially ordered by $K^+$. This fact enables us to give a characterization of $\mathcal{M}[f]$ in terms of vector optimization.

**Lemma 3.3** (see [15, p. 111; 11, p. 86], and also [52, 54])**.** *If $f$ satisfies* (3.1)*, then $\mathcal{M}[f] = \mathrm{Max}(\mathcal{E}(f)|K^+)$.*

*Proof.* Since

$$\begin{aligned} g(f, \mu, x) \leq \|x\|^{-1} |\ln f(x) + \mu x| \leq \\ \leq \|x\|^{-1}(|\ln f(x_0)| + \|\mu\|\,\|x\| + L\|x - x_0\|) \leq \\ \leq \|x\|^{-1} |\ln f(x_0)| + \|\mu\| + L(1 + \|x\|^{-1}\|x_0\|) \end{aligned}$$

for all $x, x_0 \in D_f$ and $\mu \in E^*$, we assume that $G(f, \mu)$ is defined and finite for each $\mu \in E^*$.

It follows now from

$$\begin{aligned} |G(f, \mu) - G(f, \eta)| \leq \limsup_{\mathfrak{F}} |g(\mu, x) - g(\eta, x)| = \\ = \limsup_{\mathfrak{F}} \|x\|^{-1}\|\mu x - \eta x\| \leq \|\mu - \eta\|, \end{aligned}$$

where $\mu, \eta \in E^*$, that $G$ is continuous on $E^*$.

If $\lambda \in \mathcal{E}(f)$ is a maximal element in $\mathcal{E}(f)$ with respect to $K^+$, then $\lambda + \gamma \notin \mathcal{E}(f)$ whatever $\gamma \in K^+ \setminus \{0\}$ be taken, i.e., we have $G(f, \lambda) \leq 0$ and $G(f, \lambda + \gamma) > 0$. Choosing $\gamma$ arbitrary small, we get $G(f, \lambda) = 0$ in the limit since $G$ is continuous. This means that $\lambda \in \mathcal{M}[f]$.

Conversely, if $\lambda \in \mathcal{M}[f]$, then $G(f, \lambda) = 0$ and $G(f, \lambda + \gamma) > 0$ for any $\gamma \in K^+ \setminus \{0\}$. Hence, $\lambda \in \mathcal{E}(f)$ and $\lambda + \gamma \notin \mathcal{E}(f)$, i.e., $\lambda \in \mathrm{Max}(\mathcal{E}(f)|K^+)$. $\square$

**Corollary 3.1.** *If $f$ satisfies* (3.1) *and $\lambda \in \mathcal{M}[f]$, then the equality $G(f, \lambda) = 0$ holds.*

*Remark* 3.1. For $E = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, the analogous statement was proved in [23].

A sequence $p : \mathbb{N} \to K$ such that $\|p(j)\| \to +\infty$ as $j \to \infty$ is said to be realizing for some functional $\mu \in E^*$ if

$$\lim_{j\to\infty} g(f, \mu, p(j)) = G(u, \mu).$$

It can be easily seen that each $\mu \in E^*$ has some realizing sequence.

**Lemma 3.4.** *Let* $\dim E < +\infty$. *If* $f$ *satisfies the condition* (3.1), *then* $G(f, \mu) = \max\{\mu z + \psi[f](z) : z \in K, \|z\| = 1\}$ *for each* $\mu \in E^*$ *and* $G(f, \mu) = \mu z + \psi[f](z)$ *iff* $z$ *is a cluster point of the sequence* $b_i = \|x_i\|^{-1}x_i$, $i \in \mathbb{N}$, *where* $x_i$ *is some realizing sequence for* $\mu$.

*Proof.* Take any $\mu \in E^*$. Let $x_i, i \in \mathbb{N}$, be a realizing sequence for $\mu$. Since the sequence $b_i = \|x_i\|^{-1}x_i$ is bounded, there exists a cluster point $z$ of this sequence. Obviously, $\|z\| = \lim\limits_{i\to+\infty} \|b_i\| = 1$ and $z \in K$ as $b_i \in K$ and $K$ is closed. Let $x_{i(k)}, k \in \mathbb{N}$, be a subsequence of $x_i$ such that $b_{i(k)} = \|x_{i(k)}\|^{-1}x_{i(k)} \to z$ as $k \to +\infty$. Note that $x_{i(k)}$ is also a realizing sequence for $\mu$. Then we have

$$\begin{aligned} G(f, \mu) &= \lim_{k\to\infty} \|x_{i(k)}\|^{-1}(\mu x_{i(k)} + \ln f(x_{i(k)})) = \\ &= \mu z + \lim_{k\to\infty} t_k^{-1} \ln \|u(x_{i(k)})\|, \end{aligned}$$

where $t_k = \|x_{i(k)}\|$. By (3.1) we get

$$0 \le \lim_{k\to\infty} t_k^{-1}(\ln f(x_{i(k)}) - \ln f(t_k z)) \le \lim_{k\to\infty} t_k^{-1}M\|x_i(k) - t_k z\| = 0$$

and, therefore, we have

$$\lim_{k\to\infty} t_k^{-1} \ln f(x_{i(k)}) = \lim_{k\to\infty} t_k^{-1} \ln f(t_k z)) \le \psi[f](z).$$

Thus, we obtain

$$G(f, \mu) \le \mu z + \psi[f](z). \tag{3.3}$$

On the other hand, for each $y \in K$ such that $\|y\| = 1$ we have

$$G(f, \mu) \ge \varlimsup_{t\to+\infty} t^{-1}(t\mu y + \ln f(ty)) = \mu y + \psi[f](y). \tag{3.4}$$

Combining (3.3) with (3.4), we can write $G(f, \mu) = \max\{\mu z + \psi[f](z) : z \in K, \|z\| = 1\}$ and $G(f, \mu) = \mu z + \psi[f](z)$ if $z$ is a cluster point of some sequence $b_i$.

Now let $G(f, \mu) = \mu z_0 + \psi[f](z_0)$ for some $z_0 \in E$. Obviously, $z_0 \in K$ since $K$ is the domain of $\psi[f]$. Take a sequence $t_k \in \mathbb{R}_+, k \in \mathbb{N}$, such that $\lim\limits_{k\to\infty} t_k^{-1} \ln f(t_k z_0)) = \psi[f](z_0)$. Then we have $\varlimsup\limits_{k\to+\infty} t_k^{-1}(t_k \mu z_0 + \ln f(t_k z_0) = \mu z_0 + \psi[f](z_0) = G(f, \mu)$, i.e., the sequence $x_k = t_k z_0$ is a realizing sequence for $\mu$ and $z_0$ is a cluster point for $\|x_k\|^{-1}x_k$. $\square$

**Corollary 3.2.** *If* $\dim E < +\infty$, *then* $\mathcal{E}[f] = \{\mu \in E^* : \mu y + \psi[f](y) \le 0\}$.

**Theorem 3.1** ([52, 54]). *If $f$ satisfies* (3.1)*, then*
(i) *the inclusion $\mathcal{E}(f) \subset \mathcal{E}(\exp\psi[f])$ holds in each Banach space $E$;*
(ii) *the equality $\mathcal{E}(f) = \mathcal{E}(\exp\psi[f])$ holds in an arbitrary finite-dimensional $E$.*

*Proof.* Let $\mu \in \mathcal{E}(f)$. Then for any $\varepsilon > 0$ there exists a set $D(\varepsilon)$ such that $K \setminus D(\varepsilon)$ is bounded and $\ln f(x) \le -\mu x + \varepsilon\|x\|$ for each $x \in D(\varepsilon)$. Hence

$$\psi[f](x) = \overline{\lim_{t\to\infty}}\, t^{-1} \ln f(tx) \le$$
$$\le \overline{\lim_{t\to\infty}}\, t^{-1}(-\mu tx + \varepsilon\|tx\|) = -\mu x + \varepsilon\|x\|$$

for all $x \in K$. It follows now from the definitions that $G(\exp\psi[f], \mu) \le 0$ since $\varepsilon$ is arbitrarily small. Thus, $\mu \in \mathcal{E}(\exp\psi[f])$.

Now let $\dim E < +\infty$ and $\mu \in \mathcal{E}(\exp\psi[f])$. Since $\psi[f]$ is positively homogeneous, we can write

$$\begin{aligned} G(\exp\psi[f], \mu) &= \limsup_{\mathfrak{F}} \|x\|^{-1}(\mu x + \psi[f](x)) = \\ &= \limsup_{\mathfrak{F}}(\|x\|^{-1}\mu x + \psi[f](\|x\|^{-1}x)) = \\ &= \sup\{\mu x + \psi[f](x) : \|x\| = 1, x \in K\}, \end{aligned}$$

and by Lemma 3.4 we have $G(\exp\psi[f], \mu) = G(f, \mu)$. Hence, $\mathcal{E}(f) = \mathcal{E}(\exp\psi[f])$. □

**Corollary 3.3** ([52, 54]). *If $f$ satisfies* (3.1) *and $E$ is finite-dimensional, then $\mathcal{M}(f) = \mathcal{M}(\exp\psi[f])$.*

*Proof.* By Lemma 3.3, we have

$$\mathcal{M}(f) = \mathrm{Max}(\mathcal{E}(f)|K^+) = \mathrm{Max}(\mathcal{E}(\exp\psi[f])|K^+) = \mathcal{M}(\exp\psi[f]). \qquad \square$$

The second part of Theorem 3.1 substantially uses the compactness of the unit ball in a finite-dimensional space. If the unit ball in $E$ is not compact, the assertion (ii) fails. This fact is demonstrated by the example below.

**Example 3.1.** Let $E = \ell_1$, i.e., the space of sequences $x : \mathbb{N} \to \mathbb{R}$ (or $x = (x_1, x_2, \ldots)$) with the norm $\|x\| = \sum\limits_{k=1}^{\infty} |x_k| < +\infty$. Consider the cone $K = \mathrm{cone}(B) = \{x \in \ell_1 : 2x_1 \ge \|x\|\}$, where $B := \{z \in \ell_1 : z_1 = 1,\ \|z\| \le 2\}$. Since $0 \notin B$ and $B$ is closed, bounded, and convex, the cone $K$ is a closed convex pointed cone with the bounded base $B$. Moreover, $K$ contains the unit ball of $\ell_1$ centered at $(1, 0, 0, \ldots)$, hence $K$ is solid.

Define the mapping $\varphi : K \setminus \{0\} \to \mathbb{R}$ by

$$\varphi(x) = \sum_{k=2}^{\infty} x_k x_1^{-1/k} \tag{3.5}$$

for all $x \in K \setminus \{0\}$. The series (3.5) converges absolutely everywhere on $K \setminus \{0\}$ since $\sum\limits_{k=1}^{\infty} |x_k| = \|x\| < +\infty$ and $x_1^{-1/k} \to 1$ as $k \to +\infty$ for such $x$.

The function $\varphi$ is Lipschitzian on $D = \{x \in K : x_1 \geq 1\} \in \mathfrak{F}$. To verify this fact, we write

$$|\varphi(x) - \varphi(v)| \leq \sum_{k=2}^{\infty} |x_k x_1^{-1/k} - v_k v_1^{-1/k}| \leq$$
$$\leq \sum_{k=2}^{\infty} |x_k - v_k||x_1^{-1/k}| + \sum_{k=2}^{\infty} |v_k||x_1^{-1/k} - v_1^{-1/k}|.$$

If $x_1 \geq v_1 \geq 1$, then we get $|x_1^{-1/k}| \leq 1$, $|x_1^{-1/k} - v_1^{-1/k}| \leq |x_1 - v_1| \times \sup\{k^{-1}s^{-1-1/k} : k \geq 2, s \geq v_1\} \leq (2v_1)^{-1}|x_1 - v_1|$. Thus,

$$|\varphi(x) - \varphi(v)| \leq \sum_{k=2}^{\infty} |x_k - v_k| + \frac{1}{2} v_1^{-1}|x_1 - v_1| \sum_{k=2}^{\infty} |v_k| \leq \|x - v\|.$$

It can be easily shown that everywhere on $U = \{x \in \ell_1 : 3x_1 > \|x\|, x_1 > 1\}$ the function $f = \exp\varphi$ satisfies the equation

$$y'h = \left( \sum_{k=2}^{\infty} x_1^{-1/k} h_k - x_1^{-1} \sum_{k=2}^{\infty} k^{-1} x_k x_1^{-1/k} h_1 \right) y, \tag{3.6}$$
$$y \in \mathbb{R}, \quad x \in U. \quad h \in \ell_1.$$

Evaluating $\psi[f]$, we obtain

$$\psi[f](x) = \varlimsup_{s\to\infty} s^{-1} \ln f(sx) = \varlimsup_{s\to\infty} \sum_{k=2}^{\infty} x_k (sx_1)^{-1/k}.$$

For any $m > 2$, $s \geq 1$, we have

$$\left| \sum_{k=2}^{\infty} x_k (sx_1)^{-1/k} \right| \leq s^{-1/m} \sum_{k=2}^{m} |x_k| x_1^{-1/k} +$$
$$+ \sum_{k=m+1}^{\infty} |x_k| x_1^{-1/k} \to \sum_{k=m+1}^{\infty} |x_k| x_1^{-1/k} \tag{3.7}$$

as $s \to +\infty$. Since the series (3.5) converges absolutely, we get $\psi[f](x) = 0$ for each $x \in K$.

Suppose that $\mathcal{E}(f) \supset \mathcal{E}(\exp\psi[f])$. Then $0 \in \mathcal{E}(f)$ since $0 \in \mathcal{E}(\exp\psi[f])$ and $\exp\psi[f] \equiv 1$. This yields

$$\limsup_{\mathfrak{F}} \|x\|^{-1}\varphi(x) \leq 0. \tag{3.8}$$

Take now the sequence $v : \mathbb{N} \to \ell_1$ such that $v_1(k) = v_k(k) = 2^k, k \in \mathbb{N}$, and $v_j(k) = 0$ for all $j \neq 1, k$. It is easy to see that $v(n) \in K \setminus \{0\}$ for each $n \in \mathbb{N}$ and $\|v(n)\| = 2^{n+1} \to +\infty$ as $n \to \infty$. Evaluating the limit $\varlimsup_{n\to\infty} \|v(n)\|^{-1}\varphi(v(n)) = 1/4$, we obtain the contradiction to (3.8). Hence, $0 \notin \mathcal{E}(f)$ and, therefore, $\mathcal{E}(f) \not\supset \mathcal{E}(\exp\psi[f])$.

It can be easily seen that in order to obtain the equality $\mathcal{E}(f) = \mathcal{E}(\exp\psi[f])$, we need a uniform estimate for $|\ln f(x) - \psi[f](x)|$ on $K \setminus \{0\}$. If $\dim E <$

$+\infty$, we get such an estimate from the compactness of the unit ball in $E$. If $E$ is infinite-dimensional, we can deduce the required estimate from some other conditions.

**Theorem 3.2** ([53])**.** *If* $0 \in \mathcal{E}(f \exp(-\psi[f]))$, *then the equality* $\mathcal{E}(f) = \mathcal{E}(\exp(\psi[f]))$ *holds.*

*Proof.* The proof is an immediate consequence of the following inequalities:

$$\limsup_{\mathfrak{F}} \|x\|^{-1}(\psi[f](x) + \mu x) \le \limsup_{\mathfrak{F}} \|x\|^{-1}(\ln f(x) + \mu x)+ \\ + \limsup_{\mathfrak{F}} \|x\|^{-1} \ln(f(x) \exp(-\psi[f])) \le 0,$$

where $\mu \in \mathcal{E}(f)$ is arbitrary. □

**Characteristic exponents and functionals for solutions of TDEs.** By [15, p. 28], each nontrivial solution $u$ of (1.1) with $\sup\limits_{x\in U} \|A(x)\| \le M < +\infty$ satisfies the condition

$$|\ln\|u(x)\| - \ln\|u(y)\|| \le M\|x-y\|$$

for all $x, y \in U$ such that the segment $[x,y] = \operatorname{conv}\{x,y\}$ lies in $U$. Hence, $f = \ln\|u\|$ satisfies the condition (3.1) and, therefore, Theorem 3.1 is valid for any such $u$. This means that the characteristic functionals of $u$ are completely determined by the characteristic exponents of $u$ when $E$ is finite-dimensional. Some particular case of this result was proved for $E = \mathbb{R}^2$, $K = \mathbb{R}^2_+$ in [46].

To obtain the consequences of the established relations let us introduce some necessary definitions.

It is well known that linear TDEs admit a classification analogous to the Lyapunov classification of linear ordinary differential systems [6, p. 242]. In particular, regular TDEs were defined by E. I. Grudo [20] for $E = \mathbb{R}^n$, $K = \mathbb{R}^n_+$ and by I. V. Gaĭshun [11, p. 94, 15, p. 122] in the general case. There exist two distinct ways to define regular TDEs. To this end, we can use various generalizations of irregularity coefficients as in [20] and [11, p. 94]. However, if $F$ is infinite-dimensional, this definition fails. So, the universal definition should be given in terms of generalized reducibility as in [15, p. 123]. Surely, such a definition is equivalent to the latter one if $\dim E < +\infty$, see Theorems 10.6 and 10.7 in [11, p. 96–98] and Theorem 19.1 in [15, p. 150].

Let Q be a transformation defined by $u = Q(x)v$, where $u, v \in F$ and $Q\colon U \to GL(F)$. The set SExp of all transformations Q such that $Q$ is continuously Frechét differentiable on $U$ and

$$\limsup_{\mathfrak{F}} \|x\|^{-1} \ln\|Q(x)\| = \limsup_{\mathfrak{F}} \|x\|^{-1} \ln\|Q^{-1}(x)\| = 0$$

is called strong exponential group. The set Exp of all transformations Q such that $Q$ is continuously Frechét differentiable on $U$ and $\mathcal{M}[Q] = \mathcal{M}[Q^{-1}] = 0$

is called exponential group. (We suppose that the group operation is the usual composition of transformations.)

Note that both Exp and SExp are completely defined only when $U$ and $K$ are predetermined. It was proved in [15, p. 113] that SExp = Exp.

**Definition 3.4** (see [15])**.** The equation (1.1) is said to be regular if there exists a transformation Q $\in$ Exp reducing (1.1) to an autonomous equation.

Regular TDEs with finite-dimensional $F$ provide another known class of equations having some relation between characteristic exponents and characteristic functionals of solutions. This relation can be established on the basis of Theorem 3.2.

**Theorem 3.3** ([53])**.** *If* $\dim F < +\infty$ *and* (1.1) *is regular, then* $\mathcal{E}(u) = \mathcal{E}(\exp(\psi[u]))$ *and* $\mathcal{M}[u] = \mathcal{M}[\exp \psi[u]]$.

A particular case of this result was proved in [15, p. 153].

In [41] M. V. Kozhero introduced the concept of weak regularity for TDEs based on the notion of the (weak) characteristic exponent. It seems to be natural that there exist two distinct ways to define weak regularity. But the analogy with the usual regularity is not complete since these ways are not equivalent even if $E$ and $F$ are finite-dimensional.

**Definition 3.5** ([54, 55])**.** The set WExp($K$) of all transformations Q such that $Q$ satisfies the condition $\chi[Q](x) = \chi[Q^{-1}](x) = 0$ for all $x \in K \setminus \{0\}$ and the function $S_x$ defined by $S_x(t) := Q(tx)$, $t > t_x$, has the piecewise continuous derivative in $t$ is called weak exponential group.

One can easily show that WExp is a group with respect to the usual composition of transformations and each Q $\in$ WExp preserves (weak) characteristic exponents.

The equation (1.1) is said to be weakly exponentially equivalent (WExp-equivalent) to another equation of the same type if there exists a transformation Q $\in$ WExp taking one of these equations to another.

**Definition 3.6** ([55])**.** The equation (1.1) is said to be weakly regular if this equation is WExp-equivalent to some autonomous equation.

For each $b \in K \setminus \{0\}$, consider the linear system

$$\dot{z} = [A(tb)b]z, \quad t > t_b, \quad z \in F, \tag{3.9}$$

where $t_b = \inf\{t \in \mathbb{R} : tb \in U\}$. The system (3.9) is called restriction of (1.1) onto the ray $r(b) = \{x \in E : x = tb, t > t_b\}$. Let $y$ be any solution to (1.1). Then $z(t) = y(tb)$, $t > t_b$, defines a solution to (3.9) and $\lambda[z] = \overline{\lim\limits_{t\to+\infty}}\, t^{-1} \ln \|z(t)\| = \overline{\lim\limits_{t\to+\infty}}\, t^{-1} \ln \|y(tb)\| = \psi[y](b)$, where $\lambda[z]$ denotes the Lyapunov exponent of $z$.

The equation (1.1) is said to be regular along $b \in K \setminus \{0\}$ if the restriction (3.9) of (1.1) onto the ray $r(b)$ is Lyapunov regular [6, p. 238].

**Definition 3.7** ([41])**.** The equation (1.1) is said to be Kozhero regular if this equation is regular along each $b \in K \setminus \{0\}$.

Let $\dim F = n < +\infty$ and $\lambda_1(A,b) \leq \ldots \leq \lambda_n(A,b)$ be the Lyapunov spectrum [6, p. 63] of (3.9). Characteristic exponent of (1.1) is the mapping $\chi(A)\colon K \setminus \{0\} \to \mathbb{R}^n$ defined by $\chi(A)(x) = (\chi_1(A)(x), \ldots, \chi_n(A)(x))$, $\chi_i(A)(x) = \lambda_i(A,b)$, $i = 1, \ldots, n$.

**Proposition 3.1.** *Let* $\dim F = n < +\infty$. *The equation* (1.1) *is Kozhero regular iff*

$$\chi_1(A)(x) + \ldots + \chi_n(A)(x) - \varliminf_{t\to\infty} \frac{1}{t\|x\|} \int\limits_{t_x}^{t} \mathrm{Sp}(A(tx)x)\, dt = 0, \tag{3.10}$$

*for all* $x \in K \setminus \{0\}$.

*Proof.* The left side of (3.10) is a Lyapunov irregularity coefficient for (3.9) with $b = x$. Now the required assertion follows from the usual theorems for ordinary linear differential systems, see [27, p. 77]. ☐

Since WExp preserves (weak) characteristic exponents of TDEs, the characteristic exponent of a weakly regular TDE should coincide with the exponent of some autonomous TDE. This approach enables us to give a description of weakly regular TDEs in terms of their characteristic exponents. To this end, we use the concept of weakly normal basis (fundamental system) of solutions introduced by M. V. Kozhero in [40].

Let $\dim F = n < +\infty$. A fundamental system $Y = [y_1, \ldots, y_n]$ of solutions to (1.1) is said to be weakly normal if the sum $\omega_Y(x) = \chi[y_1](x) + \cdots + \chi[y_n](x)$ does not exceed the analogous sum for any other fundamental system of solutions to (1.1).

Note that a basis of solutions to (1.1) is weakly normal iff its restriction to each ray $r(b)$, $b \in K \setminus \{0\}$, coincides with some normal basis of solutions to (3.9).

A weakly normal basis of solutions to (1.1) is called regular if there exist $\mu_i \in E^*$, $i = 1, \ldots, n$, such that $\chi[u_i](x) = \|x\|^{-1}\mu_i x, \quad x \in K \setminus \{0\}$.

**Proposition 3.2** ([55])**.** *Let* $\dim F = n < +\infty$. *The equation* (1.1) *is weakly regular iff it is Kozhero regular and has a regular basis of solutions.*

Now it is interesting to compare weak and Kozhero regularity with usual regularity. We begin with comparison of WExp and Exp. One can easily prove that $\mathrm{WExp}(K) \supset \mathrm{Exp}(K)$. The reverse inclusion can be obtained using the relation between characteristic exponents and functionals.

**Proposition 3.3** ([55])**.** *Suppose that* $\dim E < +\infty$ *and* $Q : U \to GL(F)$ *is continuously Frechét differentiable. If* $\mathrm{Q} \in \mathrm{WExp}(K)$ *and both* $\|Q\|$ *and* $\|Q^{-1}\|$ *satisfy* (3.1), *then* $\mathrm{Q} \in \mathrm{Exp}(K)$.

*Proof.* By Theorem 3.1, we have $\mathcal{M}[Q^{\pm 1}] = \{0\}$ since $\chi[Q] \equiv 0$ on $K \setminus \{0\}$. Hence $\mathrm{Q} \in \mathrm{Exp}(K)$. ☐

**Theorem 3.4** ([55]). *Let* $\dim E < +\infty$. *If two equations of the form* (1.2) *are weakly exponentially equivalent, then these equations are exponentially equivalent.*

*Proof.* Let $X$ and $Y$ be the Cauchy operators of the given equations and $\mathrm{L} \in \mathrm{WExp}(K)$ be the transformation taking one of them to another. Then for all $x, y \in U$ we have $L(x)X(x,y)L^{-1}(y) = Y(x,y)$. Since $K$ is convex, there exists a convex $D_0 \in \mathfrak{F}$ contained in $U$. Then for any $x, y \in D_0$ we get $\|X(x,y)\| \le \exp M\|x-y\|$,$\|Y(x,y)\| \le \exp M\|x-y\|$. Hence,

$$\|L(x)\| = \|Y(x,y)L(y)X(y,x)\| \le \|L(y)\| \exp 2M\|x-y\|$$

and, analogously, $\|L(y)\| \le \|L(x)\| \exp 2M\|x-y\|$. Thus,

$$|\ln \|L(x)\| - \ln \|L(y)\|| \le 2M\|x-y\|.$$

Since the coefficients of each equation (1.2) are continuous, $L$ is continuously Frechét differentiable. Now, to finish the proof, it is sufficient to apply Proposition 3.3. □

**Corollary 3.4.** *If $E$ is finite-dimensional, then any weakly regular equation* (1.2) *is regular.*

**Corollary 3.5.** *Let* $\dim E < +\infty$. *A Kozhero regular equation* (1.2) *is regular iff it has a regular basis of solutions.*

One of the most interesting unsolved problems of the asymptotic theory of TDEs is the behavior of characteristic functionals under exponentially small perturbations of the equation. It is the more so interesting that the same problem for characteristic exponents can be solved by means of the results known for ordinary differential systems. Using the above relations between characteristic exponents and characteristic functionals, we can obtain some advances in this problem.

Let us consider a perturbed equation

$$v'h = (A(x) + Q(x))hy, \quad v \in F, \quad x \in U, \quad h \in E, \tag{3.11}$$

where $Q : U \to L(E, L(F,F))$ is continuous and bounded. It should be stressed that there are no reasons for (3.11) to be completely integrable for arbitrary $Q$ even if $Q$ is taken very small or vanishing. To avoid these difficulties, for any $\rho : K \to \mathbb{R}$ we introduce the set $P(\rho, A)$ of all perturbations $Q$ satisfying the condition $\psi_Q(x) < -\rho(x), x \in K$, and such that the equation (3.11) is completely integrable.

To formulate the result, we need the notion of normal domain for (1.1) introduced by M. V. Kozhero in [40]. A point $x_0 \in K \setminus \{0\}$ is said to be a branching point for exponents of (1.1) if there exist two solutions $y_1$ and $y_2$ to (1.1) such that $\chi[y_1](x_0) = \chi[y_2](x_0)$ and for any $\varepsilon > 0$ there exists a point $\xi \in K \setminus \{0\}$ such that $\|\xi - x_0\| < \varepsilon$ and $\chi[y_1](\xi) \ne \chi[y_2](\xi)$. A point $x \in K \setminus \{0\}$ is called normal if $x$ is not branching. Any connected component of the set of all normal points is called a normal domain for (1.1).

**Theorem 3.5** ([52, 54]). *Let* $\dim E < +\infty$ *and* $K \subset \operatorname{cl} H$*, where* $H$ *is some normal domain for* (1.2)*. Then there exists a positive and positively homogeneous function* $\sigma : K \to \mathbb{R}$ *satisfying the Lipschitz condition on* $K$ *with the Lipschitz constant* $2M$ *such that for each solution* $v$ *to* (3.11) *with any* $Q \in P(\sigma, A)$ *there exists a solution* $u$ *to* (1.2) *with* $\mathcal{M}[u] = \mathcal{M}[v]$.

## 4. Proper Characteristic Functionals

The techniques developed in Section 3 make it possible to reduce the evaluation of the characteristic set to a vector optimization problem. By Lemma 3.3 we can write $\mathcal{M}[u] = \operatorname{Max}(\mathcal{E}(u)|K^+)$, where $u$ is any nontrivial solution of (1.1). Thus, to evaluate $\mathcal{M}[u]$, we have (i) to construct the set $\mathcal{E}(u)$ and then (ii) to find the maximal elements of $\mathcal{E}(u)$.

However, both (i) and (ii) are difficult problems. To avoid the difficulties arising here, we apply a specific modification of the scalarization method [17, p. 48] commonly used in the vector optimization theory.

It turns out that there exists a set of proper characteristic functionals $\mathcal{P}[u] \subset \mathcal{M}[u]$ with many useful properties. For example, we can prove the inclusion $\mu \in \mathcal{P}[u]$ for a given $\mu \in E^*$ without global information about $\mathcal{E}(u)$. Moreover, the set $\mathcal{P}[u]$ is norm-dense in $\mathcal{M}[u]$ when $E$ is finite-dimensional. Our approach enables us to describe the set $\mathcal{P}[u]$ and to use it in studying characteristic functionals.

**General construction of proper characteristic set.** For any sequence $p : \mathbb{N} \to K$, let us consider the sequence $b_p : \mathbb{N} \to K$ such that $b_p(j) := \|p(j)\|^{-1}p(j)$. By Banach–Alaoglu theorem [68, p. 80] the unit ball of $E^*$ is compact with respect to weak-star topology $\sigma(E^{**}, E^*)$ [68, p. 80]. Since $\|b_p(j)\| = 1$, we assume that the weak-star cluster set of $b_p$ is not empty. Now let us denote the $\sigma(E^{**}, E^*)$-closure of convex hull of this cluster set by $\mathfrak{B}(p)$.

**Definition 4.1** ([53]). Let $u$ be a nontrivial solution of (1.1). We say that $\mu \in \mathcal{E}(u)$ is a proper characteristic functional of $u$ if $G(u, \mu) = 0$ and there exists a realizing sequence $p$ such that $\mathfrak{B}(p) \cap (K^+)^{+i} \neq \varnothing$.

We denote the set of all proper characteristic functionals by $\mathcal{P}[u]$. This set is called proper characteristic set.

**Theorem 4.1** ([53]). *The inclusion*

$$\mathcal{P}[u] \subset \operatorname{Pos}(\mathcal{E}(u)|(K^+)^{+i})$$

*holds for any nontrivial solution* $u$ *to* (1.1).

*Proof.* Take any $\mu \in \mathcal{P}[u]$. By definition of $\mathcal{P}[u]$, there exists a realizing sequence $p$ such that $\mathfrak{B}(p) \cap (K^+)^{+i} \neq \varnothing$. For any cluster point $a_0 \in E^{**}$ of $b_p$, there exists a directed set $\Gamma$ and a subnet $q : \Gamma \to p(\mathbb{N})$ of the sequence $p$ such that the net $a : \Gamma \to K^{++}$ defined by $a(\gamma) = \|q(\gamma)\|^{-1}q(\gamma)$ converges to $a_0$ with respect to $\sigma(E^{**}, E^*)$. Note that $q$ is not necessarily a subsequence of $p$. Hence we get

$$0 = \lim_{j\to\infty} g(u,\mu,p(j)) = \lim_{\Gamma} g(u,\mu,q(\gamma)) = \\ = \lim_{\Gamma} \|q(\gamma)\|^{-1} \ln \|u(q(\gamma))\| + \mu a_0. \tag{4.1}$$

On the other hand, for any $\nu \in \mathcal{E}(u)$ we have

$$0 \geq \limsup_{\mathfrak{F}} g(u,\mu,x) \geq \lim_{\Gamma} \|q(\gamma)\|^{-1} \ln \|u(q(\gamma))\| + \nu a_0. \tag{4.2}$$

From (4.1) and (4.2), we obtain $\mu a_0 \geq \nu a_0$, i.e., $\mu \in \mathrm{Pos}(\mathcal{E}(u)|a_0)$. Since each $\lambda \in \mathcal{E}(u)$ is linear and $\sigma(E^{**}, E^*)$-continuous functional on $E^{**}$, we get $\mu b \geq \nu b$ for any $b \in \mathfrak{B}(p)$. Now we can choose $b = b_0 \in \mathfrak{B}(p) \cap (K^+)^{+i} \neq \varnothing$. Thus, $\mu \in \mathrm{Pos}(\mathcal{E}(u)|(K^+)^{+i})$. □

**Corollary 4.1** ([53]). *The inclusion*

$$\mathcal{M}[u] \supset \mathcal{P}[u].$$

*holds for any non-zero solution $u$ to* (1.1).

*Proof.* By [17, p. 49], we have

$$\mathrm{Pos}(Q|(K^+)^{+i}) \subset \mathrm{Max}(Q|K^+)$$

for any $Q \subset E^*$. Since $\mathcal{M}[u] = \mathrm{Max}(\mathcal{E}(u)|K^+)$, we get the required inclusion. □

**Example 4.1.** Consider the equation

$$y'h = \left( \sum_{k=2}^{\infty} x_1^{-1/k} h_k - x_1^{-1} \sum_{k=2}^{\infty} k^{-1} x_k x_1^{-1/k} h_1 \right) y, \tag{4.3}$$

where $x \in U = \{x \in \ell_1 : x_1 > 1,\ \|x\| < 3x_1\}$, $y \in \mathbb{R}$, $h \in \ell_1$.

It can be easily proved that the operator coefficient of (4.3) is bounded and continuous on $U$.

Let $\varphi(x) = \sum_{k=2}^{\infty} x_k x_1^{-1/k}$. For each $C \in \mathbb{R}$, the functions $u = C \exp \varphi$ are Frechèt differentiable on $U$ and satisfy the equation (4.3). Thus, the equation (4.3) is completely integrable.

Let $K = \{x \in \ell_1 : \|x\| \leq 2x_1\}$. Pick out any nontrivial solution $u$ to (4.3) from the above family. Then for $\mu \in \ell_\infty = \ell_1^*$ with the components $\mu_k = -1/2$, $k \in \mathbb{N}$, we have

$$g(u,\mu,x) = \|x\|^{-1} \left( \ln|C| + \sum_{k=2}^{\infty} x_k x_1^{-1/k} - \sum_{k=1}^{\infty} x_k/2 \right) = \\ = \|x\|^{-1} \ln|C| + \|x\|^{-1} \left( -x_1/2 + \sum_{k=2}^{\infty} x_k (x_1^{-1/k} - 1/2) \right)$$

for all $x \in U$.

Since $x_1 > 0$, we have $|x_1^{-1/k} - 1/2| \le 1/2$ for each $k \in \mathbb{N}$ and therefore

$$\left|\sum_{k=2}^{\infty} x_k(x_1^{-1/k} - 1/2)\right| \le \max_k |x_1^{-1/k} - 1/2| \sum_{k=2}^{\infty} |x_k| \le \frac{\|x\| - x_1}{2}\,.$$

Hence for all $x \in K \cap U$ we obtain

$$g(u,\mu,x) \le \|x\|^{-1} \ln |C| + \frac{1}{2} - \frac{x_1}{\|x\|} \le \|x\|^{-1} \ln |C|. \tag{4.4}$$

Since $\|x\|^{-1} \ln |C| \to 0$ as $\|x\| \to +\infty$, (4.4) yields $G(u,\mu) \le 0$ and finally we get the inclusion $\mu \in \mathcal{E}(u)$.

On the other hand, taking the sequence $p : \mathbb{N} \to K$ with $p(n) = (n^n, 0, \ldots, 0, -n^n, 0, 0, \ldots)$, i.e. $p_1(n) = -p_n(n) = n^n$ and $p_j(n) = 0$ for all the remaining $j \in \mathbb{N}$, we obtain

$$\lim_{n\to\infty} g(u,\mu,p(n)) = 2^{-1} \lim_{n\to\infty} n^{-n}(-n^n(n^{-1} - 1/2) - n^n/2) = 0.$$

Hence $G(u,\mu) = 0$ and $p$ is a realizing sequence for $\mu$.

Analogously, taking the sequence $q : \mathbb{N} \to K$ with $q(n) = (n, 0, \ldots, 0, n, 0, 0, \ldots)$, i.e., $q_1(n) = q_n(n) = n$ and $q_j(n) = 0$ for all the remaining $j \in \mathbb{N}$, we obtain

$$\lim_{n\to\infty} g(u,\mu,q(n)) = 2^{-1} \lim_{n\to\infty} n^{-1}(n(n^{-1/n} - 1/2) - n/2) = 0,$$

i.e., the $q$ is realizing for $\mu$ too.

Since the elements $\|p(n)\|^{-1}p(n)$ and $\|q(n)\|^{-1}q(n)$ are symmetric with respect to $b_0 = (1/2,\ 0,\ 0, \ldots) \in (K^+)^{+i}$, the cluster sets of $b_p$ and $b_q$ are symmetric too and, therefore, their common convex hull contains $b_0$.

Finally, let us take the sequence $r : \mathbb{N} \to K$ such that $r(2k) = p(k)$ and $r(2k-1) = q(k)$ for all $k \in \mathbb{N}$. It can be easily proved that $\mathfrak{B}(r) \supset \overline{\mathrm{conv}}(\mathfrak{B}(p) \cup \mathfrak{B}(q)) \ni b_0$. Thus, $\mathfrak{B}(r) \cap (K^+)^{+i} \ne \varnothing$ and we immediately get $\mu \in \mathcal{P}[u]$ by definition of proper characteristic set.

**Finite-dimensional proper characteristic sets.** Up to the end of this section, we will assume $E$ to be finite-dimensional. In the finite-dimensional space the weak-star topology $\sigma(E^{**}, E^*)$ coincides with the original (norm) topology of $E$ and for any solid pointed convex cone $K$, the equality $(K^+)^{+i} = \operatorname{Int} K$ holds. These facts make the definition of proper characteristic set substantially clearer. Moreover, in this specific case the relation between characteristic exponents and characteristic functionals demonstrated in Section 3 enables us to obtain much more advanced results.

**Lemma 4.1** ([56]). *Let* $\dim E < +\infty$*. If* $\mu \in \mathcal{P}[u]$ *for some nontrivial solution* $u$ *of* (1.2)*, then* $\overline{\mathrm{conv}}\{x \in K : \mu x + \psi[u](x) = 0\} \cap \operatorname{Int} K \ne \varnothing$ *and* $\mu x + \psi[u](x) \le 0$ *for all* $x \in K$*.*

*Proof.* By definition of $\mathcal{P}[u]$, we assert that $G(u,\mu) = 0$. Take now any realizing sequence $x_i$, $i \in \mathbb{N}$, of $\mu$. If $z$ is some limiting point of the sequence

$\|x_i\|^{-1}x_i$, then there exists a subsequence $x_{i(k)}, k \in \mathbb{N}$, of $x_i$ such that $\|x_{i(k)}\|^{-1}x_{i(k)} \to z$ as $k \to +\infty$, and we can write

$$0 = G(u,\mu) = \lim_{k\to\infty} t_k^{-1}(\mu x_{i(k)} + \ln\|u(x_{i(k)})\|) =$$

$$= \mu z + \lim_{k\to\infty} t_k^{-1}\ln\|u(x_{i(k)})\| = \mu z + \lim_{k\to\infty} t_k^{-1}\ln\|u(t_k z)\| +$$

$$+ \lim_{k\to\infty} t_k^{-1}(\ln\|u(x_{i(k)})\| - \ln\|u(t_k z)\|) \leq$$

$$\leq \mu z + \psi[u](z) + \lim_{k\to\infty} t_k^{-1} M\|x_i(k) - t_k z\| = \mu z + \psi[u](z),$$

where $t_k = \|x_{i(k)}\|$, $M = \sup\limits_{x\in U}\|A(x)\|$.

Thus, any limiting point of $\|x_i\|^{-1}x_i$ satisfies the condition $\mu z + \psi[u](z) = 0$ and by definition of $\mathcal{P}[u]$ we get $\overline{\operatorname{conv}}\{x \in K\colon \mu x + \psi[u](x) = 0\} \cap \operatorname{Int} K \neq \varnothing$.

On the other hand, we have

$$0 = G(u,\mu) \geq \varlimsup_{t\to+\infty}(t\|x\|)^{-1}(t\mu x + \ln\|u(tx)\|) = \mu x + \psi[u](x).$$

for any $x \in K$ and this completes the proof. □

From Lemma 4.1 and the Separation Theorem [68, Theorem 3.4], we obtain the following statement.

**Theorem 4.2.** *If* $\dim E < +\infty$, *then*

$$\mathcal{P}[u] = \operatorname{Pos}(\mathcal{E}(u)|\operatorname{Int} K)$$

*for any nontrivial solution* $u$ *to* (1.2).

**Corollary 4.2** ([56]). *The inclusion*

$$\mathcal{M}[u] \subset \operatorname{cl}\mathcal{P}[u]$$

*is valid for any nontrivial solution* $u$ *to* (1.2).

*Proof.* Since $E$ is finite-dimensional, the set $\mathcal{E}(y) \neq \varnothing$ is closed and convex and the cone $K^+$ is closed, convex, and pointed, we can apply Theorem 5.5 from [25]. By this theorem we get

$$\operatorname{Max}(\mathcal{E}(u)|K^+) \subset \operatorname{cl}\operatorname{Pos}(\mathcal{E}(u)|(K^+)^{+i}).$$

Now the required inclusion is an immediate consequence of Theorems 3.1 and 4.2. □

From Lemma 4.1 and Theorem 4.2, we can easily obtain the following description of the sets $\mathcal{E}(u)$ and $\mathcal{P}[u]$ using some standard techniques of Convex Analysis.

**Theorem 4.3** ([53]). *For any nontrivial solution* $u$ *to* (1.2), *the equalities*

$$\mathcal{E}(u) = -\partial^{\geqslant}\psi[u](0),$$

$$\mathcal{P}[u] = \operatorname{Pos}(\mathcal{E}(u)|\operatorname{Int} K) = -\bigcup_{x\in\operatorname{Int} K}\partial^{\geqslant}\psi[u](x)$$

*hold, where* $\partial^{\geqslant}\psi[u] := -\partial(\overline{\mathrm{conv}}(-\psi[u]))$ *is the Penot superdifferential.*

**Corollary 4.3** ([53]). *For any nontrivial solution* $u$ *of* (1.2)*, the equalities*

$$\mathcal{E}(u) = \partial\varphi[u](0), \quad \mathcal{P}[u] = \bigcup_{x\in \mathrm{Int}\, K} \partial\varphi[u](x)$$

*hold, where* $\varphi[u] := \overline{\mathrm{conv}}(-\psi[u])$.

The above statements are very useful in studying boundedness and closedness of characteristic sets of TDEs.

In [19], E. I. Grudo proved that the characteristic set of each solution to (1.2) is closed when $E = \mathbb{R}^2$, $K = \mathbb{R}^2_+$. However, it was demonstrated in [2] that the set $\mathrm{Max}(Q|K)$ is closed for any convex $Q \subset \mathbb{R}^2$ and any pointed convex ordering cone $K \subset \mathbb{R}^2$, but this is not true for $\mathbb{R}^3$. Since $\mathcal{M}[u] = \mathrm{Max}(\mathcal{E}(u)|K^+)$, we can assume that the analogous problems arise for characteristic sets. The following statement shows that this assumption is true.

**Proposition 4.1** ([53]). *If* $E = \mathbb{R}^3$ *and* $K = \mathbb{R}^3_+ := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3\}$*, then there exists an equation* (1.2) *with bounded* $C^\infty(E)$ *coefficients such that* $\mathcal{M}[u]$ *is non-closed for each nontrivial solution* $u$ *to this equation.*

*Proof.* Let $\|x\| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$ be the Euclidean norm of $x = (x_1, x_2, x_3) \in E$ and $\omega_h : \mathbb{R}^+ \to \mathbb{R}$ be some averaging kernel with the averaging radius $h > 0$ [59, p. 29], i.e., a continuous function such that (i) $\int_E \omega_h(\|x\|)dx = 1$; (ii) $\omega_h(r) > 0$ for $r < h$ and $\omega_h(r) = 0$ for $r \geq h$; (iii) $\omega_h(\|x\|)$ is $C^\infty(E)$. It can be easily proved that for each $t > 0$ the function $\omega_{h/t}(r) := t^3\omega_h(tr)$ is also an averaging kernel with the averaging radius $h/t$.

Define $f : E \to \mathbb{R}$ by $f(x) = \max\{x_1 + x_2 + x_3, \sqrt{2}\,\|x\|\}$ and put

$$\varphi(x) = \bar{f}_h(x) := \int_E \omega_h(\|x - z\|) f(z)\, dz.$$

It follows from [59, p. 31] that $\varphi$ is $C^\infty(E)$ and the partial derivatives $\partial\varphi/\partial x_i$, $i = 1, 2, 3$, are bounded on $E$.

Now let us show that the equation

$$dy = -\sum_{i=1}^{3} \frac{\partial\varphi}{\partial x_i} y\, dx_i, \quad x \in E, \quad y \in \mathbb{R}, \tag{4.5}$$

possesses all the required properties.

An arbitrary solution to (4.5) can be written in the form $u(x) = C\exp(-\varphi(x))$. Evaluating the modified characteristic exponent $\psi[u]$, of $u$ we get

$$\psi[u](x) = \lim_{t\to+\infty} t^{-1} \ln|u(tx)| = -\lim_{t\to+\infty} t^{-1}\varphi(tx)$$

and

$$\varphi(tx) = \int\limits_E \omega_h(\|tx - z\|)f(z)\,dz = \int\limits_E t^3\omega_h(t\|x - y\|)f(ty)\,dy =$$

$$= t\int\limits_E \omega_{h/t}(\|x - y\|)f(y)\,dy = t\bar{f}_{h/t}(x).$$

Since $\lim\limits_{t\to+\infty} \bar{f}_{h/t}(x) = f(x)$ by Theorem 2.2.1 from [59, p. 29], we obtain $\psi[u](x) = -f(x)$ for all $x \in E$.

Let $S := \{x \in E : \|x\| = \sqrt{2},\ x_1 + x_2 + x_3 = 2\}$. It can be easily seen that $S$ is a circumference contained in $K$ and tangent to the boundary of $K$ at the points $P_1 = (0, 1, 1)$, $P_2 = (1, 0, 1)$, and $P_3 = (1, 1, 0)$ in such a manner that $S \setminus \{P_1, P_2, P_3\} \subset \operatorname{Int} K$. Note that $f$ is convex and $\partial f(x) = \operatorname{conv}\{x,\ (1, 1, 1)\}$ for all $x \in S$.

From Theorem 4.3 and Corollary 4.1, we get

$$\partial f(x) = \partial(-\psi[u](x)) \subset \mathcal{M}[u]$$

for each $x \in S \cap \operatorname{Int} K = S \setminus \{P_1, P_2, P_3\}$. Suppose now that $x$ tends to some of $P_i$, $i = 1, 2, 3$, along the set $S \cap \operatorname{Int} K$. Then for each $i = 1, 2, 3$, we obtain $\Delta_i := \operatorname{conv}\{P_i,\ (1, 1, 1)\} \subset \operatorname{cl} \bigcup\limits_{x\in\operatorname{Int} K} \partial f(x) \subset \operatorname{cl} \mathcal{M}[u]$.

Since all elements of $\Delta_i$ are comparable with $(1, 1, 1)$ with respect to $K^+ = K$ and $(1, 1, 1) = \operatorname{Max}(\Delta_i|K^+)$, we see that $(\Delta_i \setminus \{(1, 1, 1)\}) \cap \mathcal{M}[u] = \varnothing$. On the other hand, we have $(1, 1, 1) \in \partial f(1, 1, 1) \subset \mathcal{M}[u]$ since $(1, 1, 1) \in \operatorname{Int} K$.

Thus, for any nontrivial solution of (4.5) the set $\mathcal{M}[u]$ has three cuts along the vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ with common vertex at $(1, 1, 1)$ and, therefore, is not closed. □

The following statement gives a sufficient condition for a characteristic set to be closed.

**Corollary 4.4** ([53]). *Suppose that* $\dim E < +\infty$ *and there exists a convex polyhedral set* $M$ *such that* $M - K^+ \supset \mathcal{E}(u) \supset M$. *Then* $\mathcal{M}[u] = \operatorname{cl} \mathcal{P}[u]$ *for any nontrivial solution* $u$ *to* (1.2).

*Proof.* Since $M$ is polyhedral, we have

$$\operatorname{Max}(\mathcal{E}(u)|K^+) = \operatorname{cl} \operatorname{Pos}(\mathcal{E}(u)|(K^+)^{+i})$$

by Theorem 5.4 in [25]. Now the required assertion is an immediate consequence of Theorem 4.2. □

**Corollary 4.5** ([53]). *Let* $\dim E < +\infty$, *and* $\dim F < +\infty$. *If the cone* $K$ *is polyhedral, then the characteristic set of any nontrivial solution to* (1.2) *is closed.*

If $n = 2$ and $K = \mathbb{R}^2_+$, then it follows from [20] that $\mathcal{M}[u]$ is bounded for any nontrivial solution $u$ to (1.2). On the other hand, if $n > 2$, then the set $\mathcal{M}[u]$ may be unbounded. Indeed, it was shown in [15, c. 156] that a certain equation (1.2) with constant coefficients has a solution $u$ such that $\mathcal{M}[u]$ is unbounded when $n = 3$, $m = 2$, and $K = \{(x_1, x_2, x_3) \in E : x_3 \geq (x_1^2 + x_2^2)^{1/2}\}$.

In order to establish sufficient conditions for the boundedness of $\mathcal{M}[u]$ we will use the following statement.

**Theorem 4.4** ([56]). *Let* $\dim E < +\infty$. *The characteristic set of a nontrivial solution to* (1.2) *is bounded iff the proper characteristic set of this solution is bounded.*

*Proof.* By Corollary 4.1, we have $\mathcal{P}[u] \subset \mathcal{M}[u]$. Hence, the boundedness of $\mathcal{M}[u]$ implies the boundedness of $\mathcal{P}[u]$. The opposite implication follows from the inclusion $\mathcal{M}[u] \subset \operatorname{cl} \mathcal{P}(u)$ since the closure of a bounded set is bounded in a finite-dimensional space. □

Theorem 4.4 enables us to give a criterion of boundedness for characteristic sets in terms of characteristic exponents.

**Theorem 4.5** ([56]). *Let* $\dim E < +\infty$. *The characteristic set* $\mathcal{M}[u]$ *of a nontrivial solution* $u$ *to* (1.2) *is bounded iff the function* $\varphi[u] := \overline{\operatorname{conv}}(-\psi[u])$ *is Lipschitzian on* $\operatorname{Int} K$.

*Proof.* Suppose that $\varphi[u]$ is Lipschitzian on $\operatorname{Int} K$ with the Lipschitz constant $L$. Take any $x \in \operatorname{Int} K$ and $h \in E$ such that $x + h \in \operatorname{Int} K$. Then for any subgradient $\mu \in \partial\varphi[u](x) \subset E^*$ we have $\mu h \leq \varphi[u](x + h) - \varphi[u](x) \leq L\|h\|$ and $\mu(-h) \leq \varphi[u](x - h) - \varphi[u](x) \leq L\|h\|$ by definition [67, p. 230]. Hence we get $|\mu h| \leq L\|h\|$. Since $K - x$ is a neighborhood of zero in $E$, we have $\|\mu\| \leq L$. By corollary 4.3, it follows that $\mathcal{P}[u]$ is bounded. Thus, $\mathcal{M}[u]$ is bounded too by Theorem 4.4.

Conversely, suppose that $\mathcal{M}[u] \subset B := \{x \in E : \|x\| \leq L\}$. Then by Theorem 4.3 and Corollary 4.1 we have $\partial\varphi[u](x) \subset \mathcal{P}[u] \subset \mathcal{M}[u] \subset B$ for all $x \in \operatorname{Int} K$. Take any $x, y \in \operatorname{Int} K$ and $\mu \in \partial\varphi[u](y)$, $\nu \in \partial\varphi[u](x)$. Then we can write $\varphi[u](x) - \varphi[u](y) \geq \mu(x - y) \geq -L\|x - y\|$ and $\varphi[u](y) - \varphi[u](x) \geq \nu(y - x) \geq -L\|x - y\|$. It follows now that $|\varphi[u](x) - \varphi[u](y)| \leq L\|x - y\|$, and this completes the proof. □

In general case, to verify the conditions of Theorem 4.5 is a difficult problem since there is no easy way to evaluate the function $\varphi[u]$. However, in some specific cases we are able to obtain effective conditions for boundedness and unboundedness of $\mathcal{M}[u]$ using some information on geometric properties of $K$.

The cone $K$ is said to be strictly convex if $K \setminus r$ is convex for any ray $r$ contained in the boundary of $K$, i.e., each such $r$ is extremal.

**Theorem 4.6** ([56]). *Let* $\dim E < +\infty$. *If* $K$ *is strictly convex, then* $\mathcal{M}[u]$ *is bounded for some nontrivial solution* $u$ *of* (1.2) *iff there exists a* *number $L > 0$ such that for any $x \neq 0$ from the boundary of $K$ there exists a functional $\mu_x \in E^*$ such that $\|\mu_x\| \leq L$ and*

$$\psi[u](z) \leq \psi[u](x) + \mu_x(x - z). \tag{4.6}$$

*for all $z \in K$.*

*Remark* 4.1. If $n = 2$, then Theorem 4.6 is equivalent to Theorem 1.7 in [19].

**Example 4.2.** Consider the equation

$$dy = y\,dx_1,\ y \in \mathbb{R}^2,\ x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

with constant coefficients. The solution $y$ of the Cauchy problem for this equation with initial value $y(0) = (1,\,1)$ has a modified characteristic exponent $\psi[y](x) = |x_1|$. If $K = \{(x_1, x_2, x_3) \in E\ :\ x_3 \geq (x_1^2 + x_2^2)^{1/2}\}$, then $\mathcal{M}[y]$ is unbounded since the estimation (4.6) is not valid for the function $\psi[y](x) = |x_1|$ at the point $x_0 = (0,\,1,\,1)$, whatever $\mu_x$ be taken.

Indeed, suppose that there exists some $\mu \in E^*$ such that (4.6) holds. Let $z_\pm := (\pm(2t - t^2)^{1/2},\, -t,\, 1)$. Then for $0 < t < 1$ we have $x_0 + z_\pm \in \operatorname{Int} K$ and $\mu z_\pm \geq \psi[y](x_0 + z_\pm) - \psi[y](x_0) = (2t - t^2)^{1/2}$. Hence $\|\mu\| \geq \|z_+ + z_-\|^{-1}|\mu(z_+) + \mu(z_-)| = (2/t - 1)^{-1/2} \to +\infty$ as $t \to 0$.

Note that in this case the unboundedness of $\mathcal{M}[y]$ can be proved in another way presented in [15, p. 156].

**Theorem 4.7** ([56])**.** *Let $\dim E < +\infty$. If $K$ is polyhedral, then the characteristic set of any nontrivial solution to* (1.2) *is bounded.*

*Remark* 4.2. Note that any closed convex pointed cone can be approximated by some convex polyhedral cone with arbitrarily small deviation. Thus we see that unboundedness of the characteristic set is merely a local effect related with a bad behavior of the solution near the boundary of $K$.

Finally we give the following statement describing a very good property of regular equations.

**Corollary 4.6** ([56])**.** *Let $\dim E < +\infty$. If $K$ is polyhedral and the equation* (1.2) *is regular, then the characteristic set of any nontrivial solution to* (1.2) *is compact.*

*Proof.* The characteristic set is bounded by Theorem 4.7 and closed by Corollary 4.5. □

## 5. Conclusions

The results presented in the paper show that the vector optimization theory provides adequate tools for the asymptotic theory of total differential equations. It turns out that applying these tools, we can solve several problems being too difficult for traditional approach. In addition, we are enabled to simplify the proofs substantially and make the presentation of the matter more clear and concise.

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