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ON THE STABILITY OF SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Let $A = (a_{ij})_{i,j=1}^{n} \in BV_{n\times n}(a,b)$, $f = (f_i)_{i=1}^{n} \in K_n(a,b;A)$, and let $h : BV_n(a,b) \to R^n$ be a continuous operator such that the nonlinear boundary value problem

$$dx(t) = dA(t)\cdot f(t,x(t)), \quad h(x) = 0 \tag{1}$$

has a solution x°.

Consider sequences $A_k \in BV_{n\times n}(a,b)$ $(k = 1,2,\dots)$ and $f_k \in K_n(a,b;A_k)$ $(k = 1,2\dots)$ and a sequence of continuous operators $h_k : BV_n(a,b) \to R^n$ $(k = 1,2,\dots)$.

In this paper, sufficient conditions are given guaranteeing both the solvability of the problem

$$dx(t) = dA_k(t)\cdot f_k(t,x(t)), \quad h_k(x) = 0 \tag{1_k}$$

for any sufficiently large k and the convergence of its solutions as $k \to \infty$ to the solution x° of the problem (1).

We use the following notation and definitions: $R =]-\infty,\infty[$, $R_+ = [0,\infty[$; $R^{n\times m}$ is the space of all real $n\times m$-matrices $X = (x_{ij})_{i,j=1}^{n,m}$ with the norm $\|X\| = \max\{\sum_{i=1}^{n} |x_{ij}| : j = 1,\dots,m\}$; $|X| = (|x_{ij}|)_{i,j=1}^{n,m}$, $[X]_+ = (|X| + X)/2$; I is the identity $n\times n$-matrix; $R^n = R^{n\times 1}$ is the space of all matrix-functions $X = (x_{ij})_{i,j=1}^{n,m} : [a,b] \to R^{n\times m}$ such that $\operatorname{var}_a^b x_{ij} < +\infty$ with the norm $\|X\|_s = \sup\{\|X(t)\| : t \in [a,b]\}$; $V(X)(t) \equiv (\operatorname{var}_a^t x_{ij})_{i,j=1}^{n,m}$, $d_1X(t) = X(t) - X(t-0)$, $d_2X(t) = X(t+0) - X(t)$. $U(y,r) = \{x \in BV_n(a,b) : \|x-y\|_s < r\}$; $D(y,r)$ is the set of all $x \in R^n$ such that $\inf\{\|x - y(\tau)\| : \tau \in [a,b]\} < r$.

If $g \in BV_1(a,b)$, $x : [a,b] \to R$ and $a \le s < t \le b$, then $\int_s^t x(\tau)dg(\tau) = \int_{]s,t[} x(\tau)dg(\tau) + x(t)d_1g(t) + x(s)d_2g(s)$, where $\int_{]s,t[} x(\tau)dg(\tau)$ is the Lebesque–Stieltjes integral over the open interval $]s,t[$. If $G(t) = (g_{ij}(t))_{i,j=1}^{l,n}$ and $X(t) = (x_{jk}(t))_{j,k=1}^{n,m}$, then

$$\int_s^t dG(\tau)\cdot X(\tau) = \Big(\sum_{j=1}^{n}\int_s^t x_{jk}(\tau)dg_{ij}(\tau)\Big)_{i,k=1}^{l,m}.$$

$L_{n\times m}(a,b;G)$ is the set of all matrix-functions $(x_{jk}(t))_{j,k=1}^{n,m}$ such that x_{jk} is integrable with respect to g_{ij} $(i = 1,\dots,l)$. $K_{n\times m}(a,b;G)$ is the Carathéodory class, i.e., the set of all mappings $F = (f_{jk})_{j,k=1}^{n,m} : [a,b]\times R^n \to R^{n\times m}$ such that (a) $f_{ik}(\cdot,x)$ is measurable

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with respect to the measures $V(g_{ij})$ and $V(g_{ij})-g_{ij}$ for $x\in R^n$ $(i=1,\dots,l)$; (b) $F(t,\cdot)$ is continuous for $t\in[a,b]$ and $\sup\{|F(\cdot,x)|:x\in D\}\in L_{n\times m}(a,b;G)$ for every compact $D\subset R^n$.

Inequalities between the matrices are understood to be componentwise. If B_1 and B_2 are normed spaces, then an operator $g:B_1\to B_2$ is called positively homogeneous if $g(\lambda x)=\lambda g(x)$ for every $\lambda\in R_+$ and $x\in B_1$.

A vector-function $x\in BV_n(a,b)$ is said to be a solution of the problem (1) if $h(x)=0$ and $x(t)=x(s)+\int_s^t dA(\tau)\cdot x(\tau)$ for $a\le s\le t\le b$.

Let $\ell:BV_n(a,b)\to R^n$ be a linear continuous operator and let $\ell_0:BV_n(a,b)\to R^n_+$ be a positively homogeneous continuous operator. We say that a matrix-function $P\in K_{n\times n}(a,b;A)$ satisfies the Opial condition with respect to the triplet $(\ell,\ell_0;A)$ if (a) there exists a matrix-function $\Phi\in L_{n\times n}(a,b;A)$ such that $|P(t,x)|\le\Phi(t)$ on $[a,b]\times R^n;b)$ $\det(I+(-1)^j d_jB(t))\ne 0$ for $t\in[a,b]$ $(j=1,2)$ and the problem $dx(t)=dB(t)\cdot x(t)$, $|\ell(x)|\le\ell_0(x)$ has only trivial solution for every $B\in BV_{n\times n}(a,b)$ for which there exists a sequence $y_k\in BV_n(a,b)$ $(k=1,2,\dots)$ such that $\lim_{k\to\infty}\int_a^t dA(\tau)\cdot P(\tau,y_k(\tau))=B(t)$ uniformly on $[a,b]$.

x° is said to be strongly isolated in the radius r if there exist $P\in K_{n\times n}(a,b;A)$, $q\in K_n(a,b;A)$, a linear continuous operator $\ell:BV_n(a,b)\to R^n$ and a positively homogeneous operator $\widetilde{\ell}:BV_n(a,b)\to R^n$ such that (a) $f(t,x)=P(t,x)x+q(t,x)$ for $t\in[a,b]$, $\|x-x^\circ(t)\|<r$ and $h(x)=\ell(x)+\widetilde{\ell}(x)$ is fulfilled on $U(x^\circ;r)$; (b) the vector-functions $\alpha(t,\rho)=\max\{|q(t,x)|:\|x\|\le\rho\}$ and $\beta(\rho)=\sup\{[|\widetilde{\ell}(x)|-\ell_0(x)]_+:\|x\|_s\le\rho\}$ satisfy $\lim_{\rho\to\infty}\frac{1}{\rho}\int_a^b dV(A)(t)\cdot\alpha(t,\rho)=0$, $\lim_{\rho\to\infty}\beta(\rho)/\rho=0$; (c) the problem $dx(t)=dA(t)\cdot[P(t,x(t))x(t)+q(t,x(t))]$, $\ell(x)+\widetilde{\ell}(x)=0$ has no solution different from x°; (d) P satisfies the Opial condition with respect to the triplet $(\ell,\ell_0;A)$.

The notation $((A_k,f_k,h_k))_{k=1}^\infty\in W_r(A,f,h;x^\circ)$ means that (a) $\lim_{k\to\infty}\int_a^t dA_k(\tau)\times f_k(\tau,x)=\int_a^t dA(\tau)\cdot f(\tau,x)$ uniformly on $[a,b]$ for every $x\in D(x^\circ;r)$; (b) $\lim_{k\to\infty}h_k(x)=h(x)$ uniformly on $U(x^\circ;r)$; (c) $\lim_{s\to0+}\sup\{\|\int_a^b dV(A_k)(t)\cdot\omega_k(t,s)\|:k=1,2,\dots\}=0$ on $[a,b]\times D(x^\circ;r)$, where $\omega_k(t,s)=\max\{|f_k(t,x)-f_k(t,y)|:\|x\|,\|y\|\le\|x^\circ\|_s+r;\ \|x-y\|\le s\}$.

The problem (1) is said to be $(x^\circ;r)$-correct if for every $\varepsilon\in]0,r[$ and $((A_k,f_k,h_k))_{k=1}^\infty\in W_r(A,f,h;x^\circ)$ there exists a natural number k_0 such that the problem (1_k) has at least one solution contained in $U(x^\circ;r)$, and any such solution belongs to $U(x^\circ;r)$ for any $k\ge k_0$.

Theorem. *If the problem* (1) *has a solution x° which is strongly isolated in the radius r, then it is $(x^\circ;r)$-correct.*

Similar results for ordinary differential equations can be found in [1].

References

1. I. T. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Current Problems in Mathematics. Newest Results, vol.* 30 (Russian), 3–103, *Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vses. Inst. Nauchn. i Tekh. Inform., Moscow*, 1987.

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