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ON MULTI-POINT BOUNDARY VALUE PROBLEMS FOR SYSTEMS OF FUNCTIONAL DIFFERENTIAL AND DIFFERENCE EQUATIONS

New sufficient conditions of solvability and unique solvability of multi-point boundary value problems for systems of functional differential and difference equations are established. Stable difference schemes for numerical solution of multi-point differential problems are constructed.

Multi-point boundary value problem, system of functional differential equations, system of functional difference equations, stable difference scheme.

რეზიუმე. დადგენილია მრავალწერტილოვანი სასაზღვრო ამოცანების ამოხსნადობისა და ცალსახად ამოხსნადობის ახალი საკმარისი პირობები ფუნქციონალურ დიფერენციალურ და სხვაობიან განტოლებათა სისტემებისათვის. აგებულია მდგრადი სხვაობიანი სქემები მრავალწერტილოვანი დიფერენციალური ამოცანების რიცხვითი ამონახსნების საპოვნელად.

δ_{ik} is the Kronecker symbol, i.e., $\delta_{kk} = 1$ and $\delta_{ik} = 0$ for $i \neq k$.

N is the set of natural numbers, $N_n = \{1, \dots n\}$, $\widetilde{N}_n = \{0, 1, \dots, n\}$.

R is the set of real numbers, $R_+ = [0, +\infty[$.

R^m is the m-dimensional real Euclidean space of points $x = (x_k)_{k=1}^m$ with the norm

$$\|x\|_{R^m} = \sum_{k=1}^m |x_k|.$$

$R_+^m = \big\{(x_k)_{k=1}^m \in R^m : x_1 \geq 0, \dots, x_m \geq 0\big\}$.

E_n^m is the space of vector functions $x : N_n \to R^m$ with the norm

$$\|x\|_{E_n^m} = \max\big\{\|x(i)\|_{R^m} : i \in N_n\big\}.$$

$\widetilde{E}_n^m$ is the space of vector functions $x : \widetilde{N}_n \to R^m$ with the norm

$$\|x\|_{\widetilde{E}_n^m} = \max\big\{\|x(i)\|_{R^m} : i \in \widetilde{N}_n\big\}.$$

$E_n = E_n^1, \quad \widetilde{E_n} = \widetilde{E}_n^1$.

$$E_n^+ = \{x \in E_n : x(i) \geq 0 \ \text{ for } \ i \in N_n\},$$
$$\widetilde{E}_n^+ = \{x \in \widetilde{E}_n : x(i) \geq 0 \ \text{ for } \ i \in \widetilde{N}_n\}.$$

Δ is the first order difference operator, i.e.,

$$\Delta x(i-1) = x(i) - x(i-1) \quad \text{for} \quad x \in \widetilde{E}_n, \quad i \in N_n.$$

$I_0 = [a, b]$, where $a \in R$ and $b \in]a, +\infty[$ are assumed to be fixed throughout the paper.

$$t_{in} = a + \frac{b-a}{n} \cdot i.$$

$C(I_0; R^m)$ is the space of continuous vector functions $u : I_0 \to R^m$ with the norm

$$\|u\|_{C(I_0;R^m)} = \max\big\{\|u(t)\|_{R^m} : t \in I_0\big\}.$$

$C(I_0; R_+^m) = \big\{x \in C(I_0; R^m) : x(t) \in R_+^m \quad \text{for} \quad t \in I_0\big\}$.

$p_n : C(I_0; R) \to \widetilde{E}_n$ and $q_n : \widetilde{E}_n \to C(I_0; R)$ are the operators given by

$$p_n(u)(i) = u(t_{in}) \quad \text{for} \quad i \in \widetilde{N}_n$$

and

$$q_n(x)(t) = \frac{n}{b-a}\big[(t_{in} - t)x(i-1) + (t - t_{i-1\,n})x(i)\big]$$
$$\text{for} \quad t_{i-1\,n} \leq t \leq t_{in}, \quad i \in \widetilde{N}.^1$$

$L^\alpha(I_0;R)$ is the space of the functions $u : I_0 \to R$ with summable α-th power, $\alpha \geq 1$,

$$\|u\|_{L^\alpha(I_0;R)} = \left(\int_a^b |u(t)|^\alpha dt\right)^{1/\alpha}.$$

$L(I_0;R_+) = \big\{u \in L((I_0;R) : u(t) \geq 0 \quad \text{for} \quad t \in I_0\big\}$.

$C(A;B)$ is the set of continuous mappings $u : A \to B$.

$K(C(I_0;R^m);L(I_0;R))$ is the class of the operators $f : C(I_0;R^m) \to L(I_0;R)$ satisfying the Carathéodory conditions, i.e.,

$$f \in K(C(I_0;R^m);L(I_0;R))$$

means that f is continuous and for any $r \in R_+$, it admits the estimate

$$|f(u_1,\dots,u_m)(t)| \leq f_r^*(t) \quad \text{for} \quad t \in I_0, \quad \sum_{k=1}^m \|u_k\|_{C(I_0;R)} \leq r,$$

where $f_r^* \in L(I_0;R_+)$.

$K(I_0 \times R^m;R)$ is the set of the functions $g : I_0 \times R^m \to R$ satisfying the Caratéodory conditions, i.e., $g \in K(I_0 \times R^m;R)$ means that $g(\cdot,x_1,\dots,x_m) : I_0 \to R$ is measurable for all $(x_k)_{k=1}^m \in R^m$, $g(t,\cdot,\dots,\cdot) : R^m \to R$ is continuous for almost all $t \in I_0$, and for any $r \in R_+$,

$$\max\left\{|g(\cdot,x_1,\dots,x_m)| : \sum_{k=1}^m |x_k| \leq r\right\} \in L(I_0;R_+).$$

Consider the system of functional differential equations

$$\frac{du_k(t)}{dt} = f_k(u_1,\dots,u_m)(t) \quad (k=1,\dots,m) \tag{0.1}$$

with the boundary conditions

$$u_k(t_k) = \varphi_k(u_1,\dots,u_m) \quad (k=1,\dots,m), \tag{0.2}$$

where $t_k \in I_0$, $f_k \in K(C(I_0;R^m);L(I_0;R))$ $(k=1,\dots,m)$, and $\varphi_k : C(I_0;R^m) \to R$ $(k=1,\dots,m)$ are continuous functionals. By solution of the problem (0.1), (0.2) we mean an absolutely continuous vector function $(u_k)_{k=1}^m : I_0 \to R^m$ satisfying both the system (0.1) (almost everywhere on I_0) and the boundary conditions (0.2).

In the form of (0.1) can be written, for example, system of ordinary differential equations

$$\frac{du_k(t)}{dt} = g_k(t,u_1(t),\dots,u_m(t)) \quad (k=1,\dots,m), \tag{0.1_1}$$

[1]It is clear that q_n may be assumed to be given on $\widetilde{E}_{n'}$ for any $n' \geq n$.

system of differential equations with deviating arguments

$$\frac{du_k(t)}{dt} = g_k\big(t, u_1(t), \dots, u_m(t), s_{\zeta_1}(u_1)(t), \dots, s_{\zeta_m}(u_m)(t)\big) \qquad (0.1_2)$$
$$(k = 1, \dots, m),$$

where

$$s_{\zeta_k}(u)(t) = \begin{cases} u(\zeta_k(t)) & \text{for } \ \zeta_k(t) \in I_0, \\ 0 & \text{for } \ \zeta_k(t) \notin I_0, \end{cases}$$

and system of integro-differential equations

$$\frac{du_k(t)}{dt} = g_k\Bigg(t, u_1(t), \dots, u_m(t), \int\limits_a^b u_1(s)\, d_s\, \alpha_1(s,t), \dots$$
$$\dots, \int\limits_a^b u_m(s)\, d_s\, \alpha_m(s,t)\Bigg) \ \ (k = 1, \dots, m). \qquad (0.1_3)$$

Particular cases of (0.2) are Cauchy–Nicoletti boundary conditions

$$u_k(t_k) = c_k \quad (k = 1, \dots, m), \qquad (0.2_1)$$

periodic boundary conditions

$$u_k(a) = u_k(b) + c_k \quad (k = 1, \dots, m), \qquad (0.2_2)$$

and so on.

Boundary value problems of the type (0.1_1), (0.2) and (0.1_1), (0.2_k) $(k = 1, 2)$ arise in different fields of natural science and engineering and for a long time have been attracting attention of many specialists. Monographs by M. A. Krasnosel'skii [38], I. T. Kiguradze [31], N. I. Vasil'ev and Yu. A. Klokov [10], Yu. V. Trubnikov and A. I. Perov [48], and the works [28, 32, 33, 35, 36, 39 and 43] contain nonimprovable in a certain sense conditions for the existence and uniqueness of their solutions. As for the question of numerical solution of such problems, it has so far been studied insufficiently. Only the Cauchy problem for differential equations and systems with continuous right sides [5, 6, 11, 26, 50, 51] and two-point boundary value problems for second order ordinary differential equations, [5–8, 12, 13, 26, 30, 34, 44] are an exception. Of the works devoted to the numerical solution of problems of periodic type for differential systems and higher order differential equations, we may note [9, 45]. Rather interesting are also the works of S. Gupta and R. Tewarson [27, 46] in which difference schemes of high accuracy have been constructed for numerical solution of a two-point boundary value problem arising in modelling kidney's functioning, although these papers do not deal with the question of convergence and stability of the proposed schemes.

In the recent years, the interest for boundary value problems for systems of functional differential equations (0.1) has increased considerably. Here, first of all, we should note the works of N. V. Azbelev and his collaborators (see [1–4] and references therein) in which the fundamentals of a general theory of such problems have been constructed. However, the problem (0.1), (0.2) has not been studied separately, with the exception of the cases where $f_1, \dots, f_m$ are the Nemytsky operators or $t_1 = t_2 = \dots = t_m$ and

$$\sup\left\{\frac{1}{r}\sum_{k=1}^{m}|\varphi_k(u_1,\dots,u_m)| : \sum_{k=1}^{m}\|u\|_{C(I_0;R)} \le r\right\} \to 0 \quad \text{as} \quad r \to +\infty.$$

The aim of the present work is: (i) to find sufficient conditions for solvability and unique solvability of the problem (0.1), (0.2) and of analogous problems for systems of functional difference equations, and (ii) to construct converging, stable difference schemes for numerical solutions of problems of the type (0.1), (0.2).

Chapter I of the present paper is devoted to the investigation of the question of existence and uniqueness of solution of the problem (0.1), (0.2). The use is made here of a method developed in [32, 35, 36] which is based on a priori estimates of functions satisfying systems of one-sided differential inequalities

$$\begin{aligned}&\big[u_k'(t) - h_k(t)u_k(t)\big]\operatorname{sign}\big[(t-t_k)u_k(t)\big] \le \\ &\le h_0(t) + f_{0k}\big(|u_1|,\dots,|u_m|\big)(t) \quad \text{for} \quad t \in I_0 \ \ (k=1,\dots,m)\end{aligned} \tag{0.3}$$

and boundary conditions

$$u_k(t_k) \le r + \varphi_{0k}\big(|u_1|,\dots,|u_m|\big) \quad (k=1,\dots,m). \tag{0.4}$$

The results of the chapter are formulated in terms of the set $W(t_1,\dots,t_m)$. The latter is defined in §1 (see Definition 1.1), while in §2 sufficient conditions are given under which

$$\big(h_1,\dots,h_m; f_{01},\dots,f_{0m}; \varphi_{01},\dots,\varphi_{0m}\big) \in W(t_1,\dots,t_m) \tag{0.5}$$

(Lemmas 2.3–2.6). It is established that if (0.5) is fulfilled, then any solution $(u_k)_{k=1}^m$ of (0.3), (0.4) admits the estimate

$$\sum_{k=1}^{m}|u_k(t)| \le \rho\left[r + \int_a^b h_0(s)\,ds\right] \quad \text{for} \quad t \in I_0,$$

where $\rho \ge 0$ is a number independent of $(u_k)_{k=1}^m$, h_0 and r (Lemma 2.1).

By means of the formulated lemma, the following theorems are proved.

Let in $C(I_0;R^m)$ the inequalities

$$\begin{aligned}&\big[f_k(u_1,\dots,u_m)(t) - h_k(t)u_k(t)\big]\operatorname{sign}\big[(t-t_k)u_k(t)\big] \le \\ &\le h_0(t) + f_{0k}\big(|u_1|,\dots,|u_m|\big)(t) \quad \textit{for} \quad t \in I_0 \ \ (k=1,\dots,m)\end{aligned} \tag{0.6}$$

and

$$\left|\varphi_k(u_1,\dots,u_m)\right| \le r + \varphi_{0k}\big(|u_1|,\dots,|u_m|\big) \quad (k=1,\dots,m) \tag{0.7}$$

be fulfilled, where $h_0 \in L(I_0;R_+)$, $r \in R_+$, *and* h_k, f_{0k} *and* φ_{0k} $(k = 1,\dots,m)$ *satisfy* (0.5). *Then the problem* (0.1), (0.2) *has at least one solution.*

Let in $C(I_0;R^m)$ *the inequalities*

$$\begin{gathered}\big[f_k(u_1,\dots,u_m)(t) - f_k(v_1,\dots,v_m)(t) - \\ -h_k(t)(u_k(t)-v_k(t))\big]\operatorname{sign}\big[(t-t_k)(u_k(t)-v_k(t))\big] \le \\ \le f_{0k}\big(|u_1-v_1|,\dots,|u_m-v_m|\big)(t) \quad \textit{for} \quad t\in I_0 \ \ (k=1,\dots,m)\end{gathered} \tag{0.8}$$

and

$$\begin{gathered}\left|\varphi_k(u_1,\dots,u_m) - \varphi(v_1,\dots,v_m)\right| \le \\ \le \varphi_{0k}\big(|u_1-v_1|,\dots,|u_m-v_m|\big) \quad (k=1,\dots,m)\end{gathered} \tag{0.9}$$

be fulfilled, where h_k, f_{0k} *and* φ_{0k} $(k = 1,\dots,m)$ *satisfy* (0.5). *Then the problem* (0.1), (0.2) *has a unique solution.*

From these theorems, by virtue of lemmas 2.3–2.6 we get efficient sufficient conditions for the existence and uniqueness of solutions of the problems (0.1), (0.2), (0.1_k), (0.2), (0.1), (0.2_j) and (0.1_k), (0.2_j) (Corollaries $1.1'$–$1.10'$, $1.1''$–$1.10''$).

In Chapter II, we study the difference problem

$$\Delta x_k(i-1) = g_k(x_1,\dots,x_m)(i) \quad (k=1,\dots,m), \tag{0.10}$$

$$x_k(i_k) = \psi_k(x_1,\dots,x_m) \quad (k=1,\dots,m), \tag{0.11}$$

where $i_k \in \widetilde{N}_n$, $g_k \in C(\widetilde{E}_n^m;E_n)$ and $\psi_k \in C(\widetilde{E}_n^m;R)$.

Given $k \in N_m$, we denote by τ_k the function

$$\tau_k(i) = \begin{cases} i & \text{for } i > i_k, \\ i-1 & \text{for } i \le i_k. \end{cases}$$

By analogy with $W(t_1,\dots,t_m)$, we introduce the set $W_n(i_1,\dots,i_m)$ (Definition 4.1) and prove the following propositions.

Let in $\widetilde{E}_n^m$ *the inequalities*

$$\begin{gathered}\big[g_k(u_1,\dots,u_m)(i) - h_k(i)x_k(\tau_k(i))\big]\operatorname{sign}\big[(\tau_k(i)-i_k)x_k(\tau_k(i))\big] \le \\ \le h_0(i) + g_{0k}\big(|x_1|,\dots,|x_m|\big)(i) \quad \textit{for} \quad i\in N_n \ \ (k=1,\dots,m)\end{gathered}$$

and

$$\left|\psi_k(x_1,\dots,x_m)\right| \le r + \psi_{0k}\big(|x_1|,\dots,|x_m|\big) \quad (k=1,\dots,m)$$

be fulfilled, where $r \in R_+$, $h_0 \in E_n^+$, *and*

$$\big(h_1,\dots,h_m; g_{01},\dots,g_{0m}; \psi_{01},\dots,\psi_{0m}\big) \in W_n(i_1,\dots,i_m). \tag{0.12}$$

Then the problem (0.10), (0.11) *has at least one solution.*

Let in $\widetilde{E}_n^m$ *the inequalities*

$$\big[g_k(x_1,\dots,x_m)(i) - g_k(y_1,\dots,y_m)(i) - h_k(i)(x_k(\tau_k(i)) - y_k(\tau_k(i)))\big] \times$$
$$\times \operatorname{sign}\big[(\tau_k(i) - i_k)(x_k(\tau_k(i)) - y_k(\tau_k(i)))\big] \leq$$
$$\leq g_{0k}\big(|x_1 - y_1|,\dots,|x_m - y_m|\big)(i) \quad \textit{for} \quad i \in N_n \ \ (k = 1,\dots,m)$$

and

$$\big|\psi_k(x_1,\dots,x_m) - \psi(y_1,\dots,y_m)\big| \leq$$
$$\leq \psi_{0k}\big(|x_1 - y_1|,\dots,|x_m - y_m|\big) \quad (k = 1,\dots,m) \tag{0.13}$$

be fulfilled, where h_k, g_{0k} *and* ψ_{ok} *satisfy* (0.12). *Then the problem* (0.10), (0.11) *has a unique solution.*

Let the conditions of Theorem 4.2 *be fulfilled and* $(x_{k0})_{k=1}^m \in \widetilde{E}_n^m$. *Then there exists a unique sequence of vector functions* $(x_{k\nu})_{k=1}^m \in \widetilde{E}_n^m$ $(\nu = 1,2,\dots)$ *such that for any natural* ν *and* $k \in \{1,\dots,m\}$, *the function* $x_{k\nu}$ *is the solution of the Cauchy problem*

$$\Delta x_{k\nu}(i-1) = g_k(x_{1\,\nu-1},\dots,x_{k-1\,\nu-1},x_{k\nu},x_{k+1\,\nu-1},\dots,x_{m\,\nu-1})(i),$$
$$x_{k\nu}(i_k) = \psi_k(x_{1\,\nu-1},\dots,x_{m\nu-1}), \tag{0.14}$$

and

$$\lim_{\nu\to+\infty} x_{k\nu}(i) = x_k(i) \quad \textit{for} \quad i \in \widetilde{N}_n \ \ (k = 1,\dots,m), \tag{0.15}$$

where $(x_k)_{k=1}^m$ *is the solution of the problem* (0.10), (0.11).

Let in $\widetilde{E}_n^m$ *the inequalities* (0.13) *and*

$$\big|g_k(x_1,\dots,x_m)(i) - g_k(y_1,\dots,y_m)(i) - h_k(i)(x_k(\tau_k(i)) - y_k(\tau_k(i)))\big| \leq$$
$$\leq g_{0k}\big(|x_1 - y_1|,\dots,|x_m - y_m|\big)(i) \quad \textit{for} \quad i \in N_n \ \ (k = 1,\dots,m)$$

be fulfilled, where h_k, g_{0k} *and* ψ_{0k} *satisfy* (0.12). *Let moreover,* $(x_{k_0})_{k=1}^m \in \widetilde{E}_n^m$ *Then there exists a unique sequence of vector functions* $(x_{k\nu})_{k=1}^m \in \widetilde{E}_n^m$ $(\nu = 1,2,\dots)$ *such that for any natural* ν *and* $k \in \{1,\dots,m\}$, *the function* $x_{k\nu}$ *is the solution of the difference equation*

$$\Delta x_{k\nu}(i-1) = h_k(i)\big[x_{k\nu}(\tau_k(i)) - x_{k\,\nu-1}(\tau_k(i))\big] +$$
$$+ g_k(x_{1\,\nu-1},\dots,x_{m\,\nu-1})(i)$$

under the initial condition (0.14), *and* (0.15) *holds, where* $(x_k)_{k=1}^m$ *is the solution of the problem* (0.10), (0.11).

These theorems allow us to determine simple sufficient conditions which guarantee both the unique solvability of the problem (0.10), (0.11) and the convergence of the above-mentioned iteration processes.

In Chapter III which is devoted to the convergence and stability of difference schemes, we start from the notion of the class $\mathcal{D}_f$ (Definition 7.1) and from Lemmas 7.1–7.3 characterizing the latter. Proofs of the main results of the chapter are based on a priori estimates of solutions of boundary value problems for difference inequalities given in §8 (Lemmas 8.1–8.3).

In §9, we investigate difference schemes of the type

$$\Delta x_k(i-1) = f_{kn}(x_1,\dots,x_m)(i) \quad (k=1,\dots,m), \tag{0.16}$$

$$x_k(i_{kn}) = \varphi_{kn}(x_1,\dots,x_m)(i) \quad (k=1,\dots,m), \tag{0.17}$$

where

$$\big(f_{kn}\big)_{n=1}^{+\infty} \in \mathcal{D}_{f_k} \quad (k=1,\dots,m),$$

the numbers $i_{kn} \in \widetilde{N}_n$ ($k \in N_m$, $n \in N$) are chosen such that

$$t_{i_{kn}n} \le t_k < t_{i_{kn}+1\,n},$$

and the functionals $\varphi_{kn} \in C(\widetilde{E}_n^m; R)$ ($k \in N_m$, $n \in N$) for any $(u_k)_{k=1}^m \in C(I_0; R^m)$ satisfy

$$\lim_{n\to+\infty} \varphi_{kn}(x_{1n},\dots,x_{mn}) = \varphi(u_1,\dots,u_m) \quad (k=1,\dots,m)$$

whenever

$$\lim_{n\to+\infty} \|x_{kn} - p_n(u_k)\|_{\widetilde{E}_n} = 0.$$

Theorem 9.1 and its corollary contain conditions under which to any $r>0$ there exists $n_0 \in N$ such that for every $n>n_0$, the set $X_n(u_1^0,\dots,u_m^0;r)$ of solutions of the problem (0.16), (0.17) satisfying

$$\sum_{k=1}^m \|x_k - p_n(u_k^0)\|_{\widetilde{E}_n} < r$$

is non-empty and

$$\sup\Big\{\sum_{k=1}^m \|x_k - p_n(u_k^0)\|_{\widetilde{E}_n} : (x_k)_{k=1}^m \in X_n(u_1^0,\dots,u_m^0;r)\Big\} \to 0 \ \text{ as } \ n\to+\infty.$$

By τ_{kn}, σ_{ikn} and q_{ikn} we denote the following functions and operators

$$\tau_{kn}(i) = \begin{cases} i & \text{for } i > i_{kn}, \\ i-1 & \text{for } i \le i_{kn}, \end{cases}$$

$$\sigma_{ikn}(x)(j) = \begin{cases} x(\tau_{kn}(i)) & \text{for } j = \tau_{kn}(i) + \operatorname{sign}(i_{kn}-i+1), \\ x(j) & \text{for } j \ne \tau_{kn}(i) + \operatorname{sign}(i_{kn}-i+1), \end{cases}$$

$$q_{ikn}(x)(t) = q_n\big(\sigma_{ikn}(x)\big)(t).$$

Let for every natural n *the inequalities*

$$\begin{gathered}\big[f_{kn}(x_1,\dots,x_m)(i)-f_{kn}(y_1,\dots,y_m)(i)-h_{kn}(i)(x_k(\tau_{kn}(i))-y_{kn}(\tau_k(i)))\big]\times\\ \times\operatorname{sign}\big[(\tau_{kn}(i)-i_{kn})(x_k(\tau_{kn}(i))-y_k(\tau_{kn}(i)))\big]\le\\ \le f_{0kn}\big(|x_1-y_1|,\dots,|x_m-y_m|\big)(i)\quad for\quad i\in N_n\ \ (k=1,\dots,m)\end{gathered}$$

and

$$\begin{gathered}\big|\varphi_{kn}(x_1,\dots,x_m)-\varphi_{kn}(y_1,\dots,y_m)\big|\le\\ \le\varphi_{0k}\big(q_n(|x_1-y_1|),\dots,q_n(|x_m-y_m|)\big)\quad(k=1,\dots,m)\end{gathered}$$

be fulfilled in $\widetilde{E}_n^m$, *where* $f_{okn}:(\widetilde{E}_n^+)^m\to E_n^+$ $(k=1,\dots,m)$ *are positively homogeneous continuous non-decreasing operators,*

$$\big(h_{kn}\big)_{n=1}^{+\infty}\in\mathcal{D}_{h_k},\quad\big(f_{okn}\big)_{n=1}^{+\infty}\in\mathcal{D}_{f_{0k}}\quad(k=1,\dots,m),$$

and h_k, f_{0k} *and* φ_{0k} $(k=1,\dots,m)$ *satisfy* (0.5). *Then:*

(a) *the problem* (0.1), (0.2) *has a unique solution* $(u_k)_{k=1}^m$*;*

(b) *the difference scheme* (0.16), (0.17) *is stable*[2]*;*

(c) *there exist* $n_0\in N$ *and* ρ *such that for any* $n>n_0$, *the problem* (0.16), (0.17) *has a unique solution* $(x_{kn})_{k=1}^m$,

$$\begin{gathered}\sum_{k=1}^m\|x_k-p_n(u_k^0)\|_{\widetilde{E}_n}\le\rho\sum_{k=1}^m\big[|p_n(u_k^0)(i_{kn})-\varphi_{kn}(p_n(u_1^0),\dots,p_n(u_m^0))|+\\ +\sum_{i=1}^n|\Delta p_n(u_k)(i-1)-f_{kn}(p_n(u_1^0),\dots,p_n(u_m^0))(i)|\big],\end{gathered}$$

and

$$\lim_{n\to+\infty}\|x_{kn}-p_n(u_k^0)\|_{\widetilde{E}_n}=0\quad(k=1,\dots,m).$$

Let the conditions (0.5), (0.8), (0.9) *and the equalities*

$$\begin{gathered}f_{kn}(x_1,\dots,x_m)(i)=(1-\delta_{i-1\,i_{kn}})\int\limits_{t_{i-1\,n}}^{t_{in}}f_k\big(q_{i1n}(x_1),\dots,q_{imn}(x_m)\big)(s)\,ds+\\ +\delta_{i-1\,i_{kn}}\int\limits_{t_{i-1\,n}}^{t_{in}}h_k(s)\,ds\,x_k(i)\quad(n\in N;\ \ i\in N_n;\ \ k\in N_m)\end{gathered}$$

and

$$\varphi_{kn}(x_1,\dots,x_m)=\varphi_k\big(q_n(x_1),\dots,q_n(x_m)\big)\quad(k=1,\dots,m)\quad(0.18)$$

---

[2] See Definition 9.1.

be fulfilled. Then the conclusion of Theorem 9.2 *is valid. Moreover, if for every* $(u_k)_{k=1}^m$ *and* $(v_k)_{k=1}^m \in C(I_0;R^m)$ *we have*

$$h_k \in L^\alpha(I_0;R), \quad f_k(u_1,\dots,u_m) \in L^\alpha(I_0;R) \quad (k=1,\dots,m)$$

and

$$\sum_{k=1}^m \big|f_k(u_1,\dots,u_m)(t) - f_k(v_1,\dots,v_m)(t)\big| \le h(t)\sum_{k=1}^m \|u_k - v_k\|_{C(I_0;R)}$$

with $1 < \alpha \le +\infty$ *and* $h \in L(I_0;R_+)$, *then*

$$\sum_{k=1}^m \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{1/\alpha-1}).$$

Let in $C(I_0;R^m)$ *the inequalities* (0.9) *and*

$$\big|f_k(u_1,\dots,u_m)(t) - f_k(v_1,\dots,v_m)(t)\big| \le$$
$$\le f_{0k}\big(|u_1-v_1|,\dots,|u_m-v_m|\big)(t) \quad for \quad t\in I_0 \ \ (k=1,\dots,m) \tag{0.19}$$

be fulfilled, where

$$(f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m}) \in W_0(t_1,\dots,t_m). \tag{0.20}$$

Next, let

$$f_{kn}(x_1,\dots,x_m)(i) = \int\limits_{t_{i-1\,n}}^{t_{in}} f_k\big(q_n(x_1),\dots,q_n(x_m)\big)(s)\,ds$$
$$(n\in N;\ \ i\in N_n;\ \ k\in N_m)$$

and φ_{kn} $(k=1,\dots,m)$ *be given by* (0.18). *Then the conclusion of Theorem* 9.2 *is valid. Moreover, if* $u_k^{0\prime}$ $(k=1,\dots,m)$ *are absolutely continuous and* $u_k^{0\prime\prime} \in L^\alpha(I_0;R)$ $(k=1,\dots,m)$ *with* $1<\alpha\le+\infty$, *then*

$$\sum_{k=1}^m \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{1/\alpha-2}).$$

Let the conditions (0.5), (0.8), (0.9), $h_k \in C(I_0;R)$, $f_k \in C(C(I_0;R^m);C(I_0;R))$, $f_{0k} \in C(C(I_0;R_+^m);C(I_0;R_+))$ $(k=1,\dots,m)$ *and*

$$f_{kn}(x_1,\dots,x_m)(i) = \frac{b-a}{n}\, f_k\big(q_n(x_1),\dots,q_n(x_m)\big)\big(t_{\tau_{kn}(i)n}\big)$$
$$(n\in N;\ \ i\in N_n;\ \ k\in N_m)$$

be fulfilled, and let φ_{kn} $(k = 1, \dots, m)$ *be given by* (0.18). *Then the conclusion of Theorem* 9.2 *is valid. Moreover, if every function* $u_k^{0\,\prime}$ *has bounded variation, then*

$$\sum_{k=1}^{m} \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-1}).$$

Let the conditions (0.9), (0.19), (0.20) *and*

$$f_{kn}(x_1, \dots, x_m)(i) = \frac{b-a}{2n}\big[f_k\big(q_n(x_1), \dots, q_n(x_m)\big)(t_{i-1\,n}) +$$
$$+ f_k\big[(q_n(x_1), \dots, q_n(x_m)\big((t_{in})\big] \quad (n \in N; \;\; i \in N_n; \;\; k \in N_m)$$

be fulfilled, and let φ_{kn} $(k = 1, \dots, m)$ *be given by* (0.18). *Then the conclusion of Theorem* 9.2 *is valid. Moreover, if* $(u_k^0)_{k=1}^m$ *are thrice continuously differentiable, then*

$$\sum_{k=1}^{m} \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-2}).$$

If $(h_{kn})_{n=1}^{+\infty} \in \mathcal{D}_{h_k}$, $(f_{0kn})_{n=1}^{+\infty} \in \mathcal{D}_{f_{0k}}$ and

$$\varphi_{0kn}(x_1, \dots, x_m) = \varphi_{0k}(q_n(x_1), \dots, q_n(x_m)) \quad (k = 1, \dots, m),$$

then there exists $n_0 \in N$ such that

$$\big(h_{1n}, \dots, h_{mn}; f_{01n}, \dots, f_{0mn}; \varphi_{01n}, \dots, \varphi_{0mn}\big) \in W_n(i_{1n}, \dots, i_{mn})$$

for $n > n_0$. Thus, if the conditions of Theorem 9.2 (or one of its corollaries) are fulfilled, then the solution of the problem (0,16), (0.17) for any $n > n_0$ can be constructed by one of the iterative methods mentioned in Chapter II.

Corollaries 9.9–9.12 concretize Corollaries 9.5–9.8 for the problem (0.1_1), (0.2). The system (0.1_1) with the boundary conditions

$$u_k(t_k) = \sum_{j=1}^{m} \big[c_{ikj} u_j(a) + c_{2kj} u_j(b)\big] + c_k \quad (k = 1, \dots, m) \tag{0.21}$$

is considered separately. Problems of this type arise in the theory of chemical reactors [37, 41] and in modelling kidney's functioning [27, 46]. We investigate the problem (0.1_1), (0.21) under the assumption that $t_k \in \{a, b\}$, $g_k \in C(I_0 \times R^m; R)$ $(k = 1, \dots, m)$,

$$\big|g_k(t, x_1, \dots, x_m) - g_k(t, \overline{x}_1, \dots, \overline{x}_m) \le \sum_{k=1}^{m} h_{kj} |x_j - \overline{x}_j| \quad (k = 1, \dots, m),$$

and either the spectral radius of the matrix

$$\left(\ell_{kj} + (b-a)\sum_{i=1}^{m} \ell_{ki}h_{ij} + \frac{2(b-a)}{\pi}\, h_{kj}\right)_{k,j=1}^{m}$$

is less than 1 or

$$t_k = a \quad \text{and} \quad \sum_{k=1}^{m} h_{kj} < \ell^*,$$

where $\ell_{kj} = |c_{1kj}|+|c_{2kj}|$ $(k,j=1,\dots,m)$ and $\ell^* = \frac{1}{b-a}\min\{\ell n(\sum\limits_{j=1}^{m} \ell_{kj})^{-1} : k=1,\dots,m\}$. These conditions along with the unique solvability of the problem turn out to guarantee the convergence and stability of Tewarson's [46] and Runge–Kutta's difference schemes (see Corollaries 9.13–9.15).

In §10, we consider the differential system

$$\frac{du_k(t)}{dt} = \overline{f}_k(u_1,\dots,u_{k-1},u_k,u_{k+1},\dots,u_m)(t) \quad (k=1,\dots,m) \tag{0.22}$$

with the boundary conditions (0.2), and for the numerical solution of the problem (0.22), (0.2), we propose the difference scheme of the type

$$\Delta x_{kn}(i-1) = \overline{f}_{kn}(x_{1\,n-1},\dots,x_{k\,n-1},x_{kn},x_{k+1\,n-1},\dots,x_{m\,n-1})(i) \tag{0.23}$$
$$(k=1,\dots,m),$$

$$x_{kn}(i_{kn}) = \varphi_k\big(q_{n-1}(x_{1\,n-1}),\dots,q_{n-1}(x_{m\,n-1})\big) \quad (k=1,\dots,m), \tag{0.24}$$

where

$$\overline{f}_k \in K(C(I_0;R^{m+1});L(I_0;R)), \quad \big(\overline{f}_{kn}\big)_{n=1}^{+\infty} \in \mathcal{D}_{\overline{f}_k} \quad (k=1,\dots,m)$$

and the functionals $\varphi_k : C(I_0;R^m) \to R$ $(k=1,\dots,m)$ satisfy (0.9).

Let in $\widetilde{E}_n^m$ the inequality

$$\big[\overline{f}_{kn}(x_1,\dots,x_k,x,x_{k+1},\dots,x_m)(i) - \overline{f}_{kn}(y_1,\dots,y_k,y,y_{k+1},\dots,y_m)(i) -$$
$$-h_{kn}(i)\big(x(\tau_{kn}(i)) - y(\tau_{kn}(i))\big)\big]\,\mathrm{sign}\,\big[(\tau_{kn}(i)-i_{kn})\big(x(\tau_{kn}(i)) -$$
$$-y(\tau_{kn}(i))\big)\big] \le f_{0kn}\big(|x_1-y_1|,\dots,|x_m-y_m|\big)(i) \quad \textit{for} \quad i\in N_n$$

be fulfilled for any $n>n_0$ and $k\in N_m$, where $f_{0kn} : (\widetilde{E}_n^+)^m \to E_n^+$ $(k=1,\dots,m)$ are positively homogeneous continuous non-decreasing operators, $\|h_{kn}\|_{\widetilde{E}_n} < 1$, $(h_{kn})_{n=1}^{+\infty} \in \mathcal{D}_{h_k}$, $(f_{0kn})_{n=1}^{+\infty} \in \mathcal{D}_{f_{0k}}$ $(k=1,\dots,m)$, and h_k, f_{0k} and φ_{0k} $(k=1,\dots,m)$ satisfy (0.5). *Then:*

(a) *the problem* (0.22), (0.2) *has a unique solution* $(u_k^0)_{k=1}^m$;

(b) *the difference scheme* (0.23), (0.24) *is stable*;[3]

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[3]See Definition 10.1.

(c) *given* $(x_{kn_0})_{k=1}^m \in \widetilde{E}_{n_0}^m$, *there exists a unique sequence* $(x_{kn})_{k=1}^m$ $(n = n_0+1,\dots)$ *of the solutions of the problems* (0.23), (0.24), *and*

$$\lim_{n\to+\infty} \|p_n(u_k^0) - x_{kn}\|_{\widetilde{E}_n} = 0.$$

Let the conditions

$$\big[\overline{f}_k(u_1,\dots,u_k,u,u_{k+1},\dots,u_m)(t) - \overline{f}_k(v_1,\dots,v_k,v,v_{k+1},\dots,v_m)(t) -$$
$$-h_k(t)\big(u(t)-v(t)\big)\big] \operatorname{sign}\big[(t-t_k)\big(u(t)-v(t)\big)\big] \le$$
$$\le f_{0k}\big(|u_1-v_1|,\dots,|u_m-v_m|\big)(t) \quad i \in I_0 \ \ (k=1,\dots,m), \tag{0.25}$$

$$\overline{f}_{kn}(x_1,\dots,x_k,x,x_{k+1},\dots,x_m)(i) =$$
$$= \delta_{i-1\, i_{kn}} \int\limits_{t_{i-1\,n}}^{t_{in}} h_k(s)\,ds\cdot x(i) + (1-\delta_{i-1\, i_{kn}})\times$$
$$\times \int\limits_{t_{i-1\,n}}^{t_{in}} f_k\big(q_{n-1}(x_1),\dots,q_{n-1}(x_k),q_{ikn}(x),q_{n-1}(x_{k+1}),\dots,q_{n-1}(x_m)\big)(s)\,ds$$
$$(n\in N;\quad i\in N_n;\quad k\in N_m) \tag{0.26}$$

be fulfilled along with (0.5) *and* (0.9), *and let* $n_0 \in N$ *be so large that* $\big|\int\limits_{t_{i-1\,n}}^{t_{in}} h_k(s)\,ds\big| < 1$ *for* $i \in N_n$ *and* $n > n_0$. *Then the conclusion of Theorem* 10.1 *is valid.*

Let the operators $\overline{f}_k \in C(C(I_0;R^{m+1});C(I_0;R))$ $(k = 1,\dots,m)$ *be bounded on every bounded set of* $C(I_0;R^{m+1})$, *the conditions*

$$h_k \in C(I_0;R), \quad f_{0k} \in C(C(I_0;R_+^m);C(I_0;R_+)) \quad (k=1,\dots,m),$$
$$\overline{f}_{kn}(x_1,\dots,x_k,x,x_{k+1},\dots,x_m)(i) = \frac{b-a}{n}\times$$
$$\times\overline{f}_k\big(q_{n-1}(x_1),\dots,q_{n-1}(x_k),q_n(x),q_{n-1}(x_{k+1}),\dots,q_{n-1}(x_m)\big)\big(t_{\tau_{kn}(i)n}\big)$$

be fulfilled along with (0.5), (0.9) *and* (0.25), *and let* $n_0 \in N$ *be so large that* $\dfrac{b-a}{n}|h_k(t)| < 1$ *for* $n > n_0$ *and* $t \in I_0$. *Then the conclusion of Theorem* 10.1 *is valid.*

Corollaries 10.3 and 10.4 concretize Corollaries 10.1 and 10.2 for the differential system

$$\frac{du_k(t)}{dt} = \overline{g}_k\big(t,u_1(t),\dots,u_{k-1}(t),u_k(t),u_{k+1}(t),\dots,u_m(t)\big)(t)$$
$$(k=1,\dots,m).$$

CHAPTER I

§ 1. Formulation of the Existence and Uniqueness Theorems

Let

$$t_k \in I_0, \quad f_k \in K(C(I_0; R^m); L(I_0; R)) \quad (k = 1, \dots, m),$$

and let $\varphi_k : C(I_0; R^m) \to R$ $(k = 1, \dots, m)$ be continuous functionals. Consider the problem of finding an absolutely continuous vector function $(u_k)_{k=1}^m : I_0 \to R$ satisfying almost everywhere the system of functional differential equations

$$\frac{du_k(t)}{dt} = f_k(u_1, \dots, u_m)(t) \quad (k = 1, \dots, m) \tag{1.1}$$

and the boundary conditions

$$u_k(t_k) = \varphi_k(u_1, \dots, u_m) \quad (k = 1, \dots, m). \tag{1.2}$$

Particular cases of (1.1) are: system of ordinary differential equations

$$\frac{du_k(t)}{dt} = g_k\big(t, u_1(t), \dots, u_m(t)\big) \ \ (k = 1, \dots, m), \tag{1.1$_1$}$$

system of differential equations with deviating arguments

$$\frac{du_k(t)}{dt} = g_k\big(t, u_1(t), \dots, u_m(t), s_{\zeta_1}(u_1)(t), \dots, s_{\zeta_m}(u_m)(t)\big) \quad (k = 1, \dots, m), \tag{1.1$_2$}$$

where

$$s_{\zeta_k}(u)(t) = \begin{cases} u(\zeta_k(t)) & \text{for } \ \zeta_k(t) \in I_0, \\ 0 & \text{for } \ \zeta_k(t) \notin I_0, \end{cases}$$

and system of integro-differential equations

$$\frac{du_k(t)}{dt} = g_k\bigg(t, u_1(t), \dots, u_m(t), \int\limits_a^b u_1(s)\, d_s\, \alpha_1(s,t), \dots \\ \dots, \int\limits_a^b u_m(s)\, d_s\, \alpha_m(s,t)\bigg) \quad (k = 1, \dots, m). \tag{1.1$_3$}$$

Everywhere in the sequel we will assume that the right-hand sides of the system (1.1$_1$) belong to $K(I_0 \times R^m; R)$, while those of the systems (1.1$_2$) and (1.1$_3$) belong to $K(I_0 \times R^{2m}; R)$. As for the functions $\zeta_k : I_0 \to R$ $(k = 1, \dots, m)$, they will be assumed to be measurable, while the functions

$\alpha_k : I_0 \times I_0 \to R$ $(k = 1, \dots, m)$ will always be measurable in the second argument, will have bounded variation in the first one and will satisfy

$$\alpha_k^* = \operatorname{vrai\,max}\left\{ \int\limits_a^b |d_s \alpha_k(s,t)| : t \in I_0 \right\} < +\infty \quad (k = 1, \dots, m).$$

An operator $f_0 : C(I_0; R_+^m) \to L(I_0; R)$ is said to be *positively homogeneous* if for any nonnegative ρ and any $(x_k)_{k=1}^m \in C(I_0; R_+^m)$ we have

$$f_0(\rho x_1, \dots, \rho x_m) = \rho f_0(x_1, \dots, x_m).$$

If for any $(x_k)_{k=1}^m$ and $(y_k)_{k=1}^m \in C(I_0; R_+^m)$ satisfying

$$x_k(t) \le y_k(t) \quad \text{for} \quad t \in I_0 \quad (k = 1, \dots, m)$$

we have

$$f_0(x_1, \dots, x_m)(t) \le f_0(y_1, \dots, y_m)(t),$$

then f_0 is called a *non-decreasing* operator.

For the sake of convenience, we introduce the following

Let $h_k \in L(I_0; R)$ $(k = 1, \dots, m)$, and let $f_{0k} : C(I_0; R_+^m) \to L(I_0; R_+)$ and $\varphi_{0k} : C(I_0; R_+^m) \to R_+$ $(k = 1, \dots, m)$ be positively homogeneous continuous nondecreasing operators and functionals such that the system of differential inequalities

$$|u_k'(t) - h_k(t) u_k(t)| \le f_{0k}\big(|u_1|, \dots, |u_m|\big)(t) \quad (k = 1, \dots, m) \tag{1.3}$$

with the boundary conditions

$$|u_k(t_k)| \le \varphi_{0k}\big(|u_1|, \dots, |u_m|\big) \quad (k = 1, \dots, m)^4 \tag{1.4}$$

has only the zero solution. Then we say that

$$(h_1, \dots, h_m; f_{01}, \dots, f_{0m}; \varphi_{01}, \dots, \varphi_{0m}) \in W(t_1, \dots, t_m).$$

The writing

$$\big(f_{01}, \dots, f_{0m}; \varphi_{01}, \dots, \varphi_{0m}\big) \in W_0(t_1, \dots, t_m)$$

means that

$$\big(0, \dots, 0; f_{01}, \dots, d_{0m}; \varphi_{01}, \dots, \varphi_{0m}\big) \in W(t_1, \dots, t_m).$$

$'$ The writing

$$\big(h_1, \dots, h_m; h_{11}, \dots, h_{1m}, \dots, h_{m1}, \dots, h_{mm}; \varphi_{01}, \dots, \varphi_{0m}\big) \in W'(t_1, \dots, t_m)$$

[4]The solution of the problem (1.3), (1.4) is likewise sought in the class of absolutely continuous vector functions $(u_k)_{k=1}^m : I_0 \to R^m$.

means that $h_k \in L(I_0;R)$, $h_{kj} \in L(I_0;R_+)$ $(k,j = 1,\dots,m)$, and $\varphi_{0k} : C(I_0;R^m_+) \to R_+$ $(k = 1,\dots,m)$ are positively homogeneous continuous nondecreasing functionals such that the system of differential inequalities

$$|u'_k(t) - h_k(t)u_k(t)| \le \sum_{j=1}^m h_{kj}(t)|u_j(t)| \quad (k = 1,\dots,m) \tag{1.3$'$}$$

under the boundary conditions (1.4) has only the zero solution.

The writing

$$\big(h_{11},\dots,h_{1m},\dots,h_{m1},\dots,h_{mm};\varphi_{01},\dots,\varphi_{0m}\big) \in W'_0(t_1,\dots,t_m)$$

means that

$$\big(0,\dots,0;h_{11},\dots,h_{1m},\dots,h_{m1},\dots,h_{mm};\varphi_{01},\dots,\varphi_{0m}\big) \in W'(t_1,\dots,t_m).$$

Let in $C(I_0;R^m)$ the inequalities

$$\begin{gathered}\big[f_k(u_1,\dots,u_m)(t) - h_k(t)u_k(t)\big]\operatorname{sign}\big[(t-t_k)u_k(t)\big] \le \\ \le h_0(t) + f_{0k}\big(|u_1|,\dots,|u_m|\big)(t) \quad \text{for} \quad t \in I_0 \ \ (k = 1,\dots,m)\end{gathered} \tag{1.5}$$

and

$$|\varphi_k(u_1,\dots,u_m)| \le r + \varphi_{0k}\big(|u_1|,\dots,|u_m|\big) \quad (k = 1,\dots,m) \tag{1.6}$$

be fulfilled, where $r \in R_+$, $h_0 \in L(I_0;R_+)$, and

$$\big(h_1,\dots,h_m;f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m}\big) \in W(t_1,\dots,t_m). \tag{1.7}$$

Then the problem (1.1), (1.2) *is solvable.*

Let in $C(I_0;R^m)$ the inequalities

$$\begin{gathered}f_k(u_1,\dots,u_m)(t)\operatorname{sign}\big[(t-t_k)u_k(t)\big] \le \\ \le h_0(t) + \sum_{j=1}^m h_{kj}(t)\|u_j\|_{C(I_0;R)} \quad \text{for} \quad t \in I_0 \ \ (k = 1,\dots,m)\end{gathered} \tag{1.8}$$

and

$$|\varphi_k(u_1,\dots,u_m)| \le \ell_0 + \sum_{j=1}^m \ell_{kj}(t)\|u_j\|_{C(I_0;R)} \quad (k = 1,\dots,m) \tag{1.9}$$

be fulfilled, where $\ell_0 \in R_+$, $\ell_{kj} \in R_+$, $h_{kj} \in L(I_0;R_+)$ $(k,j = 1,\dots,m)$, $h_0 \in L(I_0;R_+)$, and the spectral radius of the matrix

$$\left(\ell_{kj} + \int_a^b h_{kj}(t)\,dt\right)_{k,j=1}^m \tag{1.10}$$

is less than 1. *Then the problem* (1.1), (1.2) *is solvable.*

$'$ *Let in* $I_0 \times R^m$ *the inequalities*

$$g_k(t, x_1, \dots, x_m) \operatorname{sign}\big[(t - t_k)x_k\big] \le$$
$$\le h_0(t) + \sum_{j=1}^{m} h_{kj}(t)|x_j| \quad (k = 1, \dots, m), \tag{1.11}$$

and in $C(I_0; R^m)$ *the inequalities* (1.9) *be fulfilled, where the numbers* ℓ_0 *and the functions* ℓ_{kj}, h_{kj} $(k, j = 1, \dots, m)$ *satisfy the conditions of Corollary* 1.1. *Then the problem* (1.1_1), (1.2) *is solvable.*

$''$ *Let in* $I_0 \times R^{2m}$ *the inequalities*

$$g_k(t, x_1, \dots, x_m, y_1, \dots, y_m) \operatorname{sign}\big[(t - t_k)x_k\big] \le$$
$$\le h_0(t) + \sum_{j=1}^{m} \big(h_{1kj}(t)|x_j| + h_{2kj}(t)|y_j|\big) \quad (k = 1, \dots, m), \tag{1.12}$$

and in $C(I_0; R^m)$ *the inequalities* (1.9) *be fulfilled, where* $\ell_0 \in R_+$, $\ell_{kj} \in R_+$, $h_{1kj} \in L(I_0; R_+)$, $h_{2kj} \in L(I_0; R_+)$ $(k, j = 1, \dots, m)$ *and* $h_0 \in L(I_0; R_+)$. *Let, furthermore, the spectral radius of the matrix* (1.10), *where* $h_{kj}(t) = h_{1kj}(t) + s_{\varsigma_j}(1)(t) h_{2kj}(t)$ (*where* $h_{kj}(t) = h_{1kj}(t) + \alpha_j^* h_{2kj}(t)$), *be less than* 1. *Then the problem* (1.1_2), (1.2) (*the problem* (1.1_3), (1.2)) *is solvable.*

Let in $C(I_0; R^m)$ *the inequalities*

$$f_k(u_1, \dots, u_m)(t) \operatorname{sign}\big[(t - t_k)u_k(t)\big] \le h_0(t) + h_k|u_k(t)| +$$
$$+ \sum_{j=1}^{m} h_{kj}\|u_j\|_{C(I_0;R)} \quad \textit{for} \quad t \in I_0 \ \ (k = 1, \dots, m) \tag{1.13}$$

and

$$|\varphi_k(u_1, \dots, u_m)| \le \ell_0 + \varphi_{0k}\big(|u_k|\big) \quad (k = 1, \dots, m) \tag{1.14}$$

be fulfilled, where $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1, \dots, m)$, $\ell_0 \in R_+$, $h_0 \in L(I_0; R_+)$, *and* $\varphi_{0k} : C(I_0; R_+) \to R_+$ $(k = 1, \dots, m)$ *are linear continuous functionals such that*

$$\varphi_{0k}(1) \le 1, \quad \varphi_{0k}(\widetilde{h}_k) < 1, \quad \widetilde{h} = \exp\big(h_k|t - t_k|\big) \ \ (k = 1, \dots, m). \tag{1.15}$$

Moreover, let the real parts of the eigen-values of the matrix

$$\big(h_k \delta_{kj} + h_{kj}\big)_{k,j=1}^{m} \tag{1.16}$$

be negative. Then the problem (1.1), (1.2) *is solvable.*

$'$ *Let in* $I_0 \times R^m$ *the inequalities*

$$g_k(t, x_1, \dots, x_m) \operatorname{sign}\big[(t-t_k)x_k\big] \leq$$
$$\leq h_0(t) + h_k|x_k| + \sum_{j=1}^{m} h_{kj}|x_j| \quad (k = 1, \dots, m), \tag{1.17}$$

and in $C(I_0; R^m)$ *the inequalities* (1.14) *be fulfilled, where the numbers* ℓ_0, h_k, h_{kj} $(k, j = 1, \dots, m)$, *the function* h_0 *and the functionals* φ_{0k} $(k = 1, \dots, m)$ *satisfy the conditions of Corollary* 1.2. *Then the problem* (1.1_1), (1.2) *is solvable.*

$''$ *Let in* $I_0 \in R^{2m}$ *the inequalities*

$$g_k(t, x_1, \dots, x_m, y_1, \dots, y_m) \operatorname{sign}\big[(t-t_k)x_k\big] \leq h_0(t) + h_k|x_k| +$$
$$+ \sum_{j=1}^{m} \big(h_{1kj}(t)|x_j| + h_{2kj}(t)|y_j|\big) \quad (k = 1, \dots, m), \tag{1.18}$$

and in $C(I_0; R^m)$ *the inequalities* (1.14) *be fulfilled, where* $\varphi_{0k} : C(I_0; R_+) \to R_+$ $(k = 1, \dots, m)$ *are linear continuous functionals satisfying* (1.15), $h_k < 0$ $(k = 1, \dots, m)$, $\ell_0 \in R_+$, $h_0 \in L(I_0; R_+)$, *and* h_{1kj} *and* $h_{2kj} : I_0 \to R_+$ *are measurable functions, such that*

$$h_{kj} = h_{1kj}(t) + s_{\zeta_k}(1)(t) h_{2kj}(t) \equiv const$$
$$\big(\, h_{kj} = h_{1kj}(t) + \alpha_j^* h_{2kj}(t) \equiv const\,\big).$$

Moreover, let the real parts of the eigen-values of the matrix (1.16) *be negative. Then the problem* (1.1_2), (1.2) (*the problem* (1.1_3), (1.2)) *is solvable.*

Let in $C(I_0; R^m)$ *the inequalities*

$$f_k(u_1, \dots, u_m)(t) \operatorname{sign}\big[(t-t_k)u_k(t)\big] \leq h_0(t) +$$
$$+ \sum_{j=1}^{m} \big(h_{1kj}|u_j(t)| + h_{2kj} s_{\zeta_1}(|u_j|)(t)\big) \ \ \textit{for} \ \ t \in I_0 \ \ (k = 1, \dots, m) \tag{1.19}$$

and

$$|\varphi_k(u_1, \dots, u_m) \leq \ell_0 + \sum_{j=1}^{m} \ell_{kj} \|u_j\|_{L^2(I_0; R)} \quad (k = 1, \dots, m) \tag{1.20}$$

be fulfilled, where ℓ_0, ℓ_{kj}, h_{1kj}, $h_{2kj} \in R_+$ $(k, j = 1, \dots, m)$, $h_0 \in L(I_0; R_+)$, *and* $\zeta_k : I_0 \to R$ $(k = 1, \dots, m)$ *are absolutely continuous monotone functions. Moreover, let*

$$\gamma_k = \operatorname{vrai\,min}\big\{|\zeta_k'(t)| : t \in I_0\big\} > 0 \quad (k = 1, \dots, m), \tag{1.21}$$

and let the spectral radius of the matrix

$$\left((b-a)^{1/2}\ell_{kj}+\frac{2(b-a)}{\pi}\,h_{kj}\right)_{k,j=1}^{m}, \tag{1.22}$$

where $h_{kj}=h_{1kj}+h_{2kj}/\sqrt{\gamma_j}$, *be less than* 1. *Then the problem* (1.1), (1.2) *is solvable.*

$'$ *Let in* $I_0\times R^m$ *the inequalities* (1.11), *and in* $C(I_0;R^m)$ *the inequalities* (1.20) *be fulfilled, where* ℓ_0, $\ell_{kj}\in R_+$, $h_{kj}=h_{kj}(t)=const\geq 0$ $(k,j=1,\dots,m)$, $h_0\in L(I_0;R_+)$, *and the spectral radius of the matrix* (1.22) *is less than* 1. *Then the problem* (1.1_1), (1.2) *is solvable.*

$''$ *Let the functions* ζ_k $(k=1,\dots,m)$ *be absolutely continuous, monotone and satisfy* (1.21). *Moreover, let in* $I_0\times R^{2m}$ *the inequalities* (1.12), *and in* $C(I_0;R^m)$ *the inequalities* (1.20) *be fulfilled, where* ℓ_0, $\ell_{kj}\in R_+$, $h_{1kj}=h_{1kj}(t)\equiv const\geq 0$, $h_{2kj}=s_{\zeta_k}(1)(t)h_{2kj}(t)\equiv const\geq 0$ $(k,j=1,\dots,m)$, *and the spectral radius of the matrix* (1.22) *with* $h_{kj}=h_{1kj}+h_{2kj}/\sqrt{\gamma}$ *is less than* 1. *Then the problem* (1.1_2), (1.2) *is solvable.*

Let in $C(I_0;R^m)$ *the inequalities*

$$\begin{gathered}\big[f_k(u_1,\dots,u_m)(t)-f_k(v_1,\dots,v_m)(t)-\\ -h_k(t)(u_k(t)-v_k(t))\big]\operatorname{sign}\big[(t-t_k)(u_k(t)-v_k(t))\big]\leq\\ \leq f_{0k}\big(|u_1-v_1|,\dots,|u_m-v_m|\big)(t)\ \ for\ \ t\in I_0\ \ (k=1,\dots,m)\end{gathered} \tag{1.23}$$

and

$$\begin{gathered}\big|\varphi_k(u_1,\dots,u_m)-\varphi_k(v_1,\dots,v_m)\big|\leq\\ \leq\varphi_{0k}\big(|u_1-v_1|,\dots,|u_m-v_m|\big)\quad(k=1,\dots,m)\end{gathered} \tag{1.24}$$

be fulfilled, where $(h_1,\dots,h_m;f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m})\in W(t_1,\dots,t_m)$. *Then the problem* (1.1), (1.2) *has a unique solution.*

Let in $C(I_0;R^m)$ *the inequalities*

$$\begin{gathered}\big[f_k(u_1,\dots,u_m)(t)-f_k(v_1,\dots,v_m)(t)\big]\operatorname{sign}\big[(t-t_k)(u_k(t)-v_k(t))\big]\leq\\ \leq\sum_{j=1}^{m}h_{kj}(t)\|u_j-v_j\|_{C(I_0;R)}\ \ for\ \ t\in I_0\ \ (k=1,\dots,m)\end{gathered} \tag{1.25}$$

and

$$\begin{gathered}\big|\varphi_k(u_1,\dots,u_m)-\varphi_k(v_1,\dots,v_m)\big|\leq\\ \leq\sum_{j=1}^{m}\ell_{kj}\|u_j-v_j\|_{C(I_0;R)}\quad(k=1,\dots,m)\end{gathered} \tag{1.26}$$

be fulfilled, where $\ell_{kj} \in R_+$, $h_{kj} \in L(I_0; R_+)$ $(k = 1, \dots, m)$, *and the spectral radius of the matrix* (1.10) *is less than* 1. *Then the problem* (1.1), (1.2) *has a unique solution.*

′ *Let in* $I_0 \times R^m$ *the inequalities*

$$\big[g_k(t, x_1, \dots, x_m) - g_k(t, \overline{x}_1, \dots, \overline{x}_m)\big] \operatorname{sign}\big[(t - t_k)(x_k - \overline{x}_k)\big] \leq$$
$$\leq \sum_{j=1}^{m} h_{kj}(t)|x_j - \overline{x}_j| \quad (k = 1, \dots, m), \tag{1.27}$$

and in $C(I_0; R^m)$ *the inequalities* (1.26) *be fulfilled, where* $\ell_{kj} \in R_+$, $h_{kj} \in L(I_0; R_+)$ $(k, j = 1, \dots, m)$, *and the spectral radius of the matrix* (1.10) *is less than* 1. *Then the problem* (1.1_1), (1.2) *has a unique solution.*

″ *Let in* $I_0 \times R^{2m}$ *the inequalities*

$$\big[g_k(t, x_1, \dots, x_m, y_1, \dots, y_m) - g_k(t, \overline{x}_1, \dots, \overline{x}_m, \overline{y}_1, \dots, \overline{y}_m)\big] \times$$
$$\times \operatorname{sign}\big[(t - t_k)(x_k - \overline{x}_k)\big] \leq$$
$$\leq \sum_{j=1}^{m} \big[h_{1kj}(t)|x_j - \overline{x}_j| + h_{2kj}(t)|y_j - \overline{y}_j|\big] \quad (k = 1, \dots, m), \tag{1.28}$$

and in $C(I_0; R^m)$ *the inequalities* (1.26) *be fulfilled, where* $\ell_{kj} \in R_+$, $h_{1kj} \in L(I_0; R_+)$ *and* $h_{2kj} \in L(I_0; R_+)$ $(k, j = 1, \dots, m)$. *Moreover, let the spectral radius of the matrix* (1.10), *where* $h_{kj}(t) = h_{1kj}(t) + s_{\varsigma_j}(1)(t) h_{2kj}(t)$ $(h_{kj}(t) = h_{1kj}(t) + \alpha_j^* h_{2kj}(t))$, *be less than* 1. *Then the problem* (1.1_2), (1.2) (*the problem* (1.1_3), (1.2)) *has a unique solution.*

Remark 1.1. In Corollaries 1.1′, 1.1″, 1.4′, 1.4″, the restrictions imposed on the spectral radius of the matrix (1.10) may be replaced by the requirement

$$t_k = a, \quad \sum_{j=1}^{m} \ell_{kj} \exp\bigg(\int_a^b h(t)\,dt\bigg) < 1 \quad (k = 1, \dots, m), \tag{1.29}$$

where

$$h(t) = \max\bigg\{\sum_{j=1}^{m} h_{kj}(t) : k = 1, \dots, m\bigg\}. \tag{1.30}$$

Let $\varphi_k(u_1, \dots, u_m) \equiv \varphi_k(u_k)$ *and in* $C(I_0; R^m)$ *the inequalities*

$$\big[f_k(u_1, \dots, u_m)(t) - f_k(v_1, \dots, v_m)(t)\big] \times$$
$$\times \operatorname{sign}\big[(t - t_k)(u_k(t) - v_k(t))\big] \leq h_k|u_k(t) - v_k(t)| +$$

$$+\sum_{j=1}^{m} h_{kj}\|u_j - v_j\|_{C(I_0;R)} \quad \textit{for } t \in I_0 \ \ (k = 1, \dots, m) \tag{1.31}$$

and

$$|\varphi_k(u) - \varphi_k(v)| \le \varphi_{0k}\big(|u - v|\big) \ \ (k = 1, \dots, m) \tag{1.32}$$

be fulfilled, where $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1, \dots, m)$, *and* $\varphi_{0k} : C(I_0; R_+) \to R_+$ $(k = 1, \dots, m)$ *are linear continuous functionals satisfying* (1.15). *Moreover, let the real parts of the eigen-values of the matrix* (1.16) *be negative. Then the problem* (1.1), (1.2) *has a unique solution.*

$'$ *Let* $\varphi_k(u_1, \dots, u_m) \equiv \varphi_k(u_k)$, *in* $I_0 \times R^m$ *the inequalities*

$$\begin{gathered}\big[g_k(t, x_1, \dots, x_m) - g_k(t, \overline{x}_1, \dots, \overline{x}_m)\big] \operatorname{sign}\big[(t - t_k)(x_k - \overline{x}_k)\big] \le \\ \le h_k|x_k - \overline{x}_k| + \sum_{j=1}^{m} h_{kj}|x_j - \overline{x}_j| \quad (k = 1, \dots, m)\end{gathered} \tag{1.33}$$

and in $C(I_0; R^m)$ *the inequalities* (1.32) *be fulfilled, where* $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1, \dots, m)$, *and* $\varphi_{0k} : C(I_0; R_+) \to R_+$ $(k = 1, \dots, m)$ *are linear continuous functionals satisfying* (1.15). *Moreover, let the real parts of the eigen-values of the matrix* (1.16) *be negative. Then the problem* (1.1_1), (1.2) *has a unique solution.*

$''$ *Let* $\varphi_k(u_1, \dots, u_m) \equiv \varphi_k(u_k)$, *in* $I_0 \times R^{2m}$ *the inequalities*

$$\begin{gathered}\big[g_k(t, x_1, \dots, x_m, y_1, \dots, y_m) - g_k(t, \overline{x}_1, \dots, \overline{x}_m, \overline{y}_1, \dots, \overline{y}_m)\big] \times \\ \times \operatorname{sign}\big[(t - t_k)(x_k - \overline{x}_k)\big] \le h_k|x_k - \overline{x}_k| + \\ + \sum_{j=1}^{m}\big[h_{1kj}(t)|x_j - \overline{x}_j| + h_{2kj}(t)|y_j - \overline{y}_j|\big] \quad (k = 1, \dots, m),\end{gathered} \tag{1.34}$$

and in $C(I_0; R^m)$ *the inequalities* (1.32) *be fulfilled, where* $\varphi_{0k} : C(I_0; R_+) \to R_+$ $(k = 1, \dots, m)$ *are linear continuous functionals satisfying* (1.15), $h_k < 0$ $(k = 1, \dots, m)$, *and* h_{1kj} *and* $h_{2kj} : I_0 \to R_+$ *are measurable functions such that*

$$\begin{gathered}h_{kj} = h_{1kj}(t) + s_{\zeta_k}(1)(t)h_{2kj}(t) \equiv \textit{const} \\ \big(h_{kj} = h_{1kj}(t) + \alpha_j^* h_{2kj}(t) \equiv \textit{const}\,\big).\end{gathered}$$

Moreover, let the real parts of the eigen-values of the matrix (1.16) *be negative. Then the problem* (1.1_2), (1.2) (*the problem* (1.1_3), (1.2)) *has a unique solution.*

Let in $C(I_0; R^m)$ *the inequalities*

$$\big[f_k(u_1,\dots,u_m)(t) - f_k(v_1,\dots,v_m)(t)\big] \operatorname{sign}\big[(t-t_k)(u_k(t)-v_k(t))\big] \le$$
$$\le \sum_{j=1}^{m} \big(h_{1kj}|u_j(t)-v_j(t)| + h_{2kj}s_{\zeta_j}(|u_j-v_j|)(t)\big) \tag{1.35}$$
$$\textit{for} \quad t \in I_0 \ \ (k=1,\dots,m)$$

and

$$|\varphi_k(u_1,\dots,u_m) - \varphi_k(v_1,\dots,v_m)| \le$$
$$\le \sum_{j=1}^{m} \ell_{kj}\|u_j - v_j\|_{L^2(I_0;R)} \ \ (k=1,\dots,m) \tag{1.36}$$

be fulfilled, where ℓ_{kj}, h_{1kj}, $h_{2kj} \in R_+$ $(k,j=1,\dots,m)$, *and* $\zeta_k : I_0 \to R$ $(k=1,\dots,m)$ *are absolutely continuous functions satisfying* (1.21). *Moreover, let the spectral radius of the matrix* (1.22), *where* $h_{kj} = h_{1kj} + h_{2kj}/\sqrt{\gamma_j}$, *be less than* 1. *Then the problem* (1.1), (1.2) *has a unique solution.*

′ *Let in* $I_0 \times R^m$ *the inequalities* (1.27), *and in* $C(I_0;R^m)$ *the inequalities* (1.36) *be fulfilled, where* $\ell_{kj} \in R_+$, $h_{kj} = h_{kj}(t) \equiv const \ge 0$ $(k,j=1,\dots,m)$, *and let the spectral radius of the matrix* (1.22) *be less than* 1. *Then the problem* (1.1_1), (1.2) *has a unique solution.*

″ *Let the functions* ζ_k $(k=1,\dots,m)$ *be absolutely continuous, monotone and satisfy* (1.21). *Moreover, let in* $I_0 \times R^{2m}$ *the inequalities* (1.28), *and in* $C(I_0;R^m)$ *the inequalities* (1.36) *be fulfilled, where* $\ell_{kj} \in R_+$, $h_{1kj} = h_{2kj}(t) \equiv const \ge 0$, $h_{2kj} = s_{\zeta_k}(1)(t)h_{2kj}(t) \equiv const \ge 0$ $(k=1,\dots,m)$, *and the spectral radius of the matrix* (1.22) *with* $h_{kj} = h_{1kj} + h_{2kj}/\sqrt{\gamma_j}$ *is less than* 1. *Then the problem* (1.1_2), (1.2) *has a unique solution.*

Remark 1.2. Corollaries 1.3, 1.3′, 1.3″, 1.6, 1.6′ and 1.6″ remain valid if we require that the conditions (1.9) and (1.26) be fulfilled instead of (1.20) and (1.36) and if we consider the spectral radius of the matrix

$$\left(\ell_{kj} + (b-a)\sum_{i=1}^{m}\ell_{ki}h_{ik} + \frac{2(b-a)}{\pi}\,h_{kj}\right)_{k,j=1}^{m} \tag{1.37}$$

instead of that of the matrix (1.22).

In this subsection, we study the case where the boundary conditions (1.2) have the form

$$u_k(t_k) = c_k \quad (k = 1, \dots, m) \tag{1.38}$$

with $c_k \in R$ $(k = 1, \dots, m)$.

Let one of the following three conditions be fulfilled:

1) *in* $C(I_0; R^m)$ *the inequalities* (1.8) *hold, where* $h_0 \in L(I_0; R_+)$, $h_{kj} \in L(I_0; R_+)$ $(k, j = 1, \dots, m)$, *and the spectral radius of the matrix*

$$\left(\int_a^b h_{kj}(t)\, dt \right)_{k,j=1}^m \tag{1.39}$$

is less than 1*;*

2) *in* $C(I_0; R^m)$ *the inequalities* (1.13) *hold, where* $h_0 \in L(I_0; R_+)$, $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1, \dots, m)$, *and the real parts of the eigen-values of the matrix* (1.16) *are negative;*

3) *in* $C(I_0; R^m)$ *the inequalities* (1.19) *hold, where* $\zeta_k : I_0 \to R$ $(k = 1, \dots, m)$ *are absolutely continuous monotone functions satisfying* (1.21), $h_0 \in L(I_0; R_+)$, $h_{1kj} \in R_+$, $h_{2kj} \in R_+$ $(k, j = 1, \dots, m)$, *and the spectral radius of the matrix*

$$\left(h_{1kj} + \frac{h_{2kj}}{\sqrt{\gamma_j}} \right)_{k,j=1}^m \tag{1.40}$$

is less than $\pi/2(b-a)$.

Then the problem (1.1), (1.39) *is solvable.*

$'$ *Let one of the following three conditions be fulfilled:*

1) *in* $I_0 \times R^m$ *the inequalities* (1.11) *hold, where* $h_0 \in L(I_0; R_+)$, $h_{kj} \in L(I_0; R_+)$ $(k, j = 1, \dots, m)$, *and the spectral radius of the matrix* (1.39) *is less than* 1*;*

2) *in* $I_0 \times R^m$ *the inequalities* (1.17) *hold, where* $h_k < 0$, $h_0 \in L(I_0; R_+)$, $h_{kj} \in R_+$ $(k, j = 1, \dots, m)$, *and the real parts of the eigen-values of the matrix* (1.16) *are negative;*

3) *in* $I_0 \times R^m$ *the inequalities* (1.11) *hold, where* $h_0 \in L(I_0; R_+)$, $h_{kj} = h_{kj}(t) \equiv const \geq 0$ $(k, j = 1, \dots, m)$, *and the spectral radius of the matrix* $(h_{kj})_{k.j=1}^m$ *is less than* $\pi/2(b-a)$.

Then the problem (1.1_1), (1.38) *is solvable.*

$''$ *Let one of the following three conditions be fulfilled:*

1) *in* $I_0 \times R^{2m}$ *the inequalities* (1.12) *hold, where* $h_0 \in L(I_0; R_+)$, $h_{1kj} \in L(I_0; R_+)$, $h_{2kj} \in L(I_0; R_+)$ $(k, j = 1, \dots, m)$, *and the spectral radius of the matrix* (1.39), *where* $h_{kj}(t) = h_{1kj}(t) + s_{\zeta_k}(1)(t) h_{2kj}(t)$ $(h_{kj}(t) = h_{1kj}(t) + \alpha_j^* h_{2kj}(t))$, *is less than* 1*;*

2) in $I_0 \times R^{2m}$ the inequalities (1.18) *hold, where $h_0 \in L(I_0;R_+)$, $h_k < 0$ $(k = 1,\dots,m)$, the functions h_{1kj} and $h_{2kj} : I_0 \to R_+$ are measurable,*

$$h_{kj} = h_{1kj}(t) + s_{\zeta_k}(1)(t)h_{2kj}(t) \equiv const$$
$$\big(\, h_{kj} = h_{1kj}(t) + \alpha_j^* h_{2kj}(t) \equiv const \,\big),$$

and the real parts of the eigen-values of the matrix (1.16) *are negative;*

3) the functions ζ_k $(k = 1,\dots,m)$ are absolutely continuous, monotone and satisfy (1.21)*; moreover, in $I_0 \times R^{2m}$ the inequalities* (1.18) *hold, where $h_0 \in L(I_0;R_+)$, $h_{1kj} = h_{1kj}(t) \equiv const \geq 0$, $h_{2kj} = s_{\zeta_k}(1)(t)h_{2kj}(t) \equiv const \geq 0$, and the spectral radius of the matrix* (1.40) *is less than $\pi/2(b-a)$.*

Then the problem (1.1_3), (1.38) *is solvable.*

Let one of the following three conditions be fulfilled:

1) in $C(I_0;R^m)$ the inequalities (1.25) *hold, where $h_{kj} \in L(I_0;R_+)$ $(k,j = 1,\dots,m)$, and the spectral radius of the matrix* (1.39) *is less than* 1*;*

2) in $C(I_0;R^m)$ the inequalities (1.31) *hold, where $h_k < 0$, $h_{kj} \in R_+$ $(k,j = 1,\dots,m)$, and the real parts of the eigen-values of the matrix* (1.16) *are negative;*

3) in $C(I_0;R^m)$ the inequalities (1.35) *hold, where $\zeta_k : I_0 \to R$ $(k = 1,\dots,m)$ are absolutely continuous monotone functions satisfying* (1.21), *$h_{1kj} \in R_+$, $h_{2kj} \in R_+$ $(k,j = 1,\dots,m)$, and the spectral radius of the matrix* (1.40) *is less than $\pi/2(b-a)$.*

Then the problem (1.1), (1.38) *has a unique solution.*

$'$ *Let one of the following three conditions be fulfilled:*

1) in $I_0 \times R^m$ the inequalities (1.27) *hold, where $h_{kj} \in L(I_0;R_+)$ $(k,j = 1,\dots,m)$, and the spectral radius of the matrix* (1.39) *is less than* 1*;*

2) in $I_0 \times R^m$ the inequalities (1.33) *hold, where $h_k < 0$, $h_{kj} \in R_+$ $(k,j = 1,\dots,m)$, and the real parts of the eigen-values of the matrix* (1.16) *are negative;*

3) in $I_0 \times R^m$ the inequalities (1.27) *hold, where $h_{kj} = h_{kj}(t) \equiv const \geq 0$ $(k,j = 1,\dots,m)$, and the spectral radius of the matrix $(h_{kj})_{k,j=1}^m$ is less than $\pi/2(b-a)$.*

Then the problem (1.1_1), (1.38) *has a unique solution.*

$''$ *Let one of the following three conditions be fulfilled:*

1) in $I_0 \times R^{2m}$ the inequalities (1.28) *hold, where $h_{1kj} \in L(I_0;R_+)$, $h_{2kj} \in L(I_0;R_+)$ $(k,j = 1,\dots,m)$, and the spectral radius of the matrix* (1.39) *with $h_{kj}(t) = h_{1kj}(t) + s_{\zeta_k}(1)(t)h_{2kj}(t)$ (with $h_{kj}(t) = h_{1kj}(t) + \alpha_j^* h_{2kj}(t)$) is less than* 1*;*

2) in $I_0 \times R^{2m}$ the inequalities (1.34) *hold, where $h_k < 0$ $(k = 1,\dots,m)$, the functions h_{1kj} and $h_{2kj} : I_0 \to R_+$ are measurable, $h_{kj}(t) = h_{1kj}(t) + s_{\zeta_k}(1)(t)h_{2kj}(t) \equiv const$ $(h_{kj}(t) = h_{1kj}(t) + \alpha_j^* h_{2kj}(t) \equiv const)$, and the real parts of the eigen-values of the matrix* (1.16) *are negative;*

3) the functions ζ_k $(k = 1,\dots,m)$ are absolutely continuous, monotone and satisfy (1.21)*; moreover, in $I_0 \times R^{2m}$ the inequalities* (1.34) *hold, where*

$h_{1kj} = h_{1kj}(t) \equiv const \geq 0$, $h_{2kj} = s_{\zeta_k}(1)(t)h_{2kj}(t) \equiv const \geq 0$, *and the spectral radius of the matrix* (1.40) *is less than* $\pi/2(b-a)$.

Then the problem (1.1_2), (1.38) (*the problem* (1.1_3), (1.38)) *has a unique solution.*

In conclusion, let us consider the case where boundary conditions (1.2) are periodic, i.e.,

$$u_k(a) = u_k(b) + c_k \quad (k = 1, \dots, m) \tag{1.41}$$

with $c_k \in R$ $(k = 1, \dots, m)$.

Let in $C(I_0; R^m)$ *the inequalities*

$$f_k(u_1, \dots, u_m)(t) \operatorname{sign}\big[\sigma_k u_k(t)\big] \leq h_0(t) + h_k|u_k(t)| + \\ + \sum_{j=1}^{m} h_{kj}\|u_j\|_{C(I_0;R)} \quad \textit{for} \quad t \in I_0 \ \ (k = 1, \dots, m) \tag{1.42}$$

be fulfilled, where $h_0 \in L(I_0; R_+)$, $\sigma \in \{-1, 1\}$, $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1, \dots, m)$, *and the real parts of the eigen-values of the matrix* (1.16) *are negative. Then the problem* (1.1), (1.41) *has at least one solution.*

′ *Let in* $I_0 \times R^m$ *the inequalities*

$$g_k(t, x_1, \dots, x_m) \operatorname{sign}(\sigma_k x_k) \leq h_0(t) + h_k|x_k| + \\ + \sum_{j=1}^{m} h_{kj}|x_j| \quad (k = 1, \dots, m)$$

be fulfilled, where $h_0 \in L(I_0; R_+)$, $\sigma_k \in \{-1, 1\}$, $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1, \dots, m)$, *and the real parts of the eigen-values of the matrix* (1.16) *are negative. Then the problem* (1.1_1), (1.41) *has at least one solution.*

″ *Let in* $I_0 \times R^{2m}$ *the inequalities*

$$g_k(t, x_1, \dots, x_m, y_1, \dots, y_m) \operatorname{sign}(\sigma_k x_k) \leq h_0(t) + h_k|x_k| + \\ + \sum_{j=1}^{m} \big(h_{1kj}(t)|x_j| + h_{2kj}(t)|y_j|\big) \quad (k = 1, \dots, m)$$

be fulfilled, where $h_0 \in L(I_0; R_+)$, $\sigma \in \{-1, 1\}$, $h_k < 0$, $(k = 1, \dots, m)$, *the functions* h_{1kj} *and* $h_{2kj} : I_0 \to R_+$ *are measurable,*

$$h_{kj} = h_{1kj}(t) + s_{\zeta_k}(1)(t)h_{2kj}(t) \equiv const \\ \big(h_{kj} = h_{1kj}(t) + \alpha_j^* h_{2kj}(t) \equiv const\big),$$

and the real parts of the eigen-values of the matrix (1.16) *are negative. Then the problem* (1.1_2), (1.41), (*the problem* (1.1_3), (1.41), *has at least one solution.*

Let in $C(I_0; R^m)$ *the inequalities*

$$\big[f_k(u_1,\dots,u_m)(t) - f_k(v_1,\dots,v_m)(t)\big] \times$$
$$\times \operatorname{sign}\big[\sigma_k(u_k(t) - v_k(t))\big] \le h_k|u_k(t) - v_k(t)| +$$
$$+\sum_{j=1}^{m} h_{kj}\|u_j - v_j\|_{C(I_0;R)} \quad \textit{for } t \in I_0 \quad (k = 1,\dots,m)$$

be fulfilled, where $\sigma_k \in \{-1, 1\}$, $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1,\dots,m)$, *and the real parts of the eigen-values of the matrix* (1.16) *are negative. Then the problem* (1.1), (1.41) *has a unique solution.*

′ *Let in* $I_0 \times R^m$ *the inequalities*

$$\big[g_k(t, x_1,\dots,x_m) - g_k(t, \overline{x}_1,\dots,\overline{x}_m)\big] \operatorname{sign}\big[\sigma_k(x_k - \overline{x}_k)\big] \le$$
$$\le h_k|x_k - \overline{x}_k| + \sum_{j=1}^{m} h_{kj}|x_j - \overline{x}_j| \quad (k = 1,\dots,m)$$

be fulfilled, where $\sigma_k \in \{-1, 1\}$, $h_k < 0$, $h_{kj} \in R_+$ $(k, j = 1,\dots,m)$, *and the real parts of the eigen-values of the matrix* (1.16) *are negative. Then the problem* (1.1_1), (1.41) *has a unique solution.*

″ *Let in* $I_0 \times R^{2m}$ *the inequalities*

$$\big[g_k(t, x_1,\dots,x_m, y_1,\dots,y_m) - g_k(t, \overline{x}_1,\dots,\overline{x}_m, \overline{y}_1,\dots,\overline{y}_m)\big] \times$$
$$\times \operatorname{sign}\big[\sigma_k(x_k - \overline{x}_k)\big] \le h_k|x_k - \overline{x}_k| +$$
$$+\sum_{j=1}^{m} \big[h_{1kj}(t)|x_j - \overline{x}_j| + h_{2kj}(t)|y_j - \overline{y}|\big] \quad (k = 1,\dots,m)$$

be fulfilled, where $\sigma_k \in \{-1, 1\}$, $h_k < 0$ $(k = 1,\dots,m)$, *the functions* h_{1kj} *and* $h_{2kj} : I_0 \to R_+$ *are measurable,*

$$h_{kj} = h_{1kj}(t) + s_{\zeta_k}(1)(t) h_{2kj}(t) \equiv const$$
$$\big(\, h_{kj} = h_{1kj}(t) + \alpha_j^* h_{2kj}(t) \equiv const \,\big),$$

and the real parts of the eigen-values of the matrix (1.16) *are negative. Then the problem* (1.1_2), (1.41) (*the problem* (1.1_3), (1.41)) *has a unique solution.*

§ 2. Auxiliary Propositions

Let the conditions (1.7) *be fulfilled. Then there exists a positive ρ such that for arbitrary $h_0 \in L(I_0; R_+)$ and $r \in R_+$, any solution of the system of differential inequalities*

$$\begin{gathered}\big[u_k'(t) - h_k(t)u_k(t)\big] \operatorname{sign}\big[(t-t_k)u_k(t)\big] \leq \\ \leq h_0(t) + f_{0k}\big(|u_1|, \dots, |u_m|\big)(t) \quad \textit{for} \quad t \in I_0 \quad (k = 1, \dots, m) \end{gathered} \tag{2.1}$$

under the boundary conditions

$$|u_k(t_k)| \leq r + \varphi_{0k}\big(|u_1|, \dots, |u_m|\big) \quad (k = 1, \dots, m) \tag{2.2}$$

admits the estimate

$$\sum_{k=1}^{m} |u_k(t)| \leq \rho\left[r + \int_a^b h_0(s)\,ds\right] \quad \textit{for} \quad t \in I_0. \tag{2.3}$$

Proof. First of all, let us prove the existence of a positive number ρ such that for any $r > 0$ and $h_0 \in L(I_0; R_+)$, an arbitrary solution $(y_k)_{k=1}^m$ of

$$\begin{gathered}|y_k'(t) - h_k(t)y_k(t)| \leq \\ \leq h_0(t) + f_{0k}\big(|y_1|, \dots, |y_m|\big)(t) \ \text{ for } \ t \in I_0 \ \ (k = 1, \dots, m), \end{gathered} \tag{2.1$'$}$$

$$|y_k(t_k)| \leq r + \varphi_{0k}\big(|y_1|, \dots, |y_m|\big) \ \ (k = 1, \dots, m) \tag{2.2$'$}$$

admits the estimate

$$\sum_{k=1}^{m} |y_k(t)| \leq \rho\left[r + \int_a^b h_0(t)\,dt\right] \quad \text{for} \quad t \in I_0. \tag{2.3$'$}$$

Suppose on the contrary that ρ does not exist. Then for any natural n, there exist $r_n > 0$, $h_{0n} \in L(I_0; R_+)$ and an absolutely continuous vector function $(y_{kn})_{k=1}^m : I_0 \to R$ such that

$$\begin{gathered}|y_{kn}'(t) - h_k(t)y_{kn}(t)| \leq \\ \leq h_{0n}(t) + f_{0k}\big(|y_{1n}|, \dots, |y_{mn}|\big)(t) \ \text{ for } \ t \in I_0 \ \ (k = 1, \dots, m), \end{gathered} \tag{2.4}$$

$$|y_{kn}(t_k)| \leq r_n + \varphi_{0k}\big(|y_{1n}|, \dots, |y_{mn}|\big) \ \ (k = 1, \dots, m), \tag{2.5}$$

and

$$\rho_n = \max\left\{\sum_{k=1}^{m} |y_{kn}(t)| : t \in I_0\right\} > n\left[r_n + \int_a^b h_{0n}(t)\,dt\right]. \tag{2.6}$$

Let

$$z_{kn}(t) = \frac{1}{\rho_n}\, y_{kn}(t) \quad (k = 1, \dots, m).$$

Then

$$\max\left\{\sum_{k=1}^{m}|z_{kn}(t)| : t\in I_0\right\}=1 \quad (n=1,2,\dots). \tag{2.7}$$

On the other hand, since f_{0k} and φ_{0k} $(k=1,\dots,m)$ are positively homogeneous, from (2.4) and (2.7) we get

$$|z'_{kn}(t)|\le \widetilde{h}_{0n}(t)+g_k(t) \quad \text{for} \quad t\in I_0 \ \ (k=1,\dots,m), \tag{2.8}$$

$$|z'_{kn}(t)-h_k(t)z_{kn}(t)|\le \widetilde{h}_{0n}(t)+ \\ +f_{0k}\big(|z_{1n}|,\dots,|z_{mn}|\big)(t) \quad \text{for} \quad t\in I_0 \ \ (k=1,\dots,m), \tag{2.9}$$

$$|z_k(t_k)|\le \frac{1}{n}+\varphi_{0k}\big(|z_{1n}|,\dots,|z_{mn}|\big) \quad (k=1,\dots,m), \tag{2.10}$$

where

$$\widetilde{h}_{0n}(t)=\frac{h_{0n}(t)}{\rho_n}, \quad g_k(t)=f_{0k}(1,\dots,1)(t)+|h_k(t)|.$$

Moreover, as it follows from (2.6),

$$\int_a^b \widetilde{h}_{0n}(t)\,dt\le\frac{1}{n} \quad (n=1,2,\dots). \tag{2.11}$$

According to (2.7), the sequences $(z_{kn})_{n=1}^{+\infty}$ $(k=1,\dots,m)$ are uniformly bounded. On the other hand, by (2.8) and (2.11) we have

$$|z_{kn}(t)-z_{kn}(s)|\le\frac{1}{n}+\left|\int_s^t g_k(\tau)\,d\tau\right| \quad \text{for} \ \ s\in I_0, \ \ t\in I_0 \ \ (k=1,\dots,m).$$

This implies that $(z_{kn})_{n=1}^{+\infty}$ $(k=1,\dots,m)$ are uniformly continuous. By the Arzella–Ascoli lemma, without loss of generality we can assume that $(z_{kn})_{n=1}^{+\infty}$ $(k=1,\dots,m)$ uniformly converge on I_0. Setting

$$\lim_{n\to+\infty} z_{kn}(t)=z_k(t) \quad (k=1,\dots,m)$$

and taking into consideration that φ_{0k} $(k=1,\dots,m)$ are continuous, from (2.7), (2.10) we obtain

$$\max\left\{\sum_{k=1}^{m}|z_k(t)| : t\in I_0\right\}=1 \tag{2.12}$$

and

$$|z_k(t_k)|\le\varphi_{0k}\big(|z_1|,\dots,|z_m|\big) \quad (k=1,\dots,m). \tag{2.13}$$

By virtue of (2.9) and (2.11), we have

$$|z_{kn}(t)| \le |z_{kn}(t_k)| \exp\left(\int\limits_{t_k}^{t} h_k(s)\,ds\right) + \frac{1}{n}\exp\left(\int\limits_{a}^{b} |h_k(s)|\,ds\right) +$$
$$+\left|\int\limits_{t_k}^{t} \exp\left(\int\limits_{\tau}^{t} h_k(s)\,ds\right) f_{0k}\big(|z_{1n}|,\dots,|z_{mn}|\big)(\tau)d\tau\right|$$
$$\text{for}\quad t\in I_0 \ \ (k=1,\dots,m).$$

Passing in these inequalities to limit as $n \to +\infty$ and taking into account the fact that f_{0k} $(k=1,\dots,m)$ are continuous, we obtain

$$|z_k(t)| \le u_k(t) \ \text{ for } t\in I_0 \ \text{ and } \ |z_k(t_k)| = u_k(t_k) \ (k=1,\dots,m), \tag{2.14}$$

where

$$u_k(t) = |z_k(t_k)| \exp\left(\int\limits_{t_k}^{t} h_k(s)\,ds\right) +$$
$$+\left|\int\limits_{t_k}^{t} \exp\left(\int\limits_{\tau}^{t} h_k(s)\,ds\right) f_{0k}\big(|z_1|,\dots,|z_m|\big)(\tau)d\tau\right| \quad (k=1,\dots,m).$$

Obviously

$$|u_k'(t) - h_k(t)u_k(t)| =$$
$$= f_{0k}\big(|z_1|,\dots,|z_m|\big)(t) \quad \text{for} \quad t\in I_0 \ \ (k=1,\dots,m). \tag{2.15}$$

Owing to the monotonicity of φ_{0k} and f_{0k} $(k=1,\dots,m)$, it follows from (2.13)–(2.15) that $(u_k)_{k=1}^m$ is a solution of (1.3), (1.4). On the other hand, from (2.12) and (2.14) we have

$$\max\left\{\sum_{k=1}^{m} u_k(t) : t\in I_0\right\} \ge 1.$$

But this contradicts to (1.7). The obtained contradiction proves the existence of a positive ρ possessing the above-mentioned property.

Assume now that $r>0$ is an arbitrary number, $h\in L(I_0;R_+)$ is and a function, $(u_k)_{k=1}^m$ is a solution of (2.1), (2.2). By (2.1),

$$|u_k(t)| \le y_k(t) \ \text{ for } \ t\in I_0 \quad \text{and} \quad |u_k(t_k)| = y_k(t_k) \ \ (k=1,\dots,m), \tag{2.16}$$

where

$$y_k(t) = |u_k(t_k)| \exp\Big(\int_{t_k}^{t} h_k(s)\,ds\Big) +$$

$$+\bigg|\int_{t_k}^{t}\Big(\int_{\tau}^{t} h_k(s)\,ds\Big)\big[h_0(\tau) + f_{0k}\big(|u_1|,\dots,|u_m|\big)(\tau)\big]\,d\tau\bigg| \quad (k=1,\dots,m).$$

Due to (2.16), it follows from (2.2) and

$$|y_k'(t) - h_k(t)y_k(t)| = h_0(t) + f_{0k}\big(|u_1|,\dots,|u_m|\big)(t)$$
$$\text{for} \quad t \in I_0 \ \ (k=1,\dots,m)$$

that $(y_k)_{k=1}^m$ is a solution of (2.1′), (2.2′). Therefore, according to the above-said, (2.3′) is valid. ■

$$W(t_1,\dots,t_m) \qquad W_0(t_1,\dots,t_m)$$

Let (1.7) *be fulfilled. Then there exists* $\gamma \in]0,1[$ *such that*

$$\Big(h_1,\dots,h_m; \frac{1}{\gamma}\, f_{01},\dots,\frac{1}{\gamma}\, f_{0m}; \frac{1}{\gamma}\,\varphi_{01},\dots,\frac{1}{\gamma}\,\varphi_{0m}\Big) \in W(t_1,\dots,t_m). \tag{2.17}$$

Proof. Let ρ be the positive number appearing in Lemma 2.1. Choose $\gamma \in]0,1[$ such that

$$\frac{1-\gamma}{\gamma}\,\rho \sum_{k=1}^{m}\Big[\varphi_{0k}(1,\dots,1) + \int_a^b f_{0k}(1,\dots,1)(s)\,ds\Big] < \frac{1}{2}. \tag{2.18}$$

Consider an arbitrary solution $(u_k)_{k=1}^m$ of

$$|u_k'(t) - h_k(t)u_k(t)| \le$$
$$\le \frac{1}{\gamma}\, f_{0k}\big(|u_1|,\dots,|u_m|\big)(t) \quad \text{for} \quad t \in I_0 \ \ (k=1,\dots,m), \tag{2.19}$$

$$|u_k(t_k)| \le \frac{1}{\gamma}\,\varphi_{0k}\big(|u_1|,\dots,|u_m|\big) \quad (k=1,\dots,m). \tag{2.20}$$

It is evident that this solution satisfies (2.1) and (2.2), where

$$h_0(t) = \frac{1-\gamma}{\gamma}\sum_{k=1}^{m} f_{0k}\big(|u_1|,\dots,|u_m|\big)(t)$$

and

$$r = \frac{1-\gamma}{\gamma}\sum_{k=1}^{m}\varphi_{0k}\big(|u_1|,\dots,|u_m|\big).$$

Therefore, due to our choice of ρ, the estimate (2.3) is valid, that is,

$$\sum_{k=1}^{m} |u_k(t)| \le \frac{1-\gamma}{\gamma}\,\rho \sum_{k=1}^{m} \Big[\varphi_{0k}\big(|u_1|,\dots,|u_m|\big) + \\ + \int_a^b f_{0k}\big(|u_1|,\dots,|u_m|\big)(s)\,ds\Big].$$

Putting

$$u^* = \max\Big\{\sum_{k=1}^{m} |u_k(t)| : t \in I_0\Big\},$$

by (2.18) from the last inequality we find

$$u^* \le \frac{1}{2}\,u^*,$$

and hence $u^* = 0$. Thus we have proved that the problem (2.19), (2.20) has only the trivial solution, i.e., (2.17) is fulfilled. ■

Let

$$h_k(t) \equiv 0, \quad f_{0k}(u_1,\dots,u_m)(t) = \\ = \sum_{j=1}^{m} h_{kj}(t)\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m), \tag{2.21}$$

$$\varphi_{0k}(u_1,\dots,u_m) = \sum_{j=1}^{m} \ell_{kj}\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m), \tag{2.22}$$

where $\ell_{kj} \in R_+$, $h_{kj} \in L(I_0;R_+)$ $(k,j=1,\dots,m)$, *and the moduli of the eigen-values of the matrix* (1.10) *are less than* 1. *Then*

$$\big(f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m}\big) \in W_0(t_1,\dots,t_m).$$

Proof. By (2.21) and (2.22), the problem (1.3), (1.4) takes the form

$$|u_k'(t)| \le \sum_{j=1}^{m} h_{kj}(t)\|u_j\|_{C(I_0;R)} \quad \text{for} \quad t \in I_0 \ \ (k=1,\dots,m),$$

$$|u_k(t_k)| \le \sum_{j=1}^{m} \ell_{kj}\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m).$$

Let $(u_k)_{k=1}^m$ be an arbitrary solution of this problem. Then

$$|u_k(t)| \le \sum_{j=1}^m \ell_{kj} \|u_j\|_{C(I_0;R)} +$$
$$+\sum_{j=1}^m \left| \int_{t_k}^t h_{kj}(s)\,ds \right| \|u_j\|_{C(I_0;R)} \quad \text{for} \quad t \in I_0 \ \ (k=1,\dots,m).$$

Putting

$$\rho_0 = \big(\|u_k\|_{C(I_0;R)}\big)_{k=1}^m \quad \text{and} \quad \Lambda = \left(\ell_{kj} + \int_a^b h_{kj}(s)\,ds\right)_{k,j=1}^m,$$

from the last inequalities we find

$$\rho_0 \le \Lambda\rho_0, \quad \text{that is,} \quad (E-\Lambda)\rho_0 \le 0.$$

Since the spectral radius of Λ is less than 1, we have $\rho_0 \le 0$. Hence $u_k(t) \equiv 0$ $(k=1,\dots,m)$. ■

Let $g_k(t) = h_k \operatorname{sign}(t-t_k)$, $h_k < 0$ $(k=1,\dots,m)$,

$$f_{0k}(u_1,\dots,u_m)(t) = \sum_{j=1}^m h_{kj}\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m),$$

$h_{kj} \in R_+$, *and let* $\varphi_{0k} : C(I_0;R_+) \to R_+$ $(k=1,\dots,m)$ *be linear continuous functionals satisfying* (1.15). *Moreover, let the real parts of the eigen-values of the matrix* (1.16) *be negative. Then*

$$\big(g_1,\dots,g_m; f_{01},\dots,f_{0m}; \varphi_{01},\dots,\varphi_{0m}\big) \in W(t_1,\dots,t_m).$$

Proof. To prove the lemma, it suffices to show that the problem

$$u_k'(t)\operatorname{sign}\big[(t-t_k)u_k(t)\big] \le h_k|u_k(t)| +$$
$$+\sum_{j=1}^m h_{kj}\|u_j\|_{C(I_0;R)} \quad \text{for} \quad t\in I_0 \ \ (k=1,\dots,m), \tag{2.23}$$
$$u_k(t_k) \le \varphi_{0k}\big(|u_k|\big) \quad (k=1,\dots,m) \tag{2.24}$$

has only the zero solution.

Let $(u_k)_{k=1}^m$ be an arbitrary solution of (2.23), (2.24). Then

$$
\begin{aligned}
|u_k(t)| \le |u_k(t_k)| \exp\big(h_k|t-t_k|\big) + \\
+\exp\big(h_k|t-t_k|\big) \sum_{j=1}^m h_{kj}\|u_j\|_{C(I_0;R)} \bigg| \int_{t_k}^{t} \exp\big(-h_k|s-t_k|\big)\,ds \bigg| = \\
= |u_k(t_k)| \exp\big(h_k|t-t_k|\big) + \\
+\sum_{j=1}^m \frac{h_{kj}}{|h_k|}\,\|u_j\|_{C(I_0;R)} \big[1-\exp\big(h_k|t-t_k|\big)\big] \qquad (2.25) \\
\text{for} \quad t\in I_0 \ \ (k=1,\dots,m).
\end{aligned}
$$

Owing to (1.15) and (2.24), this implies that

$$
|u_k(t_k)| \le \alpha_k |u_k(t_k)| + (1-\alpha_k)\sum_{j=1}^m \frac{h_{kj}}{|h_k|}\,\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m),
$$

where $\alpha_k=\varphi_{0k}(\widetilde{h}_k)<1$ $(k=1,\dots,m)$. Therefore

$$
|u_k(t_k)| \le \sum_{j=1}^m \frac{h_{kj}}{|h_k|}\,\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m).
$$

With regard for these estimates, from (2.25) we obtain

$$
\|u_k\|_{C(I_0;R)} \le \sum_{j=1}^m \frac{h_{kj}}{|h_k|}\,\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m),
$$

and hence

$$
(E-\Lambda)\rho_0 \le 0, \tag{2.26}
$$

where

$$
\rho_0=\big(\|u_k\|_{C(I_0;R)}\big)_{k=1}^m, \quad \Lambda=\Big(\frac{h_{kj}}{|h_k|}\Big)_{k,j=1}^m .
$$

From the fact that $h_k<0$, $h_{kj}\ge 0$ $(k,j=1,\dots,m)$ and the real parts of the eigen-values of the matrix (1.16) are negative, it follows that the moduli of the eigen-values of Λ are less than 1.[5] Therefore from (2.26) we find $\rho_0=0$, that is, $u_k(t)\equiv 0$ $(k=1,\dots,m)$. ■

Let

$$
f_{0k}(u_1,\dots,u_m)(t)=\sum_{j=1}^m \big(h_{1kj}u_j(t)+h_{2kj}s_{\varsigma_j}(u_j)(t)\big) \quad (k=1,\dots,m)
$$

[5] See [14], pp. 369–371.

and

$$\varphi_{0k}(u_1,\dots,u_m)=\sum_{j=1}^m \ell_{kj}\|u_j\|_{L^2(I_0;R)} \quad (k=1,\dots,m),$$

where h_{1kj}, h_{2kj}, $\ell_{kj}\in R_+$ $(k,j=1,\dots,m)$, *and* $\zeta_k : I_0\to R$ $(k=1,\dots,m)$ *are absolutely continuous functions satisfying* (1.21). *Moreover, let the moduli of the eigen-values of the matrix* (1.22) *be less than* 1. *Then*

$$\big(f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m}\big)\in W_0(t_1,\dots,t_m).$$

Proof. We have to prove that the problem

$$|u_k'(t)|\le\sum_{j=1}^m\big(h_{1kj}|u_j(t)|+h_{2kj}s_{\zeta_j}(|u_j|)(t)\big)\quad\text{for}\quad t\in I_0\ \ (k=1,\dots,m),$$

$$|u_k(t_k)|\le\sum_{j=1}^m \ell_{kj}\|u_j\|_{L^2(I_0;R)}\quad (k=1,\dots,m)$$

has only the trivial solution. Let $(u_k)_{k=1}^m$ be an arbitrary solution of this problem. Then

$$\begin{aligned}|u_k(t)|&\le|u_k(t_k)|+\sum_{j=1}^m\left[h_{1kj}\left|\int_{t_k}^t|u_j(\tau)|\,d\tau\right|+h_{2kj}\left|\int_{t_k}^t s_{\zeta_j}(|u_j|)(\tau)\,d\tau\right|\right]\le\\&\le\sum_{j=1}^m\ell_{kj}\|u_j\|_{L^2(I_0;R)}+\sum_{j=1}^m\left[h_{1kj}\left|\int_{t_k}^t|u_j(\tau)|\,d\tau\right|+\right.\\&\left.+h_{2kj}\left|\int_{t_k}^t s_{\zeta_j}(|u_j|)(\tau)\,d\tau\right|\right]\quad\text{for}\quad t\in I_0\ \ (k=1,\dots,m).\end{aligned}$$

From this, by virtue of Minkowski's inequality we find

$$\begin{aligned}\|u_k\|_{L^2(I_0;R)}\le(b-a)^{1/2}&\sum_{j=1}^m\ell_{kj}\|u_j\|_{L^2(I_0;R)}+\\&+\sum_{j=1}^m h_{1kj}\left(\int_a^b\left|\int_{t_k}^t|u_j(\tau)|\,d\tau\right|^2dt\right)^{1/2}+\\&+\sum_{j=1}^m h_{2kj}\left(\int_a^b\left|\int_{t_k}^t s_{\zeta_j}(|u_j|)(\tau)\,d\tau\right|^2dt\right)^{1/2}\Bigg]\\&\qquad(k=1,\dots,m).\end{aligned}\tag{2.27}$$

According to Wirtinger's inequality [49],

$$\int\limits_a^b \left| \int\limits_{t_k}^t |u_j(\tau)|\, d\tau \right|^2 dt \le \Big[\frac{2(b-a)}{\pi}\Big]^2 \int\limits_a^b |u_j(t)|^2 dt,$$

$$\int\limits_a^b \left| \int\limits_{t_k}^t s_{\varsigma_j}(|u_j|)(\tau)\, d\tau \right|^2 dt \le \Big[\frac{2(b-a)}{\pi}\Big]^2 \int\limits_a^b \big|s_{\varsigma_j}(|u_j|)(t)\big|^2 dt \quad (k=1,\dots,m).$$

On the other hand, by (1.21) we have

$$\int\limits_a^b \big|s_{\varsigma_j}(|u_j|)(t)\big|^2 dt \le \frac{1}{\gamma_j} \|u_j\|^2_{L^2(I_0;R)} \quad (k=1,\dots,m).$$

On the basis of these estimates, we obtain from (2.27) the inequality (2.26), where

$$\rho_0 = \big(\|u_k\|_{L^2(I_0;R)}\big)_{k=1}^m,$$
$$\Lambda = \left((b-a)^{1/2}\ell_{kj} + \frac{2(b-a)}{\pi}\left(h_{1kj} + \frac{h_{2kj}}{\sqrt{\gamma_j}}\right)\right)_{k,j=1}^m.$$

Since the moduli of the eigen-values of Λ are less than 1, it follows from (2.26) that $\rho_0 = 0$, that is, $u_k(t) \equiv 0$ $(k=1,\dots,m)$. ■

′ *Let*

$$f_{0k}(u_1,\dots,u_m)(t) = \sum_{j=1}^m \big(h_{1kj}|u_j(t)| + h_{2kj} s_{\varsigma_j}(|u_j|)(t)\big) \quad (k=1,\dots,m)$$

and

$$\varphi_{0k}(u_1,\dots,u_m) = \sum_{j=1}^m \ell_{kj}\|u_j\|_{C(I_0;R)} \quad (k=1,\dots,m),$$

where h_{1kj}, h_{2kj}, $\ell_{kj} \in R_+$ $(k,j=1,\dots,m)$, *and* $\zeta_k : I_0 \to R$ $(k = 1,\dots,m)$ *are absolutely continuous functions satisfying* (1.21). *Moreover, let the moduli of the eigen-values of the matrix* (1.37), *where* $h_{kj} = h_{1kj} + h_{2kj}/\sqrt{\gamma_j}$, *be less than* 1. *Then*

$$\big(f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m}\big) \in W_0(t_1,\dots,t_m).$$

Proof. Let $(u_k)_{k=1}^m$ be an arbitrary solution of

$$|u_k'(t)| \le \sum_{j=1}^m \big[h_{1kj}|u_j(t)| + h_{2kj}s_{\varsigma_j}(|u_j|)(t)\big] \quad \text{for} \quad t \in I_0 \ \ (k = 1, \dots, m),$$

$$|u_k(t_k)| \le \sum_{j=1}^m \ell_{kj}\|u_j\|_{C(I_0;R)} \quad (k = 1, \dots, m).$$

Then

$$\|u_k\|_{C(I_0;R)} \le (b-a)^{-1/2}\|u_k\|_{L^2(I_0;R)} + \int_a^b |u_k'(t)|\,dt \le$$

$$\le (b-a)^{-1/2}\|u_k\|_{L^2(I_0;R)} + \sum_{j=1}^m \Bigg[h_{1kj}\int_a^b |u_j(t)|\,dt +$$

$$+ h_{2kj}\int_a^b s_{\varsigma_j}(|u_j|)(t)\,dt\Bigg] \le (b-a)^{-1/2}\|u_k\|_{L^2(I_0;R)} +$$

$$+ (b-a)^{-1/2}\sum_{j=1}^m h_{kj}\|u_j\|_{L^2(I_0;R)} \quad (k = 1, \dots, m),$$

and hence

$$\|u_k(t_k)| \le \widetilde{\varphi}_{0k}(u_1, \dots, u_m) \quad (k = 1, \dots, m),$$

where

$$\widetilde{\varphi}_{0k}(u_1, \dots, u_m) = \sum_{j=1}^m \widetilde{\ell}_{kj}\|u_j\|_{L^2(I_0;R)}$$

and

$$\widetilde{\ell}_{kj} = (b-a)^{-1/2}\ell_{kj} + (b-a)^{1/2}\sum_{i=1}^m \ell_{ki}h_{ij}.$$

By Lemma 2.5,

$$\big(f_{01}, \dots, f_{0m}; \widetilde{\varphi}_{01}, \dots, \widetilde{\varphi}_{0m}\big) \in W_0(t_1, \dots, t_m).$$

Therefore $u_k(t) \equiv 0$ $(k = 1, \dots, m)$. ∎

Let

$$f_{0k}(u_1, \dots, u_m)(t) = \sum_{j=1}^m h_{kj}\|u_j\|_{C([a,t];R)},$$

$$\varphi_{0k}(u_1, \dots, u_m) = \sum_{j=1}^m \ell_{kj}\|u_j\|_{C(I_0;R)},$$

$h_{kj} \in L(I_0; R_+)$, $\ell_{kj} \in R_+$ $(k,j = 1,\dots,m)$, and let (1.29) *be fulfilled, where h is the function given by* (1.30). *Then*

$$\big(f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m}\big) \in W_0(t_1,\dots,t_m).$$

Proof. Let $(u_k)_{k=1}^m$ be a solution of

$$|u_k'(t)| \le \sum_{j=1}^m h_{kj}(t)\|u_j\|_{C([a,t];R)} \quad (k = 1,\dots,m),$$
$$|u_k(a)| \le \sum_{j=1}^m \ell_{kj}(t)\|u_j\|_{C(I_0;R)} \quad (k = 1,\dots,m).$$

Putting

$$u(t) = \max\big\{|u_k(t)| : k = 1,\dots,m\big\},$$
$$\ell = \max\bigg\{\sum_{j=1}^m \ell_{kj} : k = 1,\dots,m\bigg\}$$

and taking into account (1.30), we find

$$u(t) \le \ell\|u\|_{C(I_0;R)} + \int_a^t h(s)u(s)\,ds \quad \text{for} \quad a \le t \le b.$$

From this, owing to the Gronwall–Bellman lemma, we have

$$\|u\|_{C(I_0;R)} \le \delta\|u\|_{C(I_0;R)},$$

where $\delta = \ell\exp(\int_a^b h(t)dt) < 1$. Therefore $u(t) \equiv 0$. ■

§ 3. Proof of the Existence and Uniqueness Theorems

Proof of Theorem 1.1. Let ρ be the constant appearing in Lemma 2.1,

$$\rho_0 = \rho\bigg(r + \int_a^b h(t)\,dt\bigg),$$
$$\chi(s) = \begin{cases} 1 & \text{for } |s| \le \rho_0, \\ 2 - \frac{|s|}{\rho_0} & \text{for } \rho_0 \le |s| \le 2\rho_0, \\ 0 & \text{for } |s| \ge 2\rho_0, \end{cases}$$

and let $(\widetilde{f}_k)_{k=1}^m$, $(g_k)_{k=1}^m$ and $(\widetilde{\varphi}_k)_{k=1}^m$ be the operators and functionals given by

$$\widetilde{f}_k(u_1,\dots,u_m)(t) = \chi\big(\|u\|_{C(I_0;R^m)}\big) \times$$
$$\times\big[f_k(u_1,\dots,u_m)(t) - h_k(t)u_k(t)\big] \quad (k = 1,\dots,m), \tag{3.1}$$
$$\widetilde{\varphi}_k(u_1,\dots,u_m) =$$

$$= \chi\big(\|u\|_{C(I_0;R^m)}\big)\varphi_k(u_1,\dots,u_m) \quad (k=1,\dots,m), \tag{3.2}$$

$$g_k(u_1,\dots,u_m(t) = \exp\bigg(\int\limits_{t_k}^{t} h_k(s)\,ds\bigg)\widetilde{\varphi}_k(u_1,\dots,u_m) +$$

$$+\int\limits_{t_k}^{t} \exp\bigg(\int\limits_{\tau}^{t} h_k(s)\,ds\bigg)\widetilde{f}_k(u_1,\dots,u_m)(\tau)\,d\tau \ \ (k=1,\dots,m), \tag{3.3}$$

where $u=(u_k)_{k=1}^m$.

According to (3.1) and (3.2), there exist a positive r_1 and $f_0 \in L(I_0;R_+)$ such that in $C(I_0;R^m)$ the inequalities

$$|\widetilde{\varphi}_k(u_1,\dots,u_m)| \le r_1 \quad (k=1,\dots,m) \tag{3.4}$$

and

$$\sum_{k=1}^{m} |\widetilde{f}_k(u_1,\dots,u_m)(t)| \le f_0(t) \quad \text{for} \quad t \in I_0 \tag{3.5}$$

are fulfilled.

Assume

$$r_2 = \bigg(r_1 + \int\limits_a^b f_0(t)\,dt\bigg)\exp\bigg(\sum_{k=1}^{m}\int\limits_a^b |h_k(t)|\,dt\bigg).$$

Proceeding from (3.4) and (3.5) as well as from the continuity of $(f_k)_{k=1}^m : C(I_0;R^m) \to L(I_0;R^m)$ and $(\varphi_k)_{k=1}^m : C(I_0;R^m) \to R^m$, we can easily show that $(g_k)_{k=1}^m : C(I_0;R^m) \to C(I_0;R^m)$ is a completely continuous operator mapping the ball

$$\big\{u=(u_k)_{k=1}^m \in C(I_0;R^m) : \|u\|_{C(I_0;R^m)} \le r_2\big\}$$

into itself.

Therefore, by Schauder's principle [29], there exists $(u_k)_{k=1}^m \in C(I_0;R^m)$ such that

$$u_k(t) = g_k(u_1,\dots,u_m)(t) \quad \text{for} \quad t \in I_0 \ \ (k=1,\dots,m).$$

From this, owing to (3.3) we have that $u=(u_k)_{k=1}^m$ is a solution of the boundary value problem

$$u_k'(t) = h_k(t)u_k(t) + \widetilde{f}_k(u_1,\dots,u_m)(t) \quad (k=1,\dots,m), \tag{3.6}$$

$$u_k(t_k) = \widetilde{\varphi}_k(u_1,\dots,u_m) \quad (k=1,\dots,m). \tag{3.7}$$

According to the conditions (1.5), (1.6), (3.1) and (3.2), it follows from (3.6) and (3.7) that $u=(u_k)_{k=1}^m$ satisfies (2.1) and (2.2). Therefore (2.3), that is,

$$\|u\|_{C(I_0;R^m)} \le \rho_0$$

is valid owing to Lemma 2.1. Taking along with this estimate (3.1) and (3.2) into account, we can see that $u = (u_k)_{k=1}^m$ is a solution of (1.1), (1.2). ■

Corollaries 1.1–1.3, 1.1′–1.3′, 1.1″–1.3″, 1.7, 1.7′, 1.7″, 1.9, 1.9′, 1.9″ follow from this theorem by Lemmas 2.3–2.5.

Proof of Theorem 1.2. From (1.23) and (1.24), we have the inequalities (1.5) and (1.6), where

$$h_0(t) = \sum_{k=1}^m |f_k(0,\dots,0)(t)| \quad \text{and} \quad r = \sum_{k=1}^m |\varphi_k(0,\dots,0)|.$$

Consequently, all the conditions of Theorem 1.1 are fulfilled, which guarantees the solvability of the problem (1.1), (1.2). It remains to show that the problem has at most one solution.

Let $(u_k)_{k=1}^m$ and $(v_k)_{k=1}^m$ be arbitrary solutions of (1.1), (1.2), and put

$$u_{0k}(t) = u_k(t) - v_k(t) \quad (k = 1,\dots,m).$$

Because of (1.23) and (1.24),

$$\begin{aligned}\big[u'_{0k}(t) - h_k(t)u_{0k}(t)\big] \operatorname{sign}\big[(t-t_k)u_{0k}(t)\big] \le \\ \le f_{0k}\big(|u_{01}|,\dots,|u_{0m}|\big)(t) \quad \text{for} \quad t \in I_0 \ \ (k = 1,\dots,m)\end{aligned}$$

and

$$|u_{0k}(t_k)| \le \varphi_{0k}\big(|u_{01}|,\dots,|u_{0m}|\big) \quad (k = 1,\dots,m).$$

Since (1.7) is fulfilled, from these inequalities by Lemma 2.1 we have $u_{0k}(t) \equiv 0$ $(k = 1,\dots,m)$. ■

Corollaries 1.4–1.6, 1.4′–1.6′, 1.4″–1.6″, 1.8, 1.8′, 1.8″, 1.10, 1.10′, 1.10″ follow directly from this theorem and lemmas 2.3–2.5.

Validity of Remarks 1.1 and 1.2 follows from Lemmas 2.6 and 2.5′.

CHAPTER II

§ 4. Formulation of Main Results

Let m and n be natural numbers, $i_k \in \widetilde{N}_n$ $(k = 1, \dots, m)$, and let $g_k : \widetilde{E}_n^m \to E_n$ and $\psi_k : \widetilde{E}_n^m \to R$ $(k = 1, \dots, m)$ be continuous operators and functionals. Consider the problem of finding a vector function $(x_k)_{k=1}^m \in \widetilde{E}_n^m$ satisfying on N_n the system of functional difference equations

$$\Delta x_k(i-1) = g_k(x_1, \dots, x_m)(i) \quad (k = 1, \dots, m) \tag{4.1}$$

and the boundary conditions

$$x_k(i_k) = \psi_k(x_1, \dots, x_m) \quad (k = 1, \dots, m). \tag{4.2}$$

Assume

$$\tau_k(i) = \begin{cases} i & \text{for} \quad i > i_k \\ i-1 & \text{for} \quad i \le i_k \end{cases} \quad (k = 1, \dots, m). \tag{4.3}$$

Let $h_k \in E_n$, $h_k(i) \operatorname{sign}(\tau_k(i) - i_k) < 1$, and let $g_{0k} : (\widetilde{E}_n^+)^m \to E_n^+$ and $\psi_{0k} : (\widetilde{E}_n^+)^m \to R_+$ $(k = 1, \dots, m)$ be positively homogeneous continuous nondecreasing operators and functionals such that the system of difference inequalities

$$\big|\Delta x_k(i-1) + h_k(i) x_k(\tau_k(i))\big| \le g_{0k}(|x_1| \dots, |x_m|)(i) \tag{4.4}$$
$$(k = 1, \dots, m)$$

under the boundary conditions

$$|x_k(i_k)| \le \psi_{0k}(|x_1| \dots, |x_m|) \quad (k = 1, \dots, m) \tag{4.5}$$

has only the zero solution. Then we say that the vector $(h_1, \dots, h_m; g_{01}, \dots, g_{0m}; \psi_{01}, \dots, \psi_{0m})$belongs to $W_n(i_1, \dots, i_m)$.

The writing

$$(g_{01}, \dots, g_{0m}; \psi_{01} \dots, \psi_{0m}) \in W_{0n}(i_1, \dots, i_m)$$

means that

$$(0, \dots, 0; g_{01}, \dots, g_{0m}; \psi_{01} \dots, \psi_{0m}) \in W_n(i_1, \dots, i_m).$$

Let in $\widetilde{E}_n^m$ the inequalities

$$\big[g_k(x_1, \dots, x_m)(i) - h_k(i) x_k(\tau_k(i))\big] \operatorname{sign}\big[(\tau_k(i) - i_k) x_k(\tau_k(i))\big] \le$$
$$\le h_0(i) + g_{0k}(|x_1|, \dots, |x_m|)(i) \quad \textit{for} \quad i \in N_n \quad (k = 1, \dots, m) \tag{4.6}$$

and

$$|\psi_k(x_1, \dots, x_m)| \le r + \psi_{0k}(|x_1|, \dots, |x_m|) \quad (k = 1, \dots, m) \tag{4.7}$$

be fulfilled, where $r \in R_+$, $h_0 \in E_n^+$, *and*

$$(h_1, \dots, h_m; g_{01}, \dots, g_{0m}; \psi_{01} \dots, \psi_{0m}) \in W_n(i_1, \dots, i_m). \tag{4.8}$$

Then the problem (4.1), (4.2) *is solvable.*

Let in $\widetilde{E}_n^m$ *the inequalities*

$$g_k(x_1, \dots, x_m)(i) \operatorname{sign}\big[(\tau_k(i) - i_k) x_k(\tau_k(i))\big] \le$$
$$\le h_0(i) + \sum_{j=1}^m h_{kj}(i) \|x_j\|_{\widetilde{E}_n} \quad \textit{for} \quad i \in N_n \quad (k = 1, \dots, m)$$

and

$$|\psi_k(x_1, \dots, x_m)| \le r + \sum_{j=1}^m l_{kj}(i) \|x_j\|_{\widetilde{E}_n} \quad (k = 1, \dots, m)$$

be fulfilled, where $h_0 \in E_n^+$, $h_{kj} \in E_n^+$, $r \in R_+$, $l_{kj} \in R_+$ $(k, j = 1, \dots, m)$, *and the spectral radius of the matrix*

$$\Big(l_{kj} + \sum_{j=1}^n h_{kj}(i)\Big)_{k,j=1}^m \tag{4.9}$$

is less than 1. *Then the problem* (4.1), (4.2) *is solvable.*

Let in $\widetilde{E}_n^m$ *the inequalities*

$$g_k(x_1, \dots, x_m)(i) \operatorname{sign}\big[(\tau_k(i) - i_k) x_k(\tau_k(i))\big] \le$$
$$\le h_{0i} + h_{0k}|x_k(\tau_k(i))| + \sum_{j=1}^m h_{kj} \|x_j\|_{\widetilde{E}_n} \quad \textit{for} \quad i \in N_n \quad (k = 1, \dots, m)$$

and

$$|\psi_k(x_1, \dots, x_m)| \le r + \sum_{i=0}^n l_k(i) |x_k(i)| \quad (k = 1, \dots, m)$$

be fulfilled, where $h_0 \in E_n^+$, $h_{0k} < 0$, $h_{kj} \in E_+$, $r \in R_+$, $l_k \in \widetilde{E}_n$ $(k, j = 1, \dots, m)$,

$$\sum_{i=0}^n l_k(i) \le 1, \quad \sum_{i=0}^n (1 - h_{0k})^{-|i - i_k|} l_k(i) < 1 \quad (k = 1, \dots, m), \tag{4.10}$$

and the real parts of the eigen-values of the matrix

$$\big(h_{0k}\delta_{kj} + h_{kj}\big)_{k,j=1}^m \tag{4.11}$$

are negative. Then the problem (4.1), (4.2) *is solvable.*

Let in $\widetilde{E}_n^m$ the inequalities

$$g_k(x_1,\dots,x_m)(i)\,\mathrm{sign}\,\big[(\tau_k(i)-i_k)x_k(\tau_k(i))\big]\le$$
$$\le h_0(i)+\sum_{j=1}^m h_{kj}s_{kj}(|x_j|)(i)\quad for\quad i\in N_n\quad (k=1,\dots,m)$$

and

$$|\psi_k(x_1,\dots,x_m)|\le r+\sum_{j=1}^m l_{kj}\Big[\sum_{i=0}^n x_j^2(i)\Big]^{\frac12}\quad (k=1,\dots,m)$$

be fulfilled, where $h_0\in E_n^+$, $h_{kj}\in R_+$, $r\in R_+$, $l_{kj}\in R_+$, and $s_{kj}:\widetilde{E}_n^+\to\widetilde{E}_n^+$ $(k,j=1,\dots,m)$ are positively homogeneous nondecreasing operators such that

$$\sum_{i=0}^n\big[s_{kj}(x)(i)\big]^2\le\sum_{i=0}^n x^2(i)\quad (k=1,\dots,m).\tag{4.12}$$

Moreover, let the moduli of the eigen-values of the matrix

$$\left(\sqrt{n+1}l_{kj}+\frac{h_{kj}}{2\sin\frac{\pi}{4n+2}}\right)_{k,j=1}^m\tag{4.13}$$

be less than 1. *Then the problem* (4.1), (4.2) *is solvable.*

Let in $\widetilde{E}_n^m$ the inequalities

$$\begin{gathered}\big[g_k(x_1,\dots,x_m)(i)-g_k(y_1,\dots,y_m)(i)-h_k(i)\big(x_k(\tau_k(i))-\\-y_k(\tau_k(i))\big)\big]\,\mathrm{sign}\,\big[(\tau_k(i)-i_k)\big(x_k(\tau_k(i))-y_k(\tau_k(i))\big)\big]\le\\\le g_{0k}\big(|x_1-y_1|,\dots,|x_m-y_m|\big)(i)\\ for\quad i\in N_n\quad (k=1,\dots,m)\end{gathered}\tag{4.14}$$

and

$$\begin{gathered}\big|\psi_k(x_1,\dots,x_m)-\psi_k(y_1,\dots,y_m)\big|\le\\\le\psi_{0k}(|x_1-y_1|,\dots,|x_m-y_m|)\quad (k=1,\dots,m)\end{gathered}\tag{4.15}$$

be fulfilled, where h_k, g_{0k} and ψ_{0k} $(k=1,\dots,m)$ satisfy (4.8). *Then the problem* (4.1), (4.2) *has a unique solution.*

Let in $\widetilde{E}_n^m$ the inequalities

$$\begin{gathered}\big[g_k(x_1,\dots,x_m)(i)-g_k(y_1,\dots,y_m)(i)\big]\times\\\times\,\mathrm{sign}\,\big[(\tau_k(i)-i_k)\big(x_k(\tau_k(i))-y_k(\tau_k(i))\big)\big]\le\sum_{j=1}^m h_{kj}(i)\|x_j-y_j\|_{\widetilde{E}_n}\\ for\quad i\in N_n\quad (k=1,\dots,m)\end{gathered}$$

and

$$\bigl|\psi_k(x_1,\dots,x_m)-\psi_k(y_1,\dots,y_m)\bigr|\le$$
$$\le\sum_{j=1}^m l_{kj}\|x_j-y_j\|_{\widetilde{E}_n}\quad(k=1,\dots,m)\tag{4.16}$$

be fulfilled, where $h_{kj}\in\widetilde{E}_n^+$, $l_{kj}\in R_+$ $(k,j=1,\dots,m)$, *and the spectral radius of the matrix* (4.9) *is less than* 1. *Then the problem* (4.1), (4.2) *has a unique solution.*

Let in $\widetilde{E}_n^m$ *the inequalities*

$$\bigl[g_k(x_1,\dots,x_m)(i)-g_k(y_1,\dots,y_m)(i)\bigr]\times$$
$$\times\operatorname{sign}\bigl[(\tau_k(i)-i_k)\bigl(x_k(\tau_k(i))-y_k(\tau_k(i))\bigr)\bigr]\le h_{0k}\bigl|x_k(\tau_k(i))-y_k(\tau_k(i))\bigr|+$$
$$+\sum_{j=1}^m h_{kj}(i)\|x_j-y_j\|_{\widetilde{E}_n}\quad for\quad i\in N_n\quad(k=1,\dots,m)$$

be fulfilled, where $h_{0k}<0$, $h_{kj}\in R_+$ $(k,j=1,\dots,m)$, *and the real parts of the eigen-values of the matrix* (4.11) *are negative. Then for any numbers* c_k $(k=1,\dots,m)$ *and any functions* $\lambda_k\in\widetilde{E}_n$ $(k=1,\dots,m)$ *such that*

$$\sum_{i=0}^n|\lambda_k(i)|\le 1$$

and

$$\sum_{i=0}^n(1-h_{0k})^{-|i-i_k|}|\lambda_k(i)|<1\quad(k=1,\dots,m),\tag{4.17}$$

the system (4.1) *under the boundary conditions*

$$x_k(i_k)=\sum_{i=0}^n\lambda_k(i)x_k(i)+c_k\quad(k=1,\dots,m).\tag{4.18}$$

has a unique solution.

Let in $\widetilde{E}_n^m$ *the inequalities*

$$\bigl[g_k(x_1,\dots,x_m)(i)-g_k(y_1,\dots,y_m)(i)\bigr]\times$$
$$\times\operatorname{sign}\bigl[(\tau_k(i)-i_k)\bigl(x_k(\tau_k(i))-y_k(\tau_k(i))\bigr)\bigr]\le\sum_{j=1}^m h_{kj}s_{kj}(|x_j-y_j|)(i)$$
$$for\quad i\in N_n\quad(k=1,\dots,m)$$

and

$$\bigl|\psi_k(x_1,\dots,x_m)-\psi_k(y_1,\dots,y_m)\bigr|\le$$

$$\leq \sum_{j=1}^{m} l_{kj} \left[\sum_{i=0}^{n} |x_j(i) - y_j(i)|^2 \right]^{\frac{1}{2}} \quad (k = 1, \dots, m) \tag{4.19}$$

be fulfilled, where $h_{kj} \in R_+$, $l_{kj} \in R_+$, *and* $s_{kj} : \widetilde{E}_n^+ \to \widetilde{E}_n$ $(k, j = 1, \dots, m)$ *are positively homogeneous nondecreasing functionals satisfying* (4.12). *Moreover, let the spectral radius of the matrix* (4.13) *be less than* 1. *Then the problem* (4.1), (4.2) *has a unique solution.*

Let in $\widetilde{E}_n^m$ *the inequalities* (4.14) *and* (4.15) *be fulfilled, where the functions* h_k, g_{0k} *and* ψ_{0k} $(k = 1, \dots, m)$ *satisfy* (4.8). *Then, given arbitrary* $(x_{k0})_{k=1}^m \in \widetilde{E}_n^m$, *there exists a unique sequence of vector functions* $(x_{k\nu})_{k=1}^m \in \widetilde{E}_n^m$ $(\nu = 1, 2, \dots)$ *such that for any natural* ν *and* $k \in \{1, \dots, m\}$, *the function* $x_{k\nu}$ *is the solution of the Cauchy problem*

$$\Delta x_{k\nu}(i-1) =$$
$$= g_k(x_{1\,\nu-1}, \dots, x_{k-1\,\nu-1}, x_{k\nu}, x_{k+1\,\nu-1}, \dots, x_{m\,\nu-1})(i), \tag{4.20}$$
$$x_{k\nu}(i_k) = \psi_k(x_{1\,\nu-1}, \dots, x_{m\,\nu-1}), \tag{4.21}$$

and

$$\lim_{\nu \to +\infty} x_{k\nu}(i) = x_k(i) \quad \textit{for} \quad i \in N_n \quad (k = 1, \dots, m), \tag{4.22}$$

where $(x_k)_{k=1}^m$ *is the solution of the problem* (4.1), (4.2).

Remark. Under the conditions of Theorem 4.3, for any $(x_{k0})_{k=1}^m \in \widetilde{E}_n^m$ there exist $r_0 > 0$ and $\gamma \in]0, 1[$ such that

$$\sum_{k=1}^{m} |x_{k\nu}(i) - x_k(i)| \leq r_0 \gamma^\nu \quad \text{for} \quad i \in N_n \quad (\nu = 1, 2, \dots). \tag{4.23}$$

If the conditions of Corollary 4.4 *or Corollary* 4.6 *are fulfilled, then the conclusion of Theorem* 4.3 *is valid.*

Let the conditions of Corollary 4.5 *be fulfilled. Then, given arbitrary* $(x_{k0})_{k=1}^m \in \widetilde{E}_n^m$, *there exists a unique sequence of vector functions* $(x_{k\nu})_{k=1}^m \in \widetilde{E}_n^m$ $(\nu = 1, 2, \dots)$ *such that for any natural* ν *and* $k \in \{1, \dots, m\}$, *the function* $x_{k\nu}$ *is the solution of* (4.20) *under the initial condition*

$$x_{k\nu}(i_k) = \sum_{i=0}^{n} \lambda_{k\nu}(i) x_{k\,\nu-1}(i) + c_k \quad (k = 1, \dots, m). \tag{4.24}$$

Moreover, (4.22) *holds, where* $(x_k)_{k=1}^m$ *is the solution of the problem* (4.1), (4.18).

Let in $\widetilde{E}_n^m$ *the inequalities*

$$\begin{gathered}
\big|g_k(x_1,\dots,x_m)(i)-g_k(y_1,\dots,y_m)(i)-h_k(i)\big(x_k(\tau_k(i))-\\
-y_k(\tau_k(i))\big)\big|\le g_{0k}\big(|x_1-y_1|,\dots,|x_m-y_m|\big)(i)\\
\text{for}\quad i\in N_n\quad (k=1,\dots,m)
\end{gathered}\tag{4.25}$$

and (4.15) *be fulfilled, where the functions* h_k, g_{0k} *and* ψ_{0k} $(k=1,\dots,m)$ *satisfy* (4.8). *Then, given arbitrary* $(x_{k0})_{k=1}^m\in\widetilde{E}_n^m$, *there exists a unique sequence of vector functions* $(x_{k\nu})_{k=1}^m\in\widetilde{E}_n^m$ *such that for any natural* ν *and* $k\in N_m$, *the function* $x_{k\nu}$ *is the solution of the difference equation*

$$\begin{gathered}
\Delta x_{k\nu}(i-1)=\\
=h_k(i)\big[x_{k\nu}\big(\tau_k(i)\big)-x_{k\,\nu-1}\big(\tau_k(i)\big)\big]+g_k(x_{1\,\nu-1},\dots,x_{m\,\nu-1})(i)
\end{gathered}\tag{4.26}$$

under the initial condition (4.21). *Moreover,* (4.22) *holds, where* $(x_k)_{k=1}^m$ *is the solution of the problem* (4.1), (4.2).

Let in $\widetilde{E}_n^m$ *the inequalities*

$$\begin{gathered}
\big|g_k(x_1,\dots,x_m)(i)-g_k(y_1,\dots,y_m)(i)-h_k(i)\big(x_k(\tau_k(i))-y_k(\tau_k(i))\big)\big|\le\\
\le\sum_{j=1}^m h_{kj}\|x_j-y_j\|_{\widetilde{E}_n}\quad\text{for}\quad i\in N_n\quad(k=1,\dots,m)
\end{gathered}$$

be fulfilled, where

$$h_k(i)\,\mathrm{sign}(\tau_k(i)-i_k)\le h_{0k}<0\quad\text{for}\quad i\in N_n\quad(k=1,\dots,m),$$

$h_{kj}\in R_+$ $(k,j=1,\dots,m)$, *and the real parts of the eigen-values of the matrix* (4.11) *are negative. Moreover, let* $c_k\in R$, *and let the functions* $\lambda_k\in\widetilde{E}_n$ $(k=1,\dots,m)$ *satisfy* (4.17). *Then, given arbitrary* $(x_{k0})_{k=1}^m\in\widetilde{E}_n^m$, *there exists a unique sequence of vector functions* $(x_{k\nu})_{k=1}^m\in\widetilde{E}_n^m$ $(\nu=1,2,\dots)$ *such that for any natural* ν *and* $k\in\{1,\dots,m\}$, *the function* $x_{k\nu}$ *is the solution of* (4.26) *under the initial condition* (4.24). *Moreover, the equalities* (4.22) *holds, where* $(x_k)_{k=1}^m$ *is the solution of the problem* (4.1), (4.18).

Let in $\widetilde{E}_n^m$ *the inequalities* (4.16) *and*

$$\begin{gathered}
\big|g_k(x_1,\dots,x_m)(i)-g_k(y_1,\dots,y_m)(i)\big|\le\sum_{j=1}^m h_{kj}\|x_j-y_j\|_{\widetilde{E}_n}\\
\text{for}\quad i\in N_n\quad(k=1,\dots,m)
\end{gathered}$$

be fulfilled, where $h_{kj}\in E_n^+$, $l_{kj}\in R_+$ $(k,j=1,\dots,m)$, *and the spectral radius of the matrix* (4.9) *is less than* 1. *Then, given arbitrary* $(x_{k0})_{k=1}^m\in\widetilde{E}_n^m$, *there exists a unique sequence of vector functions* $(x_{k\nu})_{k=1}^m\in\widetilde{E}_n^m$ $(\nu=$

$1, 2, \dots)$ *such that for any natural* ν *and* $k \in \{1, \dots, m\}$, *the function* $x_{k\nu}$ *is the solution of the difference equation*

$$\Delta x_{k\nu}(i-1) = g_k(x_{1\,\nu-1}, \dots, x_{m\,\nu-1})(i)$$

under the initial condition (4.21). *Moreover,* (4.22) *holds, where* $(x_k)_{k=1}^m$ *is the solution of the problem* (4.1), (4.2).

Let in $\widetilde{E}_n^m$ *the inequalities* (4.19) *and*

$$\bigl|g_k(x_1, \dots, x_m)(i) - g_k(y_1, \dots, y_m)(i)\bigr| \le \sum_{j=1}^m h_{kj} s_{kj}(|x_j - y_j|)(i)$$
$$\text{for} \quad i \in N_n \quad (k = 1, \dots, m)$$

be fulfilled, where $h_{kj} \in R_+$, $l_{kj} \in R_+$, *and* s_{kj} $(k, j = 1, \dots, m)$ *are positively homogeneous nondecreasing functionals satisfying* (4.12). *Moreover, let the spectral radius of the matrix* (4.13) *be less than* 1. *Then the conclusion of Corollary* 4.10 *is valid.*

§ 5. Auxiliary Propositions

Let $i_0 \in \widetilde{N}_n$, $r_0 \in R_+$, $h \in E_n$, $g \in E_n^+$,

$$\tau_0(i) = \begin{cases} i & \text{for} \quad i > i_0, \\ i-1 & \text{for} \quad i \le i_0, \end{cases}$$
$$h(i)\operatorname{sign}(\tau_0(i) - i_0) < 1, \tag{5.1}$$

and let $x \in \widetilde{E}_n$ *satisfy*

$$\begin{aligned}\bigl[\Delta x(i-1) - h(i)x(\tau_0(i))\bigr] \operatorname{sign}\bigl[(\tau_0(i) - i_0)x(\tau_0(i))\bigr] \le \\ \le g(i) \quad \text{for} \quad i \in N_n\end{aligned} \tag{5.2}$$

and

$$|x(i_0)| \le r_0. \tag{5.3}$$

Then

$$|x(i)| \le y(i) \quad \text{for} \quad i \in \widetilde{N}_n, \tag{5.4}$$

where $y \in \widetilde{E}_n$ *is the solution of*

$$\begin{aligned}\Delta y(i-1) &= h(i)y(\tau_0(i)) + g(i)\operatorname{sign}[\tau_0(i) - i_0], \\ y(i_0) &= r_0.\end{aligned} \tag{5.5}$$

Proof. Since

$$\Delta|x(i-1)| = |x(i)| - |x(i-1)| \le \Delta x(i-1)\,\mathrm{sign}[x(i)]$$

and

$$\Delta|x(i-1)| \ge \Delta x(i-1)\,\mathrm{sign}[x(i-1)],$$

from (5.2) we find

$$\begin{aligned} \Delta|x(i-1)| - h(i)|x(i)| &\le g(i) \quad \text{for} \quad i > i_0, \\ -\Delta|x(i-1)| + h(i)|x(i-1)| &\le g(i) \quad \text{for} \quad i \le i_0, \end{aligned}$$

whence, taking into account (5.1) and (5.3), we obtain (5.4), where

$$y(i) = \begin{cases} \Big[\prod\limits_{j=i_0+1}^{i} \dfrac{1}{1-h(j)}\Big] r_0 + \sum\limits_{k=i_0+1}^{i} \Big[\prod\limits_{j=k}^{i} \dfrac{1}{1-h(j)}\Big] g(k) & \text{for} \quad i > i_0, \\ r_0 & \text{for} \quad i = i_0, \\ \Big[\prod\limits_{j=i+1}^{i_0} \dfrac{1}{1+h(j)}\Big] r_0 + \sum\limits_{k=i}^{i_0} \Big[\prod\limits_{j=i+1}^{k} \dfrac{1}{1+h(j)}\Big] g(k) & \text{for} \quad i < i_0 \end{cases}$$

is the solution of (5.5). ■

Let

$$i_k \in \widetilde{N}_n,$$
$$\tau_k(i) = \begin{cases} i & \text{for} \quad i > i_k \\ i-1 & \text{for} \quad i \le i_k \end{cases} \quad (k = 1, \dots, m) \tag{5.6}$$

and

$$(h_1, \dots, h_m; g_{01}, \dots, g_{0m}; \psi_{01} \dots, \psi_{0m}) \in W_n(i_1, \dots, i_m). \tag{5.7}$$

Then there exists a positive ρ such that, for arbitrary $h_0 \in E_n^+$ and $r \in R_+$, any solution of the system of difference inequalities

$$\begin{aligned} &\big[\Delta x_k(i-1) - h_k(i) x_k(\tau_k(i))\big]\,\mathrm{sign}\,\big[(\tau_k(i) - i_k)\big(x_k(\tau_k(i))\big] \le \\ &\le h_0(i) + g_{0k}(|x_1|, \dots, |x_m|)(i) \quad \textit{for} \quad i \in N_n \quad (k = 1, \dots, m) \end{aligned} \tag{5.8}$$

under the boundary conditions

$$|x_k(i_k)| \le r + \psi_{0k}(|x_1|, \dots, |x_m|) \quad \textit{for} \quad (k = 1, \dots, m) \tag{5.9}$$

admits the estimate

$$\sum_{k=1}^{m} \|x_k\|_{\widetilde{E}_n} \le \rho \Big[r + \sum_{j=1}^{n} h_0(j) \Big]. \tag{5.10}$$

Proof. First of all, let us prove the existence of a positive ρ such that for any $r \in R_+$ and $h_0 \in E_n^+$, an arbitrary solution of

$$\big|\Delta y_k(i-1) - h_k(i) y_k(\tau_k(i))\big| \le h_0(i) + g_{0k}(|y_1|, \dots, |y_m|)(i) \quad (5.11)$$
$$\text{for} \quad i \in N_n \quad (k = 1, \dots, m),$$
$$|y_k(i_k)| \le r + \psi_{0k}(|y_1|, \dots, |y_m|) \quad (k = 1, \dots, m) \qquad (5.12)$$

admits the estimate

$$\sum_{k=1}^{m} \|y_k\|_{\widetilde{E}_n} \le \rho \bigg[r + \sum_{j=1}^{n} h_0(j) \bigg]. \qquad (5.13)$$

Assume on the contrary that ρ does not exist. Then for any natural ν, there exist a $r_\nu \in]0, +\infty[$, $h_{0\nu} \in E_n^+$ and $y_{k\nu} \in \widetilde{E}_n$ $(k = 1, \dots, m)$ such that

$$\big|\Delta y_{k\nu}(i-1) - h_k(i) y_{k\nu}(\tau_k(i))\big| \le h_{0\nu}(i) +$$
$$+ g_{0k}(|y_{1\nu}|, \dots, |y_{m\nu}|)(i) \quad \text{for} \quad i \in N_n \quad (k = 1, \dots, m), \qquad (5.14)$$
$$|y_{k\nu}(i_k)| \le r_\nu + \psi_{0k}(|y_{1\nu}|, \dots, |y_{m\nu}|) \quad (k = 1, \dots, m) \qquad (5.15)$$

and

$$\rho_\nu = \sum_{k=1}^{m} \|y_{k\nu}\|_{\widetilde{E}_n} > \nu \bigg[r_\nu + \sum_{j=1}^{n} h_{0\nu}(j) \bigg]. \qquad (5.16)$$

Assume

$$z_{k\nu}(i) = \frac{1}{\rho_\nu} y_{k\nu}(i) \quad (k = 1, \dots, m).$$

Then

$$\sum_{k=1}^{m} \|z_{k\nu}\|_{\widetilde{E}_n} = 1 \quad (\nu = 1, 2, \dots). \qquad (5.17)$$

On the other hand, as g_{0k} and ψ_{0k} $(k = 1, \dots, m)$ are positively homogeneous, we find from (5.14)–(5.16) that

$$\big|\Delta z_{k\nu}(i-1) - h_k(i) z_{k\nu}(\tau_k(i))\big| \le \frac{1}{\nu} + g_{0k}(|z_{1\nu}|, \dots, |z_{m\nu}|)(i) \quad (5.18)$$
$$\text{for} \quad i \in N_n \quad (k = 1, \dots, m),$$
$$|z_{k\nu}(i_k)| \le \frac{1}{\nu} + \psi_{0k}(|z_{1\nu}|, \dots, |z_{m\nu}|) \quad (k = 1, \dots, m). \qquad (5.19)$$

Without restriction of generality, we may assume the sequences $(z_{k\nu})_{\nu=1}^{+\infty}$ $(k = 1, \dots, m)$ to be convergent. Putting

$$\lim_{\nu \to +\infty} z_{k\nu}(i) = z_k(i) \quad (k = 1, \dots, m),$$

from (5.18) and (5.19) we get

$$\left|\Delta z_k(i-1) - h_k(i) z_k(\tau_k(i))\right| \le g_{0k}(|z_1|,\dots,|z_m|)(i)$$
$$\text{for} \quad i \in N_n \quad (k=1,\dots,m)$$

and

$$|z_k(i_k)| \le \psi_{0k}(|z_1|,\dots,|z_m|) \quad (k=1,\dots,m).$$

This, according to (5.7), implies

$$z_k(i) = 0 \quad \text{for} \quad i \in \widetilde{N}_n \quad (k=1,\dots,m).$$

But this is impossible, for because of (5.17),

$$\sum_{k=1}^{m} \|z_k\|_{\widetilde{E}_n} = 1.$$

The obtained contradiction proves the existence of a number ρ which possesses the above-mentioned property.

Assume now that $r > 0$ is an arbitrary number, $h_0 \in E_n^+$ is a function, and $(x_k)_{k=1}^m$ is a solution of (5.8), (5.9). By Lemma 5.1, for any $k \in \{1,\dots,m\}$ the inequality

$$|x_k(i)| \le y_k(i) \quad \text{for} \quad i \in \widetilde{N}_n \tag{5.20}$$

is fulfilled, where y_k is the solution of

$$\Delta y_k(i-1) - h_k(i) y_k(\tau_k(i)) =$$
$$= \big[h_0(i) + g_{0k}(|x_1|,\dots,|x_m|)(i)\big] \operatorname{sign}[\tau_k(i) - i_k],$$
$$y_k(i_k) = r + \psi_{0k}(|x_1|,\dots,|x_m|).$$

From this it is clear that $(y_k)_{k=1}^m$ is a solution of (5.11), (5.12). Therefore, due to our choice of ρ, (5.13) holds from which, taking into account (5.20), we obtain (5.10). ■

Let (5.6) *and* (5.7) *be fulfilled. Then there exists* $\gamma \in]0,1[$ *such that*

$$\Big(h_1,\dots,h_m; \frac{1}{\gamma} g_{01},\dots,\frac{1}{\gamma} g_{0m}; \frac{1}{\gamma}\psi_{01},\dots,\psi_{0m}\Big) \in W_n(i_1,\dots,i_m). \tag{5.21}$$

Proof. Let ρ be the positive number whose existence has been established in Lemma 5.2. We choose $\gamma \in]0,1[$ such that

$$\frac{1-\gamma}{\gamma}\rho \sum_{k=1}^{m} \Big[\psi_{0k}(1,\dots,1) + \sum_{j=1}^{n} g_{0k}(1,\dots,1)\Big] < \frac{1}{2}\,. \tag{5.22}$$

Consider an arbitrary solution $(x_k)_{k=1}^m$ of

$$\left|\Delta x_k(i-1) - h_k(i) x_k(\tau_k(i))\right| \le \frac{1}{\gamma} g_{0k}(|z_1|,\dots,|z_m|)(i) \tag{5.23}$$

$$\text{for} \quad i \in N_n \quad (k = 1, \dots, m),$$

$$|x_k(i_k)| \le \frac{1}{\gamma}\psi_{0k}(|x_1|, \dots, |x_m|) \quad (k = 1, \dots, m). \tag{5.24}$$

It is evident that $(x_k)_{k=1}^m$ is at the same time a solution of (5.8), (5.9), where

$$h_0(i) = \frac{1-\gamma}{\gamma} \sum_{k=1}^m g_{0k}(|x_1|, \dots, |x_m|)(i),$$

$$r = \frac{1-\gamma}{\gamma} \sum_{k=1}^m \psi_{0k}(|x_1|, \dots, |x_m|).$$

By Lemma 5.2, (5.10) is valid, i.e.,

$$\sum_{k=1}^m \|x_k\|_{\widetilde{E}_n} \le \frac{1-\gamma}{\gamma}\rho \sum_{k=1}^m \Big[\psi_{0k}(|x_1|, \dots, |x_m|) + \sum_{j=1}^n g_{0k}(|x_1|, \dots, |x_m|)(j)\Big].$$

Assuming

$$x^* = \sum_{k=1}^m \|x_k\|_{\widetilde{E_n}},$$

from the last inequality and (5.22) we find that $x^* \le \frac{1}{2}x^*$ and $x^* = 0$. Consequently, the problem (5.23), (5.24) has only the zero solution. Thus we have proved that (5.21) is fulfilled. ■

Let (5.6) *and*

$$(h_1, \dots, h_m; \widetilde{g}_{01} \dots, \widetilde{g}_{0m}; \widetilde{\psi}_{01}, \dots, \widetilde{\psi}_{0m}) \in W_n(i_1, \dots, i_m) \tag{5.25}$$

be fulfilled. Then there exists a positive number ρ^* *such that an arbitrary sequence of vector functions* $(y_{k\nu})_{k=1}^m \in \widetilde{E}_n^m$ $(\nu = 1, 2, \dots)$ *satisfying for any* $\nu \in N$ *and* $k \in N_m$ *the inequalities*

$$\big|\Delta y_{k\nu}(i-1) - h_k(i) y_{k\nu}(\tau_k(i))\big| \le 1 +$$

$$+\widetilde{g}_{0k}\big(|y_{1\,\nu-1}|, \dots, |y_{k-1\,\nu-1}|, |y_{k\nu}|, |y_{k+1\,\nu-1}|, \dots, |y_{m\,\nu-1}|\big)(i) \tag{5.26}$$

$$\textit{for} \ \ i \in N_n,$$

$$|y_{k\nu}(i_k)| \le 1 + \widetilde{\psi}_{0k}(|y_{1\,\nu-1}|, \dots, |y_{m\,\nu-1}|) \tag{5.27}$$

with $y_{j0}(i) \equiv 1$ $(j = 1, \dots, m)$, *admits the estimate*

$$\sup\left\{\sum_{k=1}^m \|y_{k\nu}\|_{\widetilde{E}_n} \,:\, \nu \in N\right\} \le \rho^*. \tag{5.28}$$

Proof. Let $y_{j0}^*(i) \equiv 0$ $(j = 1, \dots, m)$, $k \in N_m$, and let Y_{k1} be the set of all functions $y \in \widetilde{E}_n$ satisfying

$$\big|\Delta y(i-1) - h_k(i) y(\tau_k(i))\big| \le 1 + \widetilde{g}_{0k}(y_{10}^*, \dots, y_{k-1\,0}^*, |y|, y_{k+1\,0}^*, \dots, y_{m0}^*)(i)$$

$$\text{for} \quad i \in N_n,$$

$$|y(i_k)| \le 1 + \widetilde{\psi}_{0k}(y^*_{10}, \dots, y^*_{m0}).$$

It easily follows from (5.25) that[6]

$$y^*_{k1}(i) = \sup\Big\{|y(i)| : \; y \in Y_{k1}\Big\} < +\infty \quad \text{for} \quad i \in \widetilde{N}_n.$$

On the other hand, according to Lemma 5.1,

$$y^*_{k1}(i) \le z(i) \quad \text{for} \quad i \in \widetilde{N}_n,$$

where z is the solution of the Cauchy problem

$$\Delta z(i-1) - h_k(i) z(\tau_k(i)) =$$
$$= \big[1 + \widetilde{g}_{0k}(y^*_{10}, \dots, y^*_{k-1\,0}, y^*_{k1}, y^*_{k+1\,0}, \dots, y^*_{m0})(i)\big] \operatorname{sign}(\tau_k(i) - i_k),$$
$$z(i_k) = 1 + \widetilde{\psi}_{0k}(y^*_{10}, \dots, y^*_{m0}).$$

However, $z \in Y_{k1}$. Therefore it is clear that $z(i) = y^*_{k1}(i)$. Continuing this process, we can construct a sequence of vector functions $(y^*_{k\nu})^m_{k=1} \in (\widetilde{E}^+_n)^m$ $(\nu = 1, 2, \dots)$ such that for any $k \in N_m$ and $\nu \in N$, the function $y^*_{k\nu}$ is the solution of the Cauchy problem

$$\Delta y^*_{k\nu}(i-1) - h_k(i) y^*_{k\nu}(\tau_k(i)) =$$
$$= \big[1 + \widetilde{g}_{0k}(y^*_{1\,\nu-1}, \dots, y^*_{k-1\,\nu-1}, y^*_{k\nu}, y^*_{k+1\,\nu-1}, \dots, y^*_{m\,\nu-1})(i)\big] \times$$
$$\times \operatorname{sign}[\tau_k(i) - i_k], \tag{5.26*}$$

$$y^*_{k\nu}(i_k) = 1 + \widetilde{\psi}_{0k}(y^*_{1\,\nu-1}, \dots, y^*_{m\,\nu-1}), \tag{5.27*}$$

and any $y \in \widetilde{E}_n$ satisfying

$$\big|\Delta y(i-1) - h_k(i) y(\tau_k(i))\big| \le$$
$$\le 1 + \widetilde{g}_{0k}(y^*_{1\,\nu-1}, \dots, y^*_{k-1\,\nu-1}, |y|, y^*_{k+1\,\nu-1}, \dots, y^*_{m\,\nu-1})(i) \quad \text{for} \;\; i \in N_n,$$
$$|y(i)| \le 1 + \widetilde{\psi}_{0k}(y^*_{1\,\nu-1}, \dots, y^*_{m\,\nu-1})$$

admits the estimate

$$|y(i)| \le y^*_{k\nu}(i) \quad \text{for} \quad i \in \widetilde{N}_n.$$

It is easily seen from the above-said that any sequence $(y_{k\nu})^m_{k=1} \in \widetilde{E}^m_n$ satisfying the conditions of Lemma 5.4 also satisfies

$$|y_{k\nu}(i)| \le y^*_{k\nu}(i) \quad \text{for} \;\; i \in \widetilde{N}_n \;\; (k = 1, \dots, m; \; \nu = 1, 2, \dots).$$

Therefore, to prove the lemma it suffices to show that

$$\rho^* = \sup\bigg\{\sum_{k=1}^m \|y^*_{k\nu}\|_{\widetilde{E}_n} \; : \; \nu \in N\bigg\} < +\infty.$$

[6] See the proof of Lemma 5.2.

Put

$$\rho_0 = 1,$$
$$\rho_\nu = \max\left\{1, \sum_{k=1}^m \|y^*_{k1}\|_{\widetilde{E}_n}, \dots, \sum_{k=1}^m \|y^*_{k\nu}\|_{\widetilde{E}_n}\right\} \quad (\nu = 1, 2, \dots).$$

Our aim is to prove that the sequence $(\rho_\nu)_{\nu=1}^{+\infty}$ is bounded. Assume on the contrary that

$$\lim_{\nu\to+\infty} \rho_\nu = +\infty. \tag{5.29}$$

Let

$$z_{k\nu}(i) = \frac{1}{\rho_\nu} y^*_{k\nu}(i),$$
$$\lim_{\nu\to+\infty} \sup z_{k\nu}(i) = z_k(i) \quad (k = 1, \dots, m).$$

From (5.26*) and (5.27*) we find

$$\big|\Delta z_{k\nu}(i-1) - h_k(i) z_{k\nu}(\tau_k(i))\big| \le$$
$$\le \widetilde{g}_{0k}(z_{1\,\nu-1}, \dots, z_{k-1\,\nu-1}, z_{k\nu}, z_{k+1\,\nu-1}, \dots, z_{m\,\nu-1})(i)$$
$$\text{for } \ i \in N_n \ \ (k = 1, \dots, m),$$
$$|z_{k\nu}(i_k)| \le \frac{1}{\rho_\nu} + \psi_{0k}(z_{1\,\nu-1}, \dots, z_{m\,\nu-1}) \quad (k = 1, \dots, m).$$

On the other hand, it is clear that

$$\sum_{k=1}^m \|z_k\|_{\widetilde{E}_n} = 1. \tag{5.30}$$

By Lemma 5.1, for any $k \in \{1, \dots, m\}$ and $\nu \in N$ the inequality

$$0 < z_{k\nu}(i) \le \widetilde{z}_{k\nu}(i) \quad \text{for} \quad i \in \widetilde{N}_n, \tag{5.31}$$

is fulfilled, where $\widetilde{z}_{k\nu}$ is the solution of

$$\Delta\widetilde{z}_{k\nu}(i-1) - h_k(i)\widetilde{z}_{k\nu}(\tau_k(i)) =$$
$$= \Big[\frac{1}{\rho_\nu} + \widetilde{g}_{0k}(z_{1\,\nu-1}, \dots, z_{k-1\,\nu-1}, z_{k\nu}, z_{k+1\,\nu-1}, \dots, z_{m\,\nu-1})(i)\Big] \times$$
$$\times \operatorname{sign}[\tau_k(i) - i_k],$$
$$\widetilde{z}_{k\nu}(i_k) = \frac{1}{\rho_\nu} + \psi_{0k}(z_{1\,\nu-1}, \dots, z_{m\,\nu-1}).$$

It is easily seen that

$$\lim_{\nu\to+\infty} \sup \widetilde{z}_{k\nu}(i) \le \widetilde{z}_k(i) \quad \text{for} \quad i \in \widetilde{N}_n \quad (k = 1, \dots, m), \tag{5.32}$$

where $\widetilde{z}_k$ is the solution of

$$\Delta\widetilde{z}_k(i-1) - h_k(i)\widetilde{z}_k(\tau_k(i)) = \\ = \widetilde{g}_{0k}(z_1,\dots,z_m)(i)\,\mathrm{sign}[\tau_k(i)-i_k], \tag{5.33}$$

$$\widetilde{z}_k(i_k) = \widetilde{\psi}_{0k}(z_1,\dots,z_m). \tag{5.34}$$

Owing to (5.30)–(5.32),

$$0 < z_k(i) \le \widetilde{z}_k(i) \quad \text{for} \quad i \in \widetilde{N}_n \quad (k=1,\dots,m) \tag{5.35}$$

and

$$\sum_{k=1}^m \|\widetilde{z}_k\|_{E_n} \ge 1. \tag{5.36}$$

It follows from (5.33)–(5.35) that

$$\big|\Delta\widetilde{z}_k(i-1) - h_k(i)\widetilde{z}_k(\tau_k(i))\big| \le \widetilde{g}_{0k}(|\widetilde{z}_1|,\dots,|\widetilde{z}_m|)(i) \\ \text{for} \quad i \in N_n \quad (k=1,\dots,m),$$

$$|\widetilde{z}_k(i_k)| \le \widetilde{\psi}_{0k}(|\widetilde{z}_1|,\dots,|\widetilde{z}_m|) \quad (k=1,\dots,m)$$

whence, because of (5.25), the identities $\widetilde{z}_k(i) \equiv 0$ $(k=1,\dots,m)$ follow. But this contradicts to (5.36). The obtained contradiction proves the lemma. ■

Let the conditions (5.6) *and* (5.7) *be fulfilled. Then there exist* $\gamma \in]0,1[$ *and* $\rho^* > 0$ *such that for any sequences* $r_\nu \in]0,+\infty[$ $(\nu = 1,2,\dots)$, $x_{k\nu} \in \widetilde{E}_n$ $(k=1,\dots,m;\ \nu=1,2,\dots)$ *and* $h_{0\nu} \in E_n^+$ $(\nu = 1,2,\dots)$ *satisfying for any natural* ν

$$\big[\Delta x_{k\nu}(i-1) - h_k(i)x_{k\nu}(\tau_k(i))\big]\,\mathrm{sign}\,\big[(\tau_k(i)-i_k)x_{k\nu}(\tau_k(i))\big] \le \\ \le h_{0\nu}(i) + g_{0k}\big(|x_{1\,\nu-1}|,\dots,|x_{k-1\,\nu-1}|,|x_{k\nu}|,|x_{k+1\,\nu-1}|,\dots \\ \dots,|x_{m\,\nu-1}|\big)(i) \quad i \in N_n \quad (k=1,\dots,m) \tag{5.37}$$

and

$$|x_{k\nu}(i_k)| \le r_\nu + \psi_{0k}(|x_{1\,\nu-1}|,\dots,|x_{m\,\nu-1}|) \quad (k=1,\dots,m), \tag{5.38}$$

the estimates

$$\sum_{k=1}^m \|x_{k\nu}\|_{\widetilde{E}_n} \le \rho^*\gamma^\nu\Big[\sum_{p=1}^\nu \gamma^{-p}\Big(\sum_{i=1}^n h_{0p}(i) + r_p\Big) + \sum_{k=1}^m \|x_{k0}\|_{\widetilde{E}_n}\Big] \tag{5.39}$$
$$(\nu = 1,2,\dots)$$

are valid.

Proof. By Lemma 5.3, there exists and $\gamma \in]0,1[$ such that (5.25) is fulfilled, where

$$\widetilde{g}_{0k} = \frac{1}{\gamma} g_{0k}, \quad \text{and} \quad \widetilde{\psi}_{0k} = \frac{1}{\gamma}\psi_{0k} \quad (k = 1, \dots, m). \tag{5.40}$$

Let ρ^* be the number appearing in Lemma 5.4,

$$\alpha_0 = \sum_{k=1}^{m} \|x_{k0}\|_{\widetilde{E}_n},$$

$$\alpha_\nu = \sum_{p=1}^{\nu} \gamma^{-p}\Big(\sum_{i=1}^{n} h_{0p}(i) + r_p\Big) + \alpha_0 \quad (\nu = 1, 2, \dots),$$

and

$$\widetilde{x}_{k\nu}(i) = \begin{cases} \dfrac{x_{k\nu}(i)}{\gamma^\nu \alpha_\nu} & \text{for} \quad \alpha_\nu \neq 0, \\ 0 & \text{for} \quad \alpha_\nu = 0. \end{cases} \tag{5.41}$$

Taking into account the fact that

$$\alpha_\nu \geq \alpha_{\nu-1} \quad (\nu = 1, 2, \dots)$$

as well as (5.40), from (5.37) and (5.38) we find

$$\big[\Delta\widetilde{x}_{k\nu}(i-1) - h_k(i)\widetilde{x}_{k\nu}(\tau_k(i))\big] \operatorname{sign}\big[(\tau_k(i) - i_k)\widetilde{x}_{k\nu}(\tau_k(i))\big] \leq$$
$$\leq 1 + g_{0k}\big(|\widetilde{x}_{1\,\nu-1}|, \dots, |\widetilde{x}_{k-1\,\nu-1}|, |\widetilde{x}_{k\nu}|, |\widetilde{x}_{k+1\,\nu-1}|, \dots, |\widetilde{x}_{m\,\nu-1}|\big)(i)$$
$$i \in N_n \quad (k = 1, \dots, m)$$

and

$$|\widetilde{x}_{k\nu}(i_k)| \leq 1 + \widetilde{\psi}_{0k}(|\widetilde{x}_{1\,\nu-1}|, \dots, |\widetilde{x}_{m\,\nu-1}|) \quad (k = 1, \dots, m).$$

By Lemma 5.1, for any $\nu \in N$ and $k \in N_m$ we have

$$|\widetilde{x}_{k\nu}(i)| \leq y_{k\nu}(i) \quad \text{for} \quad i \in \widetilde{N}_n,$$

where $y_{k\nu}$ is the solution of the Cauchy problem

$$\Delta y_{k\nu}(i-1) - h_k(i) y_{k\nu}(\tau_k(i)) =$$
$$= \big[1 + \widetilde{g}_{0k}\big(|\widetilde{x}_{1\,\nu-1}|, \dots, |\widetilde{x}_{k-1\,\nu-1}|, |\widetilde{x}_{k\nu}|, |\widetilde{x}_{k+1\,\nu-1}|, \dots, |\widetilde{x}_{m\,\nu-1}|\big)(i)\big] \times$$
$$\times \operatorname{sign}[\tau_k(i) - i_k],$$
$$y_{k\nu}(i_k) = 1 + \widetilde{\psi}_{0k}(|\widetilde{x}_{1\,\nu-1}|, \dots, |\widetilde{x}_{m\,\nu-1}|).$$

Obviously the sequences $(y_{k\nu})_{k=1}^{m}$ $(\nu = 1, 2, \dots)$ satisfy the conditions of Lemma 5.4. Therefore (5.28) is valid. Hence

$$\sum_{k=1}^{m} \|\widetilde{x}_{k\nu}\|_{\widetilde{E}_n} \leq \rho^* \quad (\nu = 1, 2, \dots)$$

whence, owing to (5.41), it follows (5.39). ■

The lemma below is proved analogously.

′ *Let* (5.6) *and* (5.7) *be fulfilled. Then there exist* $\gamma \in]0,1[$ *and* $\rho^* > 0$ *such that for any sequences* $r_\nu \in]0,+\infty[$ $(\nu = 1,2,\dots)$, $x_{k\nu} \in \widetilde{E}_n$ $(k = 1,\dots,m;\ \nu = 0,1,\dots)$ *and* $h_{0\nu} \in E_n^+$ $(\nu = 1,2,\dots)$ *satisfying for any natural* ν (5.38) *and*

$$\bigl|\Delta x_{k\nu}(i-1) - h_k(i)x_{k\nu}(\tau_k(i))\bigr| \le$$
$$\le h_{0\nu}(i) + g_{0k}\bigl(|x_{1\,\nu-1}|,\dots,|x_{m\,\nu-1}|\bigr)(i) \quad \text{for} \quad i \in N_n \quad (k = 1,\dots,m),$$

the estimates (5.39) *are valid.*

$$W_n(i_1,\dots,i_m) \qquad W_{0n}(i_1,\dots,i_m)$$

Let

$$g_{0k}(x_1,\dots,x_m)(i) \equiv \sum_{j=1}^m h_{kj}(i)\|x_j\|_{\widetilde{E}_n} \quad (k = 1,\dots,m) \tag{5.42}$$

and

$$\psi_{0k}(x_1,\dots,x_m) \equiv \sum_{j=1}^m l_{kj}(i)\|x_j\|_{\widetilde{E}_n} \quad (k = 1,\dots,m), \tag{5.43}$$

where $h_{kj} \in E_n^+$, $l_{kj} \in R_+$ $(k,j = 1,\dots,m)$, *and the spectral radius of the matrix*

$$\Lambda = \Bigl(l_{kj} + \sum_{i=1}^n h_{kj}(i)\Bigr)_{k,j=1}^m$$

is less than 1. *Then for any* $i_k \in \widetilde{N}_n$ $(k = 1,\dots,m)$, *we have*

$$(g_{01},\dots,g_{0m};\psi_{01},\dots,\psi_{0m}) \in W_{0n}(i_1,\dots,i_m). \tag{5.7$_0$}$$

Proof. Let $h_k(i) \equiv 0$ $(k = 1,\dots,m)$, and let $(x_k)_{k=1}^m$ be an arbitrary solution of the problem (4.4), (4.5), the latter according to (5.42) and (5.43), having the form

$$|\Delta x_k(i-1)| \le \sum_{j=1}^m h_{kj}(i)\|x_j\|_{\widetilde{E}_n} \quad \text{for} \quad i \in N_n \quad (k = 1,\dots,m),$$
$$|x_k(i_k)| \le \sum_{j=1}^m l_{kj}\|x_j\|_{\widetilde{E}_n} \quad (k = 1,\dots,m).$$

Then

$$\|x_k\|_{\widetilde{E}_n} \le \sum_{j=1}^m \Bigl(l_{kj} + \sum_{i=1}^n h_{kj}(i)\Bigr)\|x_j\|_{\widetilde{E}_n} \quad (k = 1,\dots,m),$$

that is,

$$(E - \Lambda)\Bigl(\|x_k\|_{\widetilde{E}_n}\Bigr)_{k=1}^m \le 0.$$

Since the spectral radius of Λ is less than 1, we have $x_k(i) \equiv 0$ $(k = 1,\dots,m)$. ■

Let $h_k(i) \equiv h_{0k}\operatorname{sign}(\tau_k(i) - i_k)$, $h_{0k} \equiv const < 0$ $(k = 1,\dots,m)$,

$$g_{0k}(x_1,\dots,x_m)(i) \equiv \sum_{j=1}^{m} h_{kj}\|x_j\|_{\widetilde{E}_n} \quad (k = 1,\dots,m), \tag{5.44}$$

and

$$\psi_{0k}(x_1,\dots,x_m) \equiv \sum_{i=0}^{n} l_k(i)|x_k(i)| \quad (k = 1,\dots,m), \tag{5.45}$$

where $h_{kj} \in R_+$ *and* $l_k \in \widetilde{E}_n^+$ $(k, j = 1,\dots,m)$. *Moreover, let*

$$\begin{gathered}\alpha_k = \sum_{i=0}^{n} l_k(i)(1 - h_{0k})^{-|i-i_k|} < 1,\\ \sum_{i=0}^{n} l_k(i) \le 1 \quad (k = 1,\dots,m),\end{gathered} \tag{5.46}$$

and the real parts of the eigen-values of the matrix

$$\big(h_{0k}\delta_{kj} + h_{kj}\big)_{k,j=1}^{m} \tag{5.47}$$

be negative. Then (5.7) *is fulfilled.*

Proof. To prove the lemma, it suffices to show that the problem

$$\begin{gathered}\Delta x_k(i-1)\operatorname{sign}\big[(\tau_k(i) - i_k)x_k(\tau_k(i))\big] \le\\ \le h_{0k}|x_k(\tau_k(i))| + \sum_{j=1}^{m} h_{kj}\|x_j\|_{\widetilde{E}_n} \quad \text{for} \quad i \in N_n \quad (k = 1,\dots,m),\\ |x_k(i_k)| \le \sum_{i=0}^{n} l_k(i)|x_k(i)| \quad (k = 1,\dots,m)\end{gathered}$$

has only the zero solution. Let $(x_k)_{k=1}^{m}$ be an arbitrary solution of this problem. By Lemma 5.1,

$$|x_k(i)| \le y_k(i) \quad \text{for} \quad i \in \widetilde{N}_n, \tag{5.48}$$

where y_k is the solution of

$$\begin{aligned}\Delta y_k(i-1) = h_{0k}y_k(\tau_k(i)) &+\\ &+ \sum_{j=1}^{m} h_{kj}\|x_j\|_{\widetilde{E}_n}\operatorname{sign}(\tau_k(i) - i_k),\end{aligned}$$

$$y_k(i_k) = \sum_{i=0}^{n} l_k(i)|x_k(i)|. \tag{5.49}$$

However,

$$\begin{aligned} y_k(i) = (1-h_{0k})^{-|i-i_k|} y_k(i_k) + \\ + \left[1-(1-h_{0k})^{-|i-i_k|}\right] \sum_{j=1}^{m} \frac{h_{kj}}{|h_{0k}|} \|x_j\|_{\widetilde{E}_n}. \end{aligned} \tag{5.50}$$

Taking along with this equality the inequalities (5.46), (5.48) into account, from (5.49) we find

$$y_k(i_k) \le \alpha_k y_k(i_k) + (1-\alpha_k) \sum_{j=1}^{m} \frac{h_{kj}}{|h_{0k}|} \|y_j\|_{\widetilde{E}_n},$$

and hence

$$y_k(i_k) \le \sum_{j=1}^{m} \frac{h_{kj}}{|h_{0k}|} \|y_j\|_{\widetilde{E}_n} \quad (k = 1, \dots, m).$$

Therefore, by virtue of (5.50), it holds

$$\|y_k\|_{\widetilde{E}_n} \le \sum_{j=1}^{m} \frac{h_{kj}}{|h_{0k}|} \|y_j\|_{\widetilde{E}_n} \quad (k = 1, \dots, m). \tag{5.51}$$

Since the real parts of the eigen-values of the matrix (4.47) are negative, the spectral radius of the matrix $\left(\frac{h_{kj}}{|h_{0k}|}\right)_{k,j=1}^{m}$ is less than 1. Therefore, by (5.51), we have $y_k(i) \equiv 0$ $(k = 1, \dots, m)$. Hence $x_k(i) \equiv 0$ $(k = 1, \dots, m)$. ■

Let $i_0 \in \widetilde{N}_n$, $x \in \widetilde{E}_n$, and

$$x(i_0) = 0.$$

Then

$$\sum_{i=0}^{n} x^2(i) \le \frac{1}{4\sin^2 \frac{\pi}{4n+2}} \sum_{i=1}^{n} \left[\Delta x(i-1)\right]^2. \tag{5.52}$$

Proof. Let $i_0 = 0$. Assume

$$y(i) = \begin{cases} x(i) & \text{for} \quad i \le n, \\ x(2n-i+1) & \text{for} \quad n < i \le 2n+1. \end{cases}$$

Then $y \in \widetilde{E}_{2n+1}$ and

$$y(0) = y(2n+1) = 0.$$

Therefore, by virtue of Theorem 1.1 from [40],

$$\sum_{i=0}^{2n+1} y^2(i) \le \frac{1}{4\sin^2\frac{\pi}{4n+2}} \sum_{i=1}^{2n+1} \big[\Delta y(i-1)\big]^2.$$

However,

$$\sum_{i=0}^{2n+1} y^2(i) = 2\sum_{i=0}^{n} x^2(i)$$

and

$$\begin{aligned}\sum_{i=1}^{2n+1} \big[\Delta y(i-1)\big]^2 = \sum_{i=1}^{n} \big[\Delta x(i-1)\big]^2 + \\ + \big[y(n+1)-y(n)\big]^2 + \sum_{i=n+2}^{2n+1} \big[\Delta y(i-1)\big]^2 = \\ = \sum_{i=1}^{n} \big[\Delta x(i-1)\big]^2 + \big[x(n)-x(n)\big]^2 + \\ + \sum_{i=n+2}^{2n+1} \big[\Delta x(2n+1-i)\big]^2 = 2\sum_{i=1}^{n} \big[\Delta x(i-1)\big]^2.\end{aligned}$$

Consequently, (5.52) is valid. The case where $i_0 \neq 0$ can easily be reduced to that considered above. ■

Let

$$g_{0k}(x_1,\dots,x_m)(i) \equiv \sum_{j=1}^{m} h_{kj} s_{kj}(x_j)(i) \quad (k=1,\dots,m)$$

and

$$\psi_{0k}(x_1,\dots,x_m) = \sum_{j=1}^{m} l_{kj} \left[\sum_{i=0}^{n} x_j^2(i)\right]^{\frac{1}{2}} \quad (k=1,\dots,m),$$

where $h_{kj} \in R_+$, $l_{kj} \in R_+$, and $s_{kj} : \widetilde{E}_n^+ \to \widetilde{E}_n^+$ $(k,j=1,\dots,m)$ are positively homogeneous nondecreasing operators such that for any $x \in \widetilde{E}_n$

$$\sum_{i=0}^{n} \big[s_{kj}(x)(i)\big]^2 \le \sum_{i=0}^{n} x^2(i) \quad (k=1,\dots,m). \tag{5.53}$$

Moreover, let the spectral radius of the matrix

$$\left(\sqrt{n+1}\,l_{kj} + \frac{h_{kj}}{2\sin\frac{\pi}{4n+2}}\right)_{k,j=1}^{m} \tag{5.54}$$

be less than 1. Then for any $i_k \in \widetilde{N}_n$ $(k=1,\dots,m)$, the condition (5.7_0) is fulfilled.

Proof. We have to show that the problem

$$|\Delta x_k(i-1)| \le \sum_{j=1}^{m} h_{kj} s_{kj}(x_j)(i) \quad \text{for} \quad i \in N_n \quad (k=1,\dots,m),$$

$$|x_k(i_k)| \le \sum_{j=1}^{m} l_{kj} \left[\sum_{i=0}^{n} x_j^2(i) \right]^{\frac{1}{2}} \quad (k=1,\dots,m)$$

has only the zero solution. Let $(x_k)_{k=1}^m$ be an arbitrary solution of this problem. Then

$$|x_k(i)| \le \sum_{j=1}^{m} l_{kj}\rho_j + \sum_{j=1}^{m} h_{kj} z_{kj}(i) \tag{5.55}$$

$$\text{for} \quad i \in \widetilde{N}_n \quad (k=1,\dots,m),$$

where

$$\rho_j = \left[\sum_{i=0}^{n} x_j^2(i) \right]^{\frac{1}{2}}$$

and

$$z_{kj}(i) = \begin{cases} \displaystyle\sum_{p=i_k+1}^{i} s_{kj}(|x_j|)(p) & \text{for} \quad i > i_k, \\ 0 & \text{for} \quad i = i_k, \\ \displaystyle\sum_{p=i+1}^{i_k} s_{kj}(|x_j|)(p) & \text{for} \quad i < i_k. \end{cases}$$

By Minkowski's inequality,

$$\rho_k \le \sqrt{n+1} \sum_{j=1}^{m} l_{kj}\rho_j + \sum_{j=1}^{m} h_{kj} \left[\sum_{i=0}^{n} z_{kj}^2(i) \right]^{\frac{1}{2}} \quad (k=1,\dots,m).$$

On the other hand, by Lemma 5.8 and (5.53), we have

$$\left[\sum_{i=0}^{n} z_{kj}^2(i) \right]^{\frac{1}{2}} \le \frac{1}{2\sin\frac{\pi}{4n+2}} \left[\sum_{i=0}^{n} \left| s_{kj}(|x_j|)(i) \right|^2 \right]^{\frac{1}{2}} \le$$

$$\le \frac{1}{2\sin\frac{\pi}{4n+2}} \rho_j \quad (k,j=1,\dots,m).$$

Therefore

$$\rho_k \le \sum_{j=1}^{m} \left(\sqrt{n+1}\, l_{kj} + \frac{h_{kj}}{2\sin\frac{\pi}{4n+2}} \right) \rho_j \quad (k=1,\dots,m).$$

This implies that

$$\rho_k = 0 \quad (k=1,\dots,m),$$

for the spectral radius of the matrix (5.54) is less than 1. Hence $x_k(i) \equiv 0$ $(k = 1, \dots, m)$. ■

§ 6. Proof of the Main Results

Proof of Theorem 4.1. Let ρ be the number appearing in Lemma 5.2. Assume

$$\rho_0 = \rho\Big[r + \sum_{i=1}^{n} h_0(i)\Big],$$

$$\chi(s) = \begin{cases} 1 & \text{for} \quad |s| \le \rho_0 \\ 2 - \dfrac{|s|}{\rho_0} & \text{for} \quad \rho_0 < |s| < 2\rho_0 \\ 0 & \text{for} \quad |s| \ge 2\rho_0 \end{cases}, \tag{6.1}$$

$$\widetilde{g}_k(x_1, \dots, x_m)(i) = \chi\Big(\sum_{k=1}^{m} \|x_k\|_{\widetilde{E}_n}\Big)\big[g_k(x_1, \dots, x_m)(i) - \\ - h_k(i)x_k(\tau_k(i))\big] \quad (k = 1, \dots, m), \tag{6.2}$$

$$\widetilde{\psi}_k(x_1, \dots, x_m) = \chi\Big(\sum_{k=1}^{m} \|x_k\|_{\widetilde{E}_n}\Big)\psi_k(x_1, \dots, x_m) \tag{6.3}$$
$$(k = 1, \dots, m),$$

and consider the boundary value problem

$$\Delta x_k(i-1) = h_k(i)x_k(\tau_k(i)) + \widetilde{g}_k(x_1, \dots, x_m(i) \quad (k = 1, \dots, m), \tag{6.4}$$

$$x_k(i_k) = \widetilde{\psi}_k(x_1, \dots, x_m) \quad (k = 1, \dots, m). \tag{6.5}$$

It is easily seen that this problem is equivalent to the system of equations

$$\mathcal{G}_k(x_1, \dots, x_m)(i) = x_k(i) \quad \text{for} \quad i \in \widetilde{N}_n \quad (k = 1, \dots, m), \tag{6.6}$$

where

$$\mathcal{G}_k(x_1, \dots, x_m)(i) =$$

$$
= \begin{cases}
\Big[\prod\limits_{j=i_k+1}^{i} \dfrac{1}{1-h_k(j)}\Big]\widetilde{\psi}_k(x_1,\dots,x_m)+ \\
\qquad + \sum\limits_{p=i_k+1}^{i} \Big[\prod\limits_{j=p}^{i} \dfrac{1}{1-h_k(j)}\Big]\widetilde{g}_k(x_1,\dots,x_m)(p) & \text{for } \ i>i_k, \\
\widetilde{\psi}_k(x_1,\dots,x_m) & \text{for } \ i=i_k, \\
\Big[\prod\limits_{j=i+1}^{i_k} \dfrac{1}{1+h_k(j)}\Big]\widetilde{\psi}_k(x_1,\dots,x_m)- \\
\qquad - \sum\limits_{p=i+1}^{i_k} \Big[\prod\limits_{j=i+1}^{p} \dfrac{1}{1+h_k(j)}\Big]\widetilde{g}_k(x_1,\dots,x_m)(p) & \text{for } \ i<i_k.
\end{cases}
$$

Owing to (6.1)–(6.3), the operator $(\mathcal{G}_k)_{k=1}^m : \widetilde{E}_n^m \to \widetilde{E}_n^m$ is continuous and

$$
\sup\Big\{ \big\|\mathcal{G}_k(x_1,\dots,x_m)\big\|_{\widetilde{E}_n} : \ (x_j)_{j=1}^m \in \widetilde{E}_n^m \Big\} < +\infty \quad (k=1,\dots,m).
$$

Therefore, according to the Bohl–Brouwer theorem [29], the system of equations (6.6), and consequently the problem (6.4), (6.5) have at least one solution.

Let $(x_k)_{k=1}^m$ be a solution of (6.4), (6.5). Because of (4.6), (4.7) and (6.1)–(6.3), it is clear that $(x_k)_{k=1}^m$ satisfies the (5.8) and (5.9). Therefore, by virtue of Lemma 5.2, (5.10) holds. Hence

$$
\sum_{k=1}^m \|x_k\|_{\widetilde{E}_n} \le \rho_0 .
$$

Due to this estimate, it follows from (6.1)–(6.3) that $(x_k)_{k=1}^m$ is a solution of (4.1), (4.2). ■

Proof of Theorem 4.2. From the inequalities (4.14) and (4.15), the inequalities (4.6) and (4.7) follow, where

$$
h_0(i) = \sum_{k=1}^m |g_k(0,\dots,0)(i)|, \quad r = \sum_{k=1}^m |\psi_k(0,\dots,0)|.
$$

Consequently, all the conditions of Theorem 4.1 are fulfilled which in fact ensures the solvability of the problem (4.1), (4.2).

Let $(x_k)_{k=1}^m$ and $(y_k)_{k=1}^m$ be arbitrary solutions of (4.1), (4.2). Putting

$$
z_k(i) = x_k(i) - y_k(i) \quad (k=1,\dots,m),
$$

because of (4.14) and (4.15) we obtain

$$
\begin{gathered}
\big[\Delta z_k(i-1) - h_k(i) z_k(\tau_k(i))\big] \operatorname{sign}\big[(\tau_k(i)-i_k) z_k(\tau_k(i))\big] \le \\
\le g_{0k}(|z_1|,\dots,|z_m|)(i) \quad \text{for} \quad i \in N_n \quad (k=1,\dots,m), \\
|z_k(i_k)| \le \psi_{0k}(|z_1|,\dots,|z_m|) \quad (k=1,\dots,m),
\end{gathered}
$$

whence, according to (4.8) and Lemma 5.2, it follows that $z_k(i) \equiv 0$ ($k = 1, \dots, m$). ■

Proof of Theorem 4.3. For any $k \in N_m$ and $z \in \widetilde{E}_n^+$, we assume

$$g_{0k}^0(z)(i) = g_{0k}(\delta_{1k}z, \dots, \delta_{mk}z)(i).$$

Because of (4.8),

$$(h_k, g_{0k}^0, 0) \in W_n(i_k) \quad (k = 1, \dots, m). \tag{6.7}$$

Let us arbitrarily take $(y_j)_{j=1}^m \in \widetilde{E}_n^m$, $k \in N_m$, and consider the Cauchy problem

$$\Delta x(i-1) = \widetilde{g}_k(x)(i), \quad x(i_k) = c_k, \tag{6.8}$$

where

$$\begin{aligned}\widetilde{g}_k(x)(i) &= g_k(y_1, \dots, y_{k-1}, x, y_{k+1}, \dots, y_m)(i),\\ c_k &= \psi_k(y_1, \dots, y_m).\end{aligned}$$

According to (4.14), in $\widetilde{E}_n$ the inequality

$$\begin{gathered}\big[\widetilde{g}_k(x)(i) - \widetilde{g}_k(y)(i) - h_k(i)\big(x(\tau_k(i) - y(\tau_k(i))\big)\big] \times\\ \times \operatorname{sign}\big[(\tau_k(i) - i_k)\big(x(\tau_k(i)) - y(\tau_k(i))\big)\big] \le g_{0k}^0(|x-y|)(i)\\ \text{for} \quad i \in N_n \quad (k = 1, \dots, m).\end{gathered}$$

is fulfilled. Since h_k and g_{0k}^0 satisfy (6.7), the unique solvability of (6.8) follows from Theorem 4.2.

It is clear from the above-said that given $(x_{k0})_{k=1}^m \in \widetilde{E}_n^m$, there exists a unique sequence of vector functions $(x_{k\nu})_{k=1}^m \in \widetilde{E}_n^M$ ($\nu = 1, 2, \dots$) such that for any natural ν and $k \in N_m$, the function $x_{k\nu}$ is the solution of the Cauchy problem (4.20), (4.21).

Let $(x_k)_{k=1}^m$ be the solution of the problem (4.1), (4.2), and let

$$y_{k\nu}(i) = x_{k\nu}(i) - x_k(i) \quad (k = 1, \dots, m).$$

Then, because of (4.14) and (4.15), for any $\nu \in N$ we will have

$$\begin{gathered}\big[\Delta y_{k\nu}(i-1) - h_k(i) y_{k\nu}(\tau_k(i))\big] \operatorname{sign}\big[(\tau_k(i) - i_k) y_{k\nu}(\tau_k(i))\big] \le\\ \le g_{0k}\big(|y_{1\,\nu-1}|, \dots, |y_{k-1\,\nu-1}|, |y_{k\nu}|, |y_{k+1\,\nu-1}|, \dots, |y_{m\,\nu-1}|\big)(i)\\ \text{for} \quad i \in \widetilde{N}_n \quad (k = 1, \dots, m),\\ |y_{k\nu}(i_k)| \le \psi_{0k}\big(|y_{1\,\nu-1}|, \dots, |y_{m\,\nu-1}|\big) \quad (k = 1, \dots, m).\end{gathered}$$

From these inequalities and (4.8), with regard for Lemma 5.5 we obtain

$$\sum_{k=1}^m \|y_{k\nu}\|_{\widetilde{E}_n} \le r_0 \gamma^\nu \quad (\nu = 1, 2, \dots),$$

where $\gamma \in]0,1[$ and $r_0 > 0$ do not depend on ν. Consequently, (4.23) and (4.22) are fulfilled. ■

Theorem 4.4 is proved in a similar way. The only difference is that instead of Lemma 5.5 we use Lemma 5.5$'$.

Corollaries 4.1–4.11 follow directly from Theorems 4.1–4.4 using Lemmas 5.6, 5.7 and 5.9.

CHAPTER III

§ 7. On the Class D_f

Let $f : C(I_0; R^m) \to L(I_0; R)$. Then

$$(f_n)_{n=1}^{+\infty} \in D_f \tag{7.1}$$

means that: (a) $f_n : \widetilde{E}_n^m \to \widetilde{E}_n$ is a continuous operator given $n \in N$; (b) for any $(u_k)_{k=1}^m \in C(I_0; R^m)$, the condition

$$\lim_{n\to+\infty} \sum_{i=1}^{n} \Big| f_n(x_{1n}, \dots, x_{mn})(i) - \int\limits_{t_{i-1n}}^{t_{in}} f(u_1, \dots, u_m)(t)dt \Big| = 0 \tag{7.2}$$

is fulfilled whenever $(x_{kn})_{k=1}^m \in \widetilde{E}_n^m$ $(n = 1, 2, \dots)$ and

$$\lim_{n\to+\infty} \|x_{kn} - p_n(u_k)\|_{\widetilde{E}_n} = 0 \quad (k = 1, \dots, m). \tag{7.3}$$

Remark 7.1. It is evident from Definition 7.1 that if (7.1) is fulfilled and $(g_n)_{n=1}^{+\infty} \in D_g$, then for any α and $\beta \in R$,

$$(\alpha f_n + \beta g_n)_{n=1}^{+\infty} \in D_{\alpha f + \beta g}.$$

In particular, if $(g_n)_{n=1}^{+\infty} \in D_0$, then from (7.1) it follows

$$(f_n + g_n)_{n=1}^{+\infty} \in D_f.$$

Remark 7.2. Let $f \in L(I_0; R)$ and $f_n \in E_n$ $(n = 1, 2, \dots)$. Then, according to Definition 7.1, (7.1) means that

$$\lim_{n=+\infty} \sum_{i=1}^{n} \Big| f_n(i) - \int\limits_{t_{i-1n}}^{t_{in}} f(t)dt \Big| = 0.$$

Let

$$f \in K\big(C(I_0; R^m); L(I_0; R)\big) \tag{7.4}$$

and

$$f_n(x_1, \dots, x_m)(i) =$$

$$= \int\limits_{t_{i-1n}}^{t_{in}} f(l_{i1n}(x_1, \dots, x_m), \dots, l_{imn}(x_1, \dots, x_m))(t)dt \tag{7.5}$$

$$\textit{for} \quad i \in N_n \quad (n = 1, 2, \dots),$$

where $l_{ikn} : \widetilde{E}_n^m \to C(I_0; R)$ $(i \in N_n;\ k \in N_m;\ n \in N)$ *are continuous operators such that*

$$\max\Big\{\big\|l_{ikn}(x_{1n},\dots,x_{mn}) - u_k\big\|_{C(I_0;R)} : i \in N_n\Big\} \to 0 \tag{7.6}$$
$$\text{as } n \to +\infty \ \ (k = 1,\dots,m)$$

whenever $(u_k)_{k=1}^m \in C(I_0; R^m)$ *and the sequence* $(x_{kn})_{k=1}^m$ $(n = 1, 2, \dots)$ *satisfies* (7.3). *Then* (7.1) *is fulfilled.*

Proof. Continuity of f_n $(n = 1, 2, \dots)$ is obvious. Let us prove that the condition (b) of Definition 7.1 is fulfilled. Let $(u_k)_{k=1}^m \in C(I_0; R^m)$ be arbitrarily fixed. For any $\gamma \in R_+$, we assume

$$\omega(t;\gamma) = \sup\Big\{\big|f(u_1,\dots,u_m)(t) - f(v_1,\dots,v_m)(t)\big| : (v_k)_{k=1}^m \in$$
$$\in C(I_0; R^m), \sum_{k=1}^m \|u_k - v_k\|_{C(I_0;R)} \le \gamma\Big\}.$$

By (7.4), we have $\omega(\cdot;\gamma) \in L(I_0; R)$ for any $\gamma \in R_+$, and

$$\lim_{\gamma\to 0}\int_a^b \omega(t;\gamma)dt = 0. \tag{7.7}$$

Consider an arbitrary sequence $(x_{kn})_{k=1}^m \in \widetilde{E}_n^m$ $(n = 1, 2, \dots)$ satisfying (7.3). Then, because of (7.5), we have

$$\sum_{i=1}^n \Big|f_n(x_{1n},\dots,x_{mn})(i) - \int_{t_{i-1n}}^{t_{in}} f(u_1,\dots,u_m)(t)dt\Big| \le$$
$$\le \sum_{i=1}^n \int_{t_{i-1n}}^{t_{in}} \Big|f\big(l_{i1n}(x_{1n},\dots,x_{mn}),\dots,l_{imn}(x_{1n},\dots,x_{mn})\big)(t) -$$
$$-f(u_1,\dots,u_m)(t)\Big|dt \le \int_a^b \omega(t;\gamma_n)dt \ \ (n = 1, 2, \dots), \tag{7.8}$$

where

$$\gamma_n = \max\Big\{\sum_{k=1}^m \big\|l_{ikn}(x_{1n},\dots,x_{mn}) - u_k\big\|_{C(I_0;R)} : i \in N_n\Big\}.$$

However, because of (7.6) and (7.7),

$$\lim_{n\to+\infty}\int_a^b \omega(t;\gamma_n)dt = 0.$$

Therefore (7.2) follows from (7.8). ■

Let

$$f \in C\big(C(I_0; R^m); C(I_0; R)\big), \tag{7.9}$$

$$f_n(x_1, \dots, x_m)(i) = \frac{b-a}{n} \sum_{j=1}^{j_0} \alpha_j f\big(l_{ij1n}(x_1, \dots, x_m), \dots, l_{ijmn}(x_1, \dots$$

$$\dots, x_m)\big)\Big(t_{in} - \frac{\beta_j}{n}\Big) \ \text{for } i \in N_n \ (n = 1, 2, \dots), \tag{7.10}$$

where $j_0 \in N$, $\alpha_j \in [0,1]$ and $\beta_j \in [0, b-a]$ not depend on n,

$$\sum_{j=1}^{j_0} \alpha_j = 1, \tag{7.11}$$

and $l_{ijkn} : \widetilde{E}_n^m \to C(I_0; R)$ $(i \in N_n;\ j \in N_{j_0};\ k \in N_m;\ n \in N)$ are continuous operators such that

$$\max\Big\{\big\|l_{ijkn}(x_{1n}, \dots, x_{mn}) - u_k\big\|_{C(I_0;R)} : i \in N_n\Big\} \to 0 \tag{7.12}$$
$$\text{as } n \to +\infty \ (k = 1, \dots, m; j = 1, 2, \dots)$$

whenever $(u_k)_{k=1}^m \in C(I_0; R^m)$ and the sequence $(x_{kn})_{k=1}^m$ $(n = 1, 2, \dots)$ satisfies (7.3). *Then* (7.1) *is fulfilled.*

Proof. Since l_{ijkn} $(i \in N_n;\ j \in N_{j_0},\ k \in N_m,\ n \in N)$ are continuous, it directly follows from (7.9) and (7.10) that $f_n : \widetilde{E}_n^m \to E_n$ is continuous for any $n \in N$.

Let $(u_k)_{k=1}^m \in C(I_0; R^m)$ and $(x_{kn})_{k=1}^m \in \widetilde{E}_n^m$ $(n = 1, 2, \dots)$ be a sequence satisfying (7.3). Assume

$$v(t) = f(u_1, \dots, u_m)(t)$$

and

$$v_{ijn}(t) = f(l_{ij1n}(x_{1n}, \dots, x_{mn}), \dots, l_{ijmn}(x_{1n}, \dots, x_{mn})(t).$$

Then, by (7.9) and (7.12),

$$\varepsilon_n = \max\Big\{\|v_{ijn} - v\|_{C(I_0;R)} : i \in N_n, j \in N_{j_0}\Big\} \to 0 \tag{7.13}$$
$$\text{as} \quad n \to +\infty.$$

For any $n \in N$ and $i \in N$, let us choose $\widetilde{t}_{in} \in [t_{i-1n}, t_{in}]$ such that

$$\int_{t_{i-1n}}^{t_{in}} v(t)dt = \frac{b-a}{n} v(\widetilde{t}_{in}).$$

Because of continuity of v, it is evident that

$$\widetilde{\varepsilon}_n = \max\left\{\left|v(\widetilde{t}_{in}) - v\left(t_{in} - \frac{\beta_j}{n}\right)\right| : i \in N_n, j \in N_{j_0}\right\} \to 0 \quad (7.14)$$
$$\text{as } \; n \to +\infty.$$

According to (7.10) and (7.11),

$$\sum_{i=1}^{n} \Big|f_n(x_{1n}, \dots, x_{mn})(i) - \int\limits_{t_{i-1n}}^{t_{in}} f(u_1, \dots, u_m)(t)dt\Big| =$$
$$= \frac{b-a}{n} \sum_{i=1}^{n} \Big| \sum_{j=1}^{j_0} \alpha_j \Big[v_{ijn}\Big(t_{in} - \frac{\beta_j}{n}\Big) - v(\widetilde{t}_{in})\Big]\Big| \leq$$
$$\leq \frac{b-a}{n} \sum_{i=1}^{n} \sum_{j=1}^{j_0} \alpha_j \Big[\Big|v_{ijn}\Big(t_{in} - \frac{\beta_j}{n}\Big) - v\Big(t_{in} - \frac{\beta_j}{n}\Big)\Big| +$$
$$+ \Big|v\Big(t_{in} - \frac{\beta_j}{n}\Big) - v(\widetilde{t}_{in})\Big|\Big] \leq (b-a)(\varepsilon_n + \widetilde{\varepsilon}_n) \quad (n = 1, 2, \dots),$$

whence, due to of (7.13) and (7.14), it follows (7.2). ■

Let

$$f(u_1, \dots, u_m)(t) \equiv g(t, u_1(t), \dots, u_m(t)), \qquad g \in K(I_0 \times R^m; R), \quad (7.15)$$

and

$$f_n(x_1, \dots, x_m)(i) =$$
$$= \int\limits_{t_{i-1n}}^{t_{in}} g\big(t, y_{i1n}(x_1, \dots, x_m), \dots, y_{imn}(x_1, \dots, x_m)\big)\, dt \quad (7.16)$$
$$\textit{for} \quad i \in N_n \quad (n = 1, 2, \dots),$$

where

$$y_{ikn}(x_1, \dots, x_m) = \sum_{\nu=1}^{\nu_k} \alpha_{k\nu} x_k(i - \beta_{k\nu}) + z_{ikn}(x_1, \dots, x_m), \quad (7.17)$$

the numbers $\nu_k \in N$, $\alpha_{k\nu} \in]0, 1]$ *and* $\beta_{k\nu} \in \{0, 1\}$ *do not depend on* n,

$$\sum_{\nu=1}^{\nu_k} \alpha_{k\nu} = 1 \quad (k = 1, \dots, m), \quad (7.18)$$

and $z_{ikn} : \widetilde{E}_n^m \to R$ $(i \in N_n; k \in N_m; n \in N)$ *are continuous functionals such that for any* $r > 0$,

$$\max\Big\{\big|z_{ikn}(x_1,\dots,x_m)\big| : \sum_{j=1}^{m} \|x_j\|_{\widetilde{E}_n} \le r, i \in N_n, k \in N_m\Big\} \to 0 \quad (7.19)$$
$$as \;\; n \to +\infty.$$

Then (7.1) *is fulfilled.*

Proof. For any $i \in N_n$, $k \in N_m$ and $n \in N$, we introduce the operators $\widetilde{y}_{ikn} : \widetilde{E}_n^m \to E_n$ and $l_{ikn} : \widetilde{E}_n^m \to C(I_0; R)$ by

$$\widetilde{y}_{ikn}(x_1,\dots,x_m)(j) = \begin{cases} y_{ikn}(x_1,\dots,x_m) & \text{for } \; j \in \{i-1, i\}, \\ x_k(j) & \text{for } \; j \notin \{i-1, i\}, \end{cases}$$
$$l_{ikn}(x_1,\dots,x_m)(t) = q_n(\widetilde{y}_{ikn}(x_1,\dots,x_m))(t) \;\text{ for }\; t \in I_0.$$

Obviously,

$$l_{ikn}(x_1,\dots,x_m)(t) = y_{ikn}(x_1,\dots,x_m) \qquad (7.20)$$
$$\text{for }\; t_{i-1n} \le t \le t_{in}.$$

Owing to (7.15), (7.16) and (7.20), the conditions (7.4) and (7.5) are fulfilled. On the other hand, it follows from (7.17)–(7.19) that l_{ikn} $(i \in N_n; k \in N_m; n \in N)$ satisfy (7.6) provided $(u_k)_{k=1}^m \in C(I_0; R^m)$, while the sequence $(x_{kn})_{k=1}^m$ $(n = 1, 2, \dots)$ satisfies (7.3).

Consequently, all the conditions of Lemma 7.1 are fulfilled, which exactly guarantees the fulfillment of (7.1). ■

Basing on Lemma 7.2 and repeating the arguments as in proving Lemma 7.3, we can prove.

Let

$$f(u_1,\dots,u_m)(t) = g(t, u_1(t),\dots,u_m(t)), \quad g \in C(I_0 \times R^m; R),$$
$$f_n(x_1,\dots,x_m)(i) =$$
$$= \frac{b-a}{n} \sum_{j=1}^{j_0} \alpha_j g\Big(t_{in} - \frac{\beta_j}{n}, y_{ij1n}(x_1,\dots,x_m),\dots,y_{ijmn}(x_1,\dots,x_m)\Big)$$
$$for \;\; i \in N_n \;\; (n = 1, 2, \dots),$$

where

$$y_{ijkn}(x_1,\dots,x_m) = \sum_{\nu=1}^{\nu_k} \alpha_{kj\nu} x_k(i - \beta_{kj\nu}) + z_{ijkn}(x_1,\dots,x_m),$$

the numbers $\alpha_j \in [0,1]$, $\beta_j \in [0, b-a]$, $j_0 \in N$, $\nu_k \in N$, $\alpha_{kj\nu} \in [0,1]$, $\beta_{kj\nu} \in \{0,1\}$ *do not depend on* n *and satisfy* (7.11) *and*

$$\sum_{\nu=1}^{\nu_k} \alpha_{kj\nu} = 1 \quad (k, j = 1, \dots, m),$$

while $z_{ijkn} : \widetilde{E}_n^m \to R$ $(i \in N_n; k \in N_m; j \in N_{j_0}, n \in N)$ *are continuous functionals such that for any* $r > 0$,

$$\max\Big\{ \big| z_{ijkn}(x_1, \dots, x_m) \big| : \sum_{p=1}^{m} \|x_p\|_{\widetilde{E}_n} \le r, i \in N_n; k \in N_m \Big\} \to 0$$
$$\text{as} \quad n \to +\infty \quad (j = 1, 2, \dots, j_0).$$

Then (7.1) *is fulfilled.*

§ 8. Lemmas on A Priori Estimates

Throughout this section, an interval $I_0 = [a, b]$ and points $t_k \in [a, b]$ $(k = 1, \dots, m)$ are assumed to be fixed. For any $n \in N$ and $k \in N_m$, there exists a unique $i_{kn} \in \widetilde{N}_n$ such that

$$t_{i_{kn} n} \le t_k < t_{i_{kn}+1 n}. \tag{8.1}$$

Assume

$$\tau_{kn}(i) = \begin{cases} i & \text{for } \; i > i_{kn}, \\ i-1 & \text{for } \; i \le i_{kn}. \end{cases} \tag{8.2}$$

Let

$$(h_1, \dots, h_m; f_{01}, \dots, f_{0m}; \varphi_{01}, \dots, \varphi_{0m}) \in W(t_1, \dots, t_m), \tag{8.3}$$
$$(h_{kn})_{n=1}^{+\infty} \in D_{h_k}, \quad (f_{0kn})_{n=1}^{+\infty} \in D_{f_{0k}} \;\; (k = 1, \dots, m), \tag{8.4}$$

where every $f_{0kn} : \widetilde{E}_n^m \to E_n$ *is a positively homogeneous continuous non-decreasing operator. Then there exist numbers* $n_0 \in N$ *and* $\rho \in]0, +\infty[$ *such that for any* $n > n_0$, $r \in R_+$ *and* $h_0 \in E_n^*$, *an arbitrary solution of the system of difference inequalities*

$$\begin{aligned} &\big[\Delta x_k(i-1) - h_{kn}(i) x_k(\tau_{kn}(i))\big] \operatorname{sign}\big[(\tau_{kn}(i) - i_{kn}) x_k(\tau_{kn}(i))\big] \le \\ &\quad \le h_0(i) + f_{0kn}(|x_1|, \dots, |x_m|)(i) \;\; \textit{for} \;\; i \in N_n \;\; (k = 1, \dots, m) \end{aligned} \tag{8.5}$$

under the boundary conditions

$$|x_k(i_{kn})| < r + \varphi_{0k}\big(q_n(|x_1|), \dots, q_n(|x_m|)\big) \quad (k = 1, \dots, m) \tag{8.6}$$

admits the estimate

$$\sum_{k=1}^{m} \|x_k\|_{\widetilde{E}_n} \le \rho \Big[r + \sum_{i=1}^{n} h_0(i) \Big]. \tag{8.7}$$

Proof. Suppose on the contrary that the lemma is invalid. Then there exist an increasing sequence of natural numbers $(n_\nu)_{\nu=1}^{+\infty}$ and sequences $r_\nu \in R_+$, $h_{0\nu} \in E_n^+$ and $(x_{k\nu})_{k=1}^m \in \widetilde{E}_{n_\nu}^m$ $(\nu = 1, 2, \dots)$ such that for any natural ν, the inequalities

$$\begin{gathered}\big[\Delta x_{k\nu}(i-1) - h_{kn_\nu}(i) x_{k\nu}(\tau_{kn_\nu}(i))\big] \operatorname{sign}\big[(\tau_{kn_\nu} - i_{kn_\nu}) \times \\ \times x_{k\nu}(\tau_{kn_\nu}(i))\big] \leq h_{0\nu}(i) + f_{0kn_\nu}(|x_{1\nu}|, \dots, |x_{m\nu}|)(i) \\ \text{for } \ i \in N_{n_\nu} \quad (k = 1, \dots, m), \\ |x_k(i_{kn_\nu}| \leq r_\nu + \varphi_{0k}\big(q_{n_\nu}(|x_{1\nu}|), \dots, q_{n_\nu}(|x_{m_\nu}|)\big) \quad (k = 1, \dots, m)\end{gathered}$$

and

$$\sum_{k=1}^m \|x_{k\nu}\|_{\widetilde{E}_{n_\nu}} > \nu\Big[r_\nu + \sum_{i=1}^{n_\nu} h_{0\nu}(i)\Big]$$

are fulfilled.

Because of (8.4), without restriction of generality we may assume that

$$\|h_{kn_\nu}\|_{\widetilde{E}_{n_\nu}} > 1 \quad (k = 1, \dots, m; \ \nu = 1, 2, \dots).$$

By Lemma 5.1, for any $k \in N_m$ and $\nu \in N$ we have

$$|x_{k\nu}(i)| \leq y_{k\nu}(i) \quad \text{for} \quad i \in N_{n_\nu},$$

where $y_{k\nu}$ is the solution of the Cauchy problem

$$\begin{gathered}\Delta y_{k\nu}(i-1) - h_{kn_\nu}(i) y_{k\nu}(\tau_{kn_\nu}(i)) = \\ = \big[h_{0\nu}(i) + f_{0kn_\nu}(|x_{1\nu}|, \dots, |x_{m\nu}|)(i)\big] \operatorname{sign}\big[\tau_{kn_\nu}(i) - i_{kn_\nu}\big], \\ y_{k\nu}(i_{kn_\nu}) = r_\nu + \varphi_{0k}\big(q_{n_\nu}(|x_{1\nu}|), \dots, q_{n_\nu}(|x_{m_\nu}|)\big).\end{gathered}$$

Thus

$$\begin{gathered}\big|\Delta y_{k\nu}(i-1) - h_{kn_\nu}(i) y_{k\nu}(\tau_{kn_\nu}(i))\big| \leq h_{0\nu}(i) + \\ + f_{0kn_\nu}(y_{1\nu}, \dots, y_{m\nu})(i) \ \text{ for } \ i \in N_{n_\nu}(k = 1, \dots, m; \nu = 1, 2, \dots), \quad (8.8) \\ y_{k\nu}(i_{kn_\nu}) \leq r_\nu + \varphi_{0k}\big(q_{n_\nu}(y_{1\nu}), \dots, q_{n_\nu}(y_{m_\nu})\big) \quad (8.9) \\ (k = 1, \dots, m; \nu = 1, 2, \dots),\end{gathered}$$

and

$$\rho_\nu = \sum_{k=1}^m \|y_{k\nu}\|_{\widetilde{E}_{n_\nu}} > \nu\Big[r_\nu + \sum_{i=1}^{n_\nu} h_{0\nu}(i)\Big] \quad (\nu = 1, 2, \dots). \tag{8.10}$$

Assume

$$z_{k\nu}(t) = q_{n_\nu}\Big(\frac{y_{k\nu}}{\rho_\nu}\Big)(t) \quad (k = 1, \dots, m),$$

$$h_{kn}^*(i) = \Big|h_{kn}(i) - \int_{t_{i-1n}}^{t_{in}} h_k(\tau)d\tau\Big| + \int_{t_{i-1n}}^{t_{in}} |h_k(\tau)|d\tau \ \ (k = 1, \dots, m),$$

$$g_\nu(t) = \frac{n_\nu}{(b-a)\rho_\nu} h_{0\nu}(i) + \frac{n_\nu}{(b-a)} \sum_{k=1}^{m} \big[h^*_{kn_\nu}(i) +$$
$$+ f_{0kn_\nu}(1,\dots,1)(i)\big] \;\text{ for }\; i \in N_{n_\nu}, \frac{i-1}{n_\nu} \le t < \frac{i}{n_\nu},$$

and

$$g(t) = \sum_{k=1}^{m} \big[|h_k(t)| + f_{0k}(1,\dots,1)(t)\big]. \tag{8.11}$$

By (8.4) and (8.10),

$$(h^*_{kn})_{n=1}^{+\infty} \in D_{|h_k|} \quad (k = 1,\dots,m) \tag{8.12}$$

and

$$\lim_{\nu\to+\infty} \int\limits_{\alpha}^{t} g_\nu(\tau)d\tau = \int\limits_{\alpha}^{t} g(\tau)d\tau \;\text{ uniformly on }\; [a,b]. \tag{8.13}$$

Taking into account that $|h_{kn}(i)| \le h^*_{kn}(i)$ for $i \in N_n$, from (8.8) we find

$$|z'_{k\nu}(t)| \le g_\nu(t) \quad \text{for} \quad t \in I_0; (k = 1,\dots,m; \nu = 1,2,\dots),$$

whence, because of (8.13), it follows the equicontinuity of the sequences $(z_{k\nu})_{\nu=1}^{+\infty}$ $(k = 1,\dots,m)$. On the other hand, by the definition of $z_{k\nu}$ $(k = 1,\dots,m)$,

$$\sum_{k=1}^{m} \|z_{k\nu}\|_{C(I_0;R)} = 1 \quad (\nu = 1,2,\dots). \tag{8.14}$$

Without restriction of generality, we may assume that $(z_{k\nu})_{\nu=1}^{+\infty}$ $(k = 1,\dots,m)$ converge uniformly. Owing to (8.14), the functions

$$z_k(t) = \lim_{\nu\to+\infty} z_{k\nu}(t) \quad (k = 1,\dots,m)$$

satisfy

$$\sum_{k=1}^{m} \|z_k\|_{C(I_0;R)} = 1. \tag{8.15}$$

By (8.8)–(8.10), for any natural ν we have

$$\big|z'_{k\nu}(t) - \widetilde{h}_{k\nu}(t) z_{k\nu}(t)\big| \le \widetilde{h}_{0\nu}(t) + \widetilde{f}_{0k\nu}(z_{1\nu},\dots,z_{m\nu})(t) \tag{8.16}$$
$$\text{for} \quad t \in I_0 \;\; (k = 1,\dots,m)$$

and

$$\big|z_{k\nu}(t_k)\big| \le \varepsilon_\nu + \frac{1}{\nu} + \varphi_{0k}(z_{1\nu}),\dots,z_{m_\nu}) \;\; (k = 1,\dots,m), \tag{8.17}$$

where

$$\varepsilon_\nu = \sum_{k=1}^m \max\Big\{\big|z_{k\nu}(t) - z_{k\nu}(s)\big| : a \le s < t \le b, t - s \le \frac{b-a}{n_\nu}\Big\},$$

$$\widetilde{h}_{0\nu}(t) = \frac{n_\nu}{b-a}\Big(\frac{h_{0\nu}(i)}{\rho_\nu} + \varepsilon_\nu \sum_{k=1}^m h^*_{kn_\nu}(i)\Big)$$

$$\text{for } \ t_{i-1n_\nu} \le t < t_{in_\nu}, \ i \in N_{n_\nu},$$

$$\widetilde{h}_{k\nu}(t) = \frac{n_\nu}{b-a} h_{kn_\nu}(i) \ \text{ for } \ t_{i-1n_\nu} \le t < t_{in_\nu}, \ i \in N_{n_\nu},$$

and

$$\widetilde{f}_{0k\nu}(z_{1\nu}, \dots, z_{m\nu})(t) = \frac{n_\nu}{b-a} f_{0kn_\nu}\big(p_{n_\nu}(z_{1\nu}), \dots, p_{n_\nu}(z_{m\nu})\big)(i)$$

$$\text{for } \ t_{i-1n_\nu} \le t < t_{in_\nu}, \ i \in N_{n_\nu}.$$

According to the uniform continuity of $(z_{k\nu})_{\nu=1}^{+\infty}$ $(k = 1, \dots, m)$ and because of (8.10) and (8.12), we have $\lim_{\nu \to +\infty} \varepsilon_\nu = 0$ and

$$\int_a^b \widetilde{h}_{0\nu}(t)dt \le \frac{1}{\nu} + \varepsilon_\nu \sum_{k=1}^m \sum_{i=1}^{n_\nu} h^*_{kn_\nu}(i) \to 0 \ \text{ as } \ \nu \to +\infty. \tag{8.18}$$

From (8.16) we have

$$z_{k\nu}(t) \le z_{k\nu}(t_k) \exp\Big(\int_{t_\nu}^t \widetilde{h}_{k\nu}(s)ds\Big) +$$

$$+ \Big|\int_{t_k}^t \exp\Big(\int_\tau^t \widetilde{h}_{k\nu}(s)ds\Big)\big[\widetilde{h}_{0\nu}(\tau) + \widetilde{f}_{0k\nu}(z_{1\nu}, \dots, z_{m\nu})(\tau)\big]d\tau\Big|$$

$$\text{for } \ t \in I_0 \ \ (k = 1, \dots, m; \ \nu = 1, 2, \dots).$$

Passing in these inequalities to limit as $\nu \to +\infty$ and taking into account (8.4) and (8.18), we obtain

$$z_k(t_k) = u_k(t_k), \ \ z_k(t) \le u_k(t) \ \text{ for } \ t \in I_0 \ \ (k = 1, \dots, m), \tag{8.19}$$

where

$$u_k(t) = z_k(t_k) \exp\Big(\int_{t_k}^t h_k(s)ds\Big) +$$

$$+ \Big|\int_{t_k}^t \exp\Big(\int_\tau^t h_k(s)ds\Big) f_{0k}(z_1, \dots, z_m)(\tau)d\tau\Big|.$$

On account of (8.19), from

$$u_k'(t) = h_k(t)u_k(t) + f_{0k}(z_1, \dots, z_m)(t)\,\mathrm{sign}(t - t_k) \quad (k = 1, \dots, m)$$

and (8.17) we find

$$\big|u_k'(t) - h_k(t)u_k(t)\big| \le f_{0k}(u_1, \dots, u_m)(t) \;\text{ for }\; t \in I_0, \tag{8.20}$$

$$0 \le u_k(t_k) \le \varphi_{0k}(u_1, \dots, u_m) \quad (k = 1, \dots, m), \tag{8.21}$$

whence by virtue of (8.3) it follows that $u_k(t) \equiv 0 \quad (k = 1, \dots, m)$. On the other hand, by (8.15) and (8.19) we have

$$\sum_{k=1}^{m} \|u_k\|_{C(I_0;R)} \ge 1.$$

The obtained contradiction proves the lemma. ■

From the lemma there immediately follows

′ *Let the conditions* (8.3), (8.4) *and*

$$\varphi_{0kn}(x_1, \dots, x_m) = \varphi_{0k}\big(q_n(|x_1|), \dots, q_n(|x_m|)\big) \quad (k = 1, \dots, m),$$

be fulfilled, where every $f_{0kn} : \widetilde{E}_n^m \to E_n$ *is a positively homogeneous continuous nondecreasing operator. Then there exists* $n_0 \in N$ *such that*

$$(h_{1n}, \dots, h_{mn}; f_{01n}, \dots, f_{0mn}; \varphi_{01n}, \dots, \varphi_{0mn}) \in w_n(i_{in}, \dots, i_{mn}) \quad \textit{for } n > n_0.$$

Let (8.3) *and* (8.4) *be fulfilled, where* $f_{0kn} : \widetilde{E}_n^m \to \widetilde{E}_n$ $(k = 1, \dots, m)$ *are positively homogeneous nondecreasing continuous operators, and* $n_0 \in N$ *is so large that*

$$\|h_{kn}\|_{E_n} < 1 \quad \textit{for} \quad n > n_0 \;\; (k = 1, \dots, m). \tag{8.22}$$

Then there exist $r > 0$ *and* $\gamma \in]0, 1[$ *such that for any sequences* $\alpha_n \in R_+$ $(n = n_0 + 1, \dots)$, $h_{0n} \in E_n^+$ $(n = n_0 + 1, \dots)$ *and* $x_{kn} \in \widetilde{E}_n$ $(k = 1, \dots, m; n = n_0 + 1, \dots)$ *satisfying*

$$\begin{aligned}\big[\Delta x_{kn}(i-1) - h_{kn}(i)x_{kn}(\tau_{kn}(i))\big]\,\mathrm{sign}\,\big[(\tau_{kn}(i) - i_{kn})x_{kn}(\tau_{kn}(i))\big] \le \\ \le h_{0n}(i) + f_{0kn}\big(|x_{1n-1}|, \dots, |x_{mn-1}|\big)(i) \\ \textit{for } n > n_0, \;\; i \in N_n \;\; (k = 1, \dots, m)^7\end{aligned} \tag{8.23}$$

and

$$\big|x_{kn}(i_{kn})\big| \le \alpha_n + \varphi_{0k}\big(q_{n-1}(|x_{1n-1}|), \dots, q_{n-1}(|x_{mn-1}|)\big) \tag{8.24}$$

[7] Since f_{0kn} are defined on $(\widetilde{E}_n^+)^m$, one has to determine the functions $x_{1n-1}, \dots, x_{mn-1}$ at the point n. Here and in similar situations encountered below, we assume that $x_{kn-1}(n) = x_{kn-1}(n-1)$ $(k = 1, \dots, m)$.

$$\textit{for } n > n_0 \ \ (k = 1, \dots, m),$$

the estimates

$$\sum_{k=1}^{m} \|x_{kn}\|_{\widetilde{E}_n} \le \tau\gamma^n \Big[\sum_{\nu=n_0+1}^{n} \gamma^{-\nu} \Big(\sum_{i=1}^{\nu} h_{0\nu}(i) + \alpha_\nu \Big) + \\ + \sum_{k=1}^{m} \|x_{kn_0}\|_{\widetilde{E}_{n_0}} \Big] \quad (n = n_0 + 1, \dots) \tag{8.25}$$

are valid.

To prove this lemma, we will need the following

Let

$$(h_1, \dots, h_m; \widetilde{f}_{01}, \dots, \widetilde{f}_{0m}; \widetilde{\varphi}_{01}, \dots, \widetilde{\varphi}_{0m}) \in W(t_1, \dots, t_m), \tag{8.26}$$

$$(h_{kn})_{n=1}^{+\infty} \in D_{h_k} \ \ \textit{and} \ \ (\widetilde{f}_{0kn})_{n=1}^{+\infty} \in D_{\widetilde{f}_{0k}} \ \ (k = 1, \dots, m), \tag{8.27}$$

where $\widetilde{f}_{0kn} : (\widetilde{E}_n^+)^m \to \widetilde{E}_n^+$ *are positively homogeneous nondecreasing continuous operators, and* $n_0 \in N$ *is so large that* (8.22) *is fulfilled. Then there exists a positive constant* r *such that for any sequence* $h_{0n} \in E_n^+$ $(n = n_0 + 1, \dots)$ *satisfying*

$$\sum_{i=1}^{n} \widetilde{h}_{0n}(i) \le 1 \ \ \textit{for } i \in N_n \ \ (n = n_0 + 1, \dots), \tag{8.28}$$

we have

$$\sum_{k=1}^{m} \|y_{kn}\|_{\widetilde{E}_n} \le r \quad (n = n_0 + 1, \dots), \tag{8.29}$$

where $y_{kn_0}(i) \equiv 1$ $(k = 1, \dots, m)$ *and every* $y_{kn} \in \widetilde{E}_n^+$ $(k = 1, \dots, m;$ $n > n_0)$ *is the solution of the difference equation*

$$\Delta y_{kn}(i-1) - h_{kn}(i) y_{kn}(\tau_{kn}(i)) = \\ = \big[\widetilde{h}_{0n}(i) + \widetilde{f}_{0kn}(y_{1n-1}, \dots, y_{mn-1})(i)\big] \operatorname{sign}\big[\tau_{kn}(i) - i_{kn}\big] \tag{8.30}$$

[8]

under the initial conditions

$$y_{kn}(i_{kn}) = \widetilde{\varphi}_{0k}\big(q_{n-1}(y_{1n-1}), \dots, q_{n-1}(y_{mn-1})\big) + 1. \tag{8.31}$$

---

[8] We assume that $y_{kn-1}(n) = y_{kn-1}(n-1)$ $(k = 1, \dots, m)$.

Proof. Denote by H_n the set of all $\widetilde{h}_{0n} \in E_n^+$ satisfying (8.28).

Owing to (8.22), for any natural $n > n_0$ and any

$$\widetilde{h}_{0n_0+1} \in H_{n_0+1}, \dots, \widetilde{h}_{0n} \in H_n$$

there exists a unique $(y_{kn})_{k=1}^m \in (\widetilde{E}_n^+)^m$ constructed by the way indicated in the lemma. It can be easily seen that

$$0 < \rho_n = \sup\Big\{ \sum_{k=1}^m \|y_{kn}\|_{\widetilde{E}_n} : \widetilde{h}_{0n_0+1} \in H_{n_0+1}, \dots, \widetilde{h}_{0n} \in H_n \Big\} < +\infty$$
$$(n = n_0+1, n_0+2, \dots).$$

Our aim is to prove that the sequence $(\rho_n)_{n=n_0+1}^{+\infty}$ is bounded. Suppose on the contrary that

$$\lim_{n\to+\infty} \rho_n^* = +\infty, \tag{8.32}$$

where

$$\rho_n^* = \max\{\rho_{n_0}, \dots, \rho_n\}.$$

Assume

$$z_{kn}(t) = \frac{1}{\rho_n^*} q_n(y_{kn})(t) \quad (k = 1, \dots, m)$$

and

$$Z_{kn} = \{z_{kn} : \widetilde{h}_{0n_0+1} \in H_{n_0+1}, \dots, \widetilde{h}_{0n} \in H_n\}.$$

Because of (8.30), for any $n > n_0$, $k \in (1, \dots, m)$ and $z_{kn} \in Z_{kn}$ we have

$$|z'_{kn}(t)| \le g_n(t) + \frac{g_{0n}(t)}{\rho_n^*} \quad \text{for} \quad t \in I_0, \tag{8.33}$$

where

$$g_{0n}(t) = \frac{n}{b-a} \widetilde{h}_{0n}(i) \ \text{ for } \ t_{i-1n} \le t < t_{in}, \ i \in N_n,$$
$$g_n(t) = g_{1n}(t) + \frac{n}{b-a} \sum_{k=1}^m \widetilde{f}_{0kn}(1, \dots, 1)(i)$$
$$\text{for } \ t_{i-1n} \le t < t_{in}, \ i \in N_n,$$

and

$$g_{1n}(t) = \frac{n}{b-a} \sum_{k=1}^m \Big[\Big| h_{kn}(i) - \int_{t_{i-1n}}^{t_{in}} h_k(\tau)d\tau \Big| + \int_{t_{i-1n}}^{t_{in}} \big|h_k(\tau)\big| d\tau \Big]$$
$$\text{for } \ t_{i-1n} \le t < t_{in}, \ i \in N_n.$$

According to (8.27) and (8.32),

$$\lim_{n\to+\infty} \int_a^t g_m(\tau)d\tau = \sum_{k=1}^m \int_a^t |h_k(\tau)| d\tau \ \text{ uniformly on } \ I_0 \tag{8.34}$$

and

$$\varepsilon_n = \max\Big\{\Big|\int_s^t [g_n(\tau) - g(\tau)]d\tau\Big| : a \le s < t \le b\Big\} +$$

$$+\frac{1}{\rho_n^*} \to 0 \quad \text{as} \quad n \to +\infty, \tag{8.35}$$

where

$$g(t) = \sum_{k=1}^m \Big[|h_k(t)| + \widetilde{f}_{0k}(1,\dots,1)(t)\Big].$$

Due to the definition of z_{kn} and because of (8.28), (8.32) and (8.33), we have

$$0 < z_{kn}(t) \le 1, \;\; \big|z_{kn}(t) - z_{kn}(s)\big| \le l_n(t-s) \tag{8.36}$$
$$\text{for} \;\; a \le s \le t \le b,$$

$$\big|z_{kn}(t) - z_{kn}(s)\big| \le \int_s^t g(\tau)d\tau + \varepsilon_n \;\; \text{for} \;\; a \le s \le t \le b \tag{8.37}$$

and

$$\big|z_{kn}(t) - z_{kn}(s)\big| \le +\varepsilon_n^* \;\; \text{for} \;\; a \le s \le t \le b, \;\; t - s \le \frac{b-a}{n}, \tag{8.38}$$

where

$$l_n = \sup\big\{g_n(t) : t \in I_0\big\} + \frac{n}{(b-a)\rho_n^*}$$

and

$$\varepsilon_n^* = \max\Big\{\int_s^t g(\tau)d\tau : a \le s \le t \le b, \; t - s \le \frac{b-a}{n}\Big\} +$$

$$+\varepsilon_n \to 0 \quad \text{as} \quad n \to 0. \tag{8.39}$$

By virtue of (8.36), Z_{kn} is a set of uniformly bounded equicontinuous functions for any $k \in \{1,\dots,m\}$ and $n > n_0$. Therefore the functions

$$z_{kn}^*(t) = \sup\big\{z_{kn}(t) : z_{kn} \in Z_{kn}\big\} \;\; (k = 1,\dots,m) \tag{8.40}$$

are continuous. On the other hand, by (8.37)

$$\big|z_{kn}^*(t) - z_{kn}^*(s)\big| \le \int_s^t g(\tau)d\tau + \varepsilon_n$$

$$\text{for} \;\; a \le s \le t \le b \;\; (k = 1,\dots,m; n = n_0 + 1,\dots),$$

whence, because of (8.35), it follows the equicontinuity of the sequences $(z^*_{kn})_{n=n_0+1}^{+\infty}$ $(k=1,\dots,m)$. Hence the functions

$$z^*_{kn}(t)=\lim_{n\to+\infty}\sup z^*_{kn}(t)\quad (k=1,\dots,m) \tag{8.41}$$

are continuous. It easily follows from (8.32) and (8.41) that

$$\max\Big\{\sum_{k=1}^m z^*_k(t): t\in I_0\Big\}=1. \tag{8.42}$$

Taking into account both monotonicity of the sequence $(\rho^*_n)_{n=n_0+1}^{+\infty}$ and the condition (8.38), from (8.30) and (8.31) we obtain

$$\begin{gathered}\big|z'_{kn}(t)-h^*_{kn}(t)z_{kn}(t)\big|\le h^*_{0n}(t)+f_{0kn}(z_{1n-1},\dots,z_{mn-1})(t)\\ \text{for }\ t\in I_0\ \ (k=1,\dots,m;n=n_0+1,\dots)\end{gathered}$$

and

$$\begin{gathered}\big|z_{kn}(t_k)\big|\le\varepsilon^*_n+\frac{1}{\rho^*_n}+\widetilde{\varphi}_{0k}(z_{1n-1},\dots,z_{mn-1})\\ (k=1,\dots,m;n=n_0+1,\dots),\end{gathered}$$

where

$$\begin{gathered}h^*_{0n}(t)=\frac{g_{0n}(t)}{\rho^*_n}+\varepsilon^*_n g_{1n}(t),\\ h^*_{kn}(t)=\frac{n}{b-a}h_{kn}(i)\ \text{ for }\ t_{i-1n}\le t<t_{in},\ i\in N_n\end{gathered}$$

and

$$\begin{gathered}f^*_{okn}(z_{1n-1},\dots,z_{mn-1})(t)=\\ =\frac{n}{b-a}\widetilde{f}_{0kn}\big(p_{n-1}(z_{1n-1}),\dots,p_{n-1}(z_{mn-1})\big)(i)\\ \text{for }\ \ t_{i-1n}\le t<t_{in},\ i\in N_n.\end{gathered}$$

From these inequalities, by means of (8.28) and (8.4) it follows that

$$\begin{gathered}0<z^*_{kn}(t)\le\\ \le\Big|\int_{t_k}^{t}\exp\Big(\int_s^t h^*_{kn}(\tau)d\tau\Big)f^*_{0kn}(z^*_{1n-1},\dots,z^*_{mn-1})(s)ds\Big|+\\ +\eta_n+z^*_{kn}(t_k)\exp\Big(\int_{t_k}^t h^*_{kn}(\tau)d\tau\Big) \qquad (8.43)\\ \text{for }\ t\in I_0\ \ (k=1,\dots,m;n=n_0+1,\dots)\end{gathered}$$

and

$$z_{kn}^*(t_k) \le \varepsilon_n^* + \frac{1}{\rho_n^*} + \varphi_{0k}(z_{1n-1}^*, \dots, z_{mn-1}^*) \tag{8.44}$$
$$(k = 1, \dots, m; n = n_0 + 1, \dots),$$

where

$$\eta_n = \Big(\frac{1}{\rho_n^*} + \varepsilon_n^* \int_a^b g_{1n}(t)dt\Big) \max\Big\{ \exp\Big[\int_s^t h_{kn}^*(\tau)d\tau\Big] : a \le s \le t \le b\Big\}.$$

By (8.4), (8.32), (8.34) and (8.39),

$$\lim_{n\to+\infty} \int_a^t h_{kn}^*(\tau)d\tau = \int_a^t h_k(\tau)d\tau \ \text{ uniformly on } \ I_0$$
$$\text{and} \quad \lim_{n\to+\infty} \eta_n = 0. \tag{8.45}$$

Let ε be an arbitrarily small positive number. Because of the uniformity of (8.41), there exists a natural $n_\varepsilon > n_0$ such that

$$z_{kn}^*(t) \le z_k^*(t) + \varepsilon \quad \text{for} \ \ t \in I_0 \ \ (k = 1, \dots, m; \ n > n_\varepsilon).$$

Therefore, from (8.43) and (8.44) we find

$$0 < z_{kn}^*(t) \le \eta_n + (z_k^*(t_k) + \varepsilon) \exp\Big(\int_{t_k}^t h_{kn}^*(\tau)d\tau\Big) +$$
$$+\Big|\int_{t_k}^t \exp\Big(\int_s^t h_{kn}^*(\tau)d\tau\Big) f_{0kn}^*(z_1^* + \varepsilon, \dots, z_m^* + \varepsilon)(s)ds\Big|$$
$$\text{for} \ \ t \in I_0, \quad (k = 1, \dots, m; n > n_\varepsilon) \tag{8.46}$$

and

$$z_{kn}^*(t_k) \le \varepsilon_n + \frac{1}{\rho_n^*} + \widetilde{\varphi}_{0k}(z_1^* + \varepsilon, \dots, z_m^* + \varepsilon) \tag{8.47}$$
$$(k = 1, \dots, m; n > n_\varepsilon).$$

According to (8.27), uniformly on I_0 we have

$$\lim_{n\to+\infty} \int_{t_k}^t \exp\Big(\int_s^t h_{kn}^*(\tau)d\tau\Big) f_{0kn}^*(z_1^* + \varepsilon, \dots, z_m^* + \varepsilon)(s)ds =$$
$$= \int_{t_k}^t \exp\Big(\int_s^t h_k(\tau)d\tau\Big) \widetilde{f}_{0k}(z_1^* + \varepsilon, \dots, z_m^* + \varepsilon)(s)ds \tag{8.48}$$
$$(k = 1, \dots, m).$$

On account of (8.32), (8.39), (8.45) and (8.48), it follows from (8.46) and (8.47) that

$$0 \le z_k^*(t) \le z_k^*(t_k) \exp\Big(\int\limits_{t_k}^{t} h_k(\tau)d\tau\Big) +$$

$$+\Big|\int\limits_{t_k}^{t} \exp\Big(\int\limits_{s}^{t} h_k(\tau)d\tau\Big) \widetilde{f}_{0k}(z_1^* + \varepsilon, \dots, z_m^* + \varepsilon)(s)ds\Big|$$

$$\text{for } \ t \in I_0 \ (k = 1, \dots, m),$$

$$z_k(t_k) \le \widetilde{\varphi}_{0k}(z_1^* + \varepsilon, \dots, z_m^* + \varepsilon) \quad (k = 1, \dots, m),$$

whence, because of the arbitrariness of ε, we find

$$0 \le z_k^*(t) \le u_k(t) \ \text{ for } \ t \in I_0 \ (k = 1, \dots, m) \tag{8.49}$$

and

$$z_k^*(t_k) \le \widetilde{\varphi}_{0k}(z_1^*, \dots, z_m^*) \quad (k = 1, \dots, m),$$

where

$$u_k(t) = z_k^*(t_k) \exp\Big(\int\limits_{t_k}^{t} h_k(\tau)d\tau\Big) +$$

$$+\Big|\int\limits_{t_k}^{t} \exp\Big(\int\limits_{s}^{t} h_k(\tau)d\tau\Big) \widetilde{f}_{0k}(z_1^*, \dots, z_m^*)(s)ds\Big| \ \ (k = 1, \dots, m).$$

Therefore

$$|u_k'(t) - h_k(t)u_k(t)| \le \widetilde{f}_{0k}(u_1, \dots, u_m)(t) \ \text{ for } \ t \in I_0 \ (k = 1, \dots, m)$$

and

$$0 \le u_k(t_k) \le \widetilde{\varphi}_{0k}(u_1, \dots, u_m) \ \ (k = 1, \dots, m).$$

From these inequalities, owing to (8.3) we have

$$u_k(t) \equiv 0 \quad (k = 1, \dots, m).$$

But this is impossible because of (8.42) and (8.49). The obtained contradiction proves the lemma. ■

Proof of Lemma 8.2. By Lemma 2.2, from (8.3) and (8.4) it follows the existence of $\gamma \in]0,1[$ such that (8.26) and (8.27) are fulfilled, where

$$\widetilde{f}_{0k}(u_1, \dots, u_m)(t) = \frac{1}{\gamma} f_{0k}(u_1, \dots, u_m)(t),$$

$$\widetilde{\varphi}_{0k}(u_1, \dots, u_m) = \frac{1}{\gamma} \varphi_{0k}(u_1, \dots, u_m) \ \ (k = 1, \dots, m),$$

and

$$\widetilde{f}_{0kn}(x_1,\dots,x_m)(i)=\frac{1}{\gamma}f_{0kn}(x_1,\dots,x_m)(i)\ \ (k=1,\dots,m).$$

For h_k, h_{kn}, $\widetilde{f}_{0k}$, $\widetilde{f}_{0kn}$ and $\widetilde{\varphi}_{0k}$ $(k=1,\dots,m)$, let us choose the number r in accordance with Lemma 8.3.

Assume

$$\zeta_{n_0}(\gamma)\equiv\sum_{k=1}^{m}\|x_{kn_0}\|_{\widetilde{E}_{n_0}},\quad\text{and}\quad\zeta_n(\gamma)=$$

$$=\sum_{\nu=n_0+1}^{n}\gamma^{-\nu}\Big(\sum_{i=1}^{\nu}h_{0\nu}(i)+\alpha_\nu\Big)+\sum_{k=1}^{m}\|x_{kn_0}\|_{\widetilde{E}_{n_0}}\ \ \text{for}\ \ n>n_0.$$

Obviously,

$$\zeta_n(\gamma)\geq\zeta_{n-1}(\gamma)\ \ \text{for}\ \ n>n_0,$$

and the functions

$$\widetilde{h}_{0n}(i)=\begin{cases}\dfrac{h_{0n}(i)}{\gamma^n\zeta_n(\gamma)} & \text{for}\ \ \zeta_n(\gamma)\neq 0\\ 0 & \text{for}\ \ \zeta_n(\gamma)=0\end{cases}\quad(n=n_0+1,\dots)$$

satisfy (8.28). On the other hand, if for some $n>n_0$

$$\zeta_n(\gamma)=0,$$

then, because of (8.22)–(8.24),

$$x_{k\nu}(i)\equiv 0\quad(k=1,\dots,m;\ \nu=n_0,n_0+1,\dots,n).$$

Let

$$\widetilde{x}_{kn}(i)=\begin{cases}\dfrac{x_{kn}(i)}{\gamma^n\zeta_n(\gamma)} & \text{for}\ \ \zeta_n(\gamma)\neq 0,\\ 0 & \text{for}\ \ \zeta_n(\gamma)=0.\end{cases}\tag{8.50}$$

According to the above-said, from (8.23) and (8.24) we find

$$\begin{gathered}\big[\Delta\widetilde{x}_{kn}(i-1)-h_{kn}(i)\widetilde{x}_{kn}(\tau_{kn}(i))\big]\operatorname{sign}\big[(\tau_{kn}(i)-\\ -i_{kn})\widetilde{x}_{kn}(\tau_{kn}(i))\big]\leq\widetilde{h}_{0n}(i)+\widetilde{f}_{0kn}(|\widetilde{x}_{1n-1}|,\dots,|\widetilde{x}_{mn-1}|)(i)\\ \text{for}\ \ i\in N_n\ \ (k=1,\dots,m)\end{gathered}$$

and

$$\begin{gathered}|x_{kn}(i_{kn})|\leq 1+\widetilde{\varphi}_{0k}\big(q_{n-1}(|\widetilde{x}_{1n-1}|),\dots,q_{n-1}(|\widetilde{x}_{mn-1}|)\big)\\ (k=1,\dots,m;n>n_0).\end{gathered}$$

Let $(y_{kn})_{n=n_0+1}^{+\infty}$ $(k=1,\dots,m)$ be the sequences of the functions appearing in Lemma 8.3. By Lemma 5.1,

$$|\widetilde{x}_{kn}(i)|\leq y_{kn}(i)\ \ \text{for}\ \ i\in\widetilde{N}_n\ \ (k=1,\dots,m;\ n>n_0).$$

From this, owing to (8.29) and (8.50) we get (8.25). ■

§ 9. Difference Schemes of the Type (0.16), (0.17)

Consider the differential boundary value problem

$$\frac{du_k(t)}{dt} = f_k(u_1,\dots,u_m)(t) \quad (k=1,\dots,m), \tag{9.1}$$

$$u_k(t_k) = \varphi_k(u_1,\dots,u_m) \quad (k=1,\dots,m) \tag{9.2}$$

and its difference analogue

$$\Delta x_k(i-1) = f_{kn}(x_1,\dots,x_m)(i) \quad (k=1,\dots,m), \tag{9.3}$$

$$x_k(i_{kn}) = \varphi_{kn}(x_1,\dots,x_m) \quad (k=1,\dots,m), \tag{9.4}$$

where

$$f_k \in K\big(C(I_0;R^m);L(I_0;R)\big) \quad (k=1,\dots,m), \tag{9.5}$$

$$(f_{kn})_{n=1}^{+\infty} \in D_{f_k} \quad (k=1,\dots,m), \tag{9.6$_1$}$$

and the functionals $\varphi_{kn} : \widetilde{E}_n^m \to R$ $(k=1,\dots,m;\ n=1,2,\dots)$ are continuous. For any $(u_k)_{k=1}^m \in C(I_0;R^m)$, we have

$$\lim_{n\to+\infty} \varphi_{kn}(x_{1n},\dots,x_{mn}) = \varphi_k(u_1,\dots,u_m) \ (k=1,\dots,m) \tag{9.6$_2$}$$

whenever

$$\lim_{n\to+\infty} \|x_{kn} - p_n(u_k)\|_{\widetilde{E}_n} = 0.$$

Let the problem (9.1), (9.2) *have a unique solution* $(u_k^0)_{k=1}^m$, *and let in* $C(I_0;R^m)$ *the inequalities*

$$\begin{aligned}\big[f_k(u_1,\dots,u_m)(t) - h_k(t)u_k(t)\big] \operatorname{sign}\big[(t-t_k)u_k(t)\big] \le \\ \le h_0(t) + f_{0k}(|u_1|,\dots,|u_m|)(t) \ \text{ for } \ t\in I_0 \ (k=1,\dots,m),\end{aligned} \tag{9.7}$$

$$\big|\varphi_k(u_1,\dots,u_m)\big| \le r_0 + \varphi_{0k}(|u_1|,\dots,|u_m|) \ (k=1,\dots,m), \tag{9.8}$$

be fulfilled, where $r_0 \in R_+$, $h_0 \in L(I_0;R_+)$ *and*

$$(h_1,\dots,h_m;f_{01},\dots,f_{0m};\varphi_{01},\dots,\varphi_{0m}) \in W(t_1,\dots,t_m). \tag{9.9}$$

Moreover, let for any $\rho > 0$ *the conditions*

$$\begin{gathered}\sum_{k=1}^m \big|\varphi_{kn}(x_1,\dots,x_m)\big| \le \eta_\rho, \\ \sum_{k=1}^m \big|f_{kn}(x_1,\dots,x_m)(i)\big| \le f_{\rho n}^*(i) \ \text{ for } \ i\in N_n, \\ \sum_{k=1}^m \|x_k\|_{\widetilde{E}_n} \le m\rho \ \ (n=1,2,\dots)\end{gathered} \tag{9.10}$$

hold, where $\eta_\rho \in R_+$ *and*

$$(f^*_{\rho n}) \in D_{f^*_\rho}, \quad f^*_\rho \in L(I_0; R_+). \tag{9.11}$$

Then, given $r > 0$*, there exists a natural number* $n_0 = n_0(r)$ *such that for every* $n > n_0$*, the set* $X_n(u^0_1, \dots, u^0_m; r)$ *of the solutions of* (9.3), (9.4) *satisfying*

$$\sum_{k=1}^{m} \|x_k - p_n(u^0_k)\|_{\widetilde{E}_n} < r \tag{9.12}$$

is non-empty and

$$\sup\Big\{ \sum_{k=1}^{m} \|x_k - p_n(u^0_k)\|_{\widetilde{E}_n} : (x_k)_{k=1}^m \in X_n(u^0_1, \dots, u^0_m; r)\Big\} \to 0 \tag{9.13}$$
$$\text{as} \quad n \to +\infty.$$

Proof. By Lemma 2.1, (9.9) ensures the existence of a positive ρ_0 such that every solution of

$$\begin{gathered}\big[u'_k(t) - h_k(t)u_k(t)\big] \operatorname{sign}\big[(t - t_k)u_k(t)\big] \le \\ \le h_0(t) + f_{0k}(|u_1|, \dots, |u_m|)(t) \ \text{ for } \ t \in I_0 \ (k = 1, \dots, m),\end{gathered} \tag{9.14}$$

$$\big|u_k(t_k)\big| \le r_0 + \varphi_{0k}(|u_1|, \dots, |u_m|) \ \ (k = 1, \dots, m) \tag{9.15}$$

admits the estimate

$$\sum_{k=1}^{m} |u_k(t)| \le \rho_0 \quad \text{for} \quad t \in I_0. \tag{9.16}$$

Let $r > 0$, $\rho = \rho_0 + r$, and define $\chi : R \to R$ by

$$\chi(s) = \begin{cases} s & \text{for} \quad |s| \le \rho, \\ \rho \operatorname{sign}(s) & \text{for} \quad |s| > \rho. \end{cases}$$

Assume

$$h_{kn}(i) = \int\limits_{t_{i-1n}}^{t_{in}} h_k(\tau) d\tau,$$

$$\widetilde{f}_{kn}(x_1, \dots, x_m)(i) = h_{kn}(i) x_k(\tau_{kn}(i)) +$$
$$+ f_{kn}(\chi(x_1), \dots, \chi(x_m))(i) - h_{kn}(i)\chi(x_k)(\tau_{kn}(i)),^{9}$$
$$\widetilde{\varphi}_{kn}(x_1, \dots, x_m) = \varphi_{kn}(\chi(x_1), \dots, \chi(x_m)),$$
$$\widetilde{f}_k(u_1, \dots, u_m)(t) = h_k(t)u_k(t) + f_k(\chi(u_1), \dots, \chi(u_m))(t) - h_k(t)\chi(u_k)(t),$$
$$\widetilde{\varphi}_k(u_1, \dots, u_m) = \varphi_k(\chi(u_1), \dots, \chi(u_m)).$$

[9] Under $\chi(x_k)$ we mean composition of the functions x_k and χ.

Because of (9.10) and (9.11), for any natural n the inequalities

$$\sum_{k=1}^{m} \left|\widetilde{f}_{kn}(x_1,\dots,x_m)(i) - h_{kn}(i)x_k(\tau_{kn}(i))\right| \le \overline{f}_{\rho n}(i) \tag{9.17}$$
$$\text{for } \ i \in N_n$$

and

$$\sum_{k=1}^{m} \left|\widetilde{\varphi}_{kn}(x_1,\dots,x_m)\right| \le \eta_\rho \tag{9.18}$$

are fulfilled on $\widetilde{E}_n^m$, where

$$(\overline{f}_{\rho n})_{n=1}^{+\infty} \in D_{\overline{f}_\rho}, \ \text{ and } \ \overline{f}_\rho = f_\rho^* + \rho \sum_{k=1}^{m} |h_k| \in L(I_0;R). \tag{9.19}$$

It is clear from (9.9) that

$$(h_1,\dots,h_m;0,\dots,0;0,\dots,0) \in W(t_1,\dots,t_m). \tag{9.20}$$

From this, by virtue of Lemma 8.1$'$ it follows the existence of a natural n' such that

$$(h_{1n},\dots,h_{mn};0,\dots,0;0,\dots,0) \in W_n(t_{1n},\dots,t_{mn}) \ \text{ for } \ n > n'. \tag{9.21}$$

By Theorem 4.1 the conditions (9.17), (9.18) and (9.21) ensure the solvability of the boundary value problem

$$\Delta x_k(i-1) = \widetilde{f}_{kn}(x_1,\dots,x_m)(i) \ \ (k=1,\dots,m), \tag{9.22}$$
$$x_k(i_{kn}) = \varphi_{kn}(x_1,\dots,x_m) \ \ (k=1,\dots,m) \tag{9.23}$$

for any $n > n'$. Denote by $\widetilde{X}_n$ the set of all solutions of (9.22), (9.23), and show that

$$\sup\left\{ \sum_{k=1}^{m} \|x_k - p_n(u_k^0)\|_{\widetilde{E}_n} : (x_k)_{k=1}^m \in \widetilde{X}_n \right\} \to 0 \ \text{ as } \ n \to +\infty. \tag{9.24}$$

Assume the contrary. Then there exist a positive ε and an increasing sequence of natural numbers $(n_\nu)_{\nu=1}^{+\infty}$ such that for any $n = n_\nu$, the problem (9.22), (9.23) has a solution $(x_{k\nu})_{k=1}^m$ satisfying

$$\sum_{k=1}^{m} \|u_{k\nu} - u_k^0\|_{C(I_0;R)} > \varepsilon, \tag{9.25}$$

where

$$u_{k\nu}(t) = q_{n_\nu}(x_{k\nu})(t) \quad (k=1,\dots,m).$$

By (9.17)–(9.20) and Lemma 8.1, there exists a positive number ρ_1 such that

$$\sum_{k=1}^{m} |u_{k\nu}(t)| \le \rho_1 \ \text{ for } \ t \in I_0 \ \ (\nu = 1, 2, \dots).$$

Because of this estimate and (9.17) and (9.19), we have

$$|u'_{k\nu}(t)| \le g_\nu(t) \quad \text{for} \quad t \in I_0,$$

where

$$g_\nu(t) = \frac{n_\nu}{b-a}\Big[\rho_1 \sum_{k=1}^{m} \int\limits_{t_{i-1 n_\nu}}^{t_{i n_\nu}} |h_k(\tau)| d\tau + \overline{f}_{\rho n_\nu}(i)\Big]$$
$$\text{for} \quad t_{i-1 n_\nu} \le t < t_{i n_\nu}, \ i \in N_{n_\nu}.$$

Moreover,

$$\lim_{\nu \to +\infty} \int\limits_a^t g_\nu(\tau) d\tau = \int\limits_a^t \Big[f^*_\rho(\tau) + (\rho_1 + \rho) \sum_{k=1}^{m} |h_k(\tau)|\Big] d\tau$$
$$\text{uniformly on } \ I_0.$$

Consequently, the sequences $(u_{k\nu})_{\nu=1}^{+\infty}$ $(k = 1, \dots, m)$ are uniformly bounded and equicontinuous. Without restriction of generality, we may assume them to be uniformly convergent. Suppose

$$\lim_{\nu \to +\infty} u_{k\nu}(t) = u_k(t) \ \ (k = 1, \dots, m),$$
$$\widehat{f}_{k\nu}(u_{1\nu}, \dots, u_{m\nu})(t) = \frac{n_\nu}{b-a} \widetilde{f}_{k n_\nu}(p_{n_\nu}(u_{1\nu}), \dots, p_{n_\nu}(u_{m\nu}))(i)$$
$$\text{for} \quad t_{i-1 n_\nu} \le t < t_{i n_\nu},$$

and

$$\widehat{\varphi}_{k\nu}(u_{1\nu}, \dots, u_{m\nu}) = \widetilde{\varphi}_{k n_\nu}(p_{n_\nu}(u_{1\nu}), \dots, p_{n_\nu}(u_{m\nu})) +$$
$$+ u_{k\nu}(t_k) - u_{k\nu}(t_{i_{k n_\nu} n_\nu}).$$

Then

$$u_{k\nu}(t) = \widehat{\varphi}_{k\nu}(u_{1\nu}, \dots, u_{m\nu}) + \int\limits_{t_k}^{t} \widehat{f}_{k\nu}(u_{1\nu}, \dots, u_{m\nu})(\tau) d\tau$$
$$(k = 1, \dots, m).$$

By (9.5) and (9.6),

$$\lim_{\nu\to+\infty}\int\limits_{t_k}^{t}\widehat{f}_{kn}(u_{1\nu},\dots,u_{m\nu})(\tau)d\tau=\int\limits_{t_k}^{t}\widetilde{f}_k(u_1,\dots,u_m)(\tau)d\tau$$
$$(k=1,\dots,m)$$

and

$$\lim_{\nu\to+\infty}\widehat{\varphi}_{kn}(u_{1\nu},\dots,u_{m\nu})=\widetilde{\varphi}_k(u_1,\dots,u_m)\ \ (k=1,\dots,m).$$

Therefore

$$u_k(t)=\widehat{\varphi}_k(u_1,\dots,u_m)+\int\limits_{t_k}^{t}\widetilde{f}_k(u_1,\dots,u_m)(\tau)d\tau\ \ (k=1,\dots,m).$$

Hence $(u_k)_{k=1}^m$ is a solution of

$$\frac{du_k(t)}{dt}=\widetilde{f}_k(u_1,\dots,u_m)(t)\ \ (k=1,\dots,m),$$
$$u_k(t_k)=\widetilde{\varphi}_k(u_1,\dots,u_m)\ \ (k=1,\dots,m).$$

By virtue of (9.7) and (9.8), $(u_k)_{k=1}^m$ is likewise a solution of (9.14), (9.15). Therefore, owing to the above-said, it admits the estimate (9.16). From this estimate it immediately follows that $(u_k)_{k=1}^m$ is a solution of (9.1), (9.2), i.e., $u_k(t)\equiv u_k^0(t)$ $(k=1,\dots,m)$. On the other hand, by (9.25) we have

$$\sum_{k=1}^m\|u_k-u_k^0\|_{C(I_0;R)}\ge\varepsilon.$$

The obtained contradiction proves the validity of (9.24). This implies the existence of a natural $n_0>n'$ such that for any $n>n_0$, every $(x_k)_{k=1}^m\in\widetilde{X}_n$ satisfies (9.12). On the other hand, since

$$\sum_{k=1}^m|u_k^0(t)|\le\rho_0\quad\text{for}\quad t\in I_0,$$

it follows from (9.12) that

$$\sum_{k=1}^m\|x_k\|_{\widetilde{E}_n}\le\rho_0+r=\rho.$$

Thus

$$X_n(u_1^0,\dots,u_m^0;r)=\widetilde{X}_n\quad\text{for}\quad n>n_0,$$

and (9.13) is fulfilled. ■

For arbitrary $n \in N$, $i \in \widetilde{N}_n$ and $k \in N_m$, let us introduce $\sigma_{ikn} : \widetilde{E}_n \to E_n$ and $q_{ikn} : \widetilde{E}_n \to C(I_0; R)$ by

$$\sigma_{ikn}(x)(j) = \begin{cases} x(\tau_{kn}(i)) & \text{for } \ j = \tau_{kn}(i) + \operatorname{sign}(i_{kn} + 1 - i), \\ x(j) & \text{for } \ j \neq \tau_{kn}(i) + \operatorname{sign}(i_{kn} + 1 - i) \end{cases} \tag{9.26}$$

and

$$q_{ikn}(x)(t) = q_n(\sigma_{ikn}(x))(t) \ \text{ for } \ t \in I_0. \tag{9.27}$$

From Theorem 9.1, we have

Let the problem (9.1), (9.2) *have a unique solution* $(u_k^0)_{k=1}^m$, *and let* (9.7)–(9.9) *be fulfilled. Let, moreover,*

$$f_{kn}(x_1, \dots, x_m)(i) = (1 - \delta_{i-1 i_{kn}}) \times$$
$$\times \int_{t_{i-1n}}^{t_{in}} f_k(q_{i1n}(x_1), \dots, q_{imn}(x_m))(s)ds +$$
$$+\delta_{i-1 i_{kn}} h_{kn}(i) x_k(i) \ \ (n \in N; i \in N_n; k = 1, \dots, m), \tag{9.28}$$

where

$$h_{kn}(i) = \int_{t_{i-1n}}^{t_{in}} h_k(s)ds \tag{9.29}$$

and

$$\varphi_{kn}(x_1, \dots, x_m) = \varphi_k(q_n(x_1), \dots, q_n(x_m)) \ \ (k = 1, \dots, m). \tag{9.30}$$

Then there exists $n_0 \in N$ *such that for every* $n > n_0$, *the set* X_n *of all solutions of* (9.3), (9.4) *is non-empty and*

$$\sup \Big\{ \sum_{k=1}^m \|x_k - p_n(u_k^0)\|_{\widetilde{E}_n} : (x_k)_{k=1}^m \in X_n \Big\} \to 0 \tag{9.31}$$
$$as \ \ n \to +\infty.$$

Proof. The conditions (9.6_1) [10], (9.6_2), (9.10) and (9.11) follow from (9.28)–(9.30). Consequently, the conclusion of Theorem 9.1 is valid.

By (9.26) and (9.27),

$$q_{ikn}(x_k)(t) = x_k(\tau_{kn}(i)) \ \text{ for } \ i \neq i_{kn} + 1 \ t_{i-1n} \le t \le t_{in}.$$

Therefore, from (9.7) and (9.28) the inequalities

$$\big[f_{kn}(x_1, \dots, x_m)(i) - h_{kn}(i) x_k(\tau_{kn}(i))\big] \operatorname{sign} \big[(\tau_{kn}(i) - i_{kn}) x_k(\tau_{kn}(i))\big] \le$$
$$\le h_{0n}(i) + f_{0kn}(|x_1|, \dots, |x_m|)(i) \ \text{ for } \ i \in N_n \ (k = 1, \dots, m) \tag{9.32}$$

[10] See Lemma 7.1 and Remark 7.1.

follow, where

$$h_{0n}(i) = \int_{t_{i-1n}}^{t_{in}} h_0(s)ds,$$

while the functions h_{kn} $(n \in N; k = 1, \dots, m)$ and the operators

$$f_{0kn}(|x_1|, \dots, |x_m|)(i) = \int_{t_{i-1n}}^{t_{in}} f_{0k}\big(q_{i1n}(|x_1|), \dots, q_{imn}(|x_m|)\big)(s)ds$$
$$(i \in N_n;\ k = 1, \dots, m)$$

satisfy (8.4). On the other hand, because of (9.8) and (9.30) we have

$$\big|\varphi_{kn}(x_1, \dots, x_m)\big| \le r_0 + \varphi_{0k}\big(q_n(|x_1|), \dots, q_n(|x_m|)\big) \qquad (9.33)$$
$$(k = 1, \dots, m).$$

By virtue of Lemma 8.1, (9.32) and (9.33) ensure the existence of $n_0 \in N$ and $\rho > 0$ such that for any $n > n_0$, an arbitrary solution of (9.3), (9.4) admits the estimate

$$\sum_{k=1}^{m} \|x_k\|_{\widetilde{E}_n} \le \rho\Big[r_0 + \sum_{k=1}^{n} h_{0n}(i)\Big] = r_1,$$

where

$$r_1 = \Big[r_0 + \int_a^b h_0(s)ds\Big]\rho.$$

From this we get (9.12), where

$$r = r_1 + \sum_{k=1}^{m} \max\big\{|u_k^0(t)| : t \in I_0\big\}.$$

Hence

$$X_n(u_1^0, \dots, u_m^0; r) = X_n \quad \text{for} \quad n > n_0.$$

Taking into account the above equality, we deduce (9.31) from (9.13). ■

The following propositions are proved in a similar way. [11]

Let the problem (9.1), (9.2) *have a unique solution* $(u_k^0)_{k=1}^m$*, and let in* $C(I_0; R^m)$ *the inequalities* (9.8) *and*

$$\big|f_k(u_1, \dots, u_m)(t)\big| \le h_0(t) + f_{0k}(|u_1|, \dots, |u_m|)(t) \qquad (9.34)$$
$$\textit{for} \quad t \in I_0 \quad (k = 1, \dots, m)$$

be fulfilled, where $r_0 \in R_+$, $h_0 \in L(I_0; R_+)$ *and*

$$(f_{01}, \dots, f_{0m}; \varphi_{01}, \dots, \varphi_{0m}) \in W_0(t_1, \dots, t_m). \qquad (9.35)$$

[11] See Lemmas 7.1 and 7.2.

Moreover, let

$$f_{kn}(x_1,\dots,x_m)(i)=\int\limits_{t_{i-1n}}^{t_{in}} f_k(q_n(x_1),\dots,q_n(x_m))(s)ds \tag{9.36}$$

$$(n\in N;\ i\in N_n;\ k=1,\dots,m),$$

and let the functionals φ_{kn} $(k=1,\dots,m;n\in N)$ *be given by* (9.30). *Then the conclusion of Corollary* 9.1 *is valid.*

Let the operators

$$f_k\in C\big(C(I_0;R^m);C(I_0;R)\big)\quad (k=1,\dots,m) \tag{9.37}$$

be bounded on every bounded set of the space $C(I_0;R^m)$*, the problem* (9.1), (9.2) *have a unique solution and* (9.7)–(9.9) *be fulfilled, where* $r_0\in R_+$, $h_0\in C(I_0;R_+)$*, and*

$$h_k\in C(I_0;R),\quad f_{0k}\in C\big(C(I_0;R_+^m);C(I_0;R_+)\big) \tag{9.38}$$

$$(k=1,\dots,m).$$

Moreover, let

$$f_{kn}(x_1,\dots,x_m)(i)=\frac{b-a}{n}f_k\big(q_n(x_1),\dots,q_n(x_m)\big)\big(t_{\tau_{kn}(i)n}\big)$$

$$(n\in N;\ i\in N_n;\ k=1,\dots,m), \tag{9.39}$$

and let the functionals φ_{kn} $(k=1,\dots,m;n\in N)$ *be given by* (9.30). *Then the conclusion of Corollary* 9.1 *is valid.*

Let the problem (9.1), (9.2) *have a unique solution* $(u_k^0)_{k=1}^m$*, and let* (9.8), (9.34), (9.35) *and* (9.37) *be fulfilled with* $r_0\in R_+$, $h_0\in L(I_0;R_+)$ *and*

$$f_{0k}\in C\big(C(I_0;R_+^m);C(I_0;R_+)\big)\ \ (k=1,\dots,m).$$

Let further

$$f_{kn}(x_1,\dots,x_m)(i)=\frac{b-a}{2n}\Big[f_k(q_n(x_1),\dots,q_n(x_m))(t_{i-1n})+$$

$$+f_k(q_n(x_1),\dots,q_n(x_m))(t_{in})\Big]\ \ (n\in N;\ i\in N_n;\ k=1,\dots,m), \tag{9.40}$$

and let the functionals φ_{kn} $(k=1,\dots,m;n\in N)$ *be given by* (9.30). *Then the conclusion of Corollary* 9.1 *is valid.*

The difference process (9.3), (9.4) is said to be stable if there exist $n_0\in N$ and $\rho>0$ such that for any natural $n>n_0$ and any vector functions $(y_k)_{k=1}^m$ and $(z_k)_{k=1}^m\in \widetilde{E}_n^m$, the estimates

$$\sum_{k=1}^m \|z_k-y_k\|_{\widetilde{E}_n}\le \rho\sum_{k=1}^m\Big[|\Phi_{kn}(z_1,\dots,z_m)-\Phi_{kn}(y_1,\dots,y_m)|+$$

$$+\sum_{i=1}^{n} |F_{kn}(z_1,\dots,z_m)-F_{kn}(y_1,\dots,y_m)(i)|\Big] \tag{9.41}$$

are valid, where

$$\begin{gathered} F_{kn}(x_1,\dots,x_m)(i)=\Delta x_k(i-1)-f_{kn}(x_1,\dots,x_m)(i), \\ \Phi_{kn}(x_1,\dots,x_m)=x_k(i_{kn})-\varphi_{kn}(x_1,\dots,x_m) \ \ (k=1,\dots,m). \end{gathered} \tag{9.42}$$

Let for any natural n in the space $\widetilde{E}_n^m$ the inequalities

$$\begin{gathered} \big[f_{kn}(x_1,\dots,x_m)(i)-f_{kn}(y_1,\dots,y_m)(i)-h_{kn}(i)(x_k(\tau_{kn}(i))- \\ -y_k(\tau_{kn}(i)))\big]\,\mathrm{sign}\,\big[(\tau_{kn}(i)-i_{kn})(x_k(\tau_{kn}(i))-y_k(\tau_{kn}(i)))\big]\le \\ \le f_{0kn}(|x_1-y_1|,\dots,|x_m-y_m|)(i) \ \ \textit{for} \ \ i\in N_n \ \ (k=1,\dots,m) \end{gathered} \tag{9.43}$$

and

$$\begin{gathered} |\varphi_{kn}(x_1,\dots,x_m)-\varphi_{kn}(y_1,\dots,y_m)|\le \\ \le \varphi_{0k}(q_n(|x_1-y_1|),\dots,q_n(|x_m-y_m|)) \ \ (k=1,\dots,m) \end{gathered} \tag{9.44}$$

be fulfilled, where $f_{0kn}:(\widetilde{E}_n^+)^m\to E_n^+$ $(k=1,\dots,m)$ are positively homogeneous continuous nondecreasing operators,

$$(h_{kn})_{n=1}^{+\infty}\in D_{h_k}, \ \ (f_{0kn})_{n=1}^{+\infty}\in D_{f_{0k}} \ \ (k=1,\dots,m), \tag{9.45}$$

and h_k, f_{0k} and φ_{0k} $(k=1,\dots,m)$ satisfy (9.9). *Then:* (*a*) *the problem* (9.1), (9.2) *has a unique solution $(u_k^0)_{k=1}^m$;* (*b*) *the difference scheme* (9.3), (9.4) *is stable;* (*c*) *there exist $n_0\in N$ and $\rho>0$ such that for any $n>n_0$, the problem* (9.3), (9.4) *has a unique solution $(x_{kn})_{k=1}^m$,*

$$\begin{gathered} \sum_{k=1}^{m}\|x_{kn}-p_n(u_k^0)\|_{\widetilde{E}_n}\le\rho\sum_{k=1}^{m}\Big[|\Phi_{kn}(p_n(u_1^0),\dots,p_n(u_m^0))|+ \\ +\sum_{i=1}^{n}|F_{kn}(p_n(u_1^0),\dots,p_n(u_m^0))(i)|\Big], \end{gathered} \tag{9.46}$$

and

$$\lim_{n\to+\infty}\|x_{kn}-p_n(u_k^0)\|_{\widetilde{E}_n}=0 \ \ (k=1,\dots,m). \tag{9.47}$$

Proof. Owing to (9.6_1), (9.6_2) and (9.43)–(9.45), in $C(I_0;R^m)$ the inequalities

$$\begin{gathered} \big[f_k(u_1,\dots,u_m)(t)-f_k(v_1,\dots,v_m)(t)-h_k(t)(u_k(t)- \\ -v_k(t))\big]\,\mathrm{sign}\,\big[(t-t_k)(u_k(t)-v_k(t))\big]\le \\ \le f_{0k}(|u_1-v_1|,\dots,|u_m-v_m|)(t) \ \ \text{for} \ \ t\in I_0 \ \ (k=1,\dots,m) \end{gathered} \tag{9.48}$$

and

$$|\varphi_k(u_1,\dots,u_m)-\varphi_k(v_1,\dots,v_m)|\le$$

$$\leq \varphi_{0k}(|u_1 - v_1|, \dots, |u_m - v_m|) \ \ (k = 1, \dots, m) \tag{9.49}$$

are fulfilled. But, by virtue of Theorem 1.2, the conditions (9.9), (9.48) and (9.49) guarantee the existence of a unique solution $(u_k^0)_{k=1}^m$ of (9.1), (9.2).

Let ρ and n_0 be the numbers appearing in Lemma 8.1, let $n > n_0$, and $(y_k)_{k=1}^m$ and $(z_k)_{k=1}^m$ be arbitrary vector functions from $\widetilde{E}_n^m$. Assume

$$x_k(i) = z_k(i) - y_k(i), \quad r = \sum_{k=1}^m |\Phi_{kn}(z_1, \dots, z_m) - \Phi_{kn}(y_1, \dots, y_m)|,$$

$$h_0(i) = \sum_{k=1}^m |F_{kn}(z_1, \dots, z_m)(i) - F_{kn}(y_1, \dots, y_m)(i)|.$$

Then, because of (9.43) and (9.44), the inequalities (8.5) and (8.6) are fulfilled. The estimate (8.7), i.e., the estimate (9.41) holds by virtue of Lemma 8.1. Thus the stability of the process (9.3), (9.4) is proved.

According to Lemma 8.1$'$,

$$\begin{aligned}(h_{1n}, \dots, h_{mn}; f_{01n}, \dots, f_{0mn}; \varphi_{01n}, \dots, \varphi_{0mn}) \in \\ \in W_n(i_{1n}, \dots, i_{mn}) \ \text{ for } \ n > n_0.\end{aligned} \tag{9.50}$$

By Theorem 4.2, from (9.43), (9.44) and (9.50) it follows that for any natural $n > n_0$, the problem (9.3), (9.4) has a unique solution $(x_{kn})_{k=1}^m$. Owing to (9.41), it is evident that (9.46) holds. On the other hand, according to (9.6_1) and (9.6_2),

$$\lim_{n \to +\infty} \sum_{i=1}^n \big| F_{kn}(p_n(u_1^0), \dots, p_n(u_m^0))(i) \big| =$$

$$= \lim_{n \to +\infty} \sum_{i=1}^n \left| f_{kn}(p_n(u_1^0), \dots, p_n(u_m^0))(i) - \int_{t_{i-1n}}^{t_{in}} f_k(u_1^0, \dots, u_m^0)(t)dt \right| = 0$$

and

$$\lim_{n \to +\infty} \Phi_{kn}(p_n(u_1^0), \dots, p_n(u_m^0)) = u_k(t_k) - \varphi_k(u_1^0, \dots, u_m^0) = 0$$
$$(k = 1, \dots, m).$$

Hence (9.47) is fulfilled. Thus the theorem is proved. ■

Let the conditions (9.9), (9.48) *and* (9.49) *be fulfilled, and let f_{kn} and φ_{kn} be given by* (9.26)–(9.30). *Then the conclusion of Theorem* 9.2 *is valid. Besides, if for any $(u_k)_{k=1}^m$ and $(v_k)_{k=1}^m \in C(I_0; R^m)$ we have*

$$h_k \in L^\alpha(I_0; R), \ \ f_k(u_1, \dots, u_m) \in L^\alpha(I_0; R) \ \ (k = 1, \dots, m) \tag{9.51}$$

and

$$\sum_{k=1}^{m} |f_k(u_1,\dots,u_m)(t) - f_k(v_1,\dots,v_m)(t)| \le$$
$$\le h(t)\sum_{k=1}^{m} \|u_k - v_k\|_{C(I_0;R)} \tag{9.52}$$

with $1 < \alpha \le +\infty$ *and* $h \in L(I_0;R_+)$, *then*

$$\sum_{k=1}^{m} \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = 0\big(n^{\frac{1}{\alpha}-1}\big). \tag{9.53}$$

Proof. To see that the first part of the above corollary is valid, it is sufficient to note that (9.43)–(9.45) follow from (9.26)–(9.30), (9.48) and (9.49).

Assume now that (9.51) and (9.52) are fulfilled.

Because of (9.51),

$$\int\limits_{t_{i-1n}}^{t_{in}} |h_k(\tau)|d\tau \le \frac{r_1}{n^{1-\frac{1}{\alpha}}} \ \ (i \in N;\ k=1,\dots,m),$$

$$|q_{ikn}(p_n(u_k^0))(t) - u_k^0(t)| \le \int\limits_{t_{j-1n}}^{t_{jn}} |u_k^{0\prime}(s)|ds \le \frac{r_2}{n^{1-\frac{1}{\alpha}}}$$
$$\text{for} \quad t_{j-1n} \le t \le t_{jn} \ \ (i \in N_n;\ k=1,\dots,m)$$

and

$$\|q_{ikn}(p_n(u_k^0)) - u_k^0\|_{C(I_0;R)} \le \frac{r_2}{n^{1-\frac{1}{\alpha}}} \ \ (k=1,\dots,m),$$

where

$$r_1 = (b-a)^{1-\frac{1}{\alpha}} \max\big\{\|h_k\|_{L^\alpha(I_0;R)} : k \in N_m\big\},$$
$$r_2 = (b-a)^{1-\frac{1}{\alpha}} \max\big\{\|u_k^{0\prime}\|_{L^\alpha(I_0;R)} : k \in N_m\big\}.$$

It is also clear that

$$|u_k^0(t) - u_k^0(s)| \le \frac{r_2}{n^{1-\frac{1}{\alpha}}} \ \text{ for } \ |t-s| \le \frac{b-a}{n}$$

and

$$\|q_n(p_n(u_k^0)) - u_k^0\|_{C(I_0;R)} \le \frac{r_2}{n^{1-\frac{1}{\alpha}}} \ \ (k=1,\dots,m).$$

Taking into account these estimates and (9.49) and (9.52), we find

$$\sum_{i=1}^{n} |F_{kn}(p_n(u_1^0) - p_n(u_m^0))(i)| \le$$

$$\leq \sum_{i=1}^{n} \int\limits_{t_{i-1n}}^{t_{in}} |f_k\big(q_{ikn}(p_n(u_1^0)), \dots, q_{ikn}(p_n(u_m^0))\big)(s) -$$

$$-f_k(u_1^0, \dots, u_m^0)(s)|ds + \int\limits_{t_{i_{kn}n}}^{t_{i_{kn}+1n}} |u_k^{0\prime}(t)|dt +$$

$$+|u_k^0(t_{i_{kn}+1n})| \int\limits_{t_{i_{kn}n}}^{t_{i_{kn}+1n}} |h_k(s)|ds \leq \frac{r_3}{n^{1-\frac{1}{\alpha}}}$$

and

$$|\Phi_{kn}(p_n(u_1^0), \dots, p_n(u_m^0))| \leq |u_k^0(t_k) - u_k^0(t_{i_{kn}n})| +$$
$$+\varphi_{0k}\big(|q_n(p_n(u_1^0)) - u_1^0|, \dots, |q_n(p_n(u_m^0)) - u_m^0|\big) \leq \frac{r_4}{n^{1-\frac{1}{\alpha}}},$$

where

$$r_3 = mr_2 \int\limits_a^b h(s)ds + r_1 \max\big\{\|u_k^0\|_{C(I_0;R)} : k \in N_m\big\} + r_2$$

and

$$r_4 = r_2 + r_2 \max\{\varphi_{0k}(1, \dots, 1) : k \in N_m\}.$$

By virtue of the above obtained inequalities, the estimate (9.53) with $\rho_0 = m\rho(r_3 + r_4)$ follows from (9.46). ■

Let in $C(I_0; R^m)$ *the inequalities* (9.49) *and*

$$|f_k(u_1, \dots, u_m)(t) - f_k(v_1, \dots, v_m)(t)| \leq$$
$$\leq f_{0k}(|u_1 - v_1|, \dots, |u_m - v_m|)(t) \ \ \textit{for} \ \ t \in I_0 \ \ (k = 1, \dots, m) \tag{9.54}$$

be fulfilled, where f_{0k} *and* φ_{0k} $(k = 1, \dots, m)$ *satisfy* (9.35). *Moreover, let* f_{kn} *and* φ_{kn} *be given by* (9.36) *and* (9.30). *Then the conclusion of Theorem* 9.2 *is valid. Besides, if* $u_k^{0\prime}$ $(k = 1, \dots, m)$ *are absolutely continuous and*

$$u_k^{0\prime\prime} \in L^\alpha(I_0; R) \quad (k = 1, \dots, m) \tag{9.55}$$

with $1 < \alpha \leq +\infty$, *then*

$$\sum_{k=1}^{m} \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = 0\big(n^{\frac{1}{\alpha}-2}\big). \tag{9.56}$$

The proof is similar to that of Corollary 9.5. One should only take into account that because of (9.55),

$$\|q_n(p_n(u_k^0)) - u_k^0\|_{C(I_0;R)} \leq \frac{r}{n^{2-\frac{1}{\alpha}}} \quad (k = 1, \dots, m)$$

with
$$r = (b-a)^{2-\frac{1}{\alpha}} \max\left\{\|u_k^{0''}\|_{L^\alpha(I_0;R)} : k \in N_m\right\}.$$

Let the conditions (9.9), (9.37), (9.38), (9.48) *and* (9.49) *be fulfilled, and let* f_{kn} *and* φ_{kn} *be given by* (9.39) *and* (9.30). *Then the conclusion of Theorem* 9.2 *is valid. Besides, if* (9.52) *is fulfilled, where* $h(t) \equiv h = const$ *and every* $u_k^{0'}$ *has bounded variation, then*

$$\sum_{k=1}^m \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-1}).$$

Let the conditions (9.35), (9.49) *and* (9.54) *be fulfilled, and let* f_{kn} *and* φ_{kn} *be given by* (9.40) *and* (9.30). *Then the conclusion of Theorem* 9.2 *is valid. Besides, if* $(u_k^0)_{k=1}^m$ *is thrice continuously differentiable, then*

$$\sum_{k=1}^m \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-2}).$$

Let the conditions of Theorem 9.2 be fulfilled. Then, as it has been mentioned above, starting from some n_0 the condition (9.50) is fulfilled. Therefore, owing to Theorem 4.3, for any $n > n_0$ and $(y_{k0})_{k=1}^m \in \widetilde{E}_n^m$ the solution $(x_{kn})_{k=1}^m$ of (9.3), (9.4) admits the representation

$$\lim_{\nu \to +\infty} y_{k\nu}(i) = x_{kn}(i) \quad (i \in \widetilde{N}_n;\ k = 1, \dots, m), \tag{9.57}$$

where every $y_{k\nu}$ is the solution of the Cauchy problem

$$\begin{gathered} \Delta y_{k\nu}(i-1) = \\ = f_{kn}(y_{1\nu-1}, \dots, y_{k-1\nu-1}, y_{k\nu}, y_{k+1\nu-1}, \dots, y_{m\nu-1})(i), \qquad (9.58) \\ y_{k\nu}(i_{kn}) = \varphi_{kn}(y_{1\nu-1}, \dots, y_{m\nu-1}). \qquad (9.59) \end{gathered}$$

If instead of (9.43)

$$\begin{gathered} |f_{kn}(x_1, \dots, x_m)(i) - f_{kn}(y_1, \dots, y_m)(i) - h_{kn}(i)(x_k(\tau_{kn}(i)) - \\ -y_k(\tau_{kn}(i)))| \le f_{0kn}(|x_1 - y_1|, \dots, |x_m - y_m|)(i) \qquad (9.43') \\ \text{for } i \in N_n \ \ (k = 1, \dots, m) \end{gathered}$$

is fulfilled, then we can replace (9.58) by

$$\begin{gathered} \Delta y_{k\nu}(i-1) = h_{kn}(i)(y_{k\nu}(\tau_{kn}(i)) - y_{k\nu-1}(\tau_{kn}(i))) + \\ +f_{kn}(y_{1\nu-1}, \dots, y_{m\nu-1})(i), \qquad (9.58') \end{gathered}$$

but if

$$\begin{gathered} |f_{kn}(x_1, \dots, x_m)(i) - f_{kn}(y_1, \dots, y_m)(i)| \le \\ \le f_{0kn}(|x_1 - y_1|, \dots, |x_m - y_m|)(i) \qquad (9.43'') \\ \text{for } i \in N_n \ \ (k = 1, \dots, m), \end{gathered}$$

then it can be replaced by

$$\Delta y_{k\nu}(i-1) = f_{kn}(y_{1\nu-1},\dots,y_{m\nu-1})(i). \tag{9.58''}$$

Hence if the conditions of Corollaries 9.6 and 9.8 are fulfilled, then for any $n > n_0$ the problem (9.3), (9.4) can be solved by the method of successive approximations (9.58″), (9.59), the zero approximation $(y_{k0})_{k=1}^m \in \widetilde{E}_n^m$ being prescribed arbitrarily.

Consider the differential system

$$\frac{du_k(t)}{dt} = g_k(t, u_1(t),\dots,u_m(t)) \ \ (k=1,\dots,m). \tag{9.60}$$

with the boundary conditions (9.2), where

$$g_k \in K(I_0 \times R^m; R) \ \ (k=1,\dots,m).$$

Corollaries 9.5–9.8 take the following form for this problem.

Let in $C(I_0; R^m)$ *and in* $I_0 \times R^m$ *the inequalities* (9.49) *and*

$$\begin{gathered}\big[g_k(t,x_1,\dots,x_m) - g_k(t,\overline{x}_1,\dots,\overline{x}_m) - h_k(t)(x_k - \overline{x}_k)\big] \times \\ \times \operatorname{sign}\big[(t-t_k)(x_k-\overline{x}_k)\big] \le \sum_{j=1}^m h_{kj}(t)|x_j - \overline{x}_j|, \\ (k=1,\dots,m),\end{gathered} \tag{9.61}$$

respectively, be fulfilled, where

$$\begin{gathered}(h_1,\dots,h_m; h_{11},\dots,h_{1m},\dots,h_{m1},\dots,h_{mn}; \varphi_{01},\dots,\varphi_{om}) \in \\ \in W'(t_1,\dots,t_m).^{12}\end{gathered} \tag{9.62}$$

Moreover, let

$$\begin{gathered}f_{kn}(x_1,\dots,x_m)(i) = \\ = (1-\delta_{i-1 i_{kn}}) \int_{t_{i-1n}}^{t_{in}} g_k(t, x_1(\tau_{1n}(i),\dots,x_m(\tau_{mn}(i)))dt + \\ +\delta_{i-1i_{kn}} \int_{t_{i-1n}}^{t_{in}} h_k(t)dt\, x_k(i) \ \ (n\in N;\ i \in N_n;\ k=1,\dots,m),\end{gathered}$$

and let the functionals φ_{kn} $(k=1,\dots,m)$ *be given by* (9.30). *Then:* (*a*) *the problem* (9.60), (9.2) *has a unique solution* $(u_k^0)_{k=1}^m$; (*b*) *the difference*

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[12] See Definition 1.1′.

scheme (9.3), (9.4) *is stable; (c) there exist* $n_0 \in N$ *and* $\rho \in R_+$ *such that for any* $n > n_0$, *the problem* (9.3), (9.4) *has a unique solution* $(x_{kn})_{k=1}^m$ *and* (9.46) *and* (9.47) *are fulfilled.*

Remark 9.1. If the conditions of Corollary 9.9 are fulfilled, $h_k \in L^\alpha(I_0; R)$, $1 < \alpha \leq +\infty$, and for any $r \in R_+$ we have

$$g^*(\cdot; r) \in L^\alpha(I_0; R)$$

and

$$\sum_{k=1}^m |g_k(t, x_1, \dots, x_m) - g_k(t, \overline{x}_1, \dots, \overline{x}_m) \leq h(t, r) \sum_{k=1}^m |x_k - \overline{x}_k|$$

with $h(\cdot; r) \in L(I_0; R_+)$ and

$$g^*(t, r) = \max\Big\{ \sum_{k=1}^m |g_k(t, x_1, \dots, x_m)| : \sum_{k=1}^m |x_k| \leq r \Big\},$$

then the estimate

$$\sum_{k=1}^m \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-1+\frac{1}{\alpha}})$$

is valid.

Let in $C(I_0; R^m)$ *and in* $I_0 \times R^m$ *the inequalities* (9.49) *and*

$$\begin{aligned} \big|g_k(t, x_1, \dots, x_m) - g_k(t, \overline{x}_1, \dots, \overline{x}_m)\big| \leq \\ \leq \sum_{j=1}^m h_{kj}(t) |x_j - \overline{x}_j| \quad (k = 1, \dots, m), \end{aligned} \tag{9.63}$$

respectively, be fulfilled, where

$$\begin{aligned} (h_{11}, \dots, h_{1m}, \dots, h_{m1}, \dots, h_{mm}; \varphi_{01}, \dots, \varphi_{om}) \in \\ \in W_0'(t_1, \dots, t_m). \end{aligned} \tag{9.64}$$

Moreover, let

$$f_{kn}(x_1, \dots, x_m)(i) = \int\limits_{t_{i-1n}}^{t_{in}} g_k(t, q_n(x_1)(t), \dots, q_n(x_m)(t)) dt$$
$$(n \in N;\ i \in N_n;\ k = 1, \dots, m),$$

and let the functionals φ_{kn} $(k = 1, \dots, m)$ *be given by* (9.30). *Then the conclusion of Corollary* 9.9 *is valid. Besides, if* ${u_k^0}'$ $(k = 1, \dots, m)$ *are* *absolutely continuous and* $u_k^{0\prime\prime} \in L^\alpha(I_0;R)$ $(k=1,\dots,m)$ *with* $1<\alpha\le +\infty$, *then*

$$\sum_{k=1}^m \|x_{kn}-p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-2+\frac{1}{\alpha}}).$$

Let

$$g_k \in C(I_0\times R^m;R) \quad (k=1,\dots,m),$$

and let the conditions (9.49), (9.61) *and* (9.62) *be fulfilled, where*

$$h_k\in C(I_0;R),\ \ h_{kj}\in C(I_0;R_+) \quad (k=1,\dots,m).$$

Moreover, let

$$f_{kn}(x_1,\dots,x_m)(i) = \\ = \frac{b-a}{n} g_k\big(t_{\tau_{kn}(i)n}, x_1(\tau_{kn}(i)),\dots,x_m(\tau_{kn}(i))\big) \\ (n\in N;\ i\in N_n;\ k=1,\dots,m),$$

and let the functionals φ_{kn} $(k=1,\dots,m)$ *be given by* (9.30). *Then the conclusion of Corollary* 9.9 *is valid. Besides, if every* $u_k^{0\prime}$ *has bounded variation, then*

$$\sum_{k=1}^m \|x_{kn}-p_n(u_k^0)\|_{\widetilde{E}_n} = O\Big(\frac{1}{n}\Big).$$

Let

$$g_k \in C(I_0\times R^m;R) \quad (k=1,\dots,m),$$

and let the conditions (9.49), (9.63) *and* (9.64) *be fulfilled, where* $h_{kj}\in C(I_0;R_+)$ $(k,j=1,\dots,m)$. *Moreover, let*

$$f_{kn}(x_1,\dots,x_m)(i) = \frac{b-a}{2n}\big[g_k(t_{in},x_1(i),\dots,x_m(i)) + \\ +g_k(t_{i-1n},x_1(i-1),\dots,x_m(i-1))\big] \ \ (k=1,\dots,m),$$

and let the functionals φ_{kn} $(k=1,\dots,m)$ *be given by* (9.30). *Then the conclusion of Corollary* 9.9 *is valid. Besides, if every* u_k^0 *is twice continuously differentiable and* $u_k^{0\prime\prime}$ *has bounded variation, then*

$$\sum_{k=1}^m \|x_{kn}-p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-2}).$$

In conclusion, let us consider the boundary value problem

$$u_k(t_k)=\sum_{j=1}^{m}\big[c_{1kj}u_j(a)+c_{2kj}u_j(b)\big]+c_k \quad (k=1,\dots,m) \tag{9.65}$$

for the system (9.60), where $t_k\in\{a,b\}$ and $c_{1kj},\ c_{2kj},\ c_k\in R$ $(k,j=1,\dots,m)$. Assume

$$l_{kj}=|c_{1kj}|+|c_{2kj}| \quad (k,j=1,\dots,m)$$

and

$$l^*=\frac{1}{b-a}\min\Big\{\ln\Big(\sum_{j=1}^{m}l_{kj}\Big)^{-1}:k=1,\dots,m\Big\}.$$[13]

Owing to Remarks 1.1 and 1.2, the following corollary is valid.

Let $g_k\in C(I_0\times R^m;R)$ $(k=1,\dots,m)$, and let in $I_0\times R^m$ the inequalities

$$|g_k(t,x_1,\dots,x_m)-g_k(t,\overline{x}_1,\dots,\overline{x}_m)|\le \\ \le\sum_{j=1}^{m}h_{kj}|x_j-\overline{x}_j| \quad (k=1,\dots,m) \tag{9.66}$$

be fulfilled, where either the spectral radius of the matrix

$$\Big(l_{kj}+(b-a)\sum_{i=1}^{m}l_{ki}h_{ij}+\frac{2(b-a)}{\pi}h_{kj}\Big)_{k,j=1}^{m} \tag{9.67}$$

is less than 1, *or*

$$t_k=a,\quad \sum_{j=1}^{m}h_{kj}<l^* \quad (k=1,\dots,m).$$

Then the problem (9.60), (9.65) *has a unique solution.*

Let the conditions of Corollary 9.13 *be fulfilled,*

$$z_{kn}(x_1,\dots,x_m)(i)=\frac{1}{2}\big[x_k(i)-x_k(i-1)\big]+ \\ +\frac{b-a}{8n}\Big[g_k(t_{i-1n},x_1(i-1),\dots,x_m(i-1))- \\ -g_k(t_{in},x_1(i),\dots,x_m(i))\Big] \quad (k=1,\dots,m),$$

$$f_{kn}(x_1,\dots,x_m)(i)=\frac{b-a}{6n}\big[g_k(t_{in},x_1(i),\dots,x_m(i))+$$

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[13] If $\sum_{k=1}^{m}l_{kj}=0$ $(k=1,\dots,m)$, then $l^*=+\infty$

$$+4g_k\Big(t_{in} - \frac{b-a}{2n}, z_{1n}(x_1,\dots,x_m)(i),\dots,z_{mn}(x_1,\dots,x_m)(i)\big) +$$
$$+g_k(t_{i-1n}, x_1(i-1),\dots,x_m(i-1))\Big] \quad (k=1,\dots,m), \tag{9.68}$$

and

$$\varphi_{kn}(x_1,\dots,x_m) = \sum_{j=1}^m \big[c_{1kj}x_j(0) + c_{2kj}x_j(n)\big] + c_k \tag{9.69}$$
$$(k=1,\dots,m).$$

Then: (a) the problem (9.60), (9.65) *has a unique solution* $(u_k^0)_{k=1}^m$*; (b) the difference scheme* (9.3), (9.4) *is stable; (c) there exist* $n_0 > N$ *and* $\rho > 0$ *such that for any* $n > n_0$*, the problem* (9.3), (9.4) *has a unique solution* $(x_{kn})_{k=1}^m$ *and the conditions* (9.46), (9.47) *are fulfilled. Besides, if* u_k^0 $(k = 1,\dots,m)$ *have continuous derivatives up to the fifth order inclusively, then*

$$\sum_{k=1}^m \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O(n^{-4}). \tag{9.70}$$

Proof. Because of (9.66) and (9.67)–(9.69), the inequalities (9.44) and

$$|f_{kn}(x_1,\dots,x_m)(i) - f_{kn}(\overline{x}_1,\dots,\overline{x}_m)(i)| \le$$
$$\le f_{0kn}(|x_1-\overline{x}_1|,\dots,|x_m-\overline{x}_m|)(i) \quad (k=1,\dots,m)$$

are fulfilled, where

$$f_{0kn}(x_1,\dots,x_m)(i) = \frac{b-a}{2n}\sum_{j=1}^m \Big(h_{kj} + \frac{h_0}{n}\Big)(x_j(i) - x_j(i-1)),$$
$$\varphi_{0k}(u_1,\dots,u_m) = \sum_{j=1}^m (|c_{1kj}|u_j(a) + |c_{2kj}|u_j(b)),$$

and

$$h_0 = \frac{b-a}{6}\max\Big\{\sum_{\nu=1}^m h_{k\nu}h_{\nu j} : k,j=1,\dots,m\Big\}.$$

Moreover, by Lemma 7.4,

$$(f_{kn})_{n=1}^{+\infty} \in D_{f_k} \ \text{ and } \ (f_{0kn})_{n=1}^{+\infty} \in D_{f_{0k}} \quad (k=1,\dots,m),$$

where

$$f_{0k}(u_1,\dots,u_m)(t) = \sum_{j=1}^m h_{kj}u_j(t).$$

On the other hand, by Lemmas 2.5′ and 2.6, the inclusion (9.35) is fulfilled since h_{kj}, c_{1kj} and c_{2kj} satisfy the conditions of Corollary 9.13.

Using now Theorem 9.2, we can easily see that assertions (b)–(c) of Corollary 9.14 are valid.

Assume now that every u_k^0 has continuous derivatives up to the fifth order inclusively.

For arbitrarily fixed $t \in [a, b[$ and $k \in N_m$, we put

$$\psi_{kt}(s) = u_k^0(t+s) - u_k^0(t) - \frac{s}{6}\big({u_k^0}'(t+s) + 4{u_k^0}'\Big(t+\frac{s}{2}\Big) + {u_k^0}'(t)\big).$$

Then

$$\psi_{kt}(0) = \psi'_{kt}(0) = \cdots = \psi_{kt}^{(IV)}(0) = 0,$$

and hence

$$\Big|\psi_{kt_{i-1n}}\Big(\frac{b-a}{n}\Big)\Big| = \Big|u_k^0(t_{in}) - u_k^0(t_{i-1n}) -$$
$$-\frac{b-a}{6n}\Big[{u_k^0}'(t_{in}) + 4{u_k^0}'\Big(t_{in} - \frac{b-a}{2n}\Big) + {u_k^0}'(t_{i-1n})\Big]\Big| \le \rho_0 n^{-5}$$
$$\text{for} \quad i \in N_n, \; k \in N_m, \; n > n_0,$$

where ρ_0 is a number independent of k and n. Similarly,

$$\Big|u_k^0\Big(t_{in} - \frac{b-a}{2n}\Big) - z_{kn}(p_n(u_1^0), \dots, p_n(u_m^0))(i)\Big| =$$
$$= \Big|u_k^0\Big(t_{in} - \frac{b-a}{2n}\Big) - \frac{1}{2}\big[u_k^0(t_{in}) + u_k^0(t_{i-1n})\big] -$$
$$-\frac{b-a}{8n}\big[{u_k^0}'(t_{i-1n}) - {u_k^0}'(t_{in})\big]\Big| \le \rho_1 n^{-4}$$
$$\text{for} \quad i \in N_n, \; k \in N_m, \; n > n_0.$$

Therefore

$$\big|F_{kn}(p_n(u_1^0), \dots, p_n(u_m^0))(i)\big| = \psi_{kt_{i-1n}}\Big(\frac{b-a}{n}\Big) +$$
$$+\frac{2(b-a)}{3n}\Big[g_k\Big(t_{in} - \frac{b-a}{2n}, u_1^0\Big(t_{in} - \frac{b-a}{2n}\Big), \dots, u_m^0\Big(t_{in} - \frac{b-a}{2n}\Big)\Big) -$$
$$-g_k\Big(t_{in} - \frac{b-a}{2n}, z_{1n}(p_n(u_1^0), \dots, p_n(u_m^0))(i), \dots, z_{mn}(p_n(u_1^0), \dots$$
$$\dots, p_n(u_m^0))(i)\Big)\Big] \le \rho_0 n^{-5} + + \frac{2(b-a)}{3n}\sum_{j=1}^{m} h_{kj}\Big|u_j^0\Big(t_{in} - \frac{b-a}{2n}\Big) -$$
$$-z_{jn}(p_n(u_1^0), \dots, p_n(u_m^0))(i)\Big| \le \rho_2 n^{-5} \tag{9.71}$$
$$\text{for} \quad i \in N_n, \; k \in N_m, \; n > n_0.$$

On the other hand, since $t_k \in \{a, b\}$, it is clear that

$$\Phi_{kn}(p_n(u_1^0), \dots, p_n(u_m^0)) = 0 \;\text{ for }\; k \in N_m, \;\; n > n_0. \tag{9.72}$$

Owing to (9.71) and (9.72), the estimate (9.70) follows from (9.46). ■

In a similar way, we prove

Let the conditions of Corollary 9.13 *be fulfilled, and let the functionals* φ_{kn} $(k=1,\dots,m)$ *be given by* (9.69) *and*

$$f_{1kn}(x_1,\dots,x_m)(i)=\frac{b-a}{n}g_k(t_{i-1n},x_1(i),\dots,x_m(i)),$$

$$f_{jkn}(x_1,\dots,x_m)(i)=\frac{b-a}{n}g_k\Big(t_{i-1n}+\frac{b-a}{n}\beta_j,x_1+$$

$$+\sum_{\nu=1}^{j-1}\gamma_{j\nu}f_{\nu 1n}(x_1,\dots,x_m)(i),\dots,x_m+$$

$$+\sum_{\nu=1}^{j-1}\gamma_{j\nu}f_{\nu mn}(x_1,\dots,x_m)(i)\Big)\ \ (j=2,\dots,j_0),$$

$$f_{kn}(x_1,\dots,x_m)(i)=\sum_{j=1}^{j_0}\alpha_j f_{jkn}(x_1,\dots,x_m)(i)\ \ (k=1,\dots,m),$$

where $\alpha_j\in[0,1]$, $\beta_j\in[0,1]$ *and* $\gamma_{j\nu}\in R$ *are independent of* n, *and*

$$\sum_{j=1}^{j_0}\alpha_j=1.$$

Then: (*a*) *the problem* (9.60), (9.65) *has a unique solution* $(u_k^0)_{k=1}^m$; (*b*) *the difference scheme* (9.3), (9.4) *is stable;* (*c*) *there exist* $n_0\in N$ *and* $\rho>0$ *such that for any* $n>n_0$, *the problem* (9.3), (9.4) *has a unique solution* $(x_{kn})_{k=1}^m$ *and the conditions* (9.46), (9.47) *are fulfilled. Besides, if* u_k^0 $(k=1,\dots,m)$ *have continuous derivatives up to the* ν_0+1*-th order inclusively, where* ν_0 *is the error order of the Runge-Kutta method under consideration, then*

$$\sum_{k=1}^m\|x_k-p_n(u_k^0)\|_{\widetilde{E}_n}=O\Big(n^{-\nu_0}\Big).$$

Note that if the conditions of Corollary 9.9 or those of Corollary 9.11 are fulfilled, then starting from a sufficiently large n_0, we can solve the problem (9.3), (9.4) by the method (9.58), (9.59). On the other hand, if the conditions of one of Corollaries 9.10, 9.12, 9.14 and 9.15 are fulfilled, then this problem can be solved by the iterative method (9.58″), (9.59).

§ 10. Difference Schemes of the Type (0.23), (0.24)

In this section, we consider the differential boundary value problem

$$\frac{du_k(t)}{dt}=\overline{f}_k(u_1,\dots,u_k,u_k,u_{k+1},\dots,u_m)(t)\ \ (k=1,\dots,m),\tag{10.1}$$

$$u_k(t_k)=\varphi_k(u_1,\dots,u_m)\quad(k=1,\dots,m).\tag{10.2}$$

We assume that $t_k \in I_0$,

$$\overline{f}_k \in K\big(C(I_0; R^{m+1}); L(I_0; R)\big) \ \ (k = 1, \dots, m), \tag{10.3}$$

and the functionals $\varphi_k : C(I_0; R^m_+) \to R_+$ $(k = 1, \dots, m)$ satisfy

$$\begin{gathered} |\varphi_k(u_1, \dots, u_m) - \varphi_k(v_1, \dots, v_m)| \le \\ \le \varphi_{0k}(|u_1 - v_1|, \dots, |u_m - v_m|) \ \ (k = 1, \dots, m), \end{gathered} \tag{10.4}$$

where $\varphi_{0k} : C(I_0; R^m_+) \to R_+$ $(k = 1, \dots, m)$ are positively homogeneous continuous nondecreasing functionals.

Let

$$(\overline{f}_{kn})_{n=1}^{+\infty} \in D_{\overline{f}_k} \quad (k = 1, \dots, m) \tag{10.5}$$

and there exist $n_0 \in N$ such that the difference Cauchy problem

$$\Delta y(i-1) = \overline{f}_{kn}(x_1, \dots, x_k, y, x_{k+1}, \dots, x_m)(i), \ \ y(i_{kn}) = c_0 \tag{10.6}$$

has a unique solution, for any $k \in N_m$, $n > n_0$, $c_0 \in R$ and $(x_j)_{j=1}^m \in \widetilde{E}^m_n$. Then for arbitrarily given $(x_{kn_0})_{k=1}^m \in \widetilde{E}^m_{n_0}$, there obviously exists a unique sequence $(x_{kn})_{k=1}^m \in \widetilde{E}^m_n$ $(n = n_0 + 1, n_0 + 2, \dots)$ such that for any natural $n > n_0$, $(x_{kn})_{k=1}^m$ is the solution of

$$\begin{gathered} \Delta x_{kn}(i-1) = \\ = \overline{f}_{kn}(x_{1n-1}, \dots, x_{kn-1}, x_{kn}, x_{k+1\; n-1}, \dots, x_{mn-1})(i) \\ (k = 1, \dots, m), \end{gathered} \tag{10.7}$$

under the boundary conditions

$$\begin{gathered} x_{kn}(i_{kn}) = \varphi_k(q_{n-1}(x_{1n-1}), \dots, q_{n-1}(x_{mn-1})) \\ (k = 1, \dots, m).^{14} \end{gathered} \tag{10.8}$$

It should be emphasized here that in (10.7) every function x_{jn-1}, $j \in N_m$, is assumed to be extended at the point n by

$$x_{jn-1}(n) = x_{jn-1}(n-1).$$

The difference process (10.7), (10.8) $(n = n_0 + 1, n_0 + 2, \dots)$ is said to be stable if there exist $r \in]0, +\infty[$ and $\gamma \in]0, 1[$ such that for any sequences $(y_{kn})_{k=1}^m$, $(z_{kn})_{k=1}^m \in \widetilde{E}^m_n$ $(n = n_0, n_0 + 1, \dots)$, the estimates

$$\sum_{k=1}^m \|z_{kn} - y_{kn}\|_{\widetilde{E}_n} \le r \sum_{\nu=n_0}^n \gamma^{n-\nu} \varepsilon_\nu \ \ (n = n_0 + 1, \dots) \tag{10.9}$$

[14] Clearly, for any natural n (10.7), (10.8) decomposes into m independent Cauchy problems, each one, according to the above-said, being solvable.

are valid, where

$$\varepsilon_{n_0} = \sum_{k=1}^{m} \|z_{kn_0} - y_{kn_0}\|_{\widetilde{E}_{n_0}},$$
$$\varepsilon_\nu = \sum_{k=1}^{m} \big(|\varepsilon^0_{k\nu}| + \sum_{i=1}^{\nu} |\varepsilon_{k\nu}(i)|\big) \quad (\nu = n_0+1,\dots), \tag{10.10}$$

$$\varepsilon^0_{k\nu} = z_{k\nu}(i_{k\nu}) - \varphi_k(q_{\nu-1}(z_{1\nu-1}),\dots,q_{\nu-1}(z_{m\nu-1})) -$$
$$-y_{k\nu}(i_{k\nu}) + \varphi_k(q_{\nu-1}(y_{1\nu-1}),\dots,q_{\nu-1}(y_{m\nu-1})), \tag{10.11}$$

$$\varepsilon_{k\nu}(i) = \Delta z_{k\nu}(i-1) - \overline{f}_{k\nu}(z_{1\nu-1},\dots,z_{k\nu-1},z_{k\nu},z_{k+1\nu-1},\dots$$
$$\dots,z_{m\nu-1})(i) - \Delta y_{k\nu}(i-1) +$$
$$+\overline{f}_{k\nu}(y_{1\nu-1},\dots,y_{k\nu-1},y_{k\nu},y_{k+1\nu-1},\dots,y_{m\nu-1})(i). \tag{10.12}$$

Let for any $n > n_0$ and $k \in N_m$ the inequality

$$\big[\overline{f}_{kn}(x_1,\dots,x_k,x,x_{k+1},\dots,x_m)(i) - \overline{f}_{kn}(y_1,\dots,y_k,y,y_{k+1},\dots,y_m)(i) -$$
$$-h_{kn}(i)(x(\tau_{kn}(i)) - y(\tau_{kn}(i)))\big] \operatorname{sign}\big[(\tau_{kn}(i) - i_{kn})(x(\tau_{kn}(i)) -$$
$$-y(\tau_{kn}(i)))\big] \le f_{0kn}(|x_1-y_1|,\dots,|x_m-y_m|)(i) \ \text{ for } \ i \in N_n \tag{10.13}$$

be fulfilled in $\widetilde{E}^m_n$, where $f_{0kn} : (\widetilde{E}^+_n)^m \to E^+_n$ $(k = 1,\dots,m)$ are positively homogeneous continuous nondecreasing operators,

$$\|h_{kn}\|_{\widetilde{E}_n} < 1 \quad (k = 1,\dots,m), \tag{10.14}$$
$$(h_{kn})_{n=1}^{+\infty} \in D_{h_k}, \quad (f_{0kn})_{n=1}^{+\infty} \in D_{f_{0k}} \quad (k = 1,\dots,m), \tag{10.15}$$

and

$$(h_1,\dots,h_m; f_{01},\dots,f_{0m}; \varphi_{01},\dots,\varphi_{0m}) \in W(t_1,\dots,t_m). \tag{10.16}$$

Then: (a) the problem (10.1), (10.2) has a unique solution $(u^0_k)_{k=1}^m$; (b) the difference scheme (10.7), (10.8) $(n = n_0+1,\dots)$ is stable; (c) for any $(x_{kn_0})_{k=1}^m \in \widetilde{E}^m_{n_0}$, there exists a unique sequence $(x_{kn})_{k=1}^m$ $(k = n_0+1,\dots)$ of solutions of the problems (10.7), (10.8), and

$$\lim_{n\to+\infty} \|x_{kn} - p_n(u^0_k)\|_{\widetilde{E}_n} = 0 \ \ (k = 1,\dots,m). \tag{10.17}$$

Proof. By (10.5), (10.13) and (10.15), in $C(I_0; R^m)$ the inequalities

$$\big[\overline{f}_k(u_1,\dots,u_k,u,u_{k+1},\dots,u_m)(t) -$$
$$-\overline{f}_k(v_1,\dots,v_k,v,v_{k+1},\dots,v_m)(t) -$$
$$-h_k(t)(u(t)-v(t))\big] \operatorname{sign}\big[(t-t_k)(u(t)-v(t))\big] \le$$
$$\le f_{0k}(|u_1-v_1|,\dots,|u_m-v_m|)(t) \tag{10.18}$$
$$\text{for } \ t \in I_0 \ \ (k = 1,\dots,m)$$

are fulfilled. Hence the operators

$$f_k(u_1,\dots,u_m)(t)=\overline{f}_k(u_1,\dots,u_{k-1},u_k,u_k,u_{k+1},\dots,u_m)(t) \quad (10.19)$$
$$(k=1,\dots,m)$$

and the functionals φ_k $(k=1,\dots,m)$ satisfy all the conditions of Theorem 1.2, which guarantees the existence of a unique solution $(u_k^0)_{k=1}^m$ of (10.1), (10.2).

Let us prove the stability of the schemes (10.7), (10.8) $(n=n_0+1,\dots)$. It should be first of all noted that h_{kn}, f_{0kn}, h_k, f_{0k} and φ_{0k} $(k=1,\dots,m; n=n_0+1,\dots)$ satisfy the conditions of Lemma 8.2. Let γ and r be the numbers appearing in that lemma, and let $(y_{kn})_{k=1}^m$ and $(z_{kn})_{k=1}^m\in\widetilde{E}_n^m$ $(n=n_0,n_0+1,\dots)$ be arbitrary sequences. Assume

$$v_{kn}(i)=z_{kn}(i)-y_{kn}(i) \quad (k=1,\dots,m).$$

Then, because of (10.4) and (10.13), we will have

$$\begin{gathered}\big[\Delta v_{kn}(i-1)-h_{kn}(i)v_{kn}(\tau_{kn}(i))\big]\operatorname{sign}\big[(\tau_{kn}(i)-\\ -i_{kn})v_{kn}(\tau_{kn}(i))\big]\le f_{0kn}(|v_{1n-1}|,\dots,|v_{mn-1}|)(i)+|\varepsilon_{kn}(i)|\\ \text{for } i\in N_n \quad (k=1,\dots,m)\end{gathered}$$

and

$$\begin{gathered}|v_{kn}(i_{kn})\le\varphi_{0k}(q_{n-1}(|v_{1n-1}|),\dots,q_{n-1}(|v_{mn-1}|))+|\varepsilon_{kn}^0|\\ (k=1,\dots,m),\end{gathered}$$

where ε_{kn}^0 and ε_{kn} $(k=1,\dots,m)$ are given by (10.11) and (10.12). From this, due to our choice of r and γ and because of (10.10), we obtain (10.9) which in fact means the stability of the scheme under consideration.

By virtue of Theorem 4.2, (10.13) and (10.14) guarantee the unique solvability of (10.6) for any $n>n_0$, $k\in N_m$, $c_0\in R$ and $(x_j)_{k=1}^m\in\widetilde{E}_n^m$. Consequently, for arbitrarily given $(x_{kn_0})_{k=1}^m\in\widetilde{E}_{n_0}^m$, there exists a unique sequence $(x_{kn})_{k=1}^m$ $(n=n_0+1,\dots)$ of solutions of (10.7), (10.8).

To prove the theorem, it remains to show that (10.17) is fulfilled.

Since the scheme is stable, we have

$$\sum_{k=1}^m\|p_n(u_k^0)-x_{kn}\|_{\widetilde{E}_n}\le r\sum_{\nu=n_0}^n\gamma^{n-\nu}\varepsilon_\nu \quad (n=n_0+1,\dots), \quad (10.20)$$

where

$$\varepsilon_{n_0}=\sum_{k=1}^m\|x_{kn_0}-p_{n_0}(u_k^0)\|_{\widetilde{E}_0},$$

$$\varepsilon_\nu=\sum_{k=1}^m|u_k^0(t_{i_{k\nu}\nu})-\varphi_k(q_{\nu-1}(y_{1\nu-1}),\dots,q_{\nu-1}(y_{m\nu-1}))|+$$

$$+\sum_{k=1}^{m}\sum_{i=1}^{\nu}\Big|\overline{f}_{k\nu}(y_{1\nu-1},\dots,y_{k\nu-1},y_{k\nu},y_{k+1\nu-1},\dots,y_{m\nu-1})(i)-$$

$$-\int\limits_{t_{i-1\nu}}^{t_{i\nu}} f_k(u_1^0,\dots,u_k^0,u_k^0,u_{k+1}^0,\dots,u_m^0)(s)ds\Big| \tag{10.21}$$

$$(\nu=n_0+1,\dots)$$

with $y_{k\nu}=p_\nu(u_k^0)$ $(k=1,\dots,m)$. Due to the continuity of φ_k $(k=1,\dots,m)$ and because of (10.5),

$$\lim_{\nu\to+\infty}\varepsilon_\nu=0.$$

Therefore

$$\varepsilon^*=\sup\Big\{\varepsilon_\nu:\nu\geq\frac{n}{2}\Big\}\to 0 \ \text{ as } \ n\to+\infty. \tag{10.22}$$

Let $\gamma_0\in]\gamma^{\frac{1}{2}},1[$, and let $n_1>2n_0$ be so large that

$$\Big(\frac{n}{2}\Big)^{\frac{2}{n}}\leq\gamma_0^2 \quad\text{for}\quad n\geq n_1.$$

Then from (10.20) we find

$$\sum_{k=1}^{m}\|p_n(u_k^0)-x_{kn}\|_{\widetilde{E}_n}\leq r\sum_{\nu=n_0}^{[\frac{n}{2}]}\gamma^{n-\nu}\varepsilon_\nu+$$

$$+r\sum_{\nu=[\frac{n}{2}]+1}^{n}\gamma^{n-\nu}\varepsilon_\nu\leq r\varepsilon_{2n_0}^*\frac{n}{2}\gamma^{\frac{n}{2}}+\varepsilon_n^*\frac{r}{1-\gamma}\leq$$

$$\leq r\varepsilon_{2n_0}^*\gamma_0^n+\frac{r}{1-\gamma}\varepsilon_n^* \quad (n=n_1+1,\dots),$$

whence, owing to (10.22), it follows (10.17). ■

Remark 10.1. We have incidentally proved that under the conditions of Theorem 10.1, the estimate

$$\|x_{kn}-p_n(u_k^0)\|_{\widetilde{E}_n}\leq r_0(\varepsilon_{2n_0}^*\gamma_0^n+\varepsilon_n^*) \ \ (n=n_1+1,\dots) \tag{10.23}$$

is valid, where ε_n^* $(n=2n_0,2n_0+1,\dots)$ are defined by (10.21) and (10.22), $r_0=r/(1-\gamma)$ and $\gamma_0\in]0,1[$ are independent of n and $(x_{kn_0})_{k=1}^m$.

Let along with (10.4) *the conditions* (10.16), (10.18) *and*

$$\overline{f}_{kn}(x_1,\dots,x_k,x,x_{k+1},\dots,x_m)(i)=\delta_{i-1i_{kn}}h_{kn}(i)x(i)+$$

$$+(1-\delta_{i-1i_{kn}})\int\limits_{t_{i-1n}}^{t_{in}} f_k(q_{k-1}(x_1),\dots,q_{n-1}(x_k),q_{ikn}(x),q_{n-1}(x_{k+1}),\dots$$

$$\dots,q_{n-1}(x_m))(s)ds \ \ (n\in N;\ i\in N_n;\ k=1,\dots,m), \tag{10.24}$$

be fulfilled, where

$$h_{kn}(i) = \int\limits_{t_{i-1n}}^{t_{in}} h_k(s)ds, \tag{10.25}$$

while q_{ikn} $(k \in N_m; i \in N)$ *are the operators given by* (9.26) *and* (9.27). *Let further* $n_0 \in N$ *be so large that for any* $n > n_0$, *the inequalities* (10.14) *are fulfilled. Then the conclusion of Theorem* 10.1 *is valid.*

To get sure that the corollary is valid, it suffices to note that (10.18), (10.24) and (10.25) imply (10.5), (10.13) and (10.15), where

$$f_{0kn}(x_1, \dots, x_m)(i) = \int\limits_{t_{i-1n}}^{t_{in}} f_{0k}(q_{n-1}(x_1), \dots, q_{n-1}(x_m))(s)ds$$
$$(k = 1, \dots, m).$$

Remark 10.2. If along with the conditions of Corollary 10.1, we have $u_k^{0'} \in L^\alpha(I_0; R)$, $h_k \in L^\alpha(I_0; R)$ and

$$\sum_{k=1}^{m} |\overline{f}_k(u_1, \dots, u_{m+1})(t) - \overline{f}_k(v_1, \dots, v_{m+1})(t)| \leq$$
$$\leq h(t) \sum_{k=1}^{m+1} \|u_k - v_k\|_{C(I_0;R)}$$

with $1 < \alpha \leq +\infty$ and $h_k \in L(I_0; R_+)$, then

$$\sum_{k=1}^{m} \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O\Big(n^{\frac{1}{\alpha}-1}\Big) \text{ for } n \geq n_0. \tag{10.26}$$

Indeed, because of (10.21) and (10.22), in this case

$$\varepsilon_n^* = O\Big(n^{\frac{1}{\alpha}-1}\Big),$$

according to which from (10.23) we obtain (10.26).

Let the operators

$$f_k \in C\big(C(I_0; R^{m+1}); C(I_0; R)\big) \ \ (k = 1, \dots, m)$$

be bounded on every bounded set of $C(I_0; R^{m+1})$, *and let along with* (10.4) *the conditions* (10.16) *and* (10.18) *be fulfilled, where*

$$h_k \in C(I_0; R), \ f_{0k} \in C\big(C(I_0; R^m); C(I_0; R_+)\big) \ \ (k = 1, \dots, m).$$

Let next

$$\overline{f}_{kn}(x_1, \dots, x_k, x, x_{k+1}, \dots, x_m)(i) =$$

$$= \frac{b-a}{n}\overline{f}_k(q_{n-1}(x_1),\dots,q_{n-1}(x_k),q_n(x),q_{n-1}(x_{k+1}),\dots$$
$$\dots,q_{n-1}(x_m))(t_{\tau_{kn}(i)n}) \ \ (n \in N;\ i \in N_n;\ k=1,\dots,m),$$

and let $n_0 \in N$ *be so large that*

$$\frac{b-a}{n}\|h_k\|_{C(I_0;R)} < 1 \ \ \textit{for} \ \ n > n_0 \ \ (k=1,\dots,m).$$

Then the conclusion of Theorem 10.1 *is valid.*

Remark 10.3. Let the conditions of Corollary 10.2 be fulfilled, $u_k^{0\,\prime}$ $(k = 1,\dots,m)$ have bounded variation and

$$\sum_{k=1}^{m}\|\overline{f}_k(u_1,\dots,u_{m+1}) - \overline{f}_k(v_1,\dots,v_{m+1})\|_{C(I_0;R)} \le$$
$$\le h_0\sum_{k=1}^{m+1}\|u_k - v_k\|_{C(I_0;R)},$$

where $h_0 \in R_+$. Then

$$\sum_{k=1}^{m}\|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O\Big(\frac{1}{n}\Big) \ \ \text{for} \ \ n \ge n_0. \tag{10.27}$$

Consider the differential system

$$\frac{du_k(t)}{dt} = \overline{g}_k(t,u_1(t),\dots,u_k(t),u_k(t),u_{k+1}(t),\dots,u_m(t)) \tag{10.28}$$
$$(k=1,\dots,m)$$

with the boundary conditions (10.2), where

$$\overline{g}_k \in K(I_0 \times R^{m+1};R) \ \ (k=1,\dots,m).$$

We will be interested in the case where $\varphi_k : C(I_0;R^m) \to R$ $(k=1,\dots,m)$ satisfy (10.4), while the functions $\overline{g}_k$ $(k=1,\dots,m)$ satisfy

$$\begin{aligned}&\big[\overline{g}_k(t,x_1,\dots,x_k,x,x_{k+1},\dots,x_m) -\\ &-\overline{g}_k(t,\overline{x}_1,\dots,\overline{x}_k,\overline{x},\overline{x}_{k+1},\dots,\overline{x}_m) -\\ &-h_k(t)(x-\overline{x})\big]\operatorname{sign}\big[(t-t_k)(x-\overline{x})\big] \le\\ &\le \sum_{j=1}^{m}h_{kj}(t)|x_j - \overline{x}_j|, \ \ (k=1,\dots,m),\end{aligned} \tag{10.29}$$

where

$$(h_1,\dots,h_m;h_{11},\dots,h_{1m},\dots,h_{m1},\dots,h_{mm};\varphi_{01},\dots,\varphi_{0m}) \in$$
$$\in W'(t_1,\dots,t_m). \tag{10.30}$$

From Theorem 10.1, we have the following propositions.

Let the conditions (10.4), (10.29), (10.30) *and*

$$f_{kn}(x_1,\dots,x_k,x,x_{k+1},\dots,x_m)(i)=$$
$$=\delta_{i-1i_{kn}}\int\limits_{t_{i-1n}}^{t_{in}} h_k(t)dt\cdot x(i)+(1-\delta_{i-1i_{kn}})\times$$
$$\times\int\limits_{t_{i-1n}}^{t_{in}} \overline{g}_k(t,x_1(i-1),\dots,x_k(i-1),x(\tau_{kn}(i)),x_{k+1}(i-1),\dots$$
$$\dots,x_m(i-1))dt\ \ (n\in N;\ i\in N_n;\ k=1,\dots,m)$$

be fulfilled, and let $n_0\in N$ *be so large that*

$$\Big|\int\limits_{t_{i-1n}}^{t_{in}} h_k(t)dt\Big|<1\ \ \textit{for}\ \ n>n_0\ \ (i\in N_n;\ k=1,\dots,m).$$

Then: (a) the problem (10.28), (10.2) *has a unique solution* $(u_k^0)_{k=1}^m$*; (b) the difference scheme* (10.7), (10.8) $(n=n_0+1,\dots)$ *is stable; (c) for any* $(x_{kn_0})_{k=1}^m\in\widetilde{E}_{n_0}^m$*, there exists a unique sequence* $(x_{kn})_{k=1}^m$ $(n=n_0+1,\dots)$ *of solutions of the problems* (10.7), (10.8), *and*

$$\lim\|p_n(u_k^0)-x_{kn}\|_{\widetilde{E}_n}=0.$$

Let the conditions (10.4), (10.29) *and* (10.30) *be fulfilled, and let, moreover,*

$$\overline{g}_k\in C(I_0\times R^{m+1};R),\ \ h_k\in C(I_0;R),\ h_{kj}\in C(I_0;R_+)$$
$$(k,j=1,\dots,m).$$

Let further

$$\overline{f}_{kn}(x_1,\dots,x_k,x,x_{k+1},\dots,x_m)(i)=$$
$$=\frac{b-a}{n}\overline{g}_k(t_{in},x_1(i-1),\dots,x_k(i-1),x(\tau_{kn}(i)),x_{k+1}(i-1),\dots$$
$$\dots,x_m(i-1))\ \ (n\in N;\ i\in N_n;\ k=1,\dots,m)$$

and $n_0\in N$ *be so large that*

$$\frac{b-a}{n}\|h_k\|_{C(I_0;R)}<1\ \ \textit{for}\ \ n>n_0\ \ (k=1,\dots,m).$$

Then the conclusion of Corollary 10.3 *is valid.*

Remark 10.4. Let ${u_k^0}' \in L^\alpha(I_0;R)$, $h_k \in L^\alpha(I_0;R)$ $(k=1,\dots,m)$,

$$\sum_{k=1}^{m} |\overline{g}_k(t,x_1,\dots,x_{m+1}) - \overline{g}_k(t,\overline{x}_1,\dots,\overline{x}_{m+1})| \le h(t)\sum_{k=1}^{m+1} |x_k - \overline{x}_k|,$$

and the conditions of Corollary 10.3 be fulfilled, where $1 < \alpha \le +\infty$ and $h \in L(I_0;R_+)$. Then

$$\sum_{k=1}^{m} \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O\Big(n^{\frac{1}{\alpha}-1}\Big).$$

Remark 10.5. Let the conditions of Corollary 10.4 be fulfilled, ${u_k^0}'$ $(k = 1,\dots,m)$ have bounded variation, and

$$\sum_{k=1}^{m} \big|\overline{g}_k(t,x_1,\dots,x_{m+1}) - \overline{g}_k(t,\overline{x}_1,\dots,\overline{x}_{m+1})\big| \le h_0\sum_{j=1}^{m+1} |x_j - \overline{x}_j|,$$

where $h_0 \in R_+$. Then

$$\sum_{k=1}^{m} \|x_{kn} - p_n(u_k^0)\|_{\widetilde{E}_n} = O\Big(\frac{1}{n}\Big).$$

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Authors' addresses:

Sh. Gelashvili
Faculty of Mechanics and Mathematics
I. Javakhishvili Tbilisi State University
2, University Str., 380043 Tbilisi
Georgia

I. Kiguradze
A. Razmadze Mathematical Institute
Georgian Academy of Sciences
1, M. Aleksidze Str., 380093 Tbilisi
Georgia

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