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GENERALIZED PICONE IDENTITY AND COMPARISON OF HALF-LINEAR DIFFERENTIAL EQUATIONS OF ORDER $4m$

Dedicated to Professor Kusano Takaŝi
on the occasion of his 80th birthday anniversary

Abstract. A Picone-type identity and the Sturm-type comparison theorems are established for ordinary differential equations of the form

$$\big(p(t)\varphi(u^{(2m)})\big)^{(2m)} + q(t)\varphi(u) = 0$$

and

$$\big(P(t)\varphi(v^{(2m)})\big)^{(2m)} + Q(t)\varphi(v) = 0,$$

where $m \geq 1$, $p, P \in C^{2m}([a,b],(0,\infty))$, $q, Q \in C([a,b],\mathbf{R})$, $\varphi(s) := |s|^\alpha \operatorname{sgn} s$ and $\alpha > 0$.

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რეზიუმე. ჩვეულებრივი დიფერენციალური განტოლებებისათვის

$$\big(p(t)\varphi(u^{(2m)})\big)^{(2m)} + q(t)\varphi(u) = 0$$

და

$$\big(P(t)\varphi(v^{(2m)})\big)^{(2m)} + Q(t)\varphi(v) = 0,$$

სადაც $m \geq 1$, $p, P \in C^{2m}([a,b],(0,\infty))$, $q, Q \in C([a,b],\mathbf{R})$, $\varphi(s) := |s|^\alpha \operatorname{sgn} s$ და $\alpha > 0$, დადგენილია პიკონეს ტიპის იგივობა და შტურმის შედარების თეორემა.

1. Introduction

In the classical Sturm comparison theory for linear self-adjoint differential equations of the second order a fundamental role plays by the so-called Picone's formula (see [14]). It states that if x, px', y and Py' are continuously differentiable functions on an interval I with $y(t) \neq 0$, then

$$\frac{d}{dt}\left[\frac{x}{y}\left(px'y - Pxy'\right)\right] =$$

$$= -\frac{x^2}{y}\,(Py')' + x(px')' + (p-P)x'^2 + P\left(x' - \frac{x}{y}\,y'\right)^2. \qquad (1.1)$$

If, in addition, x and y solve in I the equations

$$-\big(p(t)u'\big)' + q(t)u = 0 \qquad (1.2)$$

and

$$-\big(P(t)v'\big)' + Q(t)v = 0, \qquad (1.3)$$

respectively, where $0 < P(t) \leq p(t)$ and $Q(t) \leq q(t)$ in I, and x have consecutive zeros at a and b ($a < b$), then integrating (1.1) between a and b, we obtain

$$0 = \int_a^b \left[\big(q(t) - Q(t)\big)x^2 + \big(p(t) - P(t)\big)x'^2 + P(t)\Big(x' - \frac{x}{y}\,y'\Big)^2\right] dt \qquad (1.4)$$

and the Sturmian conclusion about the existence of a zero in $[a,b]$ for any solution y of the majorant equation (1.3) readily follows from (1.4).

Generalizations and extensions of the Sturm's comparison principle and underlying Picone-type identities to nonlinear equations and higher-order (ordinary and partial) differential operators have been obtained by various authors. We refer, in particular, to the papers [1]–[17] and the references cited therein.

The purpose of the present paper is to extend (1.1) to half-linear ordinary differential operators of the form

$$l_\alpha[x] \equiv \big(p\varphi(x^{(2m)})\big)^{(2m)} + q\varphi(x) \qquad (1.5)$$

and

$$L_\alpha[y] \equiv \big(P\varphi(y^{(2m)})\big)^{(2m)} + Q\varphi(y), \qquad (1.6)$$

where $m \geq 1$, $p, P \in C^{2m}([a,b],(0,\infty))$, $q, Q \in C([a,b],\mathbf{R})$ and $\varphi(s) := |s|^{\alpha-1}s$ for $s \neq 0$, $\alpha > 0$, and $\varphi(0) = 0$. Next, in Section 3, we illustrate the usefulness of the obtained identity by deriving Sturm's comparison theorems and other qualitative results concerning half-linear differential equations of the order $4m$.

In the linear case, i.e. if (1.5) and (1.6) reduce to a pair of $4m$th-order self-adjoint operators of the form $l_1[x] \equiv (px^{(2m)})^{(2m)} + qx$ and $L_1[y] \equiv (Py^{(2m)})^{(2m)} + Qy$, respectively, two different kinds of Picone-type identities are known in the literature. The first one which can be found in Kusano et al. [12] says (when specialized to (1.5) and (1.6)), that if $x \in D_{l_1}(I)$, $y \in D_{L_1}(I)$, and none of $y, y', \dots, y^{(2m-1)}$ vanishes in I, then

$$\frac{d}{dt}\left\{\sum_{k=0}^{2m-1}(-1)^k\,\frac{x^{(k)}}{y^{(k)}}\,\Big[x^{(k)}(Py^{(2m)})^{(2m-k-1)}-y^{(k)}(px^{(2m)})^{(2m-k-1)}\Big]\right\}=$$

$$=\frac{x^2}{y}\,L_1[y]-xl_1[x]+(q-Q)x^2+(p-P)\big[x^{(2m)}\big]^2+$$

$$+P\Big[x^{(2m)}-\frac{x^{(2m-1)}}{y^{(2m-1)}}\,y^{(2m)}\Big]^2-y^{(2m-1)}\big(Py^{(2m)}\big)'\Big[\frac{x^{(2m-1)}}{y^{(2m-1)}}-\frac{x^{(2m-2)}}{y^{(2m-2)}}\Big]^2. \quad (1.7)$$

A typical comparison result based on the above formula is the following theorem (see [12]).

Theorem A. *Suppose there exists a nontrivial real-valued function* $u \in \mathcal{D}_{l_1}([a,b])$ *which satisfies*

$$\int_a^b ul_1[u]\,dt \le 0,$$

$$u(a)=u'(a)=\dots=u^{(2m-1)}(a)=u(b)=\dots=u^{(2m-1)}(b)=0$$

and

$$\int_a^b\Big[\big(p(t)-P(t)\big)\big(u^{(2m)}\big)^2+\big(q(t)-Q(t)\big)u^2\Big]\,dt\ge 0.$$

If $v \in \mathcal{D}_{L_1}([a,b])$ *satisfies*

$$vL_1[v]\ge 0 \ \ in\ \ (a,b), \quad where\ \ P(t)\ge 0,$$

$$v^{(k)}\big[P(t)v^{(2m)}\big]^{(2m-k)}\ge 0 \ \ in\ \ (a,b), \quad 1\le k\le 2m-1,$$

and

$$\big[P(t)v^{(2m)}\big]^{(2m-\nu)}\ne 0 \ \ in\ \ (a,b)\ \ for\ some\ \ \nu, \quad 1\le\nu\le 2m-1,$$

then at least one of $v, v', \dots, v^{(2m-1)}$ *has a zero in* (a,b).

Recently, Kusano–Yoshida's formula (1.7) was generalized to half-linear ordinary differential operators of an arbitrary even order (see [5]).

The second Picone type identity applied to (1.5) and (1.6) has been obtained by N. Yoshida [16]. The specialization to the one-dimensional case studied here says that if $x \in D_{l_1}(I)$, $y \in D_{L_1}(I)$ and none of $y, y', \dots, y^{(2m-2)}$ vanishes in I, then

$$\frac{d}{dt}\left\{\sum_{k=0}^{m-1}\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\,\Big[x^{(2m-2k-2)}\big(Py^{(2m)}\big)^{(2k+1)}-\right.$$

$$\left.-\,y^{(2m-2k-2)}\big(px^{(2m)}\big)^{(2k+1)}\Big]+\right.$$

$$+\sum_{k=0}^{m-1}\left[\left(px^{(2m)}\right)^{(2m-2k-2)}x^{(2k+1)}-\left(Py^{(2m)}\right)^{(2k)}\left(\frac{\left(x^{(2m-2k-2)}\right)^2}{y^{(2m-2k-2)}}\right)'\right]\Bigg\}=$$

$$=\frac{x^2}{y}L_1[y]-xl_1[x]+(p-P)\big[x^{(2m)}\big]^2+(q-Q)x^2+$$

$$+P\Big[x^{(2m)}-\frac{x^{(2m-2)}}{y^{(2m-2)}}\,y^{(2m)}\Big]^2+$$

$$+\sum_{k=1}^{m-1}\frac{\left(Py^{(2m)}\right)^{(2k)}}{y^{(2m-2k)}}\left[x^{(2m-2k)}-\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}y^{(2m-2k)}\right]^2-$$

$$-2\sum_{k=0}^{m-1}\frac{\left(Py^{(2m)}\right)^{(2k)}}{y^{(2m-2k-2)}}\left[x^{(2m-2k-1)}-\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\,y^{(2m-2k-1)}\right]^2. \quad (1.8)$$

The following comparison theorem can be easily obtained with the help of the identity (1.8) (see [16]).

Theorem B. *Assume that there exists a nontrivial function $u\in D_{l_1}([a,b])$ which satisfies*

$$\int_a^b ul_1[u]\,dt\le 0,$$

$$u(a)=u'(a)=\cdots=u^{(2m-1)}(a)=u(b)=u'(b)=\cdots=u^{(2m-1)}(b)=0$$

and

$$V[u]\equiv\int_a^b\left[(p(t)-P(t))\left(u^{(2m)}\right)^2+(q(t)-Q(t))u^2\right]dt\ge 0.$$

If $v\in D_{L_1}([a,b])$ satisfies

$$L_1[v]\ge 0 \quad in\ (a,b),$$

$$(-1)^k v^{(2k)}(t)>0 \ \ at\ some\ point\ \ t\in(a,b),\ \ 0\le k\le m-1,$$

$$(-1)^{m+k}\left(Pv^{(2m)}\right)^{(2k)}\ge 0 \ \ in\ \ (a,b),\ \ 0\le k\le m-2,$$

$$(Pv^{(2m)})^{(2m-2)}<0 \ \ in\ \ (a,b),$$

then at least one of the functions $v,v',\dots,v^{(2m-2)}$ must vanish at some point of $[a,b]$.

2. The Generalized Picone's Identity

Let $p,P\in C^{2m}([a,b],(0,\infty))$, $m\ge 1$ and $q,Q\in C([a,b],\mathbf{R})$. For a fixed $\alpha>0$ we define the function $\varphi:\mathbf{R}\to\mathbf{R}$ by $\varphi(s)=|s|^{\alpha-1}s$ for $s\ne 0$ and $\varphi(0)=0$, and consider ordinary differential operators of the form

$$l_\alpha[x]=\left(p(t)\varphi(x^{(2m)})\right)^{(2m)}+q(t)\varphi(x)$$

and

$$L_\alpha[y] = \big(P(t)\varphi(y^{(2m)})\big)^{(2m)} + Q(t)\varphi(y)$$

with the domains $D_{l_\alpha}(a,b)$ (resp., $D_{L_\alpha}(a,b)$) defined to be the sets of all functions x (resp., y) of the class $C^{2m}([a,b],\mathbf{R})$ such that $p\varphi(x^{(2m)})$ (resp., $P\varphi(y^{(2m)})$) are in $C^{2m}((a,b),\mathbf{R})\bigcap C([a,b],\mathbf{R})$.

Also, by Φ_α we denote the form defined for $X,Y\in\mathbf{R}$ and $\alpha>0$ by

$$\Phi_\alpha(X,Y) := |X|^{\alpha+1} + \alpha|Y|^{\alpha+1} - (\alpha+1)X\varphi(Y).$$

According to the Young inequality, it follows that $\Phi_\alpha(X,Y)\ge 0$ for all $X,Y\in\mathbf{R}$ and the equality holds if and only if $X=Y$.

We begin with the following lemma which can be verified by a routine computation.

Lemma 2.1. *If* $x\in C^{2m}([a,b],\mathbf{R})$, $y\in D_{L_\alpha}((a,b))$ *and none of* $y, y', \dots, y^{(2m-2)}$ *vanishes in* (a,b), *then*

$$\frac{d}{dt}\Bigg\{\sum_{k=0}^{m-1}\Bigg[-\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi\big(y^{(2m-2k-2)}\big)}\big(P\varphi(y^{(2m)})\big)^{(2k+1)} + \left(\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi\big(y^{(2m-2k-2)}\big)}\right)'\big(P\varphi(y^{(2m)})\big)^{(2k)}\Bigg]\Bigg\} =$$

$$= -\frac{|x|^{\alpha+1}}{\varphi(y)}L_\alpha[y] + Q|x|^{\alpha+1} + P|x^{(2m)}|^{\alpha+1} - P\Phi_\alpha\Big(x^{(2m)}, \frac{x^{(2m-2)}}{y^{(2m-2)}}y^{(2m)}\Big) -$$

$$- \sum_{k=1}^{m-1}\frac{\big(P\varphi(y^{(2m)})\big)^{(2k)}}{\varphi\big(y^{(2m-2k)}\big)}\Phi_\alpha\Big(x^{(2m-2k)}, \frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}y^{(2m-2k)}\Big) +$$

$$+ \alpha(\alpha+1)\sum_{k=0}^{m-1}\frac{\big(P\varphi(y^{(2m)})\big)^{(2k)}}{\varphi\big(y^{(2m-2k-2)}\big)}\big|x^{(2m-2k-2)}\big|^{\alpha-1}\times$$

$$\times\Big[x^{(2m-2k-1)} - \frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}y^{(2m-2k-1)}\Big]^2. \qquad (2.1)$$

We now establish a *stronger form* of Picone's identity in which the relatively weak hypothesis from Lemma 2.1 that x is any $2m$-times continuously differentiable function is replaced by the assumption that x is from the domain $\mathcal{D}_{l_\alpha}$ of the operator l_α.

Lemma 2.2. *If* $x\in D_{l_\alpha}((a,b))$, $y\in D_{L_\alpha}((a,b))$ *and none of* $y, y', \dots, y^{(2m-2)}$ *vanishes in* (a,b), *then*

$$\frac{d}{dt}\Bigg\{\sum_{k=0}^{m-1}\Bigg[\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi\big(y^{(2m-2k-2)}\big)}\big(P\varphi(y^{(2m)})\big)^{(2k+1)} -$$

$$- \big(P\varphi(y^{(2m)})\big)^{(2k)}\left(\frac{|x^{(2m-2k-2)}|^{\alpha+1}}{\varphi\big(y^{(2m-2k-2)}\big)}\right)' +$$

$$+\big(p\varphi(x^{(2m)})\big)^{(2m-2k-2)}x^{(2k+1)}-x^{(2m-2k-2)}\big(p\varphi(x^{(2m)})\big)^{(2k+1)}\Big]\Bigg\}=$$

$$=\frac{|x|^{\alpha+1}}{\varphi(y)}\,L_\alpha[y]-xl_\alpha[x]+(p-P)|x^{(2m)}|^{\alpha+1}+(q-Q)|u|^{\alpha+1}+$$

$$+P\Phi_\alpha\Big(x^{(2m)},\frac{x^{(2m-2)}}{y^{(2m-2)}}\,y^{(2m)}\Big)+$$

$$+\sum_{k=1}^{m-1}\frac{\Big(P\varphi(y^{(2m)})\Big)^{(2k)}}{\varphi(y^{(2m-2k)})}\,\Phi_\alpha\Big(x^{(2m-2k)},\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}\,y^{(2m-2k)}\Big)-$$

$$-\alpha(\alpha+1)\sum_{k=0}^{m-1}\frac{\big(P\varphi(y^{(2m)})\big)^{(2k)}}{\varphi(y^{(2m-2k-2)})}\,\big|x^{(2m-2k-2)}\big|^{\alpha-1}\times$$

$$\times\Big[x^{(2m-2k-1)}-\frac{x^{(2m-2k-2)}}{y^{(2m-2k-2)}}y^{(2m-2k-1)}\Big]^2. \quad (2.2)$$

3. Applications

As the first application of the identity (2.1) we obtain the following result.

Theorem 3.1. *If there exists a nontrivial function* $u\in C^{2m}([a,b],\mathbf{R})$ *such that*

$$u(a)=u'(a)=\cdots=u^{(2m-1)}(a)=u(b)=\cdots=u^{(2m-1)}(b)=0 \quad (3.1)$$

and

$$M_\alpha[u]\equiv\int_a^b\Big[P(t)|u^{(2m)}|^{\alpha+1}+Q(t)|u|^{\alpha+1}\Big]\,dt\le 0, \quad (3.2)$$

then there does not exist a $v\in\mathcal{D}_{L_\alpha}([a,b])$ *satisfying*

$$L_\alpha[v]\ge 0 \ \text{ in } \ (a,b), \quad (3.3)$$

$$v(a)>0,\quad v(b)>0, \quad (3.4)$$

$$(-1)^k v^{(2k)}>0 \ \text{ in } \ [a,b],\ \ 1\le k\le m-1, \quad (3.5)$$

$$(-1)^{m+k}\big(P\varphi(v^{(2m)})\big)^{(2k)}\ge 0 \ \text{ in } \ (a,b),\ \ 0\le k\le m-2, \quad (3.6)$$

and

$$\big(P\varphi(v^{(2m)})\big)^{(2m-2)}<0 \ \text{ in } \ (a,b). \quad (3.7)$$

Proof. Suppose to the contrary that there exists a $v\in\mathcal{D}_{L_\alpha}([a,b])$ satisfying (3.3)–(3.7). Since $v(a)>0$, $v(b)>0$ and $v''(t)<0$ in (a,b), it follows that $v(t)>0$ on $[a,b]$. Integrating the identity (2.1) on $[a,b]$, we obtain

$$0\ge M_\alpha[u]-\int_a^b\frac{|u|^{\alpha+1}}{v^\alpha}\,L_\alpha[v]\,dt\ge$$

$$\geq -\alpha(\alpha+1)\int_a^b \frac{(P\varphi(v^{(2m)}))^{(2m-2)}}{v^\alpha}\,|u|^{\alpha-1}\Big(u'-\frac{u}{v}\,v'\Big)^2\,dt \geq 0.$$

It follows that $u' - uv'/v = 0$ in (a,b) and therefore $u/v = k$ in $[a,b]$ for some nonzero constant k. Since $u(a) = u(b) = 0$ and $v(a) > 0, v(b) > 0$, we have a contradiction. Hence there can exist no v satisfying (3.3)–(3.7). □

Theorem 3.2. *If there exists a nontrivial* $u \in C^{2m}([a,b],\mathbf{R})$ *satisfying* (3.1) *and* (3.2)*, then every solution* $v \in \mathcal{D}_{L_\alpha}((a,b))$ *of the inequality* (3.3) *satisfying* (3.5)–(3.7) *and*

$$v(t_0) > 0 \;\; \textit{for some} \;\; t_0 \in (a,b) \tag{3.8}$$

has zero in $[a,b]$.

Proof. If the function v satisfies (3.3), (3.5)–(3.7) and (3.8), then either $v(a) < 0$, and hence v, must vanish somewhere in (a,b), or $v(a) \geq 0$. In the latter case, however, Theorem 3.1 implies that $v(a) = 0$ or $v(b) = 0$, and thus the proof is complete. □

As an application of the identity (2.2), we derive the Sturm-type comparison theorem. It belongs to weak comparison results in the sense that the conclusion regarding to v applies to $[a,b]$ rather than (a,b).

Theorem 3.3. *If there exists a nontrivial* $u \in \mathcal{D}_{l_\alpha}((a,b))$ *such that*

$$\int_a^b u l_\alpha[u]\,dt \leq 0, \tag{3.9}$$

$$u(a) = u'(a) = \cdots = u^{(2m-1)}(a) = u(b) = \cdots = u^{(2m-1)}(b) = 0, \tag{3.10}$$

$$V_\alpha[u] \equiv \int_a^b \Big[\big(p(t)-P(t)\big)|u^{(2m)}|^{\alpha+1} + \big(q(t)-Q(t)\big)|u|^{\alpha+1}\Big]\,dt \geq 0, \tag{3.11}$$

and if $v \in \mathcal{D}_{L_\alpha}((a,b))$ *satisfies*

$$L_\alpha[v] \geq 0 \;\; \textit{in} \;\; (a,b), \tag{3.12}$$

$$(-1)^k v^{(2k)}(t_k) > 0 \;\; \textit{at some point} \;\; t_k \in (a,b), \;\; 0 \leq k \leq m-1, \tag{3.13}$$

$$(-1)^{m+k}\big(P\varphi(v^{(2m)})\big)^{(2k)} \geq 0 \;\; \textit{in} \;\; (a,b), \;\; 0 \leq k \leq m-2, \tag{3.14}$$

and

$$\big(P\varphi(v^{(2m)})\big)^{(2m-2)} < 0 \;\; \textit{in} \;\; (a,b), \tag{3.15}$$

then at least one of $v, v'', \ldots, v^{(2m-2)}$ *vanishes somewhere in* $[a,b]$.

Proof. Suppose that none of $v, v', \ldots, v^{(2m-2)}$ vanishes in $[a,b]$. From the identity (2.2) integrated on $[a,b]$ we obtain, in view of the the conditions of the theorem, that

$$0=V_\alpha[u]+\int_a^b \frac{|u|^{\alpha+1}}{v^\alpha} L_\alpha[v]\,dt-\int_a^b u l_\alpha[u]\,dt+\int_a^b P\Phi_\alpha\Big(u^{(2m)}, \frac{u^{(2m-2)}}{v^{(2m-2)}}\,v^{(2m)}\Big)\,dt+$$

$$+\int_a^b \bigg\{\sum_{k=1}^{m-1} \frac{\big(P\varphi(v^{(2m)})\big)^{(2k)}}{\varphi(v^{(2m-2k)})}\,\Phi_\alpha\Big(u^{(2m-2k)}, \frac{u^{(2m-2k-2)}}{v^{(2m-2k-2)}}\,v^{(2m-2k)}\Big)\bigg\}\,dt-$$

$$-\alpha(\alpha+1)\int_a^b \bigg\{\sum_{k=0}^{m-1} \frac{\big(P\varphi(v^{(2m)})\big)^{(2k)}}{\varphi(v^{(2m-2k-2)})}\,|u^{(2m-2k-2)}|^{\alpha-1}\times$$

$$\times\Big[u^{2m-2k-1)}-\frac{u^{(2m-2k-2)}}{v^{(2m-2k-2)}}\,v^{(2m-2k-1)}\Big]^2\bigg\}\,dt\ge$$

$$\ge -\alpha(\alpha+1)\int_a^b \frac{\big(P\varphi(v^{(2m)})\big)^{(2m-2)}}{v^\alpha}\,|u|^{\alpha-1}\Big(u'-\frac{u}{v}\,v'\Big)^2\,dt\ge 0.$$

Consequently, $u'-uv'/v=0$ in (a,b), that is, $u/v=k$ in (a,b), and hence on $[a,b]$ by continuity, for some nonzero constant k. However, this is not the case since $u(a)=u(b)=0$, whereas $v(t)>0$ on $[a,b]$. This contradiction shows that at least one of $v, v', \dots, v^{(2m-2)}$ must vanish in $[a,b]$. □

Finally, we use the identity (2.2) to obtain a lower bound for the first eigenvalue of the nonlinear eigenvalue problem

$$l_\alpha[u]=\lambda\varphi(u) \ \text{ in } \ (a,b), \tag{3.16}$$

$$u(a)=u'(a)=\cdots=u^{(2m-1)}(a)=u(b)=\cdots=u^{(2m-1)}(b)=0. \tag{3.17}$$

Theorem 3.4. *Let λ_1 be the first eigenvalue of the problem* (3.16)–(3.17) *and $u_1\in\mathcal{D}_{l_\alpha}((a,b))$ be the corresponding eigenfunction. If there exists a function $v\in\mathcal{D}_{L_\alpha}((a,b))$ such that*

$$(-1)^k v^{(2k)}>0 \ \text{ in } \ [a,b], \ 0\le k\le m-1,$$

$$(-1)^{m+k}\big(P\varphi(v^{(2m)})\big)^{(2k)}\ge 0 \ \text{ in } \ (a,b), \ \ 0\le k\le m-1,$$

and if $V_\alpha[u_1]\ge 0$, then $\lambda_1\ge \inf\limits_{t\in(a,b)}\big[\frac{L_\alpha[v]}{v^\alpha}\big]$.

Proof. The identity (2.2) in view of the above hypotheses implies that

$$\lambda_1\int_a^b |u_1|^{\alpha+1}\,dt-\int_a^b |u_1|^{\alpha+1}\frac{L_\alpha[v]}{v^\alpha}\,dt\ge 0,$$

from which the conclusion follows readily. □

References

1. D. R. Dunninger, A Picone integral identity for a class of fourth order elliptic differential inequalities. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **50** (1971), 630–641.

2. M. S. P. EASTHAM, The Picone identity for self-adjoint differential equations of even order. *Mathematika* **20** (1973), 197–200.
3. Á. ELBERT, A half-linear second order differential equation. *Qualitative theory of differential equations, Vol. I, II* (*Szeged,* 1979), pp. 153–180, *Colloq. Math. Soc. János Bolyai,* 30, *North-Holland, Amsterdam–New York*, 1981.
4. J. JAROŠ, Comparison theorems for half-linear equations of fourth order. *Equadiff 9 – Conference on Differential Equations and Their Applications* (*Brno, Czech Republic, August* 25–29, 1997). Abstracts and Enlarged Abstracts (Ed. by Z. Došlá, J. Kalas, J. Vosmanský), *Masaryk University, Brno*, p. 233.
5. J. JAROŠ, The higher-order Picone identity and comparison of half-linear differential equations of even order. *Nonlinear Anal.* **74** (2011), No. 18, 7513–7518.
6. J. JAROŠ AND T. KUSANO, Second-order semilinear differential equations with external forcing terms. (Japanese) *Structure of functional equations and mathematical methods* (Japanese) (*Kyoto*, 1996). *Sūrikaisekikenkyūsho Kōkyūroku* No. 984 (1997), 191–197.
7. J. JAROŠ AND T. KUSANO, A Picone type identity for second order half-linear differential equations. *Acta Math. Univ. Comenian. (N.S.)* **68** (1999), No. 1, 137–151.
8. K. KREITH, A comparison theorem for fourth order differential equations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **46** (1969), 664–666.
9. K. KREITH, A Picone identity for fourth order differential equations. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **52** (1972), 455–456.
10. K. KREITH, Oscillation theory. *Lecture Notes in Mathematics.* 324. *Springer-Verlag, Berlin–Heidelberg–New York*, 1973.
11. K. KREITH, Picone's identity and generalizations. Collection of articles dedicated to Mauro Picone on the occasion of his ninetieth birthday. *Rend. Mat. (6)* **8** (1975), 251–262.
12. T. KUSANO AND N. YOSHIDA, Picone's identity for ordinary differential operators of even order. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), No. 4, 524–530.
13. W. LEIGHTON AND Z. NEHARI, On the oscillation of solutions of self-adjoint linear differential equations of the fourth order. *Trans. Amer. Math. Soc.* **89** (1958), 325–377.
14. M. PICONE, Sui valori eccezionali di un parametro da cui dipende un'equazione differenziale lineare ordinaria del second'ordine. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **11** (1910), 1–144.
15. T. TANIGAWA AND N. YOSHIDA, Picone identities for ordinary differential equations of fourth order. *Math. J. Toyama Univ.* **27** (2004), 91–99.
16. N. YOSHIDA, A Picone identity for elliptic differential operators of order $4m$ with applications. *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)* **58** (1975), No. 3, 306–317.
17. N. YOSHIDA, Nonoscillation criteria for fourth order elliptic equations. *Hiroshima Math. J.* **5** (1975), 23–31.

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