Memoirs on Differential Equations and Mathematical Physics
Volume 33, 2004, 5–23

H. Begehr

BOUNDARY VALUE PROBLEMS FOR THE BITSADZE EQUATION

Abstract. The Schwarz, Dirichlet, Neumann and some related boundary value problems are explicitly solved for the Bitsadze equation in the unit disc of the complex plane. The results are obtained from iterations of related results for the inhomogeneous Cauchy–Riemann equation. Some generalizations for the inhomogeneous polyanalytic equation are indicated.

2000 Mathematics Subject Classification. 35J25, 35C15, 30E25.
Key words and phrases. Schwarz, Dirichlet, Neumann boundary value problems, Bitsadze equation, inhomogeneous polyanalytic equation.

რეზიუმე. ნაშრომში კომპლექსური სიბრტყის ერთეულოვან წრეში ცხადი სახით ამოხსნილია შვარცის, დირიხლეს, ნეიმანის და ზოგიერთი სხვა მათთან დაკავშირებული ამოცანები. შედეგები მიიღება არაერთგვაროვანი კოში–რიმანის განტოლებისათვის მონათესავე შედეგების იტერაციების გზით. მითითებულია რამდენიმე განზოგადება არაერთგვაროვანი პოლიანალიზური განტოლებისათვის.

1. Introduction

From the three complex second order differential operators $\partial_{\bar{z}}^2, \partial_z\partial_{\bar{z}}$ and ∂_z^2, the Bitsadze operator $\partial_{\bar{z}}^2$ being the square of the Cauchy–Riemann operator is essentially not different from ∂_z^2 but differs principally from the Laplace operator $\partial_z\partial_{\bar{z}}$. This was remarked by Bitsadze [14] who showed that the Dirichlet problem as the natural boundary condition to the Laplace operator is ill-posed for the homogeneous Bitsadze equation. Nevertheless, besides the Schwarz and the Neumann boundary value problems this Dirichlet problem will also be treated. Under some solvability conditions it is shown to be solvable. In fact, the solutions to all the problems considered in the unit disc of the complex plane are given explicitly. Solutions together with the solvability conditions are attained from the iteration of results about related problems for the inhomogeneous Cauchy–Riemann equation. Their theory is developed in [6]. The same boundary value problems for the Poisson equation are investigated in [7]. Related considerations are available from [3, 9, 10, 11, 12, 13]. Basic Cauchy–Pompeiu representations are developed e.g. in [3, 5]. They originate from a hierarchy of integral operators [2, 8] which is constructed by iterating the Pompeiu operator [20]. This Pompeiu operator is the main tool in I.N. Vekua's theory of generalized analytic functions and its main properties were studied by Vekua [20]. Besides Gakhov [15], N.I. Mushelishvili [18] and Vekua [20, 21] have contributed substantially to the theory of boundary value problems for complex equations. Bitsadze and last but not least G. Manjavidze with his extensive work on boundary value problems with displacement [16] and on generalized analytic vectors [17] have complemented the work of Georgian mathematicians on the theory of complex analytic methods.

The material developed here was at first presented in a mini-course on boundary value problems in complex analysis at the University Simón Bolivar in Caracas in May 2004.

2. Cauchy–Pompeiu Representation Formulas

As from the Cauchy theorem the Cauchy formula is deduced, from the complex Gauss theorem representation formulas can be deduced.

Cauchy–Pompeiu representation *Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^1(D;\mathbb{C}) \cap C(\bar{D};\mathbb{C})$. Then using $\zeta = \xi + i\eta$ for $z \in D$,*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{D} w_{\bar{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z} \tag{1}$$

holds.

With respect to boundary value problems a modification of this Cauchy–Pompeiu formula is important. In the case of the unit disc $\mathbb{D} = \{z : |z| < 1\}$ it is as follows, see [1, 6, 8].

Theorem 1. *Any $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\bar{\mathbb{D}};\mathbb{C})$ is representable as*

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Re}\ w(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta} + \frac{1}{2\pi} \int\limits_{|\zeta|=1} \mathrm{Im}\ w(\zeta) \frac{d\zeta}{\zeta} - $$
$$- \frac{1}{\pi} \int\limits_{|\zeta|<1} \Big(\frac{w_{\bar{\zeta}}(\zeta)}{\zeta - z} + \frac{z w_{\bar{\zeta}}(\zeta)}{1 - z\bar{\zeta}} \Big) d\xi d\eta, |z| < 1\,. \tag{2}$$

Corollary. *Any $w \in C^1(\mathbb{D};\mathbb{C}) \cap C(\bar{\mathbb{D}};\mathbb{C})$ can be represented as*

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Re}\ w(\zeta) \frac{\zeta + z}{\zeta - z}\ \frac{d\zeta}{\zeta} -$$
$$- \frac{1}{2\pi} \int\limits_{|\zeta|<1} \Big(\frac{w_{\bar{\zeta}}(\zeta)}{\zeta} \frac{\zeta + z}{\zeta - z} + \frac{\bar{w}_{\bar{\zeta}}(\zeta)}{\bar{\zeta}} \frac{1 + z\bar{\zeta}}{1 - z\bar{\zeta}} \Big) d\xi d\eta +$$
$$+ i \mathrm{Im}\, w(0), \quad |z| < 1. \tag{3}$$

Remark. For analytic functions, (3) is the Schwarz–Poisson formula

$$w(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \mathrm{Re}\ w(\zeta) \Big(\frac{2\zeta}{\zeta - z} - 1 \Big) \frac{d\zeta}{\zeta} + i\ \mathrm{Im}\ w(0)\,. \tag{3'}$$

The kernel

$$\frac{\zeta + z}{\zeta - z} = \frac{2\zeta}{\zeta - z} - 1$$

is called the Schwarz kernel. Its real part

$$\frac{\zeta}{\zeta - z} + \frac{\bar{\zeta}}{\overline{\zeta - z}} - 1 = \frac{|\zeta|^2 - |z|^2}{|\zeta - z|^2}$$

is the Poisson kernel. The Schwarz operator

$$S\varphi(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \varphi(\zeta) \frac{\zeta + z}{\zeta - z} \frac{d\zeta}{\zeta}$$

for $\varphi \in C(\partial\mathbb{D};\mathbb{R})$ is known to provide an analytic function in $\mathbb{D}$ satisfying

$$\mathrm{Re}\ S\varphi = \varphi \quad \text{on} \quad \partial\mathbb{D}$$

(see [1]) in the sense

$$\lim_{z \to \zeta} S\varphi(z) = \varphi(\zeta), \quad \zeta \in \partial\mathbb{D}\ ,$$

for z in $\mathbb{D}$ tending to ζ. The operator

$$Tf(z) = -\frac{1}{\pi} \int\limits_D f(z) \frac{d\xi d\eta}{\zeta - z}, \quad z \in \mathbb{C},$$

on $L_1(D;\mathbb{C})$ is the Pompeiu operator [19].

The formula (3) is called the Cauchy–Schwarz–Poisson–Pompeiu formula. Rewriting it according to

$$w_{\bar z} = f \ \text{ in } \ \mathbb{D}, \quad \operatorname{Re} w = \varphi \ \text{ on } \ \partial\mathbb{D}, \quad \operatorname{Im} w(0) = c\,,$$

we have that

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \varphi(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \\ - \frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\Big)d\xi d\eta + ic \qquad (3'')$$

is expressed by the given data. Applying the result of Schwarz one easily sees, taking the real part on the right-hand side and letting z tend to a boundary point ζ, that this tends to $\varphi(\zeta)$.

Differentiating with respect to $\bar z$, as every term on the right-hand side is analytic besides the T-operator applied to f, one obtains $f(z)$. Also for $z = 0$ besides ic all other terms on the right-hand side are real.

Hence, $(3'')$ is a solution to the so-called Dirichlet problem

$$w_{\bar z} = f \ \text{ in } \ \mathbb{D}, \quad \operatorname{Re} w = \varphi \ \text{ on } \ \partial\mathbb{D}, \quad \operatorname{Im} w(0) = c\,.$$

This shows how integral representation formulas serve to solve boundary value problems. The method is not restricted to the unit disc but in this case the solutions to the problems are given explicitly.

3. Iteration of Integral Representation Formulas

Integral representation formulas for solutions to first order equations can be used to get such formulas for higher order equations via iteration, see [1]-[3],[5, 8].

Theorem 2. *Let $D \subset \mathbb{C}$ be a regular domain and $w \in C^2(D;\mathbb{C}) \cap C^1(\bar D;\mathbb{C})$, then*

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar\zeta}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta + \\ + \frac{1}{\pi}\int\limits_{D} w_{\bar\zeta\bar\zeta}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}\, d\xi d\eta\,. \qquad (4)$$

Proof. For proving (4), the formula (1) applied to $w_{\bar z}$ giving

$$w_{\bar\zeta}(\zeta) = \frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar{\tilde\zeta}}(\tilde\zeta)\frac{d\tilde\zeta}{\tilde\zeta - \zeta} - \frac{1}{\pi}\int\limits_{D} w_{\bar{\tilde\zeta}\bar{\tilde\zeta}}(\tilde\zeta)\frac{d\tilde\xi d\tilde\eta}{\tilde\zeta - \zeta}$$

is inserted into (1), from what after having interchanged the order of integrations

$$w(z) = \frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar{\tilde\zeta}}(\tilde\zeta)\psi(z,\tilde\zeta)d\tilde\zeta -$$
$$- \frac{1}{\pi}\int\limits_{D} w_{\bar{\tilde\zeta}\bar{\tilde\zeta}}(\tilde\zeta)\psi(z,\tilde\zeta)d\tilde\xi d\tilde\eta \tag{5}$$

follows with

$$\psi(z,\zeta) = \frac{1}{\pi}\int\limits_{D}\frac{d\xi d\eta}{(\zeta-\tilde\zeta)(\zeta - z)} = \frac{1}{\tilde\zeta - z}\frac{1}{\pi}\int\limits_{D}\left(\frac{1}{\zeta-\tilde\zeta} - \frac{1}{\zeta - z}\right)d\xi d\eta\ .$$

The formula (1) applied to the function $\bar z$ shows

$$\frac{\overline{\tilde\zeta - z}}{\tilde\zeta - z} = \frac{1}{2\pi i}\int\limits_{\partial D}\frac{\bar\zeta d\zeta}{(\zeta-\tilde\zeta)(\zeta - z)} - \frac{1}{\pi}\int\limits_{D}\frac{d\xi d\eta}{(\zeta-\tilde\zeta)(\zeta - z)} = \tilde\psi(z,\tilde\zeta) - \psi(z,\tilde\zeta) \tag{6}$$

with a function $\tilde\psi$ analytic in both its variables. Hence by the complex Gauss theorem

$$\frac{1}{2\pi i}\int\limits_{\partial D} w_{\bar{\tilde\zeta}}(\tilde\zeta)\tilde\psi(z,\tilde\zeta)d\tilde\zeta - \frac{1}{\pi}\int\limits_{D} w_{\bar{\tilde\zeta}\bar{\tilde\zeta}}(\tilde\zeta)\tilde\psi(z,\tilde\zeta)d\tilde\xi d\tilde\eta = 0\ .$$

Subtracting this from (5) and applying (6) gives (4). □

Remark. There is a dual formula to (4) resulting from interchanging the roles of z and $\bar z$ in the preceding procedure. It can be also derived by applying complex conjugation to (4) after replacing w by $\bar w$. It is

$$w(z) = -\frac{1}{2\pi i}\int\limits_{\partial D} w(\zeta)\frac{d\bar\zeta}{\overline{\zeta - z}} + \frac{1}{2\pi i}\int\limits_{\partial D} w_\zeta(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}d\bar\zeta +$$
$$+ \frac{1}{\pi}\int\limits_{D} w_{\zeta\zeta}(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}d\xi d\eta\ . \tag{4'}$$

The kernel functions $(\overline{\zeta - z})/(\zeta - z), (\zeta - z)/(\overline{\zeta - z})$ of the second order differential operators $\partial_{\bar z}^2, \partial_z^2$ respectively are thus obtained from those Cauchy and anti-Cauchy kernels $1/(\zeta - z)$ and $1/(\overline{\zeta - z})$ for the Cauchy–Riemann operator $\partial_{\bar z}$ and its complex conjugate ∂_z. The related weakly singular integral operators are

$$T_{0,2}f(z) = \frac{1}{\pi}\int\limits_{D} f(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}\,d\xi\,d\eta,\ T_{2,0}f(z) = \frac{1}{\pi}\int\limits_{D} f(\zeta)\frac{\zeta - z}{\overline{\zeta - z}}\,d\xi\,d\eta$$

acting on $L_1(D;\mathbb{C})$.

4. Boundary Value Problems for the Inhomogeneous Bitsadze Equation

There are two basic second order differential operators, the Laplace operator $\partial_z \partial_{\bar z}$ and the Bitsadze operator $\partial_{\bar z}^2$. The third one, ∂_z^2 is just the complex conjugate of the Bitsadze operator and all formulas and results for this operator can be attained by the ones for the Bitsadze operator through complex conjugation giving dual formulas and results. Here the Bitsadze operator will be investigated. For the problems for the Laplace operator compare [7].

Theorem 3. *The Schwarz problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar z\bar z}=f \text{ in } \mathbb{D},\ \ \mathrm{Re}\, w=\gamma_0,\ \ \mathrm{Re}\, w_{\bar z}=\gamma_1 \text{ on } \partial\mathbb{D},\ \ \mathrm{Im}\, w(0)=c_0,\ \ \mathrm{Im}\, w_{\bar z}(0)=c_1,$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{R}), c_0, c_1 \in \mathbb{R}$ *is uniquely solvable through*

$$\begin{aligned}
w(z) = {} & ic_0 + i(z+\bar z) + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\frac{d\zeta}{\zeta} - \\
& - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})\frac{d\zeta}{\zeta} + \\
& + \frac{1}{2\pi}\int\limits_{|\zeta|<1} \Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\Big)(\zeta - z + \overline{\zeta - z})d\xi d\eta\,. \qquad (7)
\end{aligned}$$

Proof. Rewriting the problem as the system

$$\begin{array}{lll}
w_{\bar z} = \omega \ \text{ in } \ \mathbb{D}, & \mathrm{Re}\ w = \gamma_0 \ \text{ on } \ \partial\mathbb{D}, & \mathrm{Im}\ w(0) = c_0\ , \\
w_{\bar z} = f \ \text{ in } \ \mathbb{D}, & \mathrm{Re}\ \omega = \gamma_1 \ \text{ on } \ \partial\mathbb{D}, & \mathrm{Im}\ \omega(0) = c_1\ ,
\end{array}$$

and combining its solutions

$$\begin{aligned}
w(z) = {} & ic_0 + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} - \\
& - \frac{1}{2\pi}\int\limits_{|\zeta|<1} \left(\frac{\omega(\zeta)}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{\overline{\omega(\zeta)}}{\bar\zeta}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\right) d\xi\, d\eta\ , \\
\omega(z) = {} & ic_1 + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{\zeta+z}{\zeta-z}\,\frac{d\zeta}{\zeta} - \\
& - \frac{1}{2\pi}\int\limits_{|\zeta|<1} \left(\frac{f(\zeta)}{\zeta}\,\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\,\frac{1+z\bar\zeta}{1-z\bar\zeta}\right) d\xi\, d\eta\ ,
\end{aligned}$$

one obtains formula (7). Here the relations

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\Big(\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}-\frac{1}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\Big)\,d\xi\,d\eta=-z-\bar z\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{\tilde\zeta+\zeta}{\tilde\zeta-\zeta}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\,d\xi\,d\eta=\frac{\tilde\zeta+z}{\tilde\zeta-z}\overline{(\tilde\zeta-z)}\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{1+\bar\zeta\tilde\zeta}{1-\bar\zeta\tilde\zeta}\frac{1}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\,d\xi\,d\eta=\tilde\zeta+z\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{\overline{\tilde\zeta}+\zeta}{\overline{\tilde\zeta}-\zeta}\frac{1}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\,d\xi\,d\eta=\frac{1+z\overline{\tilde\zeta}}{1-z\overline{\tilde\zeta}}(\tilde\zeta-z)\ ,$$

$$\frac{1}{2\pi}\int\limits_{|\zeta|<1}\frac{1+\zeta\overline{\tilde\zeta}}{1-\zeta\overline{\tilde\zeta}}\frac{1}{\zeta}\frac{\zeta+z}{\zeta-z}\,d\xi d\eta=\frac{1+z\overline{\tilde\zeta}}{1-z\overline{\tilde\zeta}}\ \overline{(\tilde\zeta-z)}$$

are used. The uniqueness of the solution follows from the unique solvability of the Schwarz problem for analytic functions and the inhomogeneous Cauchy–Riemann equation, see [6]. □

It is well known that the Dirichlet problem for the Poisson equation

$$w_{z\bar z}=f\ \text{ in }\ \mathbb{D},\quad w=\gamma\ \text{ on }\ \partial\mathbb{D}\,,$$

is well posed, i.e. it is solvable for any $f\in L_1(\mathbb{D};\mathbb{C})$, $\gamma\in C(\partial\mathbb{D};\mathbb{C})$ and the solution is unique. That the solution is unique is easily seen.

Lemma 1. *The Dirichlet problem for the Laplace equation*

$$w_{z\bar z}=0\ \textit{ in }\ \mathbb{D},\quad w=0\ \textit{ on }\ \partial\mathbb{D}$$

is only trivially solvable.

Proof. From the differential equation w_z is seen to be analytic. Integrating, one obtains $w=\varphi+\bar\psi$, where φ and ψ are both analytic in $\mathbb{D}$. Without loss of generality $\psi(0)=0$ may be assumed. From the boundary condition $\varphi=-\bar\psi$ on $\partial\mathbb{D}$ follows. This Dirichlet problem is solvable if and only if, see [6],

$$0=\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\overline{\psi(\zeta)}\frac{\bar z d\zeta}{1-\bar z\zeta}=\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\psi(\zeta)\frac{d\zeta}{\zeta-z}}-\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\psi(\zeta)\frac{d\zeta}{\zeta}}=\overline{\psi(z)}\ .$$

This also implies $\varphi=0$ on $\mathbb{D}$ so that $w=0$ in $\mathbb{D}$. □

As Bitsadze [14] has realized, such a result is not true for the equation $w_{\bar z\bar z}=0$.

Lemma 2. *The Dirichlet problem for the Bitsadze equation*

$$w_{\bar z\bar z} = 0 \ \text{ in } \mathbb{D}, \quad w = 0 \ \text{ on } \partial\mathbb{D}$$

has infinitely many linearly independent solutions.

Proof. Here $w_{\bar z}$ is an analytic function in $\mathbb{D}$. Integrating gives $w(z) = \varphi(z)\bar z + \psi(z)$ with some analytic functions in $\mathbb{D}$. On the boundary we have $\varphi(z) + z\psi(z) = 0$. As this is an analytic function, this relation hold in $\mathbb{D}$ too. Hence, $w(z) = (1-|z|^2)\psi(z)$ for arbitrary analytic ψ. In particular $w_k(z) = (1-|z|^2)z^k$ is a solution of the Dirichlet problem for any $k \in \mathbb{N}$ and these solutions are linearly independent over $\mathbb{C}$. □

Because of this result, the Dirichlet problem as formulated above is ill-posed for the inhomogeneous Bitsadze equation.

Since the Dirichlet problem formulated as for the Poisson equation is not uniquely solvable for the Bitsadze equation, another kind Dirichlet problem is considered which is motivated from decomposing the Bitsadze equation into a first order system.

Theorem 4. *The Dirichlet problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar z\bar z} = f \ \text{ in } \mathbb{D}, \quad w = \gamma_0, \ w_{\bar z} = \gamma_1 \ \text{ on } \partial\mathbb{D}\,,$$

for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$ is solvable if and only if for $|z|<1$

$$\frac{\bar z}{2\pi i}\int\limits_{|\zeta|=1}\left(\frac{\gamma_0(\zeta)}{1-\bar z\zeta} - \frac{\gamma_1(\zeta)}{\zeta}\right)d\zeta + \frac{\bar z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{\overline{\zeta - z}}{1-\bar z\zeta}\,d\xi\,d\eta = 0 \tag{8}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\,\frac{\bar z d\zeta}{1-\bar z\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\,\frac{\bar z d\xi d\eta}{1-\bar z\zeta} = 0\,. \tag{9}$$

The solution then is

$$\begin{aligned} w(z) = {} & \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\,\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta + \\ & + \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\,\frac{\overline{\zeta - z}}{\zeta - z}\,d\xi\,d\eta\,. \end{aligned} \tag{10}$$

Proof. Decomposing the problem into the system

$$\begin{aligned} & w_{\bar z} = \omega \ \text{ in } \mathbb{D}, \quad w = \gamma_0 \ \text{ on } \partial\mathbb{D}\,, \\ & \omega_{\bar z} = f \ \text{ in } \mathbb{D}, \quad \omega = \gamma_1 \ \text{ on } \partial\mathbb{D}\,, \end{aligned}$$

composing its solutions

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_0(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\,\frac{d\xi d\eta}{\zeta - z}\,,$$

$$\omega(z) = \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{d\zeta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{\zeta - z},$$

and using the solvability conditions

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_0(\zeta) \frac{\bar{z} d\zeta}{1 - \bar{z}\zeta} = \frac{1}{\pi} \int\limits_{|\zeta|<1} \omega(\zeta) \frac{\bar{z} d\xi d\eta}{1 - \bar{z}\zeta},$$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_1(\zeta) \frac{\bar{z} d\zeta}{1 - \bar{z}\zeta} = \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{\bar{z} d\xi d\eta}{1 - \bar{z}\zeta},$$

one proves (10) together with (8) and (9). Here

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{(\tilde{\zeta} - \zeta)(1 - \bar{z}\zeta)} = \frac{\overline{\tilde{\zeta} - z}}{1 - \bar{z}\tilde{\zeta}} - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\overline{\zeta - z}}{1 - \bar{z}\zeta} \frac{d\zeta}{\zeta - \tilde{\zeta}} = \frac{\overline{\tilde{\zeta} - z}}{1 - \bar{z}\tilde{\zeta}}$$

and

$$-\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{(\zeta - \tilde{\zeta})(\zeta - z)} = -\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{1}{\tilde{\zeta} - z} \left(\frac{1}{\zeta - \tilde{\zeta}} - \frac{1}{\zeta - z} \right) d\xi \, d\eta = \frac{\overline{\tilde{\zeta} - z}}{\tilde{\zeta} - z}$$

are used. □

Theorem 5. *The Dirichlet–Neumann problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar{z}\bar{z}} = f \ \text{ in } \mathbb{D}, \quad w = \gamma_0, \quad \partial_\nu w_{\bar{z}} = \gamma_1 \ \text{ on } \partial\mathbb{D}, \quad w_{\bar{z}}(0) = c,$$

for $f \in L_1(\mathbb{D}; \mathbb{C}) \cap C(\partial\mathbb{D}; \mathbb{C})$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D}; \mathbb{C})$, $c \in \mathbb{C}$ *is solvable if and only if for* $z \in \mathbb{D}$

$$c - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_0(\zeta) \frac{d\zeta}{1 - \bar{z}\zeta} + \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{1 - |\zeta|^2}{\zeta(1 - \bar{z}\zeta)} d\xi \, d\eta = 0 \tag{11}$$

and

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta)) \frac{d\zeta}{\zeta(1 - \bar{z}\zeta)} + \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{\bar{z} f(\zeta)}{(1 - \bar{z}\zeta)^2} d\xi \, d\eta = 0. \tag{12}$$

The solution then is

$$\begin{aligned} w(z) = c\bar{z} &+ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma_0(\zeta) \frac{d\zeta}{\zeta - z} + \\ &+ \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta)) \frac{1 - |z|^2}{z} \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} + \\ &+ \frac{1}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{|\zeta|^2 - |z|^2}{\zeta(\zeta - z)} d\xi d\eta. \end{aligned} \tag{13}$$

Proof. The problem is equivalent to the system

$$w_{\bar z} = \omega \;\text{ in }\; \mathbb{D}, \quad w = \gamma_0 \;\text{ on }\; \partial\mathbb{D}\,,$$
$$\omega_{\bar z} = f \;\text{ in }\; \mathbb{D}, \quad \partial_\nu \omega = \gamma_1 \;\text{ on }\; \partial\mathbb{D}, \quad \omega(0) = c\,.$$

The solvability conditions are

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\,\frac{d\zeta}{1-\bar z\zeta} = \frac{1}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\,\frac{d\xi d\eta}{1-\bar z\zeta}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar\zeta f(\zeta))\,\frac{d\zeta}{1-\bar z\zeta)\zeta} + \frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{\bar z f(\zeta)}{(1-\bar z\zeta)^2}\,d\xi\,d\eta = 0,$$

and the unique solutions are given by

$$w(z) = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\,\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\,\frac{d\xi d\eta}{\zeta - z}$$

and

$$\omega(z) = c - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma(\zeta) - \bar\zeta f(\zeta))\log(1 - z\bar\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{z f(\zeta)}{\zeta(\zeta - z)}\,d\xi\,d\eta$$

according to the results on the Dirichlet and Neumann problems for the inhomogeneous Cauchy–Riemann equation, see [6]. From

$$\frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{d\xi d\eta}{1-\bar z\zeta} = 1,$$
$$\frac{1}{\pi}\int\limits_{|\zeta|<1} \log(1-\zeta\bar{\tilde\zeta})\,\frac{d\xi d\eta}{1-\bar z\zeta} = \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \log(1-\zeta\bar{\tilde\zeta})\,\frac{d\zeta}{(1-\bar z\zeta)\zeta} = 0$$

and

$$\frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{\zeta}{\tilde\zeta - \zeta}\,\frac{d\xi d\eta}{1-\bar z\zeta} = \frac{|\tilde\zeta|^2 - 1}{\tilde\zeta(1-\bar z\tilde\zeta)}\,,$$

the condition (11) follows. Similarly (13) follows from

$$-\frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{d\xi d\eta}{\zeta - z} = \bar z, \quad \frac{1}{\pi}\int\limits_{|\zeta|<1} \log(1-\zeta\bar{\tilde\zeta})\frac{d\xi d\eta}{\zeta - z} = \frac{1-|z|^2}{z}\log(1 - z\bar{\tilde\zeta})$$

and

$$\frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{\zeta}{(\tilde\zeta - \zeta)}\frac{d\xi d\eta}{\zeta - z} = \frac{|\tilde\zeta|^2 - |z|^2}{(\tilde\zeta - z)}\,. \qquad \square$$

Theorem 6. *The boundary value problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar z\bar z} = f \ \text{ in } \mathbb{D}, \quad w = \gamma_0, \quad z w_{z\bar z} = \gamma_1 \ \text{ on } \partial\mathbb{D}, \quad w_{\bar z}(0) = c\,,$$

is solvable for $f \in L_1(\mathbb{D};\mathbb{C}), \gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$, $c \in \mathbb{C}$ *if and only if for* $z \in \mathbb{D}$ *the condition* (11) *together with*

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\,\frac{d\zeta}{\zeta(1-\bar z\zeta)} + \frac{\bar z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\,\frac{d\xi d\eta}{(1-\bar z\zeta)^2} = 0 \tag{14}$$

holds. The solution then is uniquely given by

$$w(z) = c\bar z + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\,\frac{1-|z|^2}{z}\log(1-z\bar\zeta)\frac{d\zeta}{\zeta} +$$
$$+ \frac{1}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\,\frac{|\zeta|^2 - |z<|^2}{\zeta(\zeta - z)} d\xi d\eta\,. \tag{15}$$

The proof is as the last one but a modification of the Neumann condition for the inhomogeneous Cauchy–Riemann equation is involved, see [6].

Theorem 7. *The Neumann problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar z\bar z} = f \ \text{ in } \mathbb{D}, \quad \partial_\nu w = \gamma_0, \quad \partial_\nu w_{\bar z} = \gamma_1 \ \text{ on } \partial\mathbb{D}, \quad w(0) = c_0, \quad w_{\bar z}(0) = c_1$$

is uniquely solvable for $f \in C^\alpha(\mathbb{D};\mathbb{C})$, $0 < \alpha < 1$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$, $c_0, c_1 \in \mathbb{C}$ *if and only if for* $z \in \partial\mathbb{D}$

$$c_1\bar z + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\bar\zeta}{\overline{\zeta - z}} - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar\zeta f(\zeta))(1 - \bar z\zeta\log(1 - z\bar\zeta))d\bar\zeta +$$
$$+ \frac{1}{\pi}\int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta}\,\Big(\frac{\bar z\zeta(\overline{\zeta - z})}{(1-\bar z\zeta)^2} - \frac{1}{\zeta - z}\Big) d\xi d\eta = 0 \tag{16}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar\zeta f(\zeta))\,\frac{d\bar\zeta}{\overline{\zeta - z}} - \frac{\bar z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{(1-\bar z\zeta)^2} = 0\,. \tag{17}$$

The solution then is given as

$$w(z) = c_0 + c_1\bar z - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\log(1 - z\bar\zeta)\frac{d\zeta}{\zeta} +$$
$$+ \frac{1}{2\pi i}\int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar\zeta f(\zeta))(\overline{\zeta - z})\log(1 - z\bar\zeta)\frac{d\zeta}{\zeta} +$$

$$+ \frac{z}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta} \frac{\overline{\zeta - z}}{\zeta - z} d\xi d\eta \,. \tag{18}$$

Proof. From the theory of the inhomogeneous Cauchy–Riemann equation [6],

$$w(z) = c_0 - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_0(\zeta) - \bar{\zeta}\omega(\zeta)) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} - \frac{z}{\pi} \int\limits_{|\zeta|<1} \omega(\zeta) \frac{d\xi d\eta}{\zeta(\zeta - z)} \,,$$

$$\omega(z) = c_1 - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta)) \log(1 - z\bar{\zeta}) \frac{d\zeta}{\zeta} - \frac{z}{\pi} \int\limits_{|\zeta|<1} f(\zeta) \frac{d\xi d\eta}{\zeta(\zeta - z)}$$

if and only if

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_0(\zeta) - \bar{\zeta}\omega(\zeta)) \frac{d\zeta}{\zeta(1 - \bar{z}\zeta)} + \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{(1 - \bar{z}\zeta)^2} \, d\xi \, d\eta = 0 \,,$$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta)) \frac{d\zeta}{\zeta(1 - \bar{z}\zeta)} + \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{(1 - \bar{z}\zeta)^2} \, d\xi \, d\eta = 0 \,.$$

Inserting ω into the first condition leads to (16) on the basis of

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \omega(\zeta) \frac{d\zeta}{\zeta^2(1 - \bar{z}\zeta)} =$$

$$= c_1 \bar{z} - \frac{1}{2\pi i} \int\limits_{|\tilde{\zeta}|=1} (\gamma_1(\tilde{\zeta}) - \bar{\tilde{\zeta}} f(\tilde{\zeta})) \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{\log(1 - \zeta\bar{\tilde{\zeta}})}{1 - \bar{z}\zeta} \frac{d\zeta}{\zeta^2} \frac{d\tilde{\zeta}}{\tilde{\zeta}} -$$

$$- \frac{1}{\pi} \int\limits_{|\tilde{\zeta}|<1} \frac{f(\tilde{\zeta})}{\tilde{\zeta}} \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \frac{d\zeta}{(\tilde{\zeta} - \zeta)\zeta(1 - \bar{z}\zeta)} d\tilde{\xi} d\tilde{\eta} =$$

$$= c_1 \bar{z} + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta)) \bar{\zeta} \frac{d\zeta}{\zeta} + \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta(1 - \bar{z}\zeta)} d\xi d\eta$$

and

$$\frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \frac{\omega(\zeta)}{(1 - \bar{z}\zeta)^2} d\xi d\eta =$$

$$= \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \partial_{\bar{\zeta}} \frac{(\overline{\zeta - z})\omega(\zeta)}{(1 - \bar{z}\zeta)^2} d\xi d\eta - \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \frac{\overline{\zeta - z}}{(1 - \bar{z}\zeta)^2} f(\zeta) d\xi d\eta =$$

$$= \frac{\bar{z}}{2\pi i} \int\limits_{|\zeta|=1} \frac{\overline{\zeta - z}}{(1 - \bar{z}\zeta)^2} \omega(\zeta) d\zeta - \frac{\bar{z}}{\pi} \int\limits_{|\zeta|<1} \frac{\overline{\zeta - z}}{(1 - \bar{z}\zeta)^2} \, f(\zeta) d\xi d\eta \,,$$

where for $|z| = 1$

$$\frac{\bar z}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\zeta - z}}{(1-\bar z\zeta)^2}\omega(\zeta)d\zeta = \frac{z}{2\pi i}\int\limits_{|\zeta|=1}\frac{\overline{\zeta - z}}{(\zeta - z)^2}\omega(\zeta)d\zeta = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\omega(\zeta)}{\zeta - z}\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int\limits_{|\tilde\zeta|=1}(\gamma_1(\tilde\zeta) - \bar{\tilde\zeta}f(\tilde\zeta))\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1-\zeta\bar{\tilde\zeta})}{\zeta(\zeta - z)}d\zeta\frac{d\tilde\zeta}{\tilde\zeta} -$$

$$-\frac{1}{\pi}\int\limits_{|\tilde\zeta|<1}\frac{f(\tilde\zeta)}{\tilde\zeta}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{d\zeta}{(\tilde\zeta - \zeta)(\zeta - z)}d\tilde\xi d\tilde\eta =$$

$$= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta) - \bar\zeta f(\zeta))\frac{1}{z}\ \log(1 - z\bar\zeta)\frac{d\zeta}{\zeta} =$$

$$= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}(\gamma_1(\zeta) - \bar\zeta f(\zeta))\bar z\log(1 - z\bar\zeta)\frac{d\zeta}{\zeta}\,.$$

From

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1 - z\bar\zeta)d\bar\zeta = -\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-\bar z\zeta)d\zeta} = 0\,,$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1 - \zeta\bar{\tilde\zeta})\log(1 - z\bar\zeta)d\bar\zeta = -\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1 - \bar\zeta\tilde\zeta)\log(1-\bar z\zeta)d\zeta} =$$

$$= \sum_{k=1}^{+\infty}\frac{\bar{\tilde\zeta}^k}{k}\overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\log(1-\bar z\zeta)\frac{d\zeta}{\zeta^2}} =$$

$$= \sum_{k=2}^{+\infty}\frac{\bar{\tilde\zeta}^k}{k!}\overline{\partial_\zeta^{k-1}\log(1-\bar z\zeta)}|_{\zeta=0} = -\sum_{k=2}^{+\infty}\frac{\bar{\tilde\zeta}^k z^{k-1}}{(k-1)k} =$$

$$= \bar{\tilde\zeta}\log(1 - z\bar{\tilde\zeta}) - \frac{1}{z}\ (\log(1 - z\bar{\tilde\zeta}) + z\bar{\tilde\zeta}) = -\frac{1 - z\bar{\tilde\zeta}}{z}\ \log(1 - z\bar{\tilde\zeta}) - \bar{\tilde\zeta}\,,$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\zeta\log(1 - z\bar\zeta)}{\tilde\zeta - \zeta}\,d\bar\zeta = -\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log(1 - z\bar\zeta)}{1 - \tilde\zeta\bar\zeta}\,d\bar\zeta =$$

$$= \overline{\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\log 1 - \bar z\zeta}{1 - \bar{\tilde\zeta}\zeta}d\zeta} = 0$$

the relation

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\omega(\zeta)\log(1 - z\bar\zeta)\,d\bar\zeta =$$

$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta))\Big(\frac{1 - z\bar{\zeta}}{z} \log(1 - z\bar{\zeta}) + \bar{\zeta}\Big)\frac{d\zeta}{\zeta} \tag{19}$$

follows. Similarly from

$$\frac{z}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{\zeta(\zeta - z)} = \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{\zeta - z} - \frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{\zeta} = -\bar{z}\,,$$

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{\log(1 - \zeta\bar{\tilde{\zeta}})}{\zeta - z} d\xi d\eta =$$

$$= (\overline{\tilde{\zeta} - z}) \log(1 - z\bar{\tilde{\zeta}}) - \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\overline{\tilde{\zeta} - \zeta}) \log(1 - \zeta\bar{\tilde{\zeta}}) \frac{d\zeta}{\zeta - z} =$$

$$= (\overline{\tilde{\zeta} - z}) \log(1 - z\bar{\tilde{\zeta}}) + \frac{1}{2\pi i} \int\limits_{|\zeta<=1} (1 - \zeta\bar{\tilde{\zeta}}) \log(1 - \zeta\bar{\tilde{\zeta}}) \frac{d\zeta}{\zeta(\zeta - z)} =$$

$$= (\overline{\tilde{\zeta} - z}) \log(1 - z\bar{\tilde{\zeta}}) + \frac{1 - z\bar{\tilde{\zeta}}}{z} \log(1 - z\bar{\tilde{\zeta}}) = \frac{1 - |z|^2}{z} \log(1 - z\bar{\tilde{\zeta}})\,,$$

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \log(1 - \zeta\bar{\tilde{\zeta}}) \frac{d\xi d\eta}{\zeta} = \frac{1}{\pi} \int\limits_{|\zeta|<1} \partial_{\bar{\zeta}} \frac{\bar{\zeta}}{\zeta} \log(1 - \zeta\bar{\tilde{\zeta}}) d\xi d\eta =$$

$$= \frac{1}{2\pi i} \int\limits_{|\zeta|=1} \log(1 - \zeta\bar{\tilde{\zeta}}) \frac{d\zeta}{\zeta^2} = -\bar{\tilde{\zeta}}\,,$$

and

$$\frac{1}{\pi} \int\limits_{|\zeta|<1} \frac{d\xi d\eta}{(\zeta - \tilde{\zeta})(\zeta - z)} = \frac{1}{\pi(\tilde{\zeta} - z)} \int\limits_{|\zeta|<1} \Big(\frac{1}{\zeta - \tilde{\zeta}} - \frac{1}{\zeta - z}\Big) d\xi\, d\eta = -\frac{\overline{\tilde{\zeta} - z}}{\tilde{\zeta} - z}$$

it follows

$$\frac{z}{\pi} \int\limits_{|\zeta|<1} \omega(\zeta) \frac{d\xi d\eta}{\zeta(\zeta - z)} =$$

$$= -c_1\bar{z} + \frac{1}{2\pi i} \int\limits_{|\zeta|=1} (\gamma_1(\zeta) - \bar{\zeta} f(\zeta))\Big(\frac{1 - |z|^2}{z} \log(1 - z\bar{\zeta}) + \bar{\zeta}\Big)\frac{d\zeta}{\zeta} -$$

$$- \frac{z}{\pi} \int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta} \frac{\overline{\zeta - z}}{\zeta - z} d\xi d\eta\,. \tag{20}$$

From (19) and (20) the representation (18) follows. □

Theorem 8. *The boundary value problem for the inhomogeneous Bitsadze equation in the unit disc*

$$w_{\bar{z}\bar{z}} = f \;\; \textit{in} \;\; \mathbb{D}, \quad zw_z = \gamma_0, \quad zw_{z\bar{z}} = \gamma_1 \;\; \textit{on} \;\; \partial\mathbb{D}, \quad w(0) = c_0, \quad w_{\bar{z}}(0) = c_1\,,$$

for $f \in L_1(\mathbb{D};\mathbb{C})$, $\gamma_0, \gamma_1 \in C(\partial\mathbb{D};\mathbb{C})$, $c_0, c_1 \in \mathbb{C}$ *is uniquely solvable if and only if for* $|z| = 1$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{(1-z\bar\zeta)\zeta} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{1}{z}\log(1-z\bar\zeta)\,\frac{d\zeta}{\zeta} =$$

$$= \frac{\bar z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{\overline{\zeta - z}}{(1-\bar z\zeta)^2}\,d\xi\,d\eta\,, \tag{21}$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{d\zeta}{(1-\bar z\zeta)\zeta} + \frac{\bar z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\frac{d\xi d\eta}{(1-\bar z\zeta)^2} = 0\,. \tag{22}$$

The solution then is

$$w(z) = c_0 + c_1\bar z - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\log(1-z\bar\zeta)\frac{d\zeta}{\zeta} +$$

$$+ \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\Big(\frac{1-|z|^2}{z}\log(1-z\bar\zeta) + \bar\zeta\Big)\frac{d\zeta}{\zeta} +$$

$$+ \frac{z}{\pi}\int\limits_{|\zeta|<1} \frac{f(\zeta)}{\zeta}\,\frac{\overline{\zeta - z}}{\zeta - z}\,d\xi\,d\eta\,. \tag{23}$$

Proof. The system

$$w_{\bar z} = \omega \ \text{ in } \ \mathbb{D}, \quad z w_z = \gamma_0 \ \text{ on } \ \partial\mathbb{D}, \quad w(0) = c_0\,,$$
$$\omega_{\bar z} = f \ \text{ in } \ \mathbb{D}, \quad z\omega_z = \gamma_1 \ \text{ on } \ \partial\mathbb{D}, \quad \omega(0) = c_1\,,$$

is uniquely solvable if and only if

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\frac{d\zeta}{(1-\bar z\zeta)\zeta} + \frac{\bar z}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\,\frac{d\xi d\eta}{(1-\bar z\zeta)^2} = 0$$

and

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\frac{d\zeta}{(1-\bar z\zeta)\zeta} + \frac{\bar z}{\pi}\int\limits_{|\zeta|=1} \omega(\zeta)\,\frac{d\xi d\eta}{(1-\bar z\zeta)^2} = 0\,.$$

The solution then is

$$w(z) = c_0 - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_0(\zeta)\log(1-z\bar\zeta)\frac{d\zeta}{\zeta} - \frac{z}{\pi}\int\limits_{|\zeta|<1} \omega(\zeta)\,\frac{d\xi d\eta}{\zeta(\zeta - z)}\,,$$

$$\omega(z) = c_1 - \frac{1}{2\pi i}\int\limits_{|\zeta|=1} \gamma_1(\zeta)\log(1-z\bar\zeta)\frac{d\zeta}{\zeta} - \frac{z}{\pi}\int\limits_{|\zeta|<1} f(\zeta)\,\frac{d\xi d\eta}{\zeta(\zeta - z)}\,.$$

Inserting the expression for ω into the first condition gives (21) because, as in the preceding proof, on $|z|=1$ one has

$$\frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1}\frac{\omega(\zeta)}{(1-\bar{z}\zeta)^2}d\xi d\eta =$$

$$= \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\frac{1}{z}\,\log(1-z\bar{\zeta})\frac{d\zeta}{\zeta} - \frac{\bar{z}}{\pi}\int\limits_{|\zeta|<1}\frac{\overline{\zeta-z}}{(1-\bar{z}\zeta)^2}f(\zeta)\,d\xi\,d\eta\,.$$

Also from

$$\frac{z}{\pi}\int\limits_{|\zeta|<1}\omega(\zeta)\frac{d\xi d\eta}{\zeta(\zeta-z)} = -\,c_1\bar{z} + \frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma_1(\zeta)\Big(\frac{1-|z|^2}{z}\log(1-z\bar{\zeta})+\bar{\zeta}\Big)\frac{d\zeta}{\zeta} -$$

$$- \frac{z}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{\zeta}\frac{\overline{\zeta-z}}{\zeta-z}\,d\xi\,d\eta$$

the formula (23) follows. □

5. The Inhomogeneous Polyanalytic Equation

As second order equations of special type were treated in the preceding section, model equations of third, forth, fifth etc. order can also be investigated. From the material presented it is clear how to proceed and what kind of boundary conditions can be posed. However, there is a variety of boundary conditions possible. All kind of combinations of the three kinds, Schwarz, Dirichlet, Neumann conditions can be posed. And there are even others, e.g., boundary conditions of mixed type which are not investigated here.

As a simple example, the Schwarz problem will be studied for the inhomogeneous polyanalytic equation. Another possibility is the Neumann problem for the inhomogeneous polyharmonic equation, see [12, 13], and the Dirichlet problem, see [5]. As the results are published elsewhere [10, 11] only statements are given.

Theorem 9. *The Schwarz problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\bar{z}}^n w = f \;\text{ in }\; \mathbb{D}, \quad \operatorname{Re}\,\partial_{\bar{z}}^{\nu} w = \gamma_\nu \;\text{ on }\; \partial\mathbb{D}, \quad \operatorname{Im}\,\partial_{\bar{z}}^{\nu} w(0) = c_\nu, \quad 0\le\nu\le n-1\,,$$

is uniquely solvable for $f\in L_1(\mathbb{D};\mathbb{C})$, $\gamma_\nu\in C(\partial\mathbb{D};\mathbb{R})$, $c_\nu\in\mathbb{R}$, $0\le\nu\le n-1$. The solution is

$$w(z) = i\sum_{\nu=0}^{n-1}\frac{c_\nu}{\nu!}(z+\bar{z})^\nu + \sum_{\nu=0}^{n-1}\frac{(-1)^\nu}{2\pi i\nu!}\int\limits_{|\zeta|=1}\gamma_\nu(\zeta)\frac{\zeta+z}{\zeta-z}(\zeta-z+\overline{\zeta-z})^\nu\frac{d\zeta}{\zeta} +$$

$$+ \frac{(-1)^n}{2\pi(n-1)!}\int\limits_{|\zeta|<1}\Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\Big)(\zeta-z+\overline{\zeta-z})^{n-1}d\xi d\eta\,. \quad (24)$$

For the proof see [11].

Theorem 10. *The Dirichlet problem for the inhomogeneous polyanalytic equation in the unit disc*

$$\partial_{\bar z}^n w = f \text{ in } \mathbb{D}, \quad \partial_{\bar z}^\nu w = \gamma_\nu \text{ on } \partial\mathbb{D}, \quad 0 \le \nu \le n-1\,,$$

is uniquely solvable for $f \in L_1(\mathbb{D};\mathbb{C})$, $\gamma_\nu \in C(\partial\mathbb{D};\mathbb{C})$, $0 \le \nu \le n-1$, *if and only if for* $0 \le \nu \le n-1$

$$\sum_{\lambda=\nu}^{n-1}(-1)^{\lambda-\nu}\frac{\bar z}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma_\lambda(\zeta)}{1-\bar z\zeta}\frac{\overline{(\zeta-z)}^{\lambda-\nu}}{(\lambda-\nu)!}\,d\zeta - \\ -\frac{\bar z}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{1-\bar z\zeta}\frac{\overline{(z-\zeta)}^{n-1-\nu}}{(n-1-\nu)!}\,d\xi\,d\eta = 0\,. \tag{25}$$

The solution then is

$$w(z) = \sum_{\nu=0}^{n-1}\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\frac{\gamma_\nu(\zeta)}{\nu!}\,\frac{\overline{(z-\zeta)}^{\nu}}{\zeta-z}d\zeta - \frac{1}{\pi}\int\limits_{|\zeta|<1}\frac{f(\zeta)}{(n-1)!}\frac{\overline{(z-\zeta)}^{n-1}}{\zeta-z}d\xi d\eta\,. \tag{26}$$

The proof is given in [10].

References

1. H. Begehr, Complex analytic methods for partial differential equations. An introductory text. *World Scientific Publishing Co., Inc., River Edge, NJ,* 1994.
2. H. Begehr, Iterations of Pompeiu operators. *Mem. Differential Equations Math. Phys.* **12**(1997), 13–21.
3. H. Begehr, Integral representations in complex, hypercomplex and Clifford analysis. *Integral Transforms Spec. Funct.* **13**(2002), No. 3, 223–241.
4. H. Begehr, Some boundary value problems for bi-bi-analytic functions. Complex Analysis, Differential Equations and Related Topics. ISAAC Conf., Yerevan, Armenia, 2002, eds. Barsegian, G. et al., Nat. Acad. Sci. Armenia, Yerevan, 2004, 233–253.
5. H. Begehr, Combined integral representations. *Proc. 4th Intern. ISAAC Congress, Toronto,* 2003 (to appear).
6. H. Begehr, Basic boundary value problems in complex analysis. *Preprint, FU Berlin,* 2004.
7. H. Begehr, Boundary value problems for the complex Poisson equation. *Preprint, FU Berlin,* 2004.
8. H. Begehr and G. N. Hile, A hierarchy of integral operators. *Rocky Mountain J. Math.* **27**(1997), No. 3, 669–706.
9. H. Begehr and A. Kumar, Boundary value problems for bi-polyanalytic functions. *Preprint, FU Berlin,* 2003.
10. H. Begehr and A. Kumar, Boundary value problems for the inhomogeneous polyanalytic equation. *Preprint, FU Berlin,* 2004.
11. H. Begehr and D. Schmersau, The Schwarz problem for polyanalytic functions. *Preprint, FU Berlin,* 2004.
12. H. Begehr and C. J. Vanegas, Iterated Neumann problem for higher order Poisson equation. *Preprint, FU Berlin,* 2003.

13. H. Begehr and C. J. Vanegas, Neumann problem in complex analysis. *Proc. 11. Intern. Conf. Finite, Infinite Dimensional Complex Analysis, Appl., eds. P. Niamsup, A. Kanantha. Chiang Mai,* 2003, 212–225.
14. A. V. Bitsadze, On the uniqueness of the solution of the Dirichlet problem for elliptic partial differential equations. (Russian) *Uspehi Mat. Nauk (N.S.)* **3**(1948), No. 6(28), 211–212.
15. F. D. Gakhov, Boundary value problems. (Translation from Russian). *Pergamon Press, Oxford-New York-Paris; Addison-Wesley Publishing Co., Inc., Reading, Mass.–London,* 1966.
16. G. Manjavidze, Boundary value problems for conjugation with shift for analytic and generalized analytic functions. (Russian) *Tbilis. Gos. Univ., Tbilisi,* 1990.
17. G. Manjavidze and G. Akhalaia, Boundary value problems of the theory of generalized analytic vectors. *Complex methods for partial differential equations* (*Ankara,* 1998), 57–95, *Int. Soc. Anal. Appl. Comput.,* 6, *Kluwer Acad. Publ., Dordrecht,* 1999.
18. N. I. Muskhelishvili, Singular integral equations. Boundary problems of function theory and their application to mathematical physics. Translated from the second (1946) Russian edition and with a preface by J. R. M. Radok. Corrected reprint of the 1953 English translation. *Dover Publications, Inc., New York,* 1992.
19. D. Pompeiu, Sur une classe de fonctions d'une variable complexe et sur certaine equations integrales. *Rend. Circ. Mat. Palermo* **35**(1913), 277–281.
20. Vekua, I. N.: Generalized analytic functions. Pergamon Press, Oxford, 1962.
21. I. N. Vekua, Generalized analytic functions. *Pergamon Press, London-Paris-Frankfurt; Addison-Wesley Publishing Co., Inc., Reading, Mass.,* 1962.

(Received 11.10.2004)

Author's address:

I. Math. Inst.
FU Berlin
Arnimallee 3, D-14195 Berlin
Germany
E-mail: begehr@math.fu-berlin.de