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I. Kiguradze and Z. Sokhadze

EXISTENCE AND CONTINUABILITY OF SOLUTIONS OF THE INITIAL VALUE PROBLEM FOR THE SYSTEM OF SINGULAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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In the present paper we consider the system of functional differential equations

$$\frac{dx(t)}{dt} = f(x)(t) \tag{1}$$

with the weighted initial condition

$$\lim_{t\to a} \frac{\|x(t) - c_0\|}{h(t)} = 0, \tag{2}$$

where $f : C([a, b]; R^n) \to L_{loc}(]a, b]; R^n)$ is a Volterra operator $c_0 \in R^n$, and $h : [a, b] \to [0, +\infty[$ is a continuous function such that $h(t) > 0$ for $a < t \le b$.

A particular case of (1) is the differential system with the delay

$$\frac{dx(t)}{dt} = f_0\Big(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))\Big), \tag{1'}$$

where $f_0 :]a, b] \times R^n \to R^n$ is a vector function satisfying the local Carathéodory conditions and $\tau_k : [a, b] \to [a, b]$ $(k = 1, \dots, m)$ are measurable functions satisfying the inequalities $\tau_k(t) \le t$ for $a \le t \le b$ $(k = 1, \dots, m)$.

The initial value problem for regular systems of the type (1) and (1′) has been studied fully enough (see, e.g., [1, 6]). We are interested in the singular systems, i.e., the systems where $f(x)(\cdot)$ and $f_0(\cdot, x_0, x_1, \dots, x_m)$ are not summable on $[a, b]$ for some $x \in C([a, b]; R^n)$ and $x_k \in R^n$ $(k = 0, \dots, m)$. So far little is known about the initial value problem for such systems. The exception is the system

$$\frac{dx(t)}{dt} = f_0(t, x(t))$$

(see [2–5]).

Below we shall give new results on the existence and continuability of solutions of the problems (1), (2) and (1′), (2). To formulate them we shall need the following notation and definitions.

R is the set of real numbers; $R_+ = [0, +\infty[$;

R^n is the space of n-dimensional vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ $(i = 1, \dots, n)$ and the norm $\|x\| = \sum_{i=1}^n |x_i|$; $x \cdot y$ is the scalar product of the vectors x and $y \in R^n$;

if $\rho \in]0, +\infty[$, then $R^n_\rho = \Big\{x \in R^n : \|x\| \le \rho\Big\}$;

if $x = (x_i)_{i=1}^n$, then $\operatorname{sgn}(x) = (\operatorname{sgn} x_i)_{i=1}^n$;

$C([a, b]; R^n)$ is the space of continuous vector functions $x : [a, b] \to R^n$ with the norm $\|x\|_C = \max\{\|x(t)\| : a \le t \le b\|$;

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if $\rho \in]0,+\infty[$, then $C_\rho([a,b];R^n) = \Big\{x \in C([a,b];R^n) : \|x\|_C \le \rho\Big\}$;

if $a \le s \le t \le b$ and $x \in C([a,b];R^n)$, then $\nu(x)(s,t) = \max\{\|x(\xi)\| : s \le \xi \le t\}$;

$L_{loc}(]a,b];R^n)$ is the space of locally summable vector functions $x :]a,b] \to R^n$ with the topology of uniform mean convergence on every segment contained in $]a,b]$.

Definition 1. An operator $f : C([a,b];R^n) \to L_{loc}(]a,b];R^n)$ is said to be *Volterra* if for every $t_0 \in]a,b]$ and any vector functions x and $y \in C([a,b];R^n)$ satisfying $x(t) = y(t)$ for $a \le t \le t_0$, the equality $f(x)(t) = f(y)(t)$ is fulfilled a.e. on $]a,t_0$.

Definition 2. If $f : C([a,b];R^n) \to L_{loc}(]a,b];R^n)$ is a Volterra operator, then:

(i) for every $x \in C([a,b];R^n)$ under $f(x)$ is understood the vector function given by

$$f(x)(t) = f(\overline{x})(t) \quad \text{for} \quad a \le t \le b_0,$$

where $\overline{x}(t) = x(t)$ for $a \le t \le b_0$ and $\overline{x} = x(b_0)$ for $b_0 < t \le b$;

(ii) a continuous vector function $x : [a,b_0] \to R^n$ is said to be the *solution of the system* (1) *on* $[a,b_0]$ if it is absolutely continuous on every segment contained in $]a,b_0]$ and satisfies (1) a.e. on $]a,b_0]$;

(iii) $x : [a,b_0[\to R^n$ is said to be the *solution of the system* (1) *in a half-open interval* $[a,b_0[$ if for every $b_1 \in]a,b_0[$ the restriction of x on $[a,b_1]$ is the solution of the same system on $[a,b_1]$.

Definition 3. We shall say that an operator $f : C([a,b];R^n) \to L_{loc}(]a,b];R^n)$ satisfies the *local Carathéodory conditions* if it is continuous and there exists a nondecreasing in the second argument function $\gamma :]a,b] \times R_+) \to R_+$ such that $\gamma(\cdot,\rho) \in L_{loc}(]a,b];R)$ for $\rho \in R_+$, and for any $x \in C([a,b];R^n)$ the inequality $\|f(x)(t)\| \le \gamma(t,\|x\|_C)$ is fulfilled a.e. on $]a,b[$.

In the sequel we shall assume that $f : C([a,b];R^n) \to L_{loc}(]a,b];R^n)$ is a Volterra operator satisfying the local Carathéodory conditions.

Definition 4. A solution x of the system (1) defined on a segment $[a,b_0] \subset [a,b[$ (on a half-open interval $[a,b_0[\subset [a,b[)$ is said to be *continuable* if for some $b_1 \in]b_0,b]$ ($b_1 \in [b_0,b]$) the system (1) has on the segment $[a,b_1]$ a solution y satisfying $x(t) = y(t)$ for $a \le t \le b_0$. Otherwise x is said to be *noncontinuable.*

Definition 5. The problem (1), (2) is said to be locally solvable if the system (1) has on a segment $[a,b_0]$ a solution x satisfying the initial condition (2).

Theorem 1. *Let there exist a positive number ρ and summable functions p and $q : [a,b] \to R_+$ such that*

$$\begin{aligned} &\limsup_{t\to a} \left(\frac{1}{h(t)} \int_a^t p(s)ds\right) < 1, \\ &\limsup_{t\to a} \left(\frac{1}{h(t)} \int_a^t q(s)ds\right) = 0 \end{aligned} \tag{3}$$

and let for any $y \in C_\rho([a,b];R^n)$ the inequality

$$f(c_0 + hy)(t) \cdot \operatorname{sgn}(y(t)) \le p(t)\nu(y)(a,t) + q(t).$$

be fulfilled a.e. on $]a,b[$. Then the problem (1), (2) *is locally solvable.*

Corollary 1. *Let there exist summable functions $p_k : [a,b] \to R_+$ $(k = 0,\dots,m)$ and $q : [a,b] \to R_+$ such that the conditions* (3) *are fulfilled, where $p(t) = \sum\limits_{k=0}^{m} p_k(t)$, and* *let for some $\rho > 0$ the inequality*

$$f_0\Big(t, c_0 + h(t)y_0, c_0 + h(\tau_1(t))y_1, \dots, c_0 + h(\tau_m(t))y_m\Big) \cdot \operatorname{sgn}(y_0) \le$$
$$\le \sum_{k=0}^{m} p_k(t)\|y_k\| + q(t).$$

be fulfilled on $]a,b[\times R_\rho^{(m+1)n}$. *Then the problem* $(1')$, (2) *is locally solvable.*

Example 1. Let α_k and $\beta_k \in R$, $\mu \in]0,+\infty[$, $\lambda_k \in]1,+\infty[$ $(k = 1,\dots,m)$, $a = 0$, $b = 1$, $n = 1$ and $\sum\limits_{k=1}^{m} |\alpha_k| < \mu$. Then because of Corollary 1, the problem

$$\frac{dx(t)}{dt} = -\sum_{k=1}^{l} \exp\left(\frac{k}{t}\right) x^{2k-1}(t) +$$
$$+\sum_{k=1}^{m}\left[\alpha_k \frac{x(t^k)}{t^{(k-1)\mu+1}} + \beta_k \frac{|x(t^k)|^{\lambda_k}}{t^{k\mu\lambda_k-\mu+1}}\right] + \frac{t^{\mu-1}}{1+|\ln t|},$$
$$\lim_{t\to 0} \frac{x(t)}{t^\mu} = 0$$

is solvable. Consequently, under the conditions of Corollary 1, the right-hand side of the system $(1')$ with respect to the first argument may have nonintegrable singularities of arbitrary order.

Example 2. Let $n = 1$, $a = 0$, $b = 1$, $\alpha_k \in R_+$ $(k = 1,\dots,m)$ and

$$q(t) = \frac{1}{\ln(2+|\ln t|)} + \frac{1}{(2+|\ln t|)\ln^2(2+|\ln t|)},$$
$$p_k(t) = \alpha_k \ln(2+|\ln t^k|)q(t), \quad p(t) = \sum_{k=1}^{m} p_k(t).$$

Then

$$\lim_{t\to 0}\left(\frac{1}{t}\int_0^t p(s)ds\right) = \sum_{k=1}^{m} \alpha_k.$$

Therefore, owing to Corollary 1, the inequality $\sum\limits_{k=1}^{m} \alpha_k < 1$ guarantees solvability of the problem

$$\frac{dx(t)}{dt} = \sum_{k=1}^{m} p_k(t)\frac{|x(t^k)|}{t^k} + q(t), \quad \lim_{t\to 0}\frac{x(t)}{t} = 0. \tag{4}$$

On the other hand, it is not difficult to show that if $\sum\limits_{k=1}^{m} \alpha_k \ge 1$, then the problem (4) has no solution. Hence, the condition

$$\limsup_{t\to a}\left(\frac{1}{t}\int_0^t p(s)ds\right) < 1$$

in Theorem 1 and Corollary 1 is optimal and it cannot be replaced by the condition

$$\limsup_{t\to a}\left(\frac{1}{h(t)}\int_0^t p(s)ds\right) \le 1.$$

Theorem 2. *Let there exist $c \in R^n$, a nondecreasing function $\tau : [a,b] \to [a,b]$ and a decreasing in the second argument function $\varphi : [a,b] \times R_+ \to R_+$ such that $\tau(t) \le t$ for $a \le t \le b$, $\varphi(\cdot,\rho) \in L_{loc}(]a,b];R)$ for $\rho \in R_+$ and let for any $y \in C([a,b];R^n)$ the inequality*

$$f(c+y)(t)\cdot \operatorname{sgn}(y(t)) \le \varphi(t,\nu(y)(\tau(t),t))$$

be fulfilled a.e. on $[a,b]$. Moreover, let x be the solution of the system (1) on an interval $[a,b_0[\subset [a,b[$. Then for the x to be noncontinuable, it is necessary and sufficient that

$$\lim_{t\to b_0}\nu(x)(\tau(t),t) = +\infty.$$

Corollary 2. *If x is a solution of the system $(1')$ on an interval $[a,b_0[\subset [a,b[$, then for its noncontinuability it is necessary and sufficient that*

$$\tau(t) = \operatorname{ess\,min}\{\tau_i(s) : t \le s \le b_0;\ i = 1,\dots,m\}.$$

Theorem 3. *If the conditions of Theorem* 1 *are fulfilled, then the problem* (1), (2) *has at least one noncontinuable solution.*

Corollary 3. *If the conditions of Corollary* 1 *are fulfilled, then the problem* $(1')$, (2) *has at least one noncontinuable solution.*

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Authors' addresses:

I. Kiguradze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze St., Tbilisi 380093
Georgia

Z. Sokhadze
A. Tsereteli Kutaisi State University
55, Queen Tamar St., Kutaisi 384000
Georgia