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ON A THEOREM OF MYSHKIS–TSALYUK

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Consider the system of differential equations with delay

$$\frac{dx(t)}{dt} = f\Big(t, x(t), x(\tau_1(t)), \dots, x(\tau_m(t))\Big), \tag{1}$$

where $f : [a, b] \times R^{(m+1)n} \to R^n$ is a vector function from the Caratheodory class and $\tau_i : [a, b] \to [a, b]$ $(i = 1, \dots, n)$ are measurable functions satisfying the inequalities

$$\tau_i(t) \le t \quad \text{for} \quad a \le t \le b \quad (i = 1, \dots, m).$$

Let $b_0 \in]a, b]$. A vector function $x : [a, b_0] \to R^n$ (a vector function $x : [a, b_0[\to R^n$) is said to be a solution of the system (1) on $[a, b_0]$ (on $[a, b_0[$) if it is absolutely continuous on $[a, b_0]$ (on every segment contained in $[a, b_0[$) and satisfies (1) a.e. on $[a, b_0]$.

The solution x of the system (1) is said to be noncontinuable if one of the following conditions is fulfilled:

i) x is defined on the segment $[a, b]$;

ii) x is defined on the segment $[a, b_0[$, where $b_0 \in [a, b[$, and

$$\lim_{t \to b_0} \sup \|x(t)\| = +\infty.$$

A. D. Myshkis and Z. B. Tsalyuk [2] have proved a theorem on nonlocal continuability of solutions of the system (1), concerning the case where growth order of the vector function f with respect to the last mn arguments exceeds 1. Below we shall give a more general theorem of the same type.

The use will be made of the following notation:

R is the set of real numbers; $R_+ = [0, +\infty[$;

R^l is the space of vectors $x = (x_i)_{\equiv 1}^l$ with the components $x_i \in R$ $(i = 1, \dots, l)$ and the norm $\|x\| = \sum_{i=1}^l |x_i|$;

$R_+^l = \{(x_i)_{i=1}^l \in R^l : x_i \in R_+ (i = 1, \dots, l)\}$;

$x \cdot y$ is the scalar product of the vectors x and $y \in R^n$;

if $x = (x_i)_{i=1}^n$, then $\mathrm{sgn}(x) = (\mathrm{sgn}(x_i))_{i=1}^n$.

Theorem. *Let for every $s \in]a, b]$ there exist a number $\delta_s \in]0, s - a[$, a vector $c_s \in R^n$ and functions $\tau_{is} : [s - \delta_s, s] \to [s - \delta_s, s]$ and $\varphi_s : [s - \delta_s, s] \times R_+^{m+1} \to R_+$ such that*

$$\tau_i(t) \le \tau_{is}(t) \quad \textit{for} \quad s - \delta_s \le t \le \delta_s \quad (i = 1, \dots, m)$$

and the inequality

$$f(t, c_s + y_0, c_s + y_1, \dots, c_s + y_m) \cdot \mathrm{sgn}(y_0) \le \rho_s(t, \|y_0\|, \dots, \|y_m\|)$$

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is fulfilled on $[s-\delta_s, s]\times R^{(m+1)n}$. *Moreover, let* τ_{is} $(i=1,\dots,m)$ *be measurable,* φ *be summable in the first argument, continuous and nondecreasing in the last* $m+1$ *arguments and let for every* $\rho\in R_+$ *the initial problem*

$$\frac{du(t)}{dt}=\varphi_s\Big(t,u(t),u(\tau_{1s}(t)),\dots,u(\tau_{ms}(t))\Big);\quad u(s-\delta(s))=\rho$$

have upper solution defined on $[s-\delta(s),s]$. *Then every noncontinuable solution of the system* (1) *is defined on* $[a,b]$.

Corollary 1. *Let for every* $s\in]a,b]$ *there exist numbers* $m_s\in\{0,\dots,m-1\}$, $\delta_s\in]0,s-a[$, *a vector* $c_s\in R^n$ *and functions* $\alpha_s:[s-\delta(s),s]\times R_+\to R_+$ *and* $\varphi_s:R_+\to]0,+\infty[$ *such that*

$$\tau_k(t)\le s-\delta(s)\quad for\quad a\le t\le b\quad (k=m_s+1,\dots,m)$$

and the inequality

$$f(t,c_s+y_0,\dots,c(s)+y_m)\cdot\operatorname{sgn}(y_0)\le\alpha_s\Big(t,\sum_{k=m_s+1}^{m}\|y_k\|\Big)\varphi_s\Big(\sum_{k=0}^{m_s}\|y_k\|\Big).$$

is fulfilled on the set $[s-\delta(s),s]\times R^{(m+1)n}$. *Moreover, let* α_s *be summable in the first argument, continuous and nondecreasing in the second argument,* φ_s *be continuous, nondecreasing and*

$$\int_0^{+\infty}\frac{du}{\varphi_s(u)}=+\infty.$$

Then every noncontinuable solution of the system (1) *is defined on* $[a,b]$.

Corollary 2. *Let for every* $s\in]a,b]$ *there exist numbers* $m_s\in\{1,\dots,m-1\}$, $\delta_s\in]0,s-a[\cap]0,1[$, $\lambda_{ks}\in[1,+\infty[$, $\beta_s\in]-1,0[$ *and a continuous function* $\alpha_s:R_+\to R_+$ *such that the inequalities*

$$\tau_k(t)\le s-(s-t)^{\frac{1}{\lambda_{ks}}}\quad (k=1,\dots,m_s),\quad \tau_k(t)\le s-\delta(s)\quad (k=m_s+1,\dots,m)$$

and

$$f(t,c_s+y_0,\dots,c_s+y_m)\cdot\operatorname{sgn}(y_0)\le$$
$$\le\alpha_s\Big(\sum_{k=m_s+1}^{m}\|y_k\|\Big)(s-t)^{\beta_s}\Big(1+\sum_{k=1}^{m_s}\|y_k\|^{\lambda_{ks}}\Big)\ln\Big(2+\sum_{k=1}^{m_s}\|y_k\|\Big).$$

are fulfilled on $[s-\delta(s),s]$ *and on* $[s-\delta(s),s]\times R^{(m+1)n}$, *respectively. Then every noncontinuable solution of the system* (1) *is defined on* $[a,b]$.

Corollary 1 is an analogue of the wellknown A. Wintner's theorem ([1], Ch. III, §3.5) for the system (1), while Corollary 2 is a generalization of the above-mentioned Myshkis-Tsalyuk theorem.

References

1. P. Hartman, Ordinary differential equations. *John Wiley & Sons, New York-London-Sydney,* 1964.

2. A. D. Myshkis, Z. B. Tsalyuk, On nonlocal continuability of solutions of differential systems with deviating argument. (Russian) *Differentsial'nye Uravneniya* **5**(1969), No. 6, 1128–1130.

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