Memoirs on Differential Equations and Mathematical Physics
Volume 53, 2011, 29–38

G. Berikelashvili, M. M. Gupta, and M. Mirianashvili

ON THE CHOICE OF INITIAL CONDITIONS OF DIFFERENCE SCHEMES FOR PARABOLIC EQUATIONS

Abstract. We consider the first initial-boundary value problem for linear heat conductivity equation with constant coefficient in $\Omega \times (0, T]$, where Ω is a unit square. A high order accuracy ADI two level difference scheme is constructed on a 18-point stencil using Steklov averaging operators. We prove that the finite difference scheme converges in the discrete L_2-norm with the convergence rate $O(h^s + \tau^{s/2})$, when the exact solution belongs to the anisotropic Sobolev space $W_2^{s,s/2}$, $s \in (2,\, 4]$.

2010 Mathematics Subject Classification. 65M06, 65M12, 65M15.
Key words and phrases. Heat equation, ADI difference scheme, high order convergence rate.

რეზიუმე. $\Omega \times (0, T]$ არეში, სადაც Ω ერთეულოვანი კვადრატია, განხილულია მუდმივკოეფიციენტიანი სითბოგამტარობის წრფივი განტოლებისათვის დასმული პირველი საწყის-სასაზღვრო ამოცანა. სტეკლოვის გასაშუალების ოპერატორების გამოყენებით 18 წერტილიან შაბლონზე აგებულია მაღალი რიგის სიზუსტის ორშრიანი ცვალებადი მიმართულებით არაცხადი სხვაობიანი სქემა. დამტკიცებულია, რომ თუ ზუსტი ამონახსნი მიეკუთვნება სობოლევის ანიზოტროპულ $W_2^{s,s/2}$, $s \in (2,4]$ სივრცეს, მაშინ სასრულ-სხვაობიანი სქემის დისკრეტული L_2-ნორმით კრებადობის სიჩქარეა $O(h^s + \tau^{s/2})$.

1. Introduction

The purpose of this paper is to study the difference schemes approximating the first initial-boundary value problem for linear second order parabolic equations and to obtain some convergence rate estimates.

The finite difference method is a basic tool for the solution of partial differential equations. When studying the convergence of the finite difference schemes, Taylor's expansion was used traditionally. Often, the Bramble-Hilbert lemma [1], [2] takes the role of Taylor's formula for the functions from the Sobolev spaces.

As a model problem, we consider the first initial-boundary value problem for linear second-order parabolic equations with constant coefficients. We suppose that the generalized solution of this problem belongs to the anisotropic Sobolev space $W_2^{s,s/2}(Q)$, $s > 2$.

In the case of difference schemes constructed for the mentioned problem, when obtaining convergence rate estimate compatible with smoothness of the solution, various authors assume that the solution of the problem can be extended to the exterior of the domain of integration, preserving the Sobolev class.

Our investigations have shown that if instead of the exact initial condition its certain approximation is taken, then this restriction can be removed.

A high order alternating direction implicit (ADI) difference scheme is constructed in the paper for which the convergence rate estimate

$$\|y-u\|_{L_2(Q_{h,\tau})} \le c(h^s+\tau^{s/2})\|u\|_{W_2^{s,s/2}(Q)}, \quad s\in(2,\,4],$$

is obtained. Here y is a solution to the difference scheme, $Q_{h,\tau}$ is a mesh in Q, c is a positive constant independent of h, τ and u, and h and τ are space and time steps, respectively.

2. The Problem and Its Approximation

Let $\Omega = \{x=(x_1,x_2):\ 0<x_\alpha<1,\ \alpha=1,2\}$ be the unit square in R^2 with boundary Γ and let T denote a positive real number. In $Q=\Omega\times(0,T]$ we consider the equation of heat conductivity

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - au + f(x,t), \quad a = const \ge 0, \quad (x,t)\in Q_T, \tag{1}$$

under the initial and first kind boundary conditions

$$u(x,0)=u^0(x), \quad x\in\Omega, \quad u(x,t)=0, \quad x\in\Gamma, \quad t\in[0,T]. \tag{2}$$

We mean that the solution to the problem (1), (2) belongs to the anisotropic Sobolev space $W_2^{s,s/2}(Q)$, $s>2$.

Throughout the paper $\|\cdot\|_{W_2^{\lambda,\lambda/2}(Q)}$ will denote the norms and $|\cdot|_{W_2^{\lambda,\lambda/2}(Q)}$ the highest semi norms of corresponding Sobolev spaces [6].

We assume that $\bar\omega$ is a uniform mesh in Ω with the step $h=1/n$. $\omega = \overline{\omega}\cap\Omega$, $\gamma=\overline{\omega}\setminus\omega$. We cover the segment $[0,T]$ with a uniform mesh $\overline{\omega}_\tau$

(with the mesh step $\tau = T/N$). Let $\omega_\tau = \overline{\omega}_\tau \cap (0,T)$, $\omega_\tau^+ = \overline{\omega}_\tau \cap (0,T]$, $\omega_\tau^- = \overline{\omega}_\tau \cap [0,T)$, $Q_{h,\tau} = \omega \times \overline{\omega}_\tau$. We assume that there exist two positive constants $c_1 h^2 \le \tau \le c_2 h^2$. For functions defined on the mesh cylinder $\overline{\omega} \times \overline{\omega}_\tau$ we use the notation:

$$y = y(x,t) = y^j, \quad x \in \overline{\omega}, \quad t = t_j \in \overline{\omega}_\tau,$$
$$\widehat{y}(x,t) = y(x,t+\tau), \quad \check{y}(x,t) = y(x,t-\tau),$$
$$y_t = \frac{\widehat{y} - \check{y}}{\tau}, \quad y_{x_\alpha} = \frac{(I^{(+\alpha)} - I)y}{h}, \quad y_{\overline{x}_\alpha} = \frac{(I - I^{(-\alpha)})y}{h}, \quad \varkappa := \frac{h^2}{12},$$

where $Iy := y$, $I^{\pm\alpha} y := y(x \pm h r_\alpha, t)$ and r_α represents the unit vector of the axis x_α.

We define also the Steklov averaging operators:

$$T_1 u(x,t) = \frac{1}{h^2} \int_{x_1-h}^{x_1+h} \big(h - |x_1 - \xi|\big) u(\xi, x_2, t)\, d\xi,$$
$$\widehat{S} u(x,t) = \frac{1}{\tau} \int_t^{t+\tau} u(x,\zeta)\, d\zeta.$$

The operator T_2 is defined similarly. Note that these operators are commutative and

$$T_\alpha \frac{\partial^2 u}{\partial x_\alpha^2} = \Lambda_\alpha u, \quad \widehat{S}\, \frac{\partial u}{\partial t} = u_t.$$

If we apply the operator $\widehat{S} T_1 T_2$ to the eq. (1), we will get

$$(T_1 T_2 u)_t = \Lambda_1(\widehat{S} T_2 u) + \Lambda_2(\widehat{S} T_1 u) - a \widehat{S} T_1 T_2 u + \widehat{S} T_1 T_2 f. \tag{3}$$

It is easy to check that on the set of sufficiently smooth functions the following operators:

$$T_\alpha \sim I + \varkappa \Lambda_\alpha \ \text{ with errors of order } \ O(h^4),$$
$$\widehat{S} \sim (I + \widehat{I})/2 \ \text{ with errors of order } \ O(\tau^2)$$

are equivalent and, therefore, within the accuracy $O(h^4 + \tau^2)$ we obtain

$$T_1 T_2 \sim (I + \varkappa\Lambda_1)(I + \varkappa\Lambda_2), \tag{4}$$
$$\widehat{S} T_1 T_2 \sim (I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\, \frac{\widehat{I} + I}{2}, \tag{5}$$
$$\widehat{S} T_\alpha \sim (I + \varkappa\Lambda_\alpha)\, \frac{\widehat{I} + I}{2}. \tag{6}$$

Taking into the account the relations (4)–(6), we denote:

$$\eta_0 = T_1T_2u - (I + \varkappa\Lambda_1)(I + \varkappa\Lambda_2)u - (\tau^2/4)\Lambda_1\Lambda_2u, \tag{7}$$

$$\eta_\alpha = \widehat{S}T_{3-\alpha}u - (I + \varkappa\Lambda_{3-\alpha})\,\frac{\widehat{u}+u}{2},\ \ \alpha = 1, 2, \tag{8}$$

$$\begin{aligned}\eta = \widehat{S}T_1T_2u - (I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\,\frac{\widehat{u}+u}{2} &+ \\ + \Big(\frac{\tau\varkappa}{4} + \frac{\tau^2}{8}\Big)\Big(\Lambda_1 + \Lambda_2\Big)u_t &- \frac{a\tau^2}{16}\,u_t.\end{aligned} \tag{9}$$

In the equalities (7), (9) the additional terms are introduced with the aim that the resulting difference scheme operator should be factorizable.

Due to (7)–(9), from (3) we get

$$\begin{gathered}(I + \varkappa\Lambda_1)(I + \varkappa\Lambda_2)u_t + \frac{\tau^2}{4}\,\Lambda_1\Lambda_2u_t + (\eta_0)_t = \\ = \Lambda_1(I + \varkappa\Lambda_2)\,\frac{\widehat{u}+u}{2} + \Lambda_2(I + \varkappa\Lambda_1)\,\frac{\widehat{u}+u}{2} + \Lambda_1\eta_1 + \Lambda_2\eta_2 - \\ -a\bigg((I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\,\frac{\widehat{u}+u}{2} - \Big(\frac{\tau\varkappa}{4} + \frac{\tau^2}{8}\Big)(\Lambda_1 + \Lambda_2)u_t + \frac{a\tau^2}{16}\,u_t + \eta\bigg) + \\ + \widehat{S}T_1T_2f,\end{gathered}$$

that is,

$$\begin{gathered}\Big(I + \varkappa\Lambda_1 - \frac{\tau}{2}\,\Lambda_1 + \frac{a\tau}{4}\,I\Big)\Big(I + \varkappa\Lambda_2 - \frac{\tau}{2}\,\Lambda_2 + \frac{a\tau}{4}\,I\Big)u_t = \\ = \Big(\Lambda_1(I + \varkappa\Lambda_2) + \Lambda_2(I + \varkappa\Lambda_1) - a(I + \varkappa\Lambda_1 + \varkappa\Lambda_2)\Big)u + \\ + \widehat{S}T_1T_2f + \psi,\end{gathered} \tag{10}$$

where

$$\psi = \Lambda_1\eta_1 + \Lambda_2\eta_2 - a\eta - (\eta_0)_t. \tag{11}$$

Finally, if in the equation (10) we reject the remainder term and change u by the mesh function y, we will come to the difference scheme

$$By_t + Ay = \varphi,\ \ (x, t) \in \omega \times \omega_\tau^-, \tag{12}$$

where

$$\begin{aligned}A &:= A_1(I - \varkappa A_2) + A_2(I - \varkappa A_1) + a(I - \varkappa A_1 - \varkappa A_2), \\ B &:= \Big(I - \varkappa A_1 + \frac{\tau}{2}\,A_1 + \frac{a\tau}{4}\,I\Big)\Big(I - \varkappa A_2 + \frac{\tau}{2}\,A_2 + \frac{a\tau}{4}\,I\Big).\end{aligned}$$

We define the initial and boundary conditions as follows:

$$By^0 = T_1T_2u_0 + \frac{\tau}{2}\,Au_0,\ \ x \in \omega,\quad y(x, t) = 0,\ \ (x, t) \in \gamma \times \overline{\omega}_\tau. \tag{13}$$

3. An a Priori Estimate of the Solution Error

Let H be the space of mesh functions defined on $\overline{\omega}$ and vanishing on γ, with inner product and norm

$$(y,v)=\sum_{x\in\omega} h^2 y(x)v(x), \quad \|y\|=\|y\|_{L_2(\omega)}=(y,y)^{1/2}.$$

Besides, let

$$\|y\|_0=\|y\|_{L_2(Q_{h,\tau})}=\Big(\sum_{t\in\overline{\omega}_\tau}\tau\|y(\cdot,t)\|^2_{L_2(\omega)}\Big)^{1/2}.$$

In the case of self-conjugate positive operators we will use the notation

$$(y,v)_D:=(Dy,v), \ \ \|y\|_D:=\sqrt{(Dy,y)}, \ \ D=D^*>0.$$

Let

$$C:=B-\frac{\tau}{2}\,A. \tag{14}$$

It is easy to verify that

$$C=(I-\varkappa A_1)(I-\varkappa A_2)+\Big(\frac{a\tau^2}{8}+\frac{a\tau\varkappa}{4}\Big)(A_1+A_2)+ \\ +\frac{a^2\tau^2}{16}\,I+\frac{\tau^2}{4}\,A_1A_2\geq\frac{4}{9}\,I+\frac{\tau^2}{4}\,A_1A_2>0. \tag{15}$$

The following lemma plays a significant role in getting the needed a priori estimate of the solution of the difference scheme.

Lemma 1. *Let $A=A^*>0$, $B=B^*>0$ be arbitrary independent on t operators and $B>(\tau/2)A$. Then for the solution of the problem*

$$Bv_t+Av=\psi_t, \ \ (x,t)\in\omega\times\omega_\tau^-, \tag{16}$$

$$Bv^0=\psi^0, \ x\in\omega \tag{17}$$

the estimate

$$\|v\|_{L_2(Q_{h,\tau})}\leq\|C^{-1}\psi\|_{L_2(Q_{h,\tau})}$$

is valid with C defined in (14).

Proof. Summing up by $t=0,\tau,\dots,(k-1)\tau$, from (16) we find

$$Bv^k-Bv^0+\sum_{j=0}^{k-1}\tau Av^j=\psi^k-\psi^0, \ \ k=1,2,\dots,$$

that is, taking into account the initial condition (17),

$$Bv^k+\sum_{j=0}^{k-1}\tau Av^j=\psi^k, \ \ k=1,2,\dots\,. \tag{18}$$

Since $C = C^* > 0$, the inverse operator $C^{-1} = (C^{-1})^* > 0$ exists. Multiply (18) scalarly by $C^{-1}v^k$:

$$(Bv^k, C^{-1}v^k) + \Big(\sum_{j=0}^{k-1} \tau A v^j, C^{-1}v^k\Big) = (\psi^k, C^{-1}v^k), \quad k = 1, 2, \dots . \qquad (19)$$

Denote

$$\chi^0 = 0, \quad \chi^k = \sum_{j=0}^{k-1} \tau v^j, \quad k = 1, 2, \dots .$$

Then (19) yields

$$(Bv^k, C^{-1}v^k) + \Big(A\chi^k, C^{-1}\frac{\chi^{k+1} - \chi^k}{\tau}\Big) = (\psi^k, C^{-1}v^k),$$

from which, after some transformations, we obtain

$$\tau\bigg(\Big(B - \frac{\tau}{2}A\Big)v^k, C^{-1}v^k\bigg) + \frac{1}{2}\,\|\chi^{k+1}\|^2_{AC^{-1}} - \frac{1}{2}\,\|\chi^k\|^2_{AC^{-1}} = \\ = \tau(\psi^k, C^{-1}v^k)$$

or

$$2\tau\|v^k\|^2 + \|\chi^{k+1}\|^2_{AC^{-1}} - \|\chi^k\|^2_{AC^{-1}} = 2\tau(C^{-1}\psi^k, v^k), \quad k = 1, 2, \dots . \qquad (20)$$

Using the Cauchy–Bunyakovski inequality, we estimate the right-hand side of (20)

$$2\tau(C^{-1}\psi^k, v^k) \le \tau\|C^{-1}\psi^k\|^2 + \tau\|v^k\|^2$$

and sum up the obtained result by $k = 1, 2, \dots, N$. We get

$$\sum_{k=1}^{N} \tau\|v^k\|^2 + \|\chi^{N+1}\|^2_{AC^{-1}} - \|\chi^1\|^2_{AC^{-1}} \le \sum_{k=1}^{N} \tau\|C^{-1}\psi^k\|^2. \qquad (21)$$

From (14) we have

$$B^2 = C^2 + \tau AC + \frac{\tau^2}{4}\,A^2 > C^2 + \tau AC.$$

Hence

$$\tau AC^{-1} \le B^2C^{-2} - I.$$

Using this inequality and taking into account the relation $\chi^1 = \tau v^0$, we get

$$\|\chi^1\|^2_{AC^{-1}} = (\tau AC^{-1}v^0, \tau v^0) \le \big((B^2C_I^{-2})v^0, \tau v^0\big) = \\ = \tau\|BC^{-1}v^0\|^2 - \tau\|v^0\|^2 = \tau\|C^{-1}\psi^0\|^2 - \tau\|v^0\|^2,$$

which together with (21) proves the lemma. □

Consider the error $z = y - u$. From (10)–(13) we get the following problem for it:

$$\begin{gathered} Bz_t + Az = A_1\eta_1 + A_2\eta_2 + a\eta + (\eta_0)_t, \quad (x,t) \in \omega \times \omega_\tau^-, \\ Bz^0 = \eta_0^0, \quad x \in \omega, \quad z \in H. \end{gathered} \qquad (22)$$

We define the functions η_1, η_2 to be zeros on $t=T$ and substitute z in (22) by the following expression

$$z = v + A^{-1}(A_1\eta_1 + A_2\eta_2 + a\eta). \tag{23}$$

Then for v we obtain the problem (16), (17), where

$$\psi = \eta_0 - BA^{-1}(A_1\eta_1 + A_2\eta_2 + a\eta).$$

Using Lemma 1 for v, we get the estimate

$$\begin{gathered}\sum_{k=0}^{N} \tau\|v^k\|^2 \le \sum_{k=0}^{N} \tau J_k^2,\\ J_k := \big\|C^{-1}\eta_0^k - C^{-1}BA^{-1}(A_1\eta_1^k + A_2\eta_2^k + a\eta^k)\big\|.\end{gathered} \tag{24}$$

Because of (14), (15) we have

$$C^{-1}BA^{-1} = A^{-1} + \frac{\tau}{2}\, C^{-1} \le A^{-1} + \frac{9\tau}{8}\, I, \ \ C^{-1} \le (9/4)I.$$

Therefore

$$J_k \le \frac{9}{4}\,\|\eta_0^k\| + \big\|A^{-1}(A_1\eta_1^k + A_2\eta_2^k + a\eta^k)\big\| + \frac{9\tau}{8}\,\big\|A_1\eta_1^k + A_2\eta_2^k + a\eta^k\big\|.$$

Taking into account the operator inequalities

$$A \ge \frac{2}{3}\,(A_1 + A_2), \ \ A \ge \frac{32}{3}\, I, \ \ A^{-1}A_\alpha \le \frac{3}{2}\, I,$$

we get

$$J_k \le \frac{9}{4}\,\|\eta_0^k\| + \frac{3}{2}\,\Big(\|\eta_1^k\| + \|\eta_2^k\| + \frac{a}{16}\,\|\eta^k\|\Big) + \frac{9\tau}{8}\,\big\|A_1\eta_1^k + A_2\eta_2^k + a\eta^k\big\|.$$

On the basis of this and the following algebraic inequalities

$$\Big\{\sum_k \Big(\sum_i a_{ik}\Big)^2\Big\}^{1/2} \le \sum_i \Big(\sum_k a_{ik}^2\Big)^{1/2}, \ \ a_{ik} \ge 0,$$

we get from (24)

$$\begin{aligned}\|v\|_0 \le \frac{9}{4}\,\|\eta_0\|_0 + \frac{3}{2}\Big(\|\eta_1\|_0 + \|\eta_2\|_0 + \frac{a}{16}\,\|\eta\|_0\Big) + \\ + \frac{9\tau}{8}\,\big(\|A_1\eta_1\|_0 + \|A_2\eta_2\|_0 + a\|\eta\|_0\big).\end{aligned} \tag{25}$$

(23), (25) enable us to assert the validity of the following

Theorem 1. *For the solution of the difference problem* (22) *the following a priori estimate is true*

$$\|z\|_0 \le \frac{9}{4}\,\|\eta_0\|_0 + 3(\|\eta_1\|_0 + \|\eta_2\|_0) + \frac{9\tau}{8}\,\big(\|A_1\eta_1\|_0 + \|A_2\eta_2\|_0\big). \tag{26}$$

4. Convergence of the Finite-Difference Scheme

Let E denote a bounded open set in R^2 with Lipschitz continuous boundary, and let $G = E \times (0, 1)$. We introduce the set of multi-indices

$$\mathcal{B}_k = \Big\{(\alpha_1, \alpha_2, \beta) : \ \alpha_i, \beta = 0, 1, 2, \ldots ; \ \alpha_1 + \alpha_2 + 2\beta \le k\Big\}.$$

Further, let $[s]^-$ denote the largest integer less than s. The convergence analysis of our finite difference scheme is based on the following lemma.

Lemma 2. *If φ is a bounded linear functional on $W_2^{s,s/2}(G)$ such that*

$$\varphi(x_1^{\alpha_1} x_2^{\alpha_2} t^{\beta}) = 0, \ \ \forall\, (\alpha_1, \alpha_2, \beta) \in \mathcal{B}_{[s]^-},$$

then there exists a positive constant $c = c(G, s)$ such that

$$|\varphi(v)| \le c|v|_{W_2^{s,s/2}(G)}, \ \ \forall\, v \in W_2^{s,s/2}(G).$$

Lemma 2 is an easy consequence of the Dupont–Scott approximation theorem [4] (see also [5]).

If we use Lemma 2 and the well-known techniques (see, e.g., [1]–[3], [5]) for estimation of the terms in the right-hand side of the equation (26), we will get convinced in the validity of the following

Theorem 2. *Assume that the solution u to the problem* (1), (2) *belongs to the space $W_2^{s,s/2}(Q_{h,\tau})$, $2 < s \le 4$. Then the rate of convergence of the difference scheme* (12), (13) *in the L_2 grid norm is described by the estimate*

$$\|y - u\|_{L_2(Q_{h,\tau})} \le ch^s \|u\|_{W_2^{s,s/2}(Q)}, \ \ s \in (2, 4],$$

where the constant c does not depend on h and u.

Remark. A more detailed analysis enables us to obtain the estimate

$$\|y - u\|_{L_2(Q_{h,\tau})} \le c(h^s + \tau^{s/2}) \|u\|_{W_2^{s,s/2}(Q)}, \ \ s \in (2,\, 4],$$

as well without restriction $\tau \sim h^2$.

The results of the paper were announced on Sixth International Congress on Industrial Applied Mathematics (ICIAM07), Zürich, 2007 [7].

References

1. A. A. Samarskiĭ, R. D. Lazarov, and V. L. Makarov, Difference schemes for differential equations with generalized solutions. (Russian) *Visshaya Shkola, Moskow*, 1987.
2. B. S. Jovanović, The finite difference method for boundary-value problems with weak solutions. *Posebna Izdanja* [*Special Editions*], 16. *Matematički Institut u Beogradu, Belgrade*, 1993.
3. R. D. Lazarov, V. L. Makarov, and A. A. Samarskiĭ, Application of exact difference schemes for constructing and investigating difference schemes on generalized solutions. (Russian) *Mat. Sb. (N.S.)* **117(159)** (1982), No. 4, 469–480.
4. T. Dupont and R. Scott, Polynomial approximation of functions in Sobolev spaces. *Math. Comp.* **34** (1980), No. 150, 441–463.

5. B. S. Jovanović, On the convergence of finite-difference schemes for parabolic equations with variable coefficients. *Numer. Math.* **54** (1989), no. 4, 395–404.
6. O. A. Ladyzhenskaya, V. A. Solonnikov, and N. N. Ural'tseva, Linear and quasilinear equations of parabolic type. (Russian) *Izdat. "Nauka", Moskcow*, 1967.
7. G. Berikelashvili, M. M. Gupta and M. Mirianashvili, On the choice if initial conditions of difference schemes for parabolic equations. *Proc. Appl. Math. Mech.* **7** (2007), 1025605–1025606.

(Received 29.09.2010)

Authors' addresses:

G. Berikelashvili
A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186
Georgia

Department of Mathematics, Georgian Technical University
77, M. Kostava Str., Tbilisi 0175
Georgia
E-mail: bergi@rmi.ge

M. M. Gupta
Department of Mathematics
The George Washington University
Washington, DC 20052
USA
E-mail: mmg@gwu.edu

M. Mirianashvili
N. Muskhelishvili Institute of Computational Mathematics
8, Akuri Str., Tbilisi 0193
Georgia