Memoirs on Differential Equations and Mathematical Physics
Volume 50, 2010, 139–162

Vakhtang Paatashvili

HASEMAN'S PROBLEM IN CLASSES OF FUNCTIONS REPRESENTABLE BY THE CAUCHY TYPE INTEGRAL WITH DENSITY FROM $L^{p(\cdot)}(\Gamma;\rho)$

Abstract. Haseman's problem $\phi^+(\alpha(t)) = a(t)\phi^-(t) + b(t)$ is investigated in the class of the Cauchy type integrals with density from the weighted Lebesgue space with a variable exponent $p(t)$. Besides that the problem is studied in a more general statement than that considered earlier, for the first time we cover the case where the boundary condition is prescribed on a non-smooth curve, namely, on a curve with the chord condition.

2010 Mathematics Subject Classification. 47B38, 30E20, 30E25, 42B20, 45P05.

Key words and phrases. Curves with a chord-arc-condition, Lebesgue space with a variable exponent, Haseman's boundary value problem, Cauchy type integral with density in $L^{p(\cdot)}(\Gamma;\rho)$.

რეზიუმე. ჰაზემანის სასაზღვრო ამოცანა $\phi^+(\alpha(t)) = a(t)\phi^-(t) + b(t)$ შესწავლილია კოშის ტიპის ისეთ ინტეგრალთა კლასში, რომელთა სიმკვრივე მიეკუთვნება ლებეგის წონიან ცვლადმაჩვენებლიან სივრცეს. ხერხდება ამოცანის გამოკვლევა არაგლუვი წირებით შემოსაზღვრული არეებისათვის. სახელდობრ, არეებისათვის რომელთა საზღვრის ნებისმიერი ორი წერტილის შემაერთებელი უმოკლესი რკალის სიგრძის შეფარდება მათი მომჭიმავი ქორდის სიგრძესთან შემოსაზღვრული ფუნქციაა.

Let Γ be a simple, closed, rectifiable, oriented curve dividing the complex plane into the domains D^+ and D^- of which the point at infinity is contained in D^-.

Haseman's boundary value problem is called the following one: find a function ϕ, analytic on the plane cut along Γ, from a given class of functions A with the boundary values $\phi^+(t)$, $t \in \Gamma$, (from D^+) and $\phi^-(t)$ (from D^-) satisfying the condition

$$\phi^+(\alpha(t)) = a(t)\phi^-(t) + b(t), \tag{1}$$

where $\alpha = \alpha(t)$ is a homeomorphism of Γ onto itself, and $a(t)$ and $b(t)$ are given on Γ functions.

C. Haseman was the first to investigate the homogeneous problem [1]

$$\phi^+(\alpha(t)) = a(t)\phi^-(t). \tag{2}$$

Later T. Carleman [2] investigated the problem with shifts. D. A. Kveselava ([3]–[4]) gave a complete solution of the problem (1). He supposed that Γ is a Lyapunov curve, $a(t)$ and $b(t)$ belong to the Hölder class H, and $\alpha(t)$ is a preserving orientation homeomorphism of Γ onto itself with $\alpha'(t) \neq 0$ and $\alpha' \in H$.

Haseman's problem in the vector case was considered by N. P. Vekua. His results, as well as those obtained by other authors in this direction can be found in [5].

In the sequel, the boundary value problems with shifts became an investigation topic for a great deal of authors. Many of their result are exposed in a book by G. S. Litvinchuk [6].

The case where A is the $K^p(\Gamma)$-class of the functions representable by the Cauchy type integrals with density from $L^p(\Gamma)$, $p > 1$, has been considered by B .V. Khvedelidze [7], B. V. Khvedelidze and G. F. Manjavidze [8], I. B. Simonenko [9], G. F. Manjavidze [10] as well as by many other authors.

When the boundary conditions involve functions which on different parts of the boundary are integrable with different powers, then it is advisable instead of the common Lebesgue spaces to introduce into consideration a class of functions integrable in a somewhat different sense, with a finer regard for local singularities of the functions. Such are the Lebesgue spaces with variable exponent.

The generalized Lebesgue spaces have been intensively studied since 1970s. Important results were obtained for various operators in these spaces and their applications. Not going into details (the reader can be referred to the works [11]–[14]), it is worth mentioning the results that are significant for the boundary value problems of the function theory in general, and for the given work in particular, which were obtained by V. M. Kokilashvili and S. G. Samko. They investigated in detail maximal operators and singular operators with the Cauchy kernel in those spaces. They found a continuity criterion and the compactness conditions ([15]–[17]), promoting further investigation of boundary value problems of the function theory and singular integral equations in new general and more appropriate statements ([18]–[23]).

Relying on the results of [16]–[17] and [20] and using the methods of solution of the problem (1) described in [10], we study Haseman's problem in a new, more general statement. The results concerning some particular cases considered in the present paper have been announced in [24].

1. The Curves with the Chord Condition and One of Their Properties

Let Γ be a simple, rectifiable curve and $z = z(\zeta)$, $0 \le \zeta \le \ell$, be its equation with respect to the arc abscissa. We say that Γ is a curve with the chord condition or the Lavrentyev curve (see, e.g., [25, p. 163]), if there exists a constant $m > 0$ such that for any t, $\tau \in \Gamma$ we have

$$|t-\tau| \ge m\, s(t,\tau), \tag{3}$$

where $s(t,\tau)$ is the length of the least of the two arcs of Γ connecting the points t and τ. The set of such curves is denoted by HC.

Let $\alpha = \alpha(t)$ be a homeomorphism of Γ onto itself, and let at every point t there exist the derivative $\alpha'(t)$ satisfying $\alpha'(t) \ne 0$ and $\alpha' \in H(\mu)$, i.e., there exist constants M and μ, $0 < \mu \le 1$, such that for any t_1, $t_2 \in \Gamma$ we have

$$\big|\alpha'(t_1) - \alpha'(t_2)\big| < M|t_1 - t_2|^{\mu}.$$

Lemma 1. *If* $\Gamma \in$ HC *and*

$$K(\tau,t) = \frac{\alpha'(t)}{\alpha(\tau)-\alpha(t)} - \frac{1}{\tau - t}\,, \quad \tau,\, t \in \Gamma,$$

then there exists a constant c such that

$$|K(\tau,t)| < c[s(\tau,t)]^{\mu-1}.$$

Proof. Let $\tau = z(\sigma)$, $t = t(s)$ and $\gamma(\tau,t)$ be the least arc of Γ with the ends τ and t. Then

$$K(\tau,t) = \frac{\alpha'(\tau)(\tau-t) - [\alpha(\tau)-\alpha(t)]}{(\tau-t)(\alpha(\tau)-\alpha(t))} = \frac{\alpha'(\tau)\int\limits_{\gamma(\tau,t)} du - \int\limits_{\gamma(\tau,t)} \alpha'(u)\,du}{(\tau-t)(\alpha(\tau)-\alpha(t))} =$$

$$= \frac{\int\limits_{\gamma(\tau,t)} [\alpha'(\tau)-\alpha'(u)]\,du}{(\tau-t)(\alpha(\tau)-\alpha(t))} = \frac{\varphi(\tau,t)}{(\tau-t)(\alpha(\tau)-\alpha(t))}\,. \tag{4}$$

Next,

$$|\varphi(\tau,t)| = \bigg|\int\limits_{\gamma(\tau,t)} \big[\alpha'(z(\sigma)) - \alpha'(z(\zeta))\big] z'(\zeta)\,d\zeta\bigg| \le$$

$$\le M \int\limits_{\gamma(\tau,t)} \big|z(\sigma) - z(\zeta)\big|^{\mu}\,d\zeta \le M[s(\tau,t)]^{1+\mu}.$$

Now, by virtue of (3), from (4) we obtain

$$|K(\tau,t)| \leq \frac{M[s(\tau,t)]^{1+\mu}}{[ms(\tau,t)]^2|\alpha(\tau)-\alpha(t)|\,|\tau-t|^{-1}} = \frac{M}{m^2}\,\frac{[s(\tau,t)]^{\mu-1}}{\psi(\tau,t)}\,,$$

where

$$\psi(\tau,t) = |\alpha(\tau)-\alpha(t)|\,|\tau-t|^{-1}.$$

Show that $\psi(\tau,t)$ tends uniformly to $\alpha'(t)$ as $\tau\to t$. This follows from the inequality

$$\Big|\frac{\alpha(\tau)-\alpha(t)}{\tau-t}-\alpha'(t)\Big| = \left|\frac{\int\limits_{\gamma(\tau,t)}[\alpha'(u)-\alpha'(t)]\,du}{\tau-t}\right| \leq \frac{M}{m}\,[s(\tau,t)]^{\mu}.$$

From the above-said and the assumption $\alpha'(t)\neq 0$ (i.e. $\min|\alpha'(t)| = k > 0$), we obtain

$$|K(\tau,t)| \leq \frac{2M}{km^2}\,[s(\tau,t)]^{\mu-1} = c[s(\tau,t)]^{\mu-1},\ \ c=\frac{2M}{km^2}\,. \qquad \square$$

2. The Classes of Functions $\mathcal{P}$, $\widetilde{K}^{p(\cdot)}(\Gamma;\rho)$ and $K^{p(\cdot)}(\Gamma;\rho)$

1^0. Let $p:\Gamma\to R^+$ be a real function satisfying the following conditions:

(1) there is a constant A such that for any $t_1,\,t_2\in\Gamma$ we have

$$\big|p(t_1)-p(t_2)\big| < \frac{A}{|\ln(t_1-t_2)|}\,; \tag{5}$$

(2) $p_0 = \min\limits_{t\in\Gamma} p(t) > 1$.

The set of all such functions is denoted by $\mathcal{P}$.

Obviously, the condition (1) can be replaced by the condition

($1'$) there exist constants A and $\delta>0$ such that the inequality (5) holds for any $t_1,\,t_2\in\Gamma$, $|t_1-t_2|<\delta$.

Let $p_\alpha(t) = p(\alpha(t))$ and $\overline{p}_\alpha(t) = \max(p(t),p_\alpha(t))$.

Lemma 2. *If* $p\in\mathcal{P}$, $\alpha'\in H$, $\alpha'\neq 0$, *then the functions* $p_\alpha(t)$ *and* $\overline{p}_\alpha(t)$ *also belong to* $\mathcal{P}$.

Proof. Let us first show that $p_\alpha\in\mathcal{P}$. We have

$$\big|p(\alpha(t_1))-p(\alpha(t_2))\big| <$$
$$< \frac{A}{\big|\ln|\alpha(t_1)-\alpha(t_2)|\big|} = \frac{A}{\big|\ln\big|\frac{\alpha(t_1)-\alpha(t_2)}{t_1-t_2}\big|+\ln|t_1-t_2|\big|}\,.$$

Since $0<m_1\leq\big|\frac{\alpha(t_1)-\alpha(t_2)}{t_1-t_2}\big|\leq m_2$, we have $\sup\big|\ln\big|\frac{\alpha(t_1)-\alpha(t_2)}{t_1-t_2}\big|\big| < M$, and hence for $|t_1-t_2|<e^{-2M}=\delta$ we have

$$\big|p_\alpha(t_1)-p_\alpha(t_2)\big| = \big|p(\alpha(t_1))-p(\alpha(t_2))\big| < \frac{2A}{|\ln(t_1-t_2)|}\,. \tag{6}$$

Moreover, $\min p_\alpha(t) = \min p(t) = p_0 > 1$, and thus $p_\alpha\in\mathcal{P}$.

Consider now the function $\overline{p}_\alpha(t)$. Let t_1 be an arbitrary point on Γ and $\overline{p}_\alpha(t_1) = p(t_1) \neq p_\alpha(t_1)$. As far as the functions p and p_α are continuous, there exists $\delta_1 = \delta_1(t_1)$ such that for $|t_1 - t_2| < \delta_1$ we have $\overline{p}_\alpha(t_2) = p(t_2)$. Then $\big|\overline{p}_\alpha(t_1) - \overline{p}_\alpha(t_2)\big| = |p(t_1) - p(t_2)|$, and the inequality (5) is fulfilled since $p \in \mathcal{P}$. Analogously, if $\overline{p}_\alpha(t_1) = p_\alpha(t_1) \neq p(t_1)$, then there exists $\delta_2 = \delta_2(t_1)$ such that for $|t_1 - t_2| < \delta_2$ we have $\big|\overline{p}_\alpha(t_1) - \overline{p}_\alpha(t_2)\big| = |p_\alpha(t_1) - p_\alpha(t_2)|$, and the inequality (5) is fulfilled since $p_\alpha \in \mathcal{P}$. When $\overline{p}_\alpha(t_1) = p(t_1) = p_\alpha(t_1)$, then if $\overline{p}_\alpha(t_2) = p(t_2)$ we write $\big|\overline{p}_\alpha(t_1) - \overline{p}_\alpha(t_2)\big| = |p(t_1) - p(t_2)|$, while if $\overline{p}_\alpha(t_2) = p_\alpha(t_2)$ we have $\big|\overline{p}_\alpha(t_1) - \overline{p}_\alpha(t_2)\big| = |p_\alpha(t_1) - p_\alpha(t_2)|$. In both cases the inequality (5) (for any t_2) is valid.

Thus for every point $t_1 \in \Gamma$ there is a neighborhood $u(t_1;\delta)$ ($\delta = \min(m\delta_1, m\delta_2)$) in which the inequality (5) is true. From the set of these intervals we select a finite covering in such a way that none of the intervals of that covering contains another one. Consider the set of all ends of those intervals. They divide Γ into a finite number of intervals. Let δ_0 be the length of the least interval. If $t_1, t_2 \in \Gamma$ and $|t_1 - t_2| < \delta < \frac{\delta_0}{4}$, then the points t_1 and t_2 get into some of the intervals of the initial covering, and hence the inequality (5) is fulfilled. Thus $\overline{p}_\alpha \in \mathcal{P}$. $\square$

2^0. We dwell especially on the case $\rho(t) = 1$. Then $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha) = L^{p_\alpha(\cdot)}(\Gamma)$. By virtue of the inequalities $|f|^{\overline{p}_\alpha(t)} < |f|^{p(t)} + |f|^{p_\alpha(t)}$ and $|f|^{p(t)} < 1 + |f|^{\overline{p}_\alpha(t)}$, $|f|^{p_\alpha(t)} < 1 + |f|^{\overline{p}_\alpha(t)}$ we conclude that

$$L^{p(\cdot)}(\Gamma) \cap L^{p_\alpha(\cdot)}(\Gamma) = L^{\overline{p}_\alpha(\cdot)}(\Gamma).$$

Consequently, the set $L^{p(\cdot)}(\Gamma) \cap L^{p_\alpha(\cdot)}(\Gamma) = L_{p,\alpha}(\Gamma)$ can be considered as the space $L^{\overline{p}_\alpha(\cdot)}(\Gamma)$.

3^0. Let $t_k \in \Gamma$ and

$$\rho(t) = \prod_{k=1}^{N} |t - t_k|^{\nu_k}, \quad \nu_k \in \mathbb{R}. \tag{7}$$

Suppose that

$$L^{p(\cdot)}(\Gamma;\rho) =$$

$$= \Big\{ f : \ I_{p,\rho}(f) = \int_\Gamma \big|f(t)\rho(t)\big|^{p(t)} |dt| = \int_0^\ell \big|f(t(\zeta))\rho(t(\zeta))\big|^{p(t(\zeta))} d\zeta \Big\}.$$

If $p \in \mathcal{P}$ and

$$-\frac{1}{p(t_k)} < \nu_k < \frac{1}{p'(t_k)}\,, \quad k = 1, \dots, N, \ \ p'(t) = \frac{p(t)}{p(t) - 1}\,, \tag{8}$$

then $L^{p(\cdot)}(\Gamma;\rho)$ is a Banach space with the norm

$$\|f\|_{p(\cdot),\rho} = \inf \Big\{ \lambda > 0 : \ I_{p,\rho}\Big(\frac{f}{\lambda}\Big) \le 1 \Big\}.$$

$L^{p(\cdot)}(\Gamma; \frac{1}{\rho})$ is the conjugate to that space. For $\rho(t) \equiv 1$, we put $L^{p(\cdot)}(\Gamma; 1) = L^{p(\cdot)}(\Gamma)$.

We denote by $\widetilde{K}^{p(\cdot)}(\Gamma; \rho)$ the set of the functions $\phi(z)$ analytic in the plane cut along Γ and representable in the form

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t-z}\, dt + q(z) = K_\Gamma(f) + q(z), \quad z \notin \Gamma, \quad f \in L^{p(\cdot)}(\Gamma; \rho),$$

where $q(z)$ is a polynomial.

The subset of those functions from $\widetilde{K}^{p(\cdot)}(\Gamma; \rho)$ for which $q(z) = 0$ is denoted by $K^{p(\cdot)}(\Gamma; \rho)$.

Assume $\widetilde{K}^{p(\cdot)}(\Gamma) = \widetilde{K}^{p(\cdot)}(\Gamma; 1)$, $K^{p(\cdot)}(\Gamma) = K^{p(\cdot)}(\Gamma; 1)$.

If Γ is a Carleson curve (in particular, if $\Gamma \in \mathrm{HC}$), $p \in \mathcal{P}$, and for the weight ρ given by equality (7) the conditions (8) are fulfilled, then the function $\phi(z) \in K^{p(\cdot)}(\Gamma; \rho)$ for almost all $t \in \Gamma$ has angular boundary values

$$\phi^\pm(t) = \pm\frac{1}{2}\, f(t) + \frac{1}{2\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma.$$

Note that the functions $\phi^+(t)$ and $\phi^-(t)$ belong to $L^{p(\cdot)}(\Gamma; \rho)$ since the singular Cauchy operator

$$S_\Gamma : f \to S_\Gamma f, \quad (S_\Gamma f)(t) = \frac{1}{\pi i} \int_\Gamma \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \Gamma,$$

is continuous in $L^{p(\cdot)}(\Gamma; \rho)$ ([15]–[17]). Moreover, the conditions (8) are necessary for S_Γ to be continuous in $L^{p(\cdot)}(\Gamma; \rho)$ ([17], [18]).

Further, $\phi(z) - q(z) = K_\Gamma f \in E^\alpha(D^\pm)$, $\alpha > 0$, and if $\phi \in K^{p(\cdot)}(\Gamma)$, then $\phi \in E^{p_0}(D^\pm)$ ($p_0 = \min p(t)$) (see [20, Theorem 3.3]), where $E^\delta(D^\pm)$ denotes Smirnov's class (for definition, see [26, p. 203]).

Here we present one more property of the functions from $K^{p(\cdot)}(\Gamma; \rho)$.

Let $\Gamma \in HC$. It follows from (8) that $\rho^{-1}(t) \in L^{p'(\cdot)+\varepsilon}(\Gamma)$ for some $\varepsilon > 0$. Moreover, for $f \in L^{p(\cdot)}(\Gamma; \rho)$, we have $f(t) = \varphi(t)\rho^{-1}(t)$, $\varphi \in L^{p(\cdot)}(\Gamma)$. To the above product we apply the generalized Hölder's inequality [27]

$$\Big| \int_\Gamma \psi(t) g(t)\, dt \Big| < c\|\psi\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

Taking as $p(t)$ the function $p(t)/(1+\eta)$, $\eta > 0$, we obtain

$$\int_\Gamma |f|^{1+\eta}\, d\zeta = \int_\Gamma |\varphi|^{1+\eta} \Big|\frac{1}{\rho}\Big|^{1+\eta} d\zeta \le c_0 \|\varphi\|_{p(\cdot)} \Big\|\frac{1}{\rho}\Big\|_{p'(\cdot)+\widetilde{\eta}(\cdot)},$$

$$\widetilde{\eta}(t) = \frac{\eta p(t)}{p(t) - 1 - \eta}\,,$$

whence it follows that if $\max \widetilde{\eta}(t) < \varepsilon$, then $f \in L^{1+\eta}(\Gamma)$. For that it suffices to take $\eta \in \big(0; \varepsilon \min \frac{1}{p'(t)}\big)$. For such η, we have $K_\Gamma f \in E^{1+\eta}(D^\pm)$ [28, p. 29].

3. The Classes $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ and $K_\alpha^{p(\cdot)}(\Gamma;\rho)$. The Statement of the Problem

1^0. A solution of the problem (1) is sought in classes of functions representable by the Cauchy type integral with density integrable with a variable exponent.

If $\phi \in K^{p(\cdot)}(\Gamma;\rho)$ satisfies the boundary condition (1), then $\phi^+(t) = a(\beta(t))\phi^-(\beta(t)) + b(\beta(t))$, where $\beta = \beta(t)$ is the function inverse to $\alpha(t)$. Since $\phi^+ \in L^{p(\cdot)}(\Gamma;\rho)$, we must have $\big[a(\beta(t))\phi^-(\beta(t)) + b(\beta(t))\big] \in L^{p(\cdot)}(\Gamma;\rho)$. Therefore besides the assumptions $\phi \in K^{p(\cdot)}(\Gamma;\rho)$, $b \in L^{p(\cdot)}(\Gamma;\rho)$, it is natural to require the fulfilment of the conditions

$$\phi^-(\beta(t)) \in L^{p(\cdot)}(\Gamma;\rho), \quad b(\beta(t)) \in L^{p(\cdot)}(\Gamma;\rho).$$

This is equivalent to the requirement that

$$\phi^- \in L^{p(\alpha(t))}(\Gamma;\rho(\alpha(t))), \quad b \in L^{p(\alpha(t))}(\Gamma;\rho(\alpha(t))).$$

Introduce the notation

$$\rho_\alpha(t) = \rho(\alpha(t)), \quad L_{p,\alpha}(\Gamma;\rho) = L^{p(\cdot)}(\Gamma;\rho) \cap L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha).$$

Since $p_\alpha \in \mathcal{P}$, for the singular Cauchy operator S_Γ to act continuously in $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$, where

$$\rho_\alpha(t) = \prod_{k=1}^N |\alpha(t) - t_k|^{\nu_k} = \prod_{k=1}^N \big|\alpha(t) - \alpha(\beta(t_k))\big|^{\nu_k} \sim \prod_{k=1}^N |t - \beta(t_k)|^{\nu_k},$$

it is necessary and sufficient that the conditions

$$-\frac{1}{p(\alpha(\beta(t_k)))} < \nu_k < \frac{1}{p'(\alpha(\beta(t_k)))}\,, \quad k = 1, \dots, N,$$

be fulfilled [18]. In view of the fact that $\alpha(\beta(t_k)) = t_k$, these conditions coincide with the conditions (8). Thus if $p \in \mathcal{P}$, $\alpha' \in H$, $\alpha' \neq 0$ and the conditions (8) are fulfilled, then the operator S_Γ is continuous in $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$.

2^0. Along with the classes $\widetilde{K}^{p(\cdot)}(\Gamma;\rho)$ and $K^{p(\cdot)}(\Gamma;\rho)$ we introduce the following classes of holomorphic functions

$$\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho) = \Big\{\phi:\ \phi \in \widetilde{K}^{p(\cdot)}(\Gamma;\rho),\ \phi^- \in L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)\Big\},$$
$$p_\alpha(t) = p(\alpha(t)),\ \ \rho_\alpha(t) = \rho(\alpha(t)),$$
$$K_\alpha^{p(\cdot)}(\Gamma;\rho) = \Big\{\phi:\ \phi \in K^{p(\cdot)}(\Gamma;\rho),\ \phi^- \in L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)\Big\}.$$

If $\rho = 1$, then according to 2^0 of Sect. 2 we have

$$K_\alpha^{p(\cdot)}(\Gamma;1) = K_\alpha^{p(\cdot)}(\Gamma) = \Big\{\phi:\ \phi \in K^{p(\cdot)}(\Gamma),\ \phi^- \in L^{\overline{p}_\alpha(\cdot)}(\Gamma)\Big\},$$
$$\overline{p}_\alpha(t) = \max\big\{p(t), p(\alpha(t))\big\}.$$

3^0. We consider Haseman's problem under the following assumptions.

Let $\Gamma \in \mathrm{HC}$, $\alpha = \alpha(t)$ be a preserving orientation homeomorphism of Γ onto itself, $\alpha'(t) \neq 0$, $\alpha' \in H$, $a(t)$ be a continuous non-zero function on Γ, $b \in L_{p,\alpha}(\Gamma;\rho)$. Find a function ϕ from $K_\alpha^{p(\cdot)}(\Gamma;\rho)$ (or $\widetilde{K}^{p(\cdot)}(\Gamma;\rho)$), whose boundary values $\phi^+(t)$ and $\phi^-(t)$ almost everywhere on Γ satisfy the condition (1).

In the sequel, we will not repeat the assumptions regarding the given and unknown elements of the problem.

4. Conditions of Solvability of the Problem (1) in the Classes $K_\alpha^{p(\cdot)}(\Gamma;\rho)$ and $K_\alpha^{p(\cdot)}(\Gamma)$

Lemma 3. *For the problem* (1) *to be solvable in the class* $K_\alpha^{p(\cdot)}(\Gamma;\rho)$ *with a continuous coefficient* $a(t)$ *and* $b(t) \in L_{p,\alpha}(\Gamma;\rho)$, *it is necessary that the condition*

$$\int_\Gamma \widetilde{b}(t)v(t)\,dt = 0 \tag{9}$$

be fulfilled, where

$$\widetilde{b}(t) = b(t) + \frac{1}{\pi i}\int_\Gamma \frac{b(\tau)}{\tau - t}\,d\tau + \int_\Gamma K(t,\tau)b(\tau)\,d\tau$$

and $v(t)$ *is an arbitrary solution of the class* $L^{p'_\alpha(\cdot)}(\Gamma;\frac{1}{\rho})$ *or* $L^{p'_\alpha(\cdot)}(\Gamma;\frac{1}{\rho_\alpha})$ *of the equation*

$$K'(a)v = -\frac{1}{\pi i}\int_\Gamma \frac{a(t)+a(t_0)}{t-t_0}\,v(t)\,dt - \frac{a(t_0)}{\pi i}\int_\Gamma K(t,t_0)v(t)\,dt = 0. \tag{10}$$

Proof. Since the functions from $E^\delta(D^+)$ and $E^\delta(D^-)$, $\delta \geq 1$, are representable by the Cauchy integrals in the domains D^+ and D^-, respectively (see, e.g., [26, pp. 205–208]), and the restrictions of the functions from $K^{p(\cdot)}(\Gamma;\rho)$ on D^+ and D^- belong to $E^{1+\eta}(D^+)$ and $E^{1+\eta}(D^-)$, $\eta > 0$ (see Sect. 2), therefore every solution $\phi(z)$ of the problem (1) is representable in the form

$$\phi(z) = \begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{\phi^-(t)}{t-z}\,dt, & z \in D^-, \\ \dfrac{1}{2\pi i}\displaystyle\int_\Gamma \frac{\phi^+(t)}{t-z}\,dt, & z \in D^+. \end{cases}$$

Denoting $\mu(t) = \phi^-(t)$, we have $\mu \in L_{p,\alpha}(\Gamma;\rho)$. By the assumptions regarding $\phi^-(t)$ and $b(t)$, the function $a(\beta(t))\mu(\beta(t)) + b(\beta(t))$ belongs to

$L^{p(\cdot)}(\Gamma;\rho)$, and taking into account the boundary condition (1), we can write

$$\phi(z)=\begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int\limits_{\Gamma}\dfrac{\mu(t)}{t-z}\,dt, & z\in D^-,\\ \dfrac{1}{2\pi i}\displaystyle\int\limits_{\Gamma}\dfrac{a(\beta(t)\mu(\beta(t))+b(\beta(t))}{t-z}\,dt, & z\in D^+. \end{cases} \tag{11}$$

Using the Plemelj–Sokhotskiĭ formula for the Cauchy type integrals, from (11) and (1) we derive

$$K(a)\mu\equiv\frac{1}{\pi i}\int\limits_{\Gamma}\frac{a(t)+a(t_0)}{t-t_0}\,\mu(t)\,dt+M(a)\mu=\widetilde{b}(t_0), \tag{12}$$

where

$$M(a)\mu=\frac{1}{\pi i}\int\limits_{\Gamma}K(t_0,t)a(t)\mu(t)\,dt,\quad K(t_0,t)=\frac{\alpha'(t)}{\alpha(t)-\alpha(t_0)}-\frac{1}{t-t_0}\,.$$

Since the operator S_Γ is continuous in the spaces $L^{p(\cdot)}(\Gamma;\rho)$ and $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$, we have $\widetilde{b}\in L_{p,\alpha}(\Gamma;\rho)$, and moreover, by the assumption, $\mu\in L_{p,\alpha}(\Gamma;\rho)$. Hence it is not difficult to show that every solution μ of the equation (12) belonging to $L_{p,\alpha}(\Gamma;\rho)$ generates (by means of the formula (11)) a function $\phi\in K_\alpha^{p(\cdot)}(\Gamma;\rho)$ satisfying the condition (1).

We write the operator $K(a)\mu$ in the form

$$\begin{aligned} K(a)\mu&=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{a(t)-a(t_0)}{t-t_0}\mu(t)\,dt+\frac{2a(t_0)}{\pi i}\int\limits_{\Gamma}\frac{\mu(t)}{t-t_0}\,dt+M(a)\mu=\\ &=2aS_\Gamma\mu+T(a)\mu+M(a)\mu, \end{aligned}$$

where

$$T(a)\mu=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{a(t)-a(t_0)}{t-t_0}\,\mu(t)\,dt.$$

According to Lemma 1, the kernel of the operator $M(a)\mu$ has a weak singularity. Therefore it is a compact operator in the spaces $L^{p(\cdot)}(\Gamma;\rho)$ and $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$ (see [16]). Moreover, since $a(t)$ is continuous, $T(a)\mu$ is a compact operator in the above-mentioned spaces. This can be proved just in the same way as for the constant p (see, e.g., [29, p. 85]).

Thus

$$K(a)\mu=2aS_\Gamma\mu+\big(T(a)\mu+M(a)\mu\big),$$

where the operator $[T(a)+M(a)]\mu$ is completely continuous in the spaces $L^{p(\cdot)}(\Gamma;\rho)$ and $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$.

Since $a(t_0)\neq 0$ and a is continuous, and Γ is a Carleson curve, the operator $K(a)\mu$ in those spaces is Noetherian, and its index in both cases is equal to zero. Hence $K(a)\mu$ is a Fredholm operator in $L^{p(\cdot)}(\Gamma;\rho)$ and $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$ (see, e.g., [19]–[20]).

For the equation (12) to have a solution $\mu \in L^{p(\cdot)}(\Gamma;\rho)$, it is necessary and sufficient that the condition (9) be fulfilled in which $v(t)$ is any solution of the class $L^{p'(\cdot)}(\Gamma;\frac{1}{\rho})$ of the conjugate equation $K'(a)v = 0$, i.e., the equation (10). For the equation (12) to have a solution $\mu \in L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$, it is necessary and sufficient that the function $\widetilde{b} = \widetilde{b}(t_0)$ satisfy the condition (9), where this time v is any solution of the equation (10) of the class $L^{p'_\alpha(\cdot)}(\Gamma;\frac{1}{\rho_\alpha})$.

Consequently, for the equation (12) to have a solution $\mu \in L_{p,\alpha}(\Gamma;\rho)$, it is necessary that the conditions formulated in the lemma be fulfilled. $\square$

2^0. Suppose now that $\rho(t) = 1$. In this case

$$K_\alpha^{p(\cdot)}(\Gamma) = \big\{\phi:\ \phi \in K^{p(\cdot)}(\Gamma),\ \phi^- \in L^{\overline{p}_\alpha(\cdot)}(\Gamma)\big\}$$

(see 2^0 of Sect. 2). Hence if we solve the equation (12) in the class $L^{\overline{p}_\alpha(\cdot)}(\Gamma)$, then using the formula (11) we obtain all solutions of the problem (1). Since the space $L^{\overline{p}'_\alpha(\cdot)}(\Gamma)$ is conjugate to the space $L^{\overline{p}_\alpha(\cdot)}(\Gamma)$, following the way of proving Lemma 3 we can prove the following

Theorem 1. *For the problem* (1) *to be solvable in the class* $K_\alpha^{p(\cdot)}(\Gamma)$ *with a continuous coefficient* $a(t)$ *and* $b(t) \in L^{\overline{p}_\alpha(\cdot)}(\Gamma)$*, it is necessary and sufficient that the conditions* (9) *be fulfilled, where* $\widetilde{b} = b + S_\Gamma b + \int\limits_\Gamma K(\cdot,\tau)\mu(\tau)\,d\tau$*, and* $v(t)$ *is an arbitrary solution of the class* $L^{\overline{p}'_\alpha(\cdot)}(\Gamma)$ *of the equation* (10).

5. The Boundary Value Problem with the Coefficient $[\alpha'(t)]^{-1}$

1^0. To investigate the problem (1) with the coefficient $[\alpha'(t)]^{-1}$, we will need the following assertion proved in [10] (here we join the results of two Lemmas 2.1 (p. 73) and 2.2 (p. 79)).

Let Γ be a simple, closed, rectifiable curve bounding the domains D^+ and D^-. If $\omega^+(z)$ is holomorphic in D^+ and continuous in $\overline{D^+}$, $\omega^-(z) = Az + \omega_0^-(z)$, where $A = \text{const} \neq 0$, $\omega_0^-(z)$ is holomorphic in D^- and continuous in $\overline{D^-}$, the set $\gamma:\ \tau = \omega^-(t)$, $t \in \Gamma$, has no inner points and

$$\omega^+(\alpha(t)) = \omega^-(t),\ \ t \in \Gamma, \tag{13}$$

then the functions $\omega^+(z)$ and $\omega^-(z)$ are schlicht in the domains D^+ and D^-, respectively, and the curve γ is simple.

If, however, $A = 0$, then $\omega^+(z) = c$, $\omega^-(z) = c$, $c = \text{const}$.

2^0. **Lemma 4.** *In the class of functions* Ψ *holomorphic in the plane cut along* Γ*, with the restrictions* $\Psi^+(z)$ *and* $\Psi^-(z)$ *in the domains* D^+ *and* D^-*, respectively, belonging to* $E^1(D^+)$ *and* $E^1(D^-)$*, the boundary problem*

$$\alpha'(t)\Psi^+(t) = \Psi^-(t) \tag{14}$$

has only the trivial solution. In particular, the problem (14) *in any class* $K^{p(\cdot)}(\Gamma;\rho)$*, and hence in the class* $K_\alpha^{p(\cdot)}(\Gamma;\rho)$*, has only the trivial solution.*

Proof. By $F^+(z)$ and $F^-(z)$ we denote the primitives of the functions $\Psi^+(z)$ and $\Psi^-(z)$, respectively. Since $\Psi^\pm \in E^1(D^\pm)$, the primitives are continuously extendable up to Γ and absolutely continuous on Γ with respect to the arc abscissa (see, e.g., [26, p. 208]), and by the condition (3), they are continuous with respect to the parameter t as well. Moreover, from the condition $\int\limits_\Gamma \Psi^-(t)\,dt = 0$ it follows that the function $F^-(z)$ is single-valued in D^-. The condition (14) yields

$$F^+(\alpha(t)) = F^-(t).$$

Thus for the function

$$F(z) = \begin{cases} F^+(z), & z \in D^+ \\ F^-(z), & z \in D^- \end{cases} \tag{15}$$

the conditions given in the previous section are fulfilled with $A = 0$. Hence $F(z) \equiv 0$. Thus $\Psi(z) = F'(z) = 0$.

The final part of the assertion of the lemma follows from the fact that $\Psi(z) = K_\Gamma f \in E^1(D^\pm)$ when $f \in L^{p(\cdot)}(\Gamma;\rho)$ (see 2^0 of Sect. 2). □

6. The Problem (1) with the Coefficient $a(t) = 1$

1^0. Following the way of investigation of the problem (1) for $a(t) = 1$ described in [10, pp. 79–80], we, first of all, prove that the following lemma is valid.

Lemma 5. *If the equation* (10) *with* $a(t) = 1$ *has a solution* $v \in L_{p',\alpha}(\Gamma; 1/\rho)$, *then* $v = 0$.

Proof. Consider the function

$$N(z) = \begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int\limits_\Gamma \dfrac{v(t)}{t-z}\,dt, & z \in D^-, \\ \dfrac{1}{2\pi i}\displaystyle\int\limits_\Gamma \dfrac{\beta'(t)v(\beta(t))}{t-z}\,dt, & z \in D^+. \end{cases} \tag{16}$$

By the assumption, $v \in L^{p'(\cdot)}(\Gamma; 1/\rho)$, $v(\beta(t)) \in L^{p'_\alpha(\cdot)}(\Gamma; 1/\rho_\alpha)$, and for ρ and ρ_α the conditions (8) are fulfilled. Hence it is not difficult to prove that $N \in E^1(D^\pm)$. Thus we have

$$\begin{aligned} N^+(t_0) &= \frac{1}{2}\,\beta'(t_0)v(\beta(t_0)) + \frac{1}{2\pi i}\int\limits_\Gamma \frac{\beta'(t)v(\beta(t))}{t-t_0}\,dt = \\ &= \frac{v(\beta(t_0))}{2\alpha'(\beta(t_0))} + \frac{1}{2\pi i}\int\limits_\Gamma \frac{v(t)}{\alpha(t)-\alpha(t_0)}\,dt, \end{aligned}$$

i.e.,

$$N^+(\alpha(t_0)) = \frac{v(t_0)}{2\alpha'(t_0)} + \frac{1}{2\pi i}\int\limits_\Gamma \frac{v(t)}{\alpha(t)-\alpha(t_0)}\,dt,$$

whence

$$\alpha'(t_0)N^+(\alpha(t_0)) = \frac{1}{2}\,v(t_0) + \frac{\alpha'(t_0)}{2\pi i}\int\limits_{\Gamma}\frac{v(t)}{\alpha(t)-\alpha(t_0)}\,dt.$$

On the other hand,

$$N^-(t_0) = \frac{1}{2}\,v(t_0) - \frac{1}{2\pi i}\int\limits_{\Gamma}\frac{v(t)}{t-t_0}\,dt$$

and

$$K'(1)v = -\frac{1}{\pi i}\int\limits_{\Gamma}\frac{2v(t)}{t-t_0}\,dt + \frac{1}{\pi i}\int\limits_{\Gamma}\Big(\frac{\alpha'(t_0)}{\alpha(t_0)-\alpha(t)} - \frac{1}{t_0-t}\Big)v(t)\,dt =$$
$$= -\frac{1}{\pi i}\int\limits_{\Gamma}\frac{v(t)}{t-t_0}\,dt - \frac{\alpha'(t_0)}{\pi i}\int\limits_{\Gamma}\frac{v(t)}{\alpha(t)-\alpha(t_0)}\,dt = 0.$$

Therefore

$$\frac{\alpha'(t_0)}{\pi i}\int\limits_{\Gamma}\frac{v(t)}{\alpha(t)-\alpha(t_0)}\,dt = -\frac{1}{\pi i}\int\limits_{\Gamma}\frac{v(t)}{t-t_0}\,dt.$$

Consequently,

$$N^-(t_0) = \frac{1}{2}\,v(t_0) + \frac{\alpha'(t_0)}{\pi i}\int\limits_{\Gamma}\frac{v(t)}{\alpha(t)-\alpha(t_0)}\,dt,$$

and from the above-said we obtain

$$\alpha'(t_0)N^+(\alpha(t_0)) = N^-(t_0).$$

By Lemma 4, we can conclude that $N(z)=0$. Then from (16) it follows the existence of functions $\varphi^+\in E^1(D^+)$, $\varphi^-\in E^1(D^-)$ such that

$$v(t) = \varphi^+(t), \quad \beta'(t)v(\beta(t)) = \varphi^-(t),$$

i.e.,

$$\beta'(t)\varphi^+(t) = \varphi^-(t).$$

Since from the equality $\beta'(t) = [\alpha'(\beta(t))]^{-1}$ it follows that $\beta'\in H$ and $\beta'(t)\neq 0$, on the basis of Lemma 4 with $\alpha(t)$ replaced by $\beta(t)$ we establish that $\varphi^\pm(z)=0$, and hence $v=0$. □

2^0. **Theorem 2.** *For $a(t)=1$ and $b\in L_{p,\alpha}(\Gamma;\rho)$, the problem* (1) *is uniquely solvable in the class $K_\alpha^{p(\cdot)}(\Gamma;\rho)$.*

Proof. First we prove that the equation $K'(1)v=0$ in the classes $L^{p'(\cdot)}(\Gamma;1/\rho)$ and $L^{p'_\alpha(\cdot)}(\Gamma;1/\rho_\alpha)$ has only the zero solution.

According to our assumptions regarding p and ρ, there exists a number $\eta>0$ such that $L^{p'(\cdot)}(\Gamma;1/\rho)\subset L^{1+\eta}(\Gamma)$ and $L^{p'_\alpha(\cdot)}(\Gamma;1/\rho_\alpha)\subset L^{1+\eta}(\Gamma)$. Therefore all the solutions of the equation $K'(1)v=0$ belonging to at least one of the above-mentioned classes lie in the set of solutions of this equation of the class $L^{1+\eta}(\Gamma)$.

Let us show that the equation $K'(1)v = 0$ has only the zero solution in this class. Towards this end, we consider the operator $K(1)\mu$ in the class $L^{\frac{1+\eta}{\eta}}(\Gamma)$. Then $p(t) = p_\alpha(t) = (1+\eta)/\eta$, $\rho(t) = 1$, $\rho_\alpha(t) = 1$, and hence $L_{p',\alpha}(\Gamma; 1/\rho) = L^{1+\eta}(\Gamma)$. By Lemma 5, $v(t) = 0$.

Since $K(1)\mu$ is a Fredholm operator in the spaces $L^\delta(\Gamma)$, $\delta > 1$, the equations

$$\begin{aligned} &K(1)\mu = 0, \ \ \mu \in L^{\frac{1+\eta}{\eta}}(\Gamma), \\ &K'(1)v = 0, \ \ v \in L^{1+\eta}(\Gamma) \end{aligned}$$

and

$$\begin{aligned} &K(1)\mu = 0, \ \ \mu \in L^{1+\eta}(\Gamma), \\ &K'(1)v = 0, \ \ v \in L^{\frac{1+\eta}{\eta}}(\Gamma) \end{aligned}$$

have the same number of linearly independent solutions.

By the above-proven, since the equation $K'(1)v = 0$ in $L^{1+\eta}(\Gamma)$ (and likewise in $L^{\frac{1+\eta}{\eta}}(\Gamma)$) has only the zero solution, we have that the equation $K(1)\mu = 0$ in $L^{1+\eta}(\Gamma)$ has the zero solution. All the more, it has the zero solution in the classes $L^{p(\cdot)}(\Gamma;\rho)$ and $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$, and thus in the class $L_{p,\alpha}(\Gamma;\rho)$ as well.

Consider now the problem (1) with $a(t) = 1$. Since the equation (12) is a Fredholm one in $L^{p(\cdot)}(\Gamma;\rho)$, it is solvable for those $b \in L^{p(\cdot)}(\Gamma;\rho)$ for which the condition (9) is fulfilled, where v is any solution of the equation $K'(1)v = 0$ of the class $L^{p'(\cdot)}(\Gamma; 1/\rho)$. But, as we have shown, $v = 0$. Therefore (12) has a solution $\mu \in L^{p(\cdot)}(\Gamma;\rho)$.

Analogously, since $b \in L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$ and $K'(1)v = 0$ has only the zero solution in $L^{p'_\alpha(\cdot)}(\Gamma; 1/\rho_\alpha)$, the equation (12) has a unique solution $\mu_\alpha \in L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$. The difference $\widetilde{\mu}(t) = \mu(t) - \mu_\alpha(t)$ is a solution of the equation $K(1)\mu = 0$ from $L^{1+\eta}(\Gamma)$, and consequently, $\widetilde{\mu} = 0$.

Thus $\mu = \mu_\alpha$ is a unique solution of the equation (12) generating by means of the formula (11) a solution of the problem (1) of the class $K^{p(\cdot)}_\alpha(\Gamma;\rho)$. □

Corollary. *The operator*

$$R = \big(\phi^+(\alpha(t)) - \phi^-(t)\big) = K(1)\mu$$

in every space $L^{p(\cdot)}(\Gamma;\rho)$, $p \in \mathcal{P}$, has the inverse operator $R^{-1}_{p,\rho}$ which is bounded in $L^{p(\cdot)}(\Gamma;\rho)$. In particular, the operators $R^{-1}_{p',1/\rho}$ and $R^{-1}_{p'_\alpha,1/\rho_\alpha}$) exist.

Remark 1. From the above-proven theorem it follows that the problem (1) for $a(t) = 1$, $b \in L_{p,\alpha}(\Gamma;\rho)$ in the class $K^{p(\cdot)}_\alpha(\Gamma;\rho)$ has only the zero solution, while in the class of the functions $\phi \in \widetilde{K}^{p(\cdot)}_\alpha(\Gamma;\rho)$ with $\phi(\infty) = 1$, it has only the solution $\phi(z) = 1$. In particular, this assertion is valid for the classes $K^\delta(\Gamma)$ and $\widetilde{K}^\delta(\Gamma)$, $\delta > 1$.

Remark 2. Following our reasoning when proving the above theorem and using Lemma 4, we can easily see that the problem

$$\Psi^+(\alpha(t)) = \frac{1}{\alpha'(t)}\,\Psi^-(t) + b(t)$$

with $b \in L_{p,\alpha}(\Gamma;\rho)$ is uniquely solvable in the class $K_\alpha^{p(\cdot)}(\Gamma;\rho)$. In particular, the problem is uniquely solvable in the classes $K^\lambda(\Gamma)$, $\lambda > 1$ (since $K_\alpha^\lambda(\Gamma;1) = K^\lambda(\Gamma)$). If the function $b(t)$ is bounded, then these solutions coincide. If ϕ is that solution, then $\phi \in \bigcap\limits_{\delta>1} K^\delta(\Gamma)$.

7. The Problem (1) with a Coefficient Close to Unity

Lemma 6. *There exists a number* $\varepsilon>0$ *such that if* $\max|a(t)–1|<\varepsilon$, *then the boundary value problem* (1) *is uniquely solvable in the class* $K_\alpha^{p(\cdot)}(\Gamma;\rho)$, *and the homogeneous problem* (2) *has a solution* $X_0 \in \widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ *possessing the property* $[X_0]^{-1} \in \widetilde{K}_\alpha^{p'(\cdot)}(\Gamma;1/\rho)$, $X_0(\infty)=1$.

Proof. Let $\Phi \in K_\alpha^{p(\cdot)}(\Gamma;\rho)$ be a solution of the problem (1). Then it is representable by the equality (11).

Assume

$$R\mu = \Phi^+(\alpha(t)) - \Phi^-(t) = \frac{1}{2}\,(a(t)-1)\big(\mu(t) - (S_\Gamma\mu)(t)\big).$$

According to the corollary of Theorem 1, there exists the bounded in $L^{p(\cdot)}(\Gamma;\rho)$ operator $R^{-1}_{p,\rho}$. The boundary condition (1) yields

$$\mu = R^{-1}_{p,\rho}\Big[\frac{1}{2}\,(a-1)(\mu + S_\Gamma\mu) + b\Big] = N\mu.$$

We have

$$\big\|N(\mu_1-\mu_2)\big\|_{p,\rho} \le \|R^{-1}_{p,\rho}\|\frac{1}{2}\,\big(1+\|S_\Gamma\|_{p,\rho}\big)\max|a(t)-1|\,\|\mu_1-\mu_2\|_{p,\rho}.$$

Consequently, if

$$\max|a(t)-1| < \frac{2\nu}{\|R^{-1}_{p,\rho}\|(1+\|S_\Gamma\|_{p,\rho})} = \varepsilon_1, \quad \nu \in (0,1),$$

then N is a contraction operator. This implies that the equation $K(a)\mu = \widetilde{b}$ is uniquely solvable in $L^{p(\cdot)}(\Gamma;\rho)$.

Analogously, if $b \in L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$ and

$$\max|a(t)-1| < \frac{2\nu}{\|R^{-1}_{p_\alpha,\rho_\alpha}\|(1+\|S_\Gamma\|_{p_\alpha,\rho_\alpha})} = \varepsilon_{1,\alpha},$$

then the equation $K(1)\mu = \widetilde{b}$ is uniquely solvable in $L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$. According to these two statements we find that the problem (1) with $\max|a(t)-1| < \min(\varepsilon_1;\varepsilon_{1,\alpha})$ is uniquely solvable in $K_\alpha^{p(\cdot)}(\Gamma;\rho)$.

Next, since $\max\left|\frac{1}{a(t)}-1\right| < \frac{\max|a(t)-1|}{\min|a(t)|}$, it is not difficult to see that if

$$\max|a(t)-1| < \frac{2\nu\min|a(t)|}{\|R^{-1}_{p',1/\rho}\|(1+\|S_\Gamma\|_{p',1/\rho})} = \varepsilon_2,$$

then the equation $K(1/a)\mu = c$, $c \in L^{p'(\cdot)}(\Gamma;1/\rho)$, is uniquely solvable. If, however,

$$\max|a(t)-1| < \frac{2\nu\min|a(t)|}{\|R_{p'_\alpha,1/\rho_\alpha}\|(1+\|S_\Gamma\|_{p'_\alpha,1/\rho_\alpha})} = \varepsilon_{2,\alpha},$$

then the equation $K(1/a)\mu = c$, $c \in L^{p'_\alpha(\cdot)}(\Gamma;1/\rho_\alpha)$, is uniquely solvable in the class $K^{p'_\alpha(\cdot)}(\Gamma;1/\rho_\alpha)$.

Consequently, if $\max|a(t)-1| < \min(\varepsilon_2,\varepsilon_{2,\alpha})$, then the problem (1) with the coefficient $[a(t)]^{-1}$ is uniquely solvable in $K^{p'(\cdot)}_\alpha(\Gamma;1/\rho)$.

Taking $\varepsilon = \min(\varepsilon_1;\varepsilon_2;\varepsilon_{1,\alpha};\varepsilon_{2,\alpha})$ and bearing in mind that $a(t)$ and $[a(t)]^{-1}$ are bounded functions, we that find the boundary value problems

$$\Phi^+(\alpha(t)) = a(t)\Phi^-(t) + a(t), \ \ \Phi \in K^{p(\cdot)}_\alpha(\Gamma;\rho),$$
$$\Psi^+(\alpha(t)) = \frac{1}{a(t)}\,\Psi^-(t) + \frac{1}{a(t)}\,, \ \ \Psi \in K^{p'(\cdot)}_\alpha(\Gamma;1/\rho)$$

are uniquely solvable when $\max|a(t)-1| < \varepsilon$.

Note that we can choose $\eta > 0$ such that the condition (8) preserves if we replace in it $p(t)$ by $p(t)+\eta$. Replacing everywhere in the sequel $p(t)$ by $p(t)+\eta$ and denoting the obtained values for ε_k, $\varepsilon_{k,\alpha}$, $k=1,2$, respectively by $\widetilde{\varepsilon}_1$, $\widetilde{\varepsilon}_2$, $\widetilde{\varepsilon}_{1,\alpha}$, $\widetilde{\varepsilon}_{2,\alpha}$, we will find that if $\max|a(t)-1| < \varepsilon(\eta) = \min(\widetilde{\varepsilon}_1,\widetilde{\varepsilon}_2,\widetilde{\varepsilon}_{1,\alpha},\widetilde{\varepsilon}_{2,\alpha})$, then the solutions $\Phi(z)$ and $\Psi(z)$ satisfy the conditions

$$\Phi \in K^{p(\cdot)+\varepsilon}_\alpha(\Gamma,;\rho), \quad \Psi \in K^{p'(\cdot)+\eta}_\alpha(\Gamma,;1/\rho).$$

Further, we have

$$\Phi^+(\alpha(t))\Psi^+(\alpha(t)) = \Phi^-(t)\Psi^-(t) + \Phi^-(t) + \Psi^-(t) + 1.$$

Assume

$$F(z) = \begin{cases} \Phi(z)\Psi(z), & z \in D^+, \\ [\Phi(z)+1]\,[\Psi(z)+1], & z \in D^-. \end{cases}$$

Then $F(z) \in E^{1+\lambda}(D^+)$, $[F(z)-1] \in E^{1+\lambda}(D^-)$, where λ is some positive number. Moreover, $F^+(\alpha(t)) = F^-(t)$, $F(\infty)=1$. Owing to Remark 1, we have $F(z) \equiv 1$. Therefore

$$\Phi(z)\Psi(z) = 1, \ \ z \in D^+, \quad [\Phi(z)+1]\,[\Psi(z)+1] = 1, \ \ z \in D^-. \tag{17}$$

The function

$$X_0(z) = \begin{cases} \Phi(z), & z \in D^+, \\ \Phi(z)+1, & z \in D^-, \end{cases}$$

is a solution of the problem (2) of the class $\widetilde{K}^{p(\cdot)}_\alpha(\Gamma;\rho)$ and $X_0(\infty)=1$. It can be seen from (17) that $X_0(z)$ possesses the required property. □

Remark 3. From the above-said we can conclude that

$$X_0 \in \widetilde{K}_\alpha^{p(\cdot)+\eta}(\Gamma;\rho), \quad [X_0]^{-1} \in K_\alpha^{p'(\cdot)+\eta}(\Gamma;1/\rho), \quad X_0(\infty)=1.$$

Remark 4. If $\rho(t)=1$, then $X_0 \in \widetilde{K}_\alpha^{p(\cdot)}(\Gamma)$ for any $p \in \mathcal{P}$, and $[X_0]^{-1} \in \widetilde{K}_\alpha^{p(\cdot)}(\Gamma)$. In particular, $X_0 \in \bigcap\limits_{\delta>1} \widetilde{K}^\delta(\Gamma)$ and $[X_0]^{-1} \in \bigcap\limits_{\delta>1} \widetilde{K}^\delta(\Gamma)$ (this follows from the fact that $\widetilde{K}_\alpha^\delta(\Gamma) = \widetilde{K}^\delta(\Gamma)$).

8. Solution of Homogeneous Haseman's Problem. Canonical Solution

1^0. In solving the problem (2) we will need good properties of the solution of the problem (13). Denote the restrictions of the solution $\omega(z)$ to the domains D^+ and D^- by $\omega^+(z)$ and $\omega^-(z)$, respectively.

Lemma 7. *There exists a solution of the problem* (13) *possessing the properties: $\omega^+(z)$ ($\omega^-(z)$) is schlicht and continuous in $\overline{D^+}$ ($\overline{D^-}$), $\omega^-(z) = z + \omega_0^-(z)$, where $\omega_0^-(z)$ is holomorphic in D^-. Moreover,*

$$[\omega^+(z)]' \in \bigcap_{\delta>1} E^\delta(D^+), \quad [\omega^-(z)-z]' \in \bigcap_{\delta>1} E^\delta(D^-), \tag{18}$$

and $\gamma = \omega^-(\Gamma)$ is a simple rectifiable curve.

Proof. Consider the boundary value problem

$$\alpha(t)\Phi^+(\alpha(t)) = \Phi^-(t) + 1.$$

Since the free term ($b(t) \equiv 1$) in this problem is a bounded function, then by Remark 2 this problem has a unique solution $\Phi \in \bigcap\limits_{\delta>1} K^\delta(\Gamma)$. By $\omega^+(z)$ ($\omega^-(z)$) we denote the primitive of the function $\Phi(z)$ ($\Phi(z)+1$) in the domain D^+ (D^-). We choose the primitive functions in such a way that the equality (13) is fulfilled. Since $\Phi \in \bigcap\limits_{\delta>1} K^\delta(D^\pm)$, we have $\Phi \in \bigcap\limits_{\delta>1} E^\delta(D^\pm)$. Hence the functions $\omega^+(z)$ and $\omega^-(z)$ are continuous in the domains $\overline{D^+}$ and $\overline{D^-} \setminus \{\infty\}$ (see [26, p. 208]). By the result obtained in [10] and cited above in Sect. 5, they are schlicht in the corresponding domains. Moreover,

$$[\omega^+(z)]' = \Phi(z), \ \ z \in D^+, \quad [\omega^-(z)-z]' = \Phi(z), \ \ z \in D^-.$$

Hence owing to the above-said regarding $\Phi(z)$, the inclusions (18) are valid.

The resctifiability of the curve $\gamma = \omega^-(\Gamma)$ follows from the summability of the function $[\omega^-(z)]'$. □

2^0. Assume

$$\varkappa = \operatorname{ind} a(t) = \frac{1}{2\pi}\,[\arg a(t)]_\Gamma,$$

where $[f(t)]_\Gamma$ is the increment of $f(t)$ for a single circuit of the curve Γ by the point t in the positive direction.

Theorem 3. *If $a(t)$ is a non-zero continuous function, then all the solutions of the problem (2) of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ are given by the equality*

$$\Phi(z)=X(z)P(\omega(z)),$$

where $\omega(z)$ is the function constructed in Lemma 7, $P(z)$ is an arbitrary polynomial, and $X(z)$ is a solution of the homogeneous problem (2) defined to within a constant multiplier and possessing the properties:

(1) $X\in\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$, $[X]^{-1}\in\widetilde{K}_\alpha^{p'(\cdot)}(\Gamma;1/\rho)$;
(2) *the order of $X(z)$ at infinity is equal to $(-\varkappa)=-\operatorname{ind}a(t)$.*

Proof. Choose a rational function $r(t)$ such that $\max|a(t)r^{-1}(t)-1|<\varepsilon$, where ε is the number mentioned in Lemma 6. Note that we can choose the function $r(z)$ to be of the type

$$r(z)=\sum_{k=-n}^{m}A_k(z-z_0)^k,\ \ z_0\in D,\ \ r(t)\neq 0,\ \ t\in\Gamma. \tag{19}$$

Since

$$\operatorname{ind}a(t)=\operatorname{ind}\Big(\frac{a(t)}{r(t)}\,r(t)\Big)=\operatorname{ind}\frac{a(t)}{r(t)}+\operatorname{ind}r(t),$$

and from the condition $|a(t)r^{-1}(t)-1|<\varepsilon$ (assuming $\varepsilon<1$) it follows $\operatorname{ind}\frac{a(t)}{r(t)}=0$, we have $\operatorname{ind}r(t)=\operatorname{ind}a(t)=\varkappa$.

Assume $a_0(t)=a(t)r^{-1}(t)$ and write the condition (2) in the form

$$\Phi^+(\alpha(t))=a_0(t)r(t)\Phi^-(t). \tag{20}$$

By Lemma 6, there exists a function $X_0(z)$ such that

$$X_0\in\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho),\ \ [X_0]^{-1}\in\widetilde{K}_\alpha^{p'(\cdot)}(\Gamma;1/\rho),$$
$$X_0(\infty)=1,\ \ X_0^+(\alpha(t))=a_0(t)X_0^-(t).$$

Therefore (20) can be written as follows:

$$[X_0^+(\alpha(t))]^{-1}\Phi^+(\alpha(t))=r(t)[X_0^-]^{-1}\Phi^-(t). \tag{21}$$

Now the use will be made of the function $\omega(z)$ constructed in Lemma 7.

Denote the domains bounded by the curve $\gamma=\omega^-(\Gamma)$ by G^+ and G^-.

Let ω_1^+ and ω_1^- be the inverse functions to ω^+ and ω^-, respectively, in the domains G^+ and G^-. Assuming $\tau=\omega^+(\alpha(t))=\omega^-(t)$, we obtain $\alpha(t)=\omega_1^+(\tau)$, $t=\omega_1(\tau)$, and (21) can be written in the form

$$\begin{aligned}&\big[X_0^+(\omega_1^+(\tau))\big]^{-1}\Phi^+(\omega_1^+(\tau))=\\&=\big[X_0^-(\omega_1^-(\tau))\big]^{-1}r(\omega_1^-(\tau))\Phi^-(\omega_1^-(\tau)),\ \ \tau\in\gamma.\end{aligned} \tag{22}$$

From Remark 3 we can easily conclude that $\Phi(z)[X_0(z)]^{-1}$ belongs to $E^{1+\eta_1}(D^+)$, $\eta_1>0$. Using now the inclusion (18), we get

$$\varphi_1(\zeta)=\big[X_0^+(\omega_1^+(\zeta))\big]^{-1}\Phi(\omega_1^+(\zeta))\in E^1(G^+). \tag{23}$$

Analogously,

$$\big[X_0^-(\omega_1^-(\zeta))\big]^{-1}\Phi(\omega_1^-(\zeta))r(\omega_1^-(\zeta))=\varphi_1(\zeta)+q(\zeta),\ \ \zeta\in G^-, \tag{24}$$

where $q(\zeta)$ is a polynomial, and $\varphi_1\in E^1(G^-)$.

Here we have used the fact that if $\mu\in L^{\delta_0}(\Gamma)$, $\delta_0>1$, and $p(z)$ is a polynomial, then

$$\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\mu(t)}{t-z}\,dt\,p(z)=\frac{1}{\pi i}\int\limits_{\Gamma}\frac{\mu_1(t)}{t-z}\,dt+p_1(z),$$

where $\mu_1\in L^{\delta_0}(\Gamma)$, and $p_1(z)$ is a polynomial whose order is less by unity than that of the polynomial $p(z)$ [29, p. 98] and $p_1(\omega_1(\zeta))=q(\zeta)+\varphi_2(\zeta)$, where $q(z)$ is a polynomial and $\varphi_2\in\bigcap\limits_{\delta>1}E^{\delta}(G^-)$. The latter follows from $\omega^-(\zeta)=\zeta+\omega_{0,1}(\zeta)$, where $\omega_{0,1}^-(\zeta)$ is a function continuous in $\overline{G}_1^{\,-}$ and holomorphic in G^-.

Note here that if $r(z)=A_{-n}(z-z_0)^{-n}+\cdots+A_0+A_1(z-z_0)+\cdots+A_m(z-z_0)^m$, $z_0\in D^+$ (see (19)), then the order of $r(z)$ at infinity is equal to m. If $\Phi\in K_\alpha^{p(\cdot)}(\Gamma;\rho)$, then $\Phi(\infty)=0$, and (24) implies that $q(z)$ is a polynomial of order $m-1$.

Getting back to the investigation of the problem (20) which is reduced to the problem (22) and taking into account (23) and (24), we see that the equality (22) yields

$$\varphi_1^+(\tau)=\varphi_1^-(\tau)+q(\tau),\ \ \tau\in\gamma,\ \ \varphi_1^{\pm}\in E^1(G^{\pm}),$$

whence $\varphi_1^+(z)=q(z)$, $z\in G^+$, $\varphi_1(z)=0$, $z\in G^-$. But then

$$\begin{aligned}&\Phi(z)=X_0(z)q(\omega^+(z)),\ \ z\in D^+,\\&\Phi(z)=r^{-1}(z)X_0(z)q(\omega^-(z)),\ \ z\in D^-.\end{aligned}\tag{25}$$

For $\Phi(z)$ to be analytic in D^-, the function $[r(z)]^{-1}q(\omega^-(z))$ must have no singularities in D^-.

We write $r(z)$ in the form

$$r(z)=(z-z_0)^n(z-z_1)\cdots(z-z_{m+n}),$$

where $z_0\in D^+$, and $z_1,z_2,\dots,z_{m+n}$ are zeroes of $r(z)$ of which $z_1,\dots,z_j$ belong to D^+ while $z_{j+1},\dots,z_{m+n}$ belong to D^-. Then $\operatorname{ind}r(t)=j-n$, and since $\operatorname{ind}r(t)=\operatorname{ind}a(t)=\varkappa$, therefore $j-n=\varkappa$. For the function Φ to be analytic in D^-, it is necessary that the function $q(\omega^-(z))$ have zeroes at the points $z_{j+1},\dots,z_{m+n}$. Since $\omega^-(z)$ is schlicht, there exist points $\zeta_{j+1},\dots,\zeta_{m+n}$ belonging to the domain G^- and such that $\omega^-(\zeta_k)=z_k$, $k=j+1,\dots,m+n$. Consequently, we must have $q(\zeta)=\widetilde{q}(\zeta)P(\zeta)$, where $\widetilde{q}(\zeta)=(\zeta-\zeta_{j+1})\cdots(\zeta-\zeta_{m+n})$, and $P(\zeta)$ is an arbitrary polynomial.

Thus from (25) we conclude that all possible solutions of the problem (20) (and hence of the problem (2)) of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ lie in the set of functions

$$\Phi(z)=\begin{cases}\big[X_0(z)\widetilde{q}(\omega^+(z))\big]P(\omega^+(z)), & z\in D^+,\\ \big[X_0(z)r^{-1}(z)\widetilde{q}(\omega^-(z))\big]P(\omega^-(z)), & z\in D^-,\end{cases} \tag{26}$$

where $X_0(z)$ is the solution constructed in Lemma 6, $\widetilde{q}(z)$ is the above-mentioned and $P(z)$ is an arbitrary polynomial. Since $X_0\in\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$, it is easily seen that the equality (26) provides us with a general solution of the problem (2) of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$.

Assuming

$$X(z)=\begin{cases}X_0(z)\widetilde{q}(\omega^+(z)), & z\in D^+,\\ X_0(z)r^{-1}(z)\widetilde{q}(\omega^-(z)), & z\in D^-,\end{cases} \tag{27}$$

we can write the equality (26) in the form $\Phi(z)=X(z)P(\omega(z))$. □

The function $X(z)$ at infinity is of the order $m-(m-n-j)=n-j=-\varkappa$. It possesses the following properties:

(1) $X^+(\alpha(t))=a(t)X^-(t)$;

(2) $X\in\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$, $1/X\in\widetilde{K}_\alpha^{p'(\cdot)}(\Gamma;1/\rho)$;

(3) at infinity it has the order $(-\varkappa)$.

The function X possessing the properties (1)–(3) will be called a canonical solution of the problem (2) of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$, or a canonical function of the function $a(t)$.

Although our formal representation of $X(z)$ (see (27)) depends on the function $r(z)$, on the basis of Theorem 3 we can easily conclude that all canonical functions differ from each other by a non-zero constant multiplier.

We have several remarks.

Remark 5. From Remark 3 regarding the function $X_0(z)$ it follows that the canonical function $X(z)$ given by the equality (27) possesses the property: there exists $\eta>0$ such that

$$X\in\widetilde{K}_\alpha^{p(\cdot)+\eta}(\Gamma;\rho),\quad 1/X\in\widetilde{K}_\alpha^{p'(\cdot)+\eta}(\Gamma;1/\rho).$$

Remark 6. If we consider the problem for $\rho(t)=1$, i.e. in the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma)$, then there is no need to coordinate the relations between p and ρ by means of the inequality (8), and the function X satisfies the conditions $X\in\bigcap\limits_{p\in\mathcal{P}}\widetilde{K}_\alpha^{p(\cdot)}(\Gamma)$, $1/X\in\bigcap\limits_{p\in\mathcal{P}}\widetilde{K}_\alpha^{p'(\cdot)}(\Gamma)$. In particular, the continuous non-zero function $a(t)$ has a canonical function X for which

$$X\in\bigcap_{\delta>1}\widetilde{K}^\delta(\Gamma),\quad 1/X\in\bigcap_{\delta>1}\widetilde{K}^\delta(\Gamma).$$

Remark 7. From the basic assertion of the theorem it follows that every solution of the problem (2) of the class $\widetilde{K}^{\delta_0}(\Gamma)$, $\delta_0>1$ belongs to the set $\bigcap\limits_{\delta>1}\widetilde{K}^\delta(\Gamma)$.

Remark 8. Observing the proof of the assertion of the theorem on the validity of the representation of all solutions of the problem (2) in the form $\Phi(z) = X(z)P(\omega(z))$, it is not difficult to see that this assertion remains valid for all coefficients $a(t)$ for which a canonical function $X(z)$ exists with properties from Remark 5.

9. Solution of the Problem (1) in the Classes $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ and $K_\alpha^{p(\cdot)}(\Gamma;\rho)$

Theorem 4. *All the solutions of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ of Haseman's problem* (1) *with the continuous coefficient $a(t)$ and the condition $b \in L_{p,\alpha}(\Gamma;\rho)$ are given by the formula*

$$\Phi(z) = X(z)\big[\Phi_0(z) + P(\omega(z))\big], \tag{28}$$

where $X(z)$ is the canonical solution of the problem (2) *given by the equality* (27)*, $\omega(z)$ is a piecewise holomorphic solution of the problem* (13) *with the principal part equal at infinity to z, $P(z)$ is an arbitrary polynomial, and $\Phi_0(z)$ is a solution of the problem*

$$\Phi_0^+(\alpha(t)) = \Phi_0^-(t) + b_0(t), \quad b_0(t) = \big[X_0^+(\alpha(t))\big]^{-1} b(t) \tag{29}$$

belonging to the set $\bigcup\limits_{\delta>1} \widetilde{K}^\delta(\Gamma)$.

Solutions vanishing at infinity, i.e., those of the class $K_\alpha^{p(\cdot)}(\Gamma;\rho)$ exist if $\varkappa = \operatorname{ind} a(t) \geq 0$, and all such solutions are given by the formula (28) *in which $P(z)$ is an arbitrary polynomial of order $\varkappa - 1$ for $\varkappa > 0$, and $P(z) = 0$ for $\varkappa = 0$.*

If $\varkappa < 0$, then for the solvability of the problem (1) *in $K_\alpha^{p(\cdot)}(\Gamma;\rho)$ it is necessary and sufficient that the conditions*

$$\int\limits_\Gamma t^k \mu(t)\, dt = 0, \quad k = 0, \dots, |\varkappa| - 1, \tag{30}$$

be fulfilled, where μ is a solution of the equation $K(1)\mu = \widetilde{b}_0(t)$, $\widetilde{b}_0 = b + S_\Gamma b + \int\limits_\Gamma K(\cdot,\tau) b(\tau)\, d\tau$.

If the conditions (30) *are fulfilled, then there exists a unique solution Φ which is given by the equality* (28) *with $P(z) \equiv 0$.*

Proof. Let us show that the inhomogeneous problem is solvable in $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ for any $b \in L_{p,\alpha}(\Gamma;\rho)$. We choose a rational function $r(z)$ of the type (19) in such a way that $\max|a(t) - 1| < \varepsilon$ and consider the problem

$$\Psi^+(\alpha(t)) = a(t) r^{-1}(t) \Psi^-(t) + b(t).$$

This problem is solvable in $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ if ε is a sufficiently small number (see Lemma 6).

Having chosen the function $r(z)$ in the way mentioned above, we see that it has no poles in D^-. Therefore the function

$$\widetilde{\Psi}(z) = \begin{cases} \Psi(z), & z \in D^+, \\ r(z)\Psi(z), & z \in D^-, \end{cases}$$

is a solution of the problem (1) of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$.

Further, let $\Phi_*(z) \in \widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$ be any solution of the problem (1). Assuming $\Phi_0(z) = [X(z)]^{-1}\Phi_*(z)$ and taking into account that $X^{-1} \in \widetilde{K}_\alpha^{p'(\cdot)}(\Gamma;1/\rho)$, we find that $\Phi_0 \in \widetilde{K}^{\delta_0}(\Gamma)$ with $\delta_0 > 1$, and Φ_0 is a solution of the problem (29). The difference $\varphi(z) = \Phi_*(z) - \widetilde{\Psi}(z) = \Phi_0(z)X(z) - \widetilde{\Psi}(z)$ is a solution of the problem (2) of the class $\widetilde{K}^{\delta_0}(\Gamma)$. Every such solution belongs, by Remark 7, to $\bigcap\limits_{\delta>1} \widetilde{K}^{\delta}(\Gamma)$, and hence $\varphi^\pm \in \bigcap\limits_{\delta>1} L^{\delta}(\Gamma)$.

Let us show that if $\varphi \in \bigcap\limits_{\delta>1} K_\alpha^{\delta}(\Gamma)$ and the weight $\rho(t)$ satisfies the condition (8), then $\varphi \in \widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$. Towards this end, it suffices to show that $\varphi^\pm(t)$ belong to $L^{p(\cdot)}(\Gamma;\rho) \cap L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$.

Using the generalized Hölder inequality [27], we obtain

$$\int\limits_\Gamma \big|\varphi^\pm(t)\rho(t)\big|^{p(t)}\,|dt| \le c_0 \|\varphi^\pm\|_{\frac{p(\cdot)+\varepsilon}{\varepsilon}\,p(\cdot)} \|\rho\|_{p(\cdot)+\varepsilon}\,.$$

Since for $p(t) \le p_1(t)$ we have $\|f\|_{p(t)} < (1+|\Gamma|)\|f\|_{p_1(\cdot)}$, we obtain the inequality

$$\int\limits_\Gamma \big|\varphi^\pm(t)\rho(t)\big|^{p(t)}\,|dt| \le c_0(1+|\Gamma|)\|\varphi^\pm\|_{\frac{P_+(p_++\varepsilon)}{\varepsilon}} \|\rho\|_{p(\cdot)+\varepsilon}\,, \tag{31}$$

where $p_+ = \max p(t)$. By (8), there exists $\varepsilon > 0$ such that $\rho \in L^{p(\cdot)+\varepsilon}(\Gamma)$. Then from (31) it follows that $\varphi^\pm \in L^{p(\cdot)+\varepsilon}(\Gamma;\rho)$, and hence $\varphi \in \widetilde{K}^{p(\cdot)+\varepsilon}(\Gamma;\rho)$. Analogously, we can prove that $\varphi^-(t) \in L^{p_\alpha(\cdot)}(\Gamma;\rho_\alpha)$.

Thus $\varphi^- \in L_{p,\alpha}(\Gamma;\rho)$ and hence $\varphi(z)$ is a solution of the problem (2) of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$. All the solutions of the above-mentioned class are given by the equality $\Phi(z) = X(z)P(\omega(z))$. Hence $\varphi(z) = X(z)P_\varphi(\omega(z))$ for some polynomial P_φ. Therefore the function $\Phi_0(z)X(z)(=\widetilde{\Psi(z)} + X(z)P_\varphi(\omega(z)))$ is a particular solution of the problem (2) of the class $\widetilde{K}_\alpha^{p(\cdot)}(\Gamma;\rho)$. This implies that the general solution of the problem (1) is given by the equality (28).

Thus the proof the first part of the theorem is complete.

From the union of functions given by the equality (28), we select the solutions belonging to $K_\alpha^{p(\cdot)}(\Gamma;\rho)$.

First, let $\varkappa \ge 0$. In this case $X(z)$ has at infinity the order $(-\varkappa)$. Therefore any function given by the equality (28) belongs to $K_\alpha^{p(\cdot)}(\Gamma;\rho)$ if the order of the polynomial $P(z)$ does not exceed $\varkappa - 1$ ($P_{-1}(z) = 0$). Thus all the solutions in this case are given by the equality

$$\Phi(z) = X(z)(\Phi_0(z) + P_{\varkappa-1}(z)). \tag{32}$$

If $\varkappa < 0$, then $X(z)$ has at infinity pole of order $(-\varkappa)$. Therefore in (28) we have to take $P(z) = 0$ and, moreover, it is necessary and sufficient that $\Phi_0(z)$ have a zero of order $(-\varkappa) + 1$ at infinity.

Let μ be a solution of the equation $K(1)\mu = \widetilde{b}_0(t)$, $b_0(t) = b(t)[X_0(\alpha(t))]^{-1}$. Then the solution Φ_0 of the problem (29) in the domain D^- is given by the equality

$$\Phi_0(z) = -\frac{1}{2\pi i}\int\limits_{\Gamma} \frac{\mu(t)}{t-z}\,dt.$$

Obviously, Φ_0 has a zero of order $(-\varkappa+1)$ if and only if the conditions (30) are fulfilled. □

Acknowledgement

This work was supported by the Grant GNSF/ST07/3-169.

References

1. C. Haseman, Anwendung der Theorie der Integralgleichungen auf einige Bandwertaufgaben der Funktionentheorie. *Göttingen*, 1907.
2. *T. Carleman*, Sur la théorie des équations intégrales et ses applications. *Verh. Internat. Math.-Kongr.* **1** (1932), 138–151.
3. D. A. Kveselava, The solution of a boundary problem of the theory of function. *C. R. (Doklady) Acad. Sci. URSS (N.S.)* **53** (1946), 679–682.
4. D. A. Kveselava, Some boundary problems of the theory of functions. (Russian) *Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze* **16** 1948, 39–80.
5. N. P. Vekua, Systems of singular integral equations and certain boundary value problems. (Russian) *Nauka, Moscow*, 1970.
6. G. S. Litvinchuk, Boundary value problems and singular integral equations with shift. (Russian) *Nauka, Moscow*, 1977.
7. B. V. Khvedelidze, On the discontinuous Riemann-Hilbert problems with a displacement. (Russian) *Soobshch. Akad. Nauk. Gruz. SSR*, **21** (1958), No. 4, 385–389.
8. G. F. Manjavidze and B. V. Khvedelidze, On Riemann–Privalov's problem with continuous coefficients. (Russian) *Dokl. Akad. Nauk SSSR* **123** (1958), 791–794.
9. I. B. Simonenko, The Riemann and Riemann–Haseman boundary value problems with continuous coefficients. (Russian) *Issled. Sovrem. Probl. Teor. Funkts. Kompleksn. Perem., IV. Vses. Konf. Mosk. Univ.* 1958, 380–389 (1961).
10. G. F. Manjavidze, Boundary value problems for conjugation with shift for analytic and generalized analytic functions. (Russian) *Tbilis. Gos. Univ., Tbilisi,* 1990.
11. L. Diening, Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$. *Math. Inequal. Appl.* **7** (2004), No. 2, 245–253.
12. A. Nekvinda, Hardy–Littlewood maximal operator on $L^{p(x)}(\mathbb{R})$. *Math. Inequal. Appl.* **7** (2004), No. 2, 255–265.
13. M. Růžička, Electrorheological fluids: modeling and mathematical theory. *Lecture Notes in Mathematics*, 1748. *Springer–Verlag, Berlin*, 2000.
14. L. Diening and M. Růžička, Calderón–Zygmund operators on generalized Lebesgue spaces $L^{p(\cdot)}$ and problems related to fluif dynamics. *J. Reine Angew. Math.* **563** (2003), 197–220.

15. V. Kokilashvili and S. Samko, Singular integrals in weighted Lebesgue spaces with variable exponent. *Georgian Math. J.* **10** (2003), No. 1, 145–156.
16. V. Kokilashvili and S. Samko, Maximal and fractional operators in weighted $L^{p(x)}$ spaces. *Rev. Mat. Iberoamericana* **20** (2004), No. 2, 493–515.
17. V. kokilashvili, V. Paatashvili and S. Samko, Boundedness in Lebesgue spaces with variable exponent of the Cauchy singular operator on Carleson curves. *Modern operator theory and applications*, 167–186, *Oper. Theory Adv. Appl.*, 170, *Birkhauser, Basel*, 2007.
18. V. Kokilashvili and S. Samko, Singular integral equations in the Lebesgue spaces with variable exponent. *Proc. A. Razmadze Math. Inst.* **131** (2003), 61–78.
19. A. Yu. Karlovich, Fredholmness of singular integral operators with piecewise continuous coefficients on weighted Banach function spaces. *J. Integral Equations Appl.* **15** (2003), No. 3, 263–320.
20. V. Kokilashvili, V. Paatashvili and S. Samko, Boundary value problems for analytic functions in the class of Cauchy-type integrals with density in $L^{p(\cdot)}(\Gamma)$. *Bound. Value Probl.* **2005**, No. 1, 43–71.
21. V. Kokilashvili and V. Paatashvili, The Riemann–Hilbert problem in weighted classes of Cauchy type integrals with density from $L^{p(\cdot)}(\Gamma)$. *Complex Anal. Oper. Theory* **2** (2008), No. 4, 569–591.
22. V. Kokilashvili and V. Paatashvili, The Dirichlet problem for harmonic functions in the Smirnov class with variable exponent. *Georgian Math. J.* **14** (2007), No. 2, 289–299.
23. V. Kokilashvili and V. Paatashvili, On the Riemann–Hilbert–Poincare problem and I. Vekua's representation of holomorphic equations. *Bull. Georgian National Acad. Sci.* **3** (2009), No. 1, 24–29.
24. V. Paatashvili, The Haseman boundary value problem in classes of functions representable by the Cauchy type integral with density in $L^{p(\cdot)}$. *Proc. A. Razmadze Math. Inst.* **150** (2009), 142–149.
25. Ch. Pommerenke, Boundary behaviour of conformal maps. *Springer–Verlag, Berlin*, 1992.
26. I. I. Privalov, Boundary properties of analytic functions. (Russian) *Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow–Leningrad*, 1950.
27. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$. *Czechoslovak Math. J.* **41(116)** (1991), No. 4, 592–618.
28. G. Khuskivadze, V. Kokilashvili and V. Paatashvili, Boundary value problems for analytic and harmonic functions in domains with nonsmooth boundaries. Applications to conformal mappings. *Mem. Differential Equations Math. Phys.* **14** (1998), 1–195.
29. B. V. Khvedelidze, The method of Cauchy type integrals in discontinuous boundary value problems of the theory of holomorphic functions of a complex variable. (Russian) *Current problems in mathematics*, Vol. 7 (Russian), pp. 5–162 (errata insert). *Akad. Nauk SSSR Vsesojuz. Inst. Naucn. i Tehn. Informacii, Moscow*, 1975.

(Received 22.06.2009)

Author's address:

A. Razmadze Mathematical Institute
1, M. Aleksidze St., Tbilisi 0193
Georgia