**Memoirs on Differential Equations and Mathematical Physics**
Volume 57, 2012, 51–74

**Liliya Koltsova and Alexander Kostin**

# THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF MONOTONE TYPE OF FIRST-ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS, UNRESOLVED FOR THE DERIVATIVE

**Abstract.** For the first-order nonlinear ordinary differential equation

$$F(t, y, y') = \sum_{k=1}^{n} p_k(t) y^{\alpha_k} (y')^{\beta_k} = 0,$$

unresolved for the derivative, asymptotic behavior of solutions of monotone type is established for $t \to +\infty$.

**2010 Mathematics Subject Classification.** 34D05, 34E10.

**Key words and phrases.** Nonlinear differential equations, monotone solutions, asymptotic properties.

**რეზიუმე.** გამოკვლეულია წარმოებულის მიმართ ამოუხსნელი პირველი რიგის არაწრფივი ჩვეულებრივი დიფერენციალური განტოლებების მონოტონური ტიპის ამონახსნების ასიმპტოტური თვისებები.

This article describes a first-order real ordinary differential equation:

$$F(t,y,y') = \sum_{k=1}^{n} p_k(t) y^{\alpha_k} (y')^{\beta_k} = 0, \tag{1}$$

$(t,y,y') \in D$, $D = \Delta(a) \times \mathbb{R}_1 \times \mathbb{R}_2$, $\Delta(a) = [a; +\infty[$, $a > 0$, $\mathbb{R}_1 = \mathbb{R}_+$, $\mathbb{R}_2 = \mathbb{R}_- \vee \mathbb{R}_+$; $p_k(t) \in \mathrm{C}_{\Delta(a)}$ $(k = \overline{1,n},\ n \geq 2)$; $\alpha_k, \beta_k \geq 0$ $(k = \overline{1,n})$, $\sum\limits_{k=1}^{n} \beta_k \neq 0$.

Further, we assume that all the expressions, appearing in the equation, make sense; and all functions we consider in the present paper are real.

We investigate the question on the existence and on the asymptotic behavior (as $t \to +\infty$) of unboundedly continuable to the right solutions ($R$-solutions) $y(t)$ of equation (1) and derivatives $y'(t)$ of these solutions which possess the following properties:

A) $0 < y(t) \in \mathrm{C}^1_{\Delta(t_1)}$, $\Delta(t_1) \subset \Delta(a)$, where $t_1$ is defined in the course of proving each theorem;

B) among the summands $p_k(t)(y(t))^{\alpha_k}(y'(t))^{\beta_k}$ $(k = \overline{1,n})$, the terms with numbers $i = \overline{1,s}$ $(2 \leq s \leq n)$ are asymptotically principal for the given $R$-solution $y(t)$, i.e., there exist:

$$\lim_{t\to+\infty} \frac{p_i(t)(y(t))^{\alpha_i}(y'(t))^{\beta_i}}{p_1(t)(y(t))^{\alpha_1}(y'(t))^{\beta_1}} \neq 0,\ \pm\infty \quad (i = \overline{1,s}),$$
$$\lim_{t\to+\infty} \frac{p_j(t)(y(t))^{\alpha_j}(y'(t))^{\beta_j}}{p_1(t)(y(t))^{\alpha_1}(y'(t))^{\beta_1}} = 0 \quad (j = \overline{s+1,n}).$$

It is obvious that $p_i(t) \neq 0$ $(i = \overline{1,s})$.

**Lemma 1.** *Let the equation*

$$\widetilde{F}(t,\xi,\eta) = 0, \tag{2}$$

$(t,\xi,\eta) \in D_1$, $D_1 = \Delta(a) \times [-h_1; h_1] \times [-h_2; h_2]$, $h_k \in \mathbb{R}_+$ $(k = 1,2)$, *satisfy the conditions:*

1) $\widetilde{F}(t,\xi,\eta) \in \mathrm{C}^{s_1 s_2 s_3}_{t\ \xi\ \eta}(D_1)$, $s_1, s_2, s_3 \in \{0,1,2,\ldots\}$, $s_2 \geq 1$, $s_3 \geq 2$;

2) $\exists\, \widetilde{F}(+\infty,0,0) = 0$;

3) $\exists\, \widetilde{F}'_\eta(+\infty,0,0) = A_1 \in \mathbb{R} \setminus \{0\}$;

4) $\sup\limits_{D_1} |\widetilde{F}''_{\eta\eta}(t,\xi,\eta)| = A_2 \in \mathbb{R}_+$.

*Then in some domain* $D_2 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1] \times [-\widetilde{h}_2; \widetilde{h}_2]$, *where* $t_0 \geq a$, $0 < \widetilde{h}_1 \leq h_1$, $0 < \widetilde{h}_2 < \min\bigl\{h_2; \frac{|A_1|}{4A_2}\bigr\}$, *the equation* (2) *defines a unique function* $\eta = \widetilde{\eta}(t,\xi)$, *such that* $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{s_1 s_2}_{t\ \xi}(D_3)$, $D_3 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1]$, $\exists\, \widetilde{\eta}(+\infty, 0) = 0$, $\widetilde{F}(t,\xi,\widetilde{\eta}(t,\xi)) \equiv 0$. *Moreover, for* $\xi = 0$, *the function* $\widetilde{\eta}(t,\xi)$

*has the property*

$$\widetilde{\eta}(t,0) \sim -\frac{\widetilde{F}(t,0,0)}{\widetilde{F}'_{\eta}(t,0,0)}\,. \tag{3}$$

*Proof.* Let us expand the function $\widetilde{F}(t,\xi,\eta)$ with respect to the variable $\eta$ for $t \in \Delta(a)$, $\xi \in [-h_1;h_1]$ by using the Maclaurin's formula. Then the equation (2) can be written as:

$$\widetilde{F}(t,\xi,\eta) = \widetilde{F}(t,\xi,0) + \widetilde{F}'_{\eta}(t,\xi,0)\eta + R(t,\xi,\eta) = 0. \tag{4}$$

Obviously,

$$R(t,\xi,0) \equiv 0.$$

The equation (4) is equivalent to the implicit equation

$$\eta(t,\xi) = \frac{-\widetilde{F}(t,\xi,0) - R(t,\xi,\eta(t,\xi))}{\widetilde{F}'_{\eta}(t,\xi,0)}\,, \tag{5}$$

where

$$R(t,\xi,\eta) = \widetilde{F}(t,\xi,\eta) - \widetilde{F}(t,\xi,0) - \widetilde{F}'_{\eta}(t,\xi,0)\eta,$$

and, therefore,

$$R'_{\eta}(t,\xi,\eta) = \widetilde{F}'_{\eta}(t,\xi,\eta) - \widetilde{F}'_{\eta}(t,\xi,0).$$

Applying the Lagrange's theorem with respect to the variable $\eta$ to the right-hand side of the above equation, we get:

$$\widetilde{F}'_{\eta}(t,\xi,\eta_2) - \widetilde{F}'_{\eta}(t,\xi,\eta_1) = \widetilde{F}''_{\eta\eta}(t,\xi,\eta^*)(\eta_2-\eta_1), \quad \eta^* \in ]\eta_1;\eta_2[\,,$$

$$\sup_{D_1}\left|\widetilde{F}'_{\eta}(t,\xi,\eta_2) - \widetilde{F}'_{\eta}(t,\xi,\eta_1)\right| \le$$

$$\le \sup_{D_1}\left|\widetilde{F}''_{\eta\eta}(t,\xi,\eta)\right||\eta_2-\eta_1| = A_2|\eta_2-\eta_1|.$$

Assuming $\eta_1 = 0$, $\eta_2 = \eta$, we obtain:

$$\sup_{D_1}\left|R'_{\eta}(t,\xi,\eta)\right| \le A_2|\eta|.$$

We consider and evaluate also the difference $R(t,\xi,\eta_2) - R(t,\xi,\eta_1)$, $(t,\xi,\eta_i) \in D_1$ $(i = 1,2)$, applying the Lagrange's theorem with respect to the variable $\eta$:

$$R(t,\xi,\eta_2) - R(t,\xi,\eta_1) = R'_{\eta}(t,\xi,\eta^{**})(\eta_2-\eta_1), \quad \eta^{**} \in ]\eta_1;\eta_2[\,,$$

$$\sup_{D_1}\left|R(t,\xi,\eta_2)-R(t,\xi,\eta_1)\right| \le \sup_{D_1}\left|R'_{\eta}(t,\xi,\eta)\right||\eta_2-\eta_1| \le A_2|\eta_2-\eta_1|^2.$$

Assuming $\eta_1 = 0$, $\eta_2 = \eta$, we get

$$\sup_{D_1}|R(t,\xi,\eta)| \le A_2|\eta|^2.$$

Consider the domain $D_2 \subset D_1$ in which

1) $\sup\limits_{D_2}|\widetilde{F}(t,\xi,0)| \le \frac{\widetilde{h}_2|A_1|}{4}$;

2) $\inf\limits_{D_2} |\widetilde{F}'_\eta(t,\xi,0)| > \frac{|A_1|}{2}$;

3) $\sup\limits_{D_2} |R(t,\xi,\eta)| \le A_2|\eta|^2 \le A_2\widetilde{h}_2^2$.

The fulfilment of conditions 1), 2) can be achieved by increasing $t_0$ and reducing $\widetilde{h}_1$ (by virtue of the conditions of the Lemma). The fulfilment of condition 3) is obvious.

To the equation (5) we put into the correspondence the operator

$$\eta(t,\xi) = T(t,\xi,\widetilde{\eta}(t,\xi)) \equiv \frac{-\widetilde{F}(t,\xi,0) - R(t,\xi,\widetilde{\eta}(t,\xi))}{\widetilde{F}'_\eta(t,\xi,0)},$$

where $\widetilde{\eta}(t,\xi) \in B_1 \subset B$, $B = \big\{\widetilde{\eta}(t,\xi) :\ \widetilde{\eta}(t,\xi) \in \mathrm{C}_{t\ \xi}^{s_1 s_2}(D_3),\ \widetilde{\eta}(+\infty,0) = 0,$ $\|\widetilde{\eta}(t,\xi)\| = \sup\limits_{D_3} |\widetilde{\eta}(t,\xi)|\big\}$ is the Banach space, $B_1 = \big\{\widetilde{\eta}(t,\xi) : \widetilde{\eta}(t,\xi) \in B,$ $\|\widetilde{\eta}(t,\xi)\| \le \widetilde{h}_2\big\}$ is a closed subset of the Banach space $B$.

We apply here the principle of contractive mappings.

1) Let us prove that if $\widetilde{\eta}(t,\xi) \in B_1$, then $\eta(t,\xi) = T(t,\xi,\widetilde{\eta}(t,\xi)) \in B_1$: $\widetilde{\eta}(t,\xi) \in \mathrm{C}_{t\ \xi}^{s_1 s_2}(D_3)$ and $\widetilde{\eta}(+\infty,0) = 0$, then by virtue of the structure of the operator, we get

$$\eta(t,\xi) \in \mathrm{C}_{t\ \xi}^{s_1 s_2}(D_3), \quad \eta(+\infty,0) = 0;$$

$$\begin{aligned} \|\widetilde{\eta}(t,\xi)\| \le \widetilde{h}_2 \Longrightarrow \|\eta(t,\xi)\| &= \big\|T(t,\xi,\widetilde{\eta}(t,\xi))\big\| = \\ &= \Big\|\frac{-\widetilde{F}(t,\xi,0) - R(t,\xi,\widetilde{\eta}(t,\xi))}{\widetilde{F}'_\eta(t,\xi,0)}\Big\| \le \\ \le \frac{1}{\inf\limits_{D_2} |\widetilde{F}'_\eta(t,\xi,\eta)|} &\Big(\sup_{D_2} |\widetilde{F}(t,\xi,0)| + \sup_{D_2} |R(t,\xi,\widetilde{\eta}(t,\xi))|\Big) \le \\ \le \frac{2}{|A_1 t|} &\Big(\sup_{D_2} |\widetilde{F}(t,\xi,0)| + A_2\widetilde{h}_2^2\Big) \le \frac{\widetilde{h}_2}{2} + \frac{\widetilde{h}_2}{2} \le \widetilde{h}_2. \end{aligned}$$

2) Let us check the condition of contraction:

$$\begin{aligned} \widetilde{\eta}_1(t,\xi), \widetilde{\eta}_2(t,\xi) \in B_1 &\Longrightarrow \big\|\eta_2(t,\xi) - \eta_1(t,\xi)\big\| = \\ &= \Big\|\frac{R(t,\xi,\widetilde{\eta}_2(t,\xi)) - R(t,\xi,\widetilde{\eta}_1(t,\xi))}{\widetilde{F}'_\eta(t,\xi,0)}\Big\| \le \\ &\le \frac{A_2}{\inf\limits_{D_2} |\widetilde{F}'_\eta(t,\xi,\eta)|} \big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\|^2 \le \\ &\le \frac{2A_2}{|A_1|} \Big(\|\widetilde{\eta}_2(t,\xi)\| + \|\widetilde{\eta}_1(t,\xi)\|\Big)\big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\| \le \\ &\le \frac{4A_2\widetilde{h}_2}{|A_1|} \big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\| = \gamma\big\|\widetilde{\eta}_2(t,\xi) - \widetilde{\eta}_1(t,\xi)\big\|, \end{aligned}$$

where $\gamma = \frac{4A_2\widetilde{h}_2}{|A_1|} < 1$.

As a result, we have found that by the contractive mapping principle the equation (5) admits a unique solution $\eta = \widetilde{\eta}(t,\xi) \in B_1$.

Since $\widetilde{F}(t,\xi,\eta) \in \mathrm{C}^{s_1 s_2 s_3}_{t\ \xi\ \eta}(D_1)$, then by a local theorem on the differentiability of an implicit function, it can be stated that $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{s_1 s_2}_{t\ \xi}(D_3)$.

Let us prove that $\widetilde{\eta}(t,\xi)$ has the property (3) for $\xi = 0$.

The function $\widetilde{\eta}(t,\xi) \in D_3$ satisfies the equation (4), which can be written as

$$\widetilde{F}(t,0,0) + \widetilde{F}'_{\eta}(t,0,0)\widetilde{\eta}(t,0) + O(\widetilde{\eta}^2) \equiv 0, \tag{6}$$

assuming $\xi = 0$.

As $O(\widetilde{\eta}^2) = O(1)\widetilde{\eta}^2 = o(1)\widetilde{\eta}$, then the equation (6) is equivalent to the equation

$$\widetilde{F}(t,0,0) + \widetilde{F}'_{\eta}(t,0,0)\widetilde{\eta}(t,0) + o(1)\widetilde{\eta}(t,0) \equiv 0.$$

Hence, taking into account that $\widetilde{F}'_{\eta}(+\infty,0,0) = A_1 \in \mathbb{R} \setminus \{0\}$, we can write

$$\widetilde{\eta}(t,0)\Big(1 + \frac{o(1)}{\widetilde{F}'_{\eta}(t,0,0)}\Big) = -\frac{\widetilde{F}(t,0,0)}{\widetilde{F}'_{\eta}(t,0,0)}\,. \tag{7}$$

The property (3) follows from the equality (7). □

**Lemma 2** ([2]). *Let the differential equation*

$$\xi' = \alpha(t) f(t,\xi), \tag{8}$$

$(t,\xi) \in D_3$, $D_3 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1]$ $(\widetilde{h}_1 \in \mathbb{R}_+)$, *satisfy the conditions:*

1) $0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_0))$, $\int\limits_{t_0}^{+\infty} \alpha(t)\,dt = \pm\infty$;
2) $f(t,\xi) \in \mathrm{C}^{01}_{t\xi}(D_3)$, $\exists\, f(+\infty,0) = 0$, $\exists\, f'_{\xi}(+\infty,0) \neq 0$;
3) $f'_{\xi}(t,\xi) \rightrightarrows f'_{\xi}(t,0)$ *under* $\xi \to 0$ *uniformly with respect to* $t \in \Delta(t_0)$.

*Then there exists* $t_1 \geq t_0$, *such that the equation* (8) *has a non-empty set of o-solutions*

$$\Omega = \big\{\xi(t) \in \mathrm{C}^1_{\Delta(t_1)} :\ \xi(+\infty) = 0\big\},$$

*where*

a) *if* $\operatorname{sign}(\alpha f'_{\xi}(+\infty,0)) = -1$, *then* $\Omega$ *is a one-parametric family of o-solutions of the equation* (8)*;*
b) *if* $\operatorname{sign}(\alpha f'_{\xi}(+\infty,0)) = 1$, *then* $\Omega$ *contains a unique element.*

## The Existence and Asymptotics of $R$-Solutions of the Equation (1) with the Condition $y(+\infty) = 0 \vee +\infty$

The supposed asymptotics (to within a constant factor) of $R$-solution $y(t)$ with the condition $y(+\infty) = 0 \vee +\infty$ can be found from the ratio of the first two summands (we consider all possible cases with respect to the values of parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$). Taking into account that $p_1(t), p_2(t) \neq 0$

($t \in \Delta(a)$), we find that $y(t) \dot{\sim} v(t) > 0^*$ ($v \in \{v_i\}$, $i = \overline{1,4}$) under the condition that $v(+\infty) = 0 \vee +\infty$:

1) $v_1 = \left|\frac{p_1(t)}{p_2(t)}\right|^{\frac{1}{\alpha_2-\alpha_1}}$ ($\alpha_1 \neq \alpha_2$, $\beta_1 = \beta_2$), moreover, $p_1(t), p_2(t) \in \mathrm{C}^1_{\Delta(a)}$.

In all the rest asymptotics is used the function

$$I(A,t) = \int\limits_A^t \left|\frac{p_1(t)}{p_2(t)}\right|^{\frac{1}{\beta_2-\beta_1}} dt, \quad A = \begin{cases} a & (I(a,+\infty) = +\infty), \\ +\infty & (I(a,+\infty) \in \mathbb{R}_+ \cup \{0\}). \end{cases}$$

2) $v_2 = |I(A,t)|$ ($\alpha_1 = \alpha_2$, $\beta_1 \neq \beta_2$).

3) $v_3 = |I(A,t)|^{(\frac{\alpha_2-\alpha_1}{\beta_2-\beta_1}+1)^{-1}}$ ($\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\alpha_1 + \beta_1 \neq \alpha_2 + \beta_2$).

4) $v_4 = e^{\ell_0 |I(a,t)|}$ ($\ell_0 \in \mathbb{R} \setminus \{0\}$ and satisfies the conditions (13), (14), (16); $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$, $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 \neq 0$; $I(a,+\infty) = +\infty$).

A solution is sought in the form

$$y(t) = v(t)(\ell + \xi(t)), \tag{9}$$

where $\ell \in \mathbb{R}_+$; $\xi(t) \in \mathrm{C}^1_{\Delta(a)}$, $\xi(+\infty) = 0$; $v(t) = v_k(t) \in \mathrm{C}^1_{\Delta(a)}$ ($k$ is fixed, $k = \overline{1,4}$).

Differentiating the equation (9), we obtain:

$$y'(t) = v'(t)(\ell + \xi(t)) + v(t)\xi'(t) = v'(t)\Big(\ell + \xi(t) + \frac{v(t)}{v'(t)}\xi'(t)\Big).$$

Having denoted

$$\xi(t) + \frac{v(t)}{v'(t)}\xi'(t) = \eta(t), \tag{10}$$

$\eta(t) \in \mathrm{C}_{\Delta(a)}$, we get

$$y'(t) = v'(t)(\ell + \eta(t)). \tag{11}$$

The condition $y'(t) \sim \ell v'(t)$ requires the assumption that $\eta(+\infty) = 0$.

Substituting (9) and (11) into the equation (1), we obtain the equality

$$F(t, v(\ell+\xi), v'(\ell+\eta)) = \\ = \sum_{k=1}^n p_k(t)(v)^{\alpha_k}(\ell+\xi)^{\alpha_k}(v')^{\beta_k}(\ell+\eta)^{\beta_k} = 0, \tag{12}$$

which is satisfied by the functions $\xi(t)$, $\eta(t)$ and $(v'(t))^{\beta_k} : \Delta(a) \to \mathbb{R}_2$ ($k = \overline{1,n}$).

---

$^*f_i \dot{\sim} f_j$ ($i \neq j$) means that $\exists \lim\limits_{t\to+\infty} \frac{f_i}{f_j} \neq 0, \pm\infty$.

According to the condition B), indicated in the statement of the problem, we assume that

$$\frac{p_i(t)(v(t))^{\alpha_i}(v'(t))^{\beta_i}}{p_1(t)(v(t))^{\alpha_1}(v'(t))^{\beta_1}} =$$
$$= c_i^* + \varepsilon_i(t), \quad c_i^* \in \mathbb{R} \setminus \{0\},\ \varepsilon_i(+\infty) = 0 \ \ (i = \overline{1, s}); \tag{13}$$
$$\frac{p_j(t)(v(t))^{\alpha_j}(v'(t))^{\beta_j}}{p_1(t)(v(t))^{\alpha_1}(v'(t))^{\beta_1}} = \varepsilon_j(t), \quad \varepsilon_j(+\infty) = 0 \ \ (j = \overline{s+1, n}). \tag{14}$$

Then, after the division by $p_1(t)(v(t))^{\alpha_1}(v'(t))^{\beta_1}$, the equation (12) takes the form

$$\widetilde{F}(t,\xi,\eta) = \sum_{i=1}^{s}(c_i^* + \varepsilon_i(t))(\ell+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i} +$$
$$+ \sum_{j=s+1}^{n} \varepsilon_j(t)(\ell+\xi)^{\alpha_j}(\ell+\eta)^{\beta_j} = 0. \tag{15}$$

Obviously, the condition $\widetilde{F}(+\infty, 0, 0) = 0$ is necessary for the existence of a solution and of its derivative of the form (9), (11), respectively.

Thus, for $v = v_k(t)$ $(k = \overline{1,4})$ it takes the form

$$\sum_{i=1}^{s} c_i^* \ell^{\alpha_i+\beta_i} = 0. \tag{16}$$

For $v = v_4(t) : \operatorname{sign}(v') = \operatorname{sign}(\ell_0)$, $c_i^* = c_i^*(\ell_0)$, $\ell_0$, $\ell_0^{\beta_i} \in \mathbb{R}\setminus\{0\}$ $(i = \overline{1,s})$.

By virtue of its structure, the functions $\widetilde{F}(t,\xi,\eta) \in \mathrm{C}^{0\,\infty\,\infty}_{t\,\xi\,\eta}(D_1)$, $\frac{\partial^n \widetilde{F}}{\partial \xi^n}$, $\frac{\partial^m \widetilde{F}}{\partial \eta^m}$, $\frac{\partial^{n+m} \widetilde{F}}{\partial \xi^n \partial \eta^m}$ $(n = \overline{1,\infty}, \quad m = \overline{1,\infty})$ are bounded in $D_1$, where $D_1 = \Delta(a) \times [-h_1; h_1] \times [-h_2; h_2]$, $0 < h_k < \ell$ $(k = 1, 2)$.

Next, we will need expressions for the first and second order derivatives of the function $\widetilde{F}(t,\xi,\eta)$ with respect to the variables $\xi$ and $\eta$:

$$\widetilde{F}'_\xi(t,\xi,\eta) = \sum_{i=1}^{s} \alpha_i c_i^*(\ell+\xi)^{\alpha_i-1}(\ell+\eta)^{\beta_i} +$$
$$+ \sum_{k=1}^{n} \alpha_k \varepsilon_k(t)(\ell+\xi)^{\alpha_k-1}(\ell+\eta)^{\beta_k},$$
$$\widetilde{F}'_\eta(t,\xi,\eta) = \sum_{i=1}^{s} \beta_i c_i^*(\ell+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-1} +$$
$$+ \sum_{k=1}^{n} \beta_k \varepsilon_k(t)(\ell+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-1},$$
$$\widetilde{F}''_{\xi\xi}(t,\xi,\eta) = \sum_{i=1}^{s} \alpha_i(\alpha_i - 1) c_i^*(\ell+\xi)^{\alpha_i-2}(\ell+\eta)^{\beta_i} +$$

$$+\sum_{k=1}^{n}\alpha_k(\alpha_k-1)\varepsilon_k(t)(\ell+\xi)^{\alpha_k-2}(\ell+\eta)^{\beta_k},$$

$$\widetilde{F}\,''_{\xi\eta}(t,\xi,\eta)=\widetilde{F}\,''_{\eta\xi}(t,\xi,\eta)=\sum_{i=1}^{s}\alpha_i\beta_i c_i^*(\ell+\xi)^{\alpha_i-1}(\ell+\eta)^{\beta_i-1}+$$

$$+\sum_{k=1}^{n}\alpha_k\beta_k\varepsilon_k(t)(\ell+\xi)^{\alpha_k-1}(\ell+\eta)^{\beta_k-1},$$

$$\widetilde{F}\,''_{\eta\eta}(t,\xi,\eta)=\sum_{i=1}^{s}\beta_i(\beta_i-1)c_i^*(\ell+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-2}+$$

$$+\sum_{k=1}^{n}\beta_k(\beta_k-1)\varepsilon_k(t)(\ell+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-2},$$

as well as the following notation:

$$\psi_{00}(t)=\sum_{k=1}^{n}\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\psi_{l0}(t)=\sum_{k=1}^{n}\alpha_k(\alpha_k-1)\cdots(\alpha_k-l+1)\varepsilon_k(t)\ell^{\alpha_k+\beta_k},$$

$$\psi_{0m}(t)=\sum_{k=1}^{n}\beta_k(\beta_k-1)\cdots(\beta_k-m+1)\varepsilon_k(t)\ell^{\alpha_k+\beta_k},$$

$$\psi_{lm}(t)=\sum_{k=1}^{n}\alpha_k(\alpha_k-1)\cdots(\alpha_k-l+1)\times$$

$$\times\,\beta_k(\beta_k-1)\cdots(\beta_k-m+1)\varepsilon_k(t)\ell^{\alpha_k+\beta_k},$$

$$S_{l0}=\sum_{i=1}^{s}\alpha_i(\alpha_i-1)\cdots(\alpha_i-l+1)c_i^*\ell^{\alpha_i+\beta_i},$$

$$S_{0m}=\sum_{i=1}^{s}\beta_i(\beta_i-1)\cdots(\beta_i-m+1)c_i^*\ell^{\alpha_i+\beta_i},$$

$$S_{lm}=\sum_{i=1}^{s}\alpha_i(\alpha_i-1)\cdots(\alpha_i-l+1)\times$$

$$\times\,\beta_i(\beta_i-1)\cdots(\beta_i-m+1)c_i^*\ell^{\alpha_i+\beta_i},$$

$$S_{l0},S_{0m},S_{lm}\in\mathbb{R}\ \ (l,m\in\mathbb{N}),\quad S=S_{10}^2S_{02}-2S_{10}S_{01}S_{11}+S_{01}^2S_{20},$$

$$\lambda_1=\frac{2S_{01}^3}{S}\in\mathbb{R},\quad \lambda_2=-\frac{2S_{01}^2\ell^2}{S}\in\mathbb{R}.$$

**Theorem 1.** *Let a function* $v(t)=v_k(t)$ $(k=\overline{1,4})$ *be a possible asymptotics of an R-solution of the equation* (1)*, which satisfies the conditions* $v(+\infty)=0\vee+\infty$, (13), *and* (14). *Let, moreover, there exist* $\ell\in\mathbb{R}_+$, *satisfying the condition* (16).

*Then in order for the R-solution* $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ *of the differential equation* (1) *with the asymptotic properties*

$$y(t) \sim \ell v(t), \quad y'(t) \sim \ell v'(t), \tag{17}$$

*to exist, it is sufficient that the two following conditions*

$$S_{01} \neq 0, \tag{18}$$

$$S_{10} + S_{01} \neq 0. \tag{19}$$

*be fulfilled. Moreover, if* $\operatorname{sign}\big(\frac{v'(S_{10}+S_{01})}{S_{01}}\big) = 1$*, then there exists a one-parameter set of R-solutions with the asymptotic properties* (17)*; if* $\operatorname{sign}\big(\frac{v'(S_{10}+S_{01})}{S_{01}}\big) = -1$*, then R-solution with the asymptotic* (17) *is unique.*

*Proof.* For the proof we will need the following properties of the function $\widetilde{F}(t,\xi,\eta)$ :

$$\widetilde{F}\,'_{\xi}(+\infty,0,0) = \frac{S_{10}}{\ell}\,;$$

$$\widetilde{F}\,'_{\eta}(+\infty,0,0) = \frac{S_{01}}{\ell} \neq 0$$

by virtue of the condition (18).

Owing to the conditions (16), (18) and to the properties of the function $\widetilde{F}(t,\xi,\eta)$, in some domain $D_2 \subset D_1$, $D_2 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1] \times [-\widetilde{h}_2; \widetilde{h}_2]$, $t_0 \geq a$, $0 < \widetilde{h}_1 \leq h_1$, $0 < \widetilde{h}_2 < \min\Big\{h_2; \frac{|S_{01}|}{4\ell \sup\limits_{D_1} |\widetilde{F}\,''_{\eta\eta}(t,\xi,\eta)|}\Big\}$, for the equation (15) the conditions of Lemma 1 are satisfied. Consequently, there exists a unique function $\eta = \widetilde{\eta}(t,\xi) \in \mathrm{C}^{0\,\infty}_{t\;\xi}(D_3)$, $D_3 = \Delta(t_0) \times [-\widetilde{h}_1; \widetilde{h}_1]$, $\sup\limits_{D_3} \big|\frac{\partial^n \widetilde{\eta}}{\partial \xi^n}\big| < +\infty$ $(n = \overline{1,\infty})$, such that $\widetilde{F}(t,\xi,\widetilde{\eta}(t,\xi)) \equiv 0$, $\widetilde{\eta}(+\infty,0) = 0$, $\|\widetilde{\eta}(t,\xi)\| \leq \widetilde{h}_2$. Moreover, we can write

$$\frac{\partial \widetilde{\eta}(t,\xi)}{\partial \xi} = -\frac{\widetilde{F}\,'_{\xi}(t,\xi,\widetilde{\eta})}{\widetilde{F}\,'_{\eta}(t,\xi,\widetilde{\eta})}\,.$$

Thus, in view of the replacement (10), we obtain the differential equation with respect to $\xi$:

$$\xi' = \frac{v'}{v}\big(-\xi + \widetilde{\eta}(t,\xi)\big). \tag{20}$$

The question on the existence of solutions of the form (9) reduces to the study of the differential equation (20).

Let us show that the conditions 1)–3) of Lemma 2 are satisfied for the equation (20). In this case we have: $\alpha(t) = \frac{v'(t)}{v(t)}$ , $f(t,\xi) = -\xi + \widetilde{\eta}(t,\xi)$.

Obviously, the conditions 1) and 2) are satisfied.

1) Since $0 < v(t) \in \mathrm{C}^1(\Delta(a))$, therefore

$$0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_0)), \quad \int\limits_{t_0}^{+\infty} \alpha(t)\,dt = \int\limits_{t_0}^{+\infty} \frac{v'(t)}{v(t)}\,dt = \pm\infty.$$

2) Since $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{0\infty}_{t\,\xi}(D_3)$, then

$$f(t,\xi) \in \mathrm{C}^{0\infty}_{t\,\xi}(D_3), \quad \exists\, f(+\infty,0) = \widetilde{\eta}(+\infty,0) = 0,$$

$$f'_\xi(t,\xi) = -1 + \widetilde{\eta}'_\xi(t,\xi) = -1 - \frac{\widetilde{F}'_\xi(t,\xi,\widetilde{\eta})}{\widetilde{F}'_\eta(t,\xi,\widetilde{\eta})}\,,$$

$$f'_\xi(+\infty,0) = -1 - \frac{\widetilde{F}'_\xi(+\infty,0,\widetilde{\eta}(+\infty,0))}{\widetilde{F}'_\eta(+\infty,0,\widetilde{\eta}(+\infty,0))} = -\frac{S_{10}+S_{01}}{S_{01}} \neq 0$$

by virtue of the condition (19).

Let us check that the condition 3) is satisfied, that is,

$$\big\|f'_\xi(t,\xi) - f'_\xi(t,0)\big\| = \left\|\frac{\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi))}{\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi))} - \frac{\widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))}{\widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))}\right\| \rightrightarrows 0$$

as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$.

Towards this end, it suffices to verify that the following properties are satisfied:

$3_1$) $\widetilde{\eta}(t,\xi) \rightrightarrows \widetilde{\eta}(t,0)$ if $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$,

$3_2$) $\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi)) \rightrightarrows \widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))$ as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$,

$3_3$) $\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi)) \rightrightarrows \widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))$, as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$ with regard for the fact that $F'_\eta(+\infty,0,\eta(+\infty,0)) = S_{01} \neq 0$.

Let us estimate the differences $\widetilde{\eta}(t,\xi) - \widetilde{\eta}(t,0)$, $\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi)) - \widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))$, $\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi)) - \widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))$, applying the Lagrange's theorem to the first difference with respect to the variable $\xi$:

$$\widetilde{\eta}(t,\xi) - \widetilde{\eta}(t,0) = \widetilde{\eta}'_\xi(t,\xi^*)\xi, \quad \xi^* \in \,]0;\xi[\,.$$

As the functions $\varepsilon_k(t)$ $(k=\overline{1,n})$ are bounded in $\Delta(a)$ and $\|\widetilde{\eta}(t,\xi)\| \le \widetilde{h}_2$ in $D_3$, then we get the estimates in the form:

$3_1$) $\quad \big|\widetilde{\eta}(t,\xi) - \widetilde{\eta}(t,0)\big| = \big|\widetilde{\eta}'_\xi(t,\xi^*)\big|\,|\xi| =$

$$= \left| - \frac{\widetilde{F}'_\xi(t,\xi^*,\widetilde{\eta}(t,\xi^*))}{\widetilde{F}'_\eta(t,\xi^*,\widetilde{\eta}(t,\xi^*))}\right| |\xi| \le O(1)|\xi| = O(\xi) \longrightarrow 0$$

as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$;

$3_2$) taking into account that $(\ell+\xi)^{\alpha_i-1} \to \ell^{\alpha_i-1}$ as $\xi \to 0$, $(\ell+\widetilde{\eta}(t,\xi))^{\beta_i} \to (\ell+\widetilde{\eta}(t,0))^{\beta_i}$ as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$ $(i=\overline{1,s})$, we get

$$\big|\widetilde{F}'_\xi(t,\xi,\widetilde{\eta}(t,\xi)) - \widetilde{F}'_\xi(t,0,\widetilde{\eta}(t,0))\big| =$$

$$= \bigg|\sum_{i=1}^s \alpha_i c_i^* \Big[(\ell+\xi)^{\alpha_i-1}(\ell+\widetilde{\eta}(t,\xi))^{\beta_i} - \ell^{\alpha_i-1}(\ell+\widetilde{\eta}(t,0))^{\beta_i}\Big] +$$

$$+ \sum_{k=1}^n \alpha_k \varepsilon_k(t) \Big[(\ell+\xi)^{\alpha_k-1}(\ell+\widetilde{\eta}(t,\xi))^{\beta_k} - \ell^{\alpha_k-1}(\ell+\widetilde{\eta}(t,0))^{\beta_k}\Big]\bigg| \longrightarrow 0$$

as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$;

$3_3$) analogously to $3_2$), we get:

$$\left|\widetilde{F}'_\eta(t,\xi,\widetilde{\eta}(t,\xi)) - \widetilde{F}'_\eta(t,0,\widetilde{\eta}(t,0))\right| =$$

$$= \Bigg| \sum_{i=1}^{s} \beta_i c_i^* \Big[(\ell+\xi)^{\alpha_i}(\ell+\widetilde{\eta}(t,\xi))^{\beta_i-1} - \ell^{\alpha_i}(\ell+\widetilde{\eta}(t,0))^{\beta_i-1}\Big] +$$

$$+ \sum_{k=1}^{n} \beta_k \varepsilon_k(t) \Big[(\ell+\xi)^{\alpha_k}(\ell+\widetilde{\eta}(t,\xi))^{\beta_k-1} - \ell^{\alpha_k}(\ell+\widetilde{\eta}(t,0))^{\beta_k-1}\Big]\Bigg| \longrightarrow 0$$

as $\xi \to 0$ uniformly with respect to $t \in \Delta(t_0)$.

Since $\widetilde{\eta}(+\infty,0)=0$, therefore $F'_\eta(+\infty,0,\widetilde{\eta}(+\infty,0)) = S_{01} \neq 0$ by virtue of the condition (18).

Consequently, condition 3) is satisfied.

Then if $\operatorname{sign}\big(\frac{v'(S_{10}+S_{01})}{S_{01}}\big) = 1$, then there exists a one-parameter set of $o$-solutions of the equation (20) in $\Delta(t_1) \subseteq \Delta(t_0)$.

If $\operatorname{sign}\big(\frac{v'(S_{10}+S_{01})}{S_{01}}\big) = -1$, then a set of $o$-solutions of the equation (20) in $\Delta(t_1)$ contains the unique element.

Finally, having the dimension of a set of $o$-solutions of the equation (20), we have obtained the dimension of a set of $R$-solutions of the equation (1) with the asymptotic properties (17) in $\Delta(t_1)$. □

**Theorem 2.** *Let the conditions of Theorem* 1*, except for* (19)*, be satisfied, and*

$$S \neq 0, \tag{21}$$

$$\psi_{00}(t)\ln^2 v(t) = o(1), \tag{22}$$

$$(\psi_{10}(t)+\psi_{01}(t))\ln v(t) = o(1). \tag{23}$$

*Then there exists a one-parameter set of $R$-solutions $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ of the differential equation* (1) *with the asymptotic properties*

$$y(t) = v(t)(\ell+\xi(t)), \quad y'(t) \sim \ell v'(t), \tag{24}$$

*where* $\xi(t) \sim \frac{\lambda_1 \ell}{\ln v(t)}$ .

*Proof.* To prove the theorem, we will need the following properties and expressions of the function $\widetilde{F}(t,\xi,\eta)$ :

$$\widetilde{F}(t,0,0) = \psi_{00}(t),$$

$$\widetilde{F}'_\xi(t,0,0) = \frac{1}{\ell}\sum_{i=1}^{s} \alpha_i c_i^* \ell^{\alpha_i+\beta_i} + \frac{1}{\ell}\sum_{k=1}^{n} \alpha_k \ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}'_\xi(+\infty,0,0) = \frac{S_{10}}{\ell};$$

$$\widetilde{F}'_\eta(t,0,0) = \frac{1}{\ell}\sum_{i=1}^{s} \beta_i c_i^* \ell^{\alpha_i+\beta_i} + \frac{1}{\ell}\sum_{k=1}^{n} \beta_k \ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}\,'_{\eta}(+\infty,0,0)=\frac{S_{01}}{\ell}\neq 0 \ \text{ by virtue of condition (18);}$$

$$\widetilde{F}\,''_{\xi\xi}(t,0,0)=\frac{1}{\ell^2}\sum_{i=1}^{s}\alpha_i(\alpha_i-1)c_i^*\ell^{\alpha_i+\beta_i}+ + \frac{1}{\ell^2}\sum_{k=1}^{n}\ \alpha_k(\alpha_k-1)\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}\,''_{\xi\xi}(+\infty,0,0)=\frac{S_{20}}{\ell^2}\,;$$

$$\widetilde{F}\,''_{\xi\eta}(t,0,0)=\widetilde{F}\,''_{\eta\xi}(t,0,0)= =\frac{1}{\ell^2}\sum_{i=1}^{s}\alpha_i\beta_i c_i^*\ell^{\alpha_i+\beta_i}+\frac{1}{\ell^2}\sum_{k=1}^{n}\ \alpha_k\beta_k\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}\,''_{\xi\eta}(+\infty,0,0)=\widetilde{F}\,''_{\eta\xi}(+\infty,0,0)=\frac{S_{11}}{\ell^2}\,;$$

$$\widetilde{F}\,''_{\eta\eta}(t,0,0)=\frac{1}{\ell^2}\sum_{i=1}^{s}\beta_i(\beta_i-1)c_i^*\ell^{\alpha_i+\beta_i}+ + \frac{1}{\ell^2}\sum_{k=1}^{n}\ \beta_k(\beta_k-1)\ell^{\alpha_k+\beta_k}\varepsilon_k(t),$$

$$\widetilde{F}\,''_{\eta\eta}(+\infty,0,0)=\frac{S_{02}}{\ell^2}\,.$$

By virtue of the condition (18) and owing to the properties of the function $\widetilde{F}(t,\xi,\eta)$, in some domain $D_2\subset D_1$, $D_2=\Delta(t_0)\times[-\widetilde{h}_1;\widetilde{h}_1]\times[-\widetilde{h}_2;\widetilde{h}_2]$, $t_0\geq a$, $0<\widetilde{h}_1\leq h_1$, $0<\widetilde{h}_2<\min\left\{h_2;\frac{|S_{01}|}{4\ell\sup\limits_{D_1}|\widetilde{F}\,''_{\eta\eta}(t,\xi,\eta)|}\right\}$, for the equation (15) the conditions of Lemma 1 are fulfilled. Consequently, there exists a unique function $\eta=\widetilde{\eta}(t,\xi)$, $\widetilde{\eta}(t,\xi)\in \mathrm{C}^{0\infty}_{t\ \xi}(D_3)$, $D_3=\Delta(t_0)\times[-\widetilde{h}_1;\widetilde{h}_1]$, $\sup\limits_{D_3}\left|\frac{\partial^n\widetilde{\eta}}{\partial\xi^n}\right|<+\infty$ $(n=\overline{1,\infty})$, such that $\widetilde{F}(t,\xi,\widetilde{\eta}(t,\xi))\equiv 0$, $\widetilde{\eta}(+\infty,0)=0$, $\|\widetilde{\eta}(t,\xi)\|\leq\widetilde{h}_2$. Moreover, we can write:

$$\widetilde{\eta}(t,0)\sim-\frac{\widetilde{F}(t,0,0)}{\widetilde{F}\,'_{\eta}(t,0,0)}\,,$$

$$\widetilde{\eta}\,'_{\xi}(t,\xi)=-\frac{\widetilde{F}\,'_{\xi}(t,\xi,\widetilde{\eta})}{\widetilde{F}\,'_{\eta}(t,\xi,\widetilde{\eta})}\,,$$

$$\frac{\partial^2\widetilde{\eta}(t,\xi)}{\partial\xi^2}=-\frac{(\widetilde{F}\,'_{\xi})^2\widetilde{F}\,''_{\eta\eta}-2\widetilde{F}\,'_{\xi}\widetilde{F}\,'_{\eta}\widetilde{F}\,''_{\xi\eta}+(\widetilde{F}\,'_{\eta})^2\widetilde{F}\,''_{\xi\xi}}{(\widetilde{F}\,'_{\eta})^3}\,.$$

Thus, taking into account the replacement (10), we obtain the differential equation with respect to $\xi$:

$$\xi' = \frac{v'}{v}\,(-\xi + \widetilde{\eta}(t,\xi)). \tag{20}$$

The question of the existence of solutions of the type (9) reduces to the study of the differential equation (20).

Let us show that the conditions 1)–3) of Lemma 2 are satisfied for the equation (20). In this case we have: $\alpha(t) = \frac{v'(t)}{v(t)}$, $f(t,\xi) = -\xi + \widetilde{\eta}(t,\xi)$.

1) Since $0 < v(t) \in \mathrm{C}^1(\Delta(a))$, therefore

$$0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_0)), \quad \int\limits_{t_0}^{+\infty} \alpha(t)\,dt = \int\limits_{t_0}^{+\infty} \frac{v'(t)}{v(t)}\,dt = \pm\infty.$$

2) Since $\widetilde{\eta}(t,\xi) \in \mathrm{C}^{0\,\infty}_{t\,\xi}(D_3)$, therefore

$$f(t,\xi) \in \mathrm{C}^{0\,\infty}_{t\,\xi}(D_3), \quad \exists\, f(+\infty,0) = \widetilde{\eta}(+\infty,0) = 0,$$

$$f'_\xi(t,\xi) = -1 + \widetilde{\eta}\,'_\xi(t,\xi) = -1 - \frac{\widetilde{F}\,'_\xi(t,\xi,\widetilde{\eta})}{\widetilde{F}\,'_\eta(t,\xi,\widetilde{\eta})}\,.$$

Taking into account the properties of the functions $\varepsilon_k(t)$ $(k = \overline{1,n})$ and also the conditions of the theorem, we obtain:

$$f'_\xi(+\infty,0) = -1 - \frac{\widetilde{F}\,'_\xi(+\infty,0,\widetilde{\eta}(+\infty,0))}{\widetilde{F}\,'_\eta(+\infty,0,\widetilde{\eta}(+\infty,0))} = -\frac{S_{10}+S_{01}}{S_{01}} = 0.$$

Thus, condition 2) is not satisfied, and we cannot apply Lemma 2 to the equation (20).

Since $f''_{\xi\xi}(t,\xi) = \widetilde{\eta}\,''_{\xi\xi}(t,\xi)$, therefore

$$f''_{\xi\xi}(+\infty,0) = \widetilde{\eta}\,''_{\xi\xi}(+\infty,0) = -\frac{S}{\ell S_{01}^3} = -\frac{2}{\lambda_1 \ell}\,.$$

Consider the auxiliary differential equation with respect to $\xi_1$:

$$\xi_1' = -\frac{v'(t)}{\lambda_1 \ell v(t)}\,\xi_1^2.$$

and find one of its non-trivial solutions:

$$\xi_1 = \frac{\lambda_1 \ell}{\ln v(t)}\,, \quad 0 \neq \xi(t)_1 \in \mathrm{C}^1_{\Delta(t_1)} \;\; (t_1 \geq t_0), \;\; \xi_1(+\infty) = 0.$$

We consider the question on the existence in the equation (20) of solutions of the form $\xi = \xi_1(1+\widetilde{\xi})$, where $\widetilde{\xi}(t) \in \mathrm{C}^1_{\Delta(t_1)}$, $\widetilde{\xi}(+\infty) = 0$. For the unknown function $\widetilde{\xi}$ we obtain the following differential equation:

$$\widetilde{\xi}\,' = \frac{v'\xi_1}{v}\biggl(-\frac{1}{\xi_1} - \frac{v\xi_1'}{v'\xi_1^2} + \Bigl(-\frac{1}{\xi_1} - \frac{v\xi_1'}{v'\xi_1^2}\Bigr)\widetilde{\xi} + \frac{\widetilde{\eta}(t,\xi_1(1+\widetilde{\xi}))}{\xi_1^2}\biggr), \tag{25}$$

$(t,\widetilde{\xi}) \in D_4$, $D_4 = \Delta(t_1) \times [-h_4; h_4]$ $(0 < h_4 \leq \widetilde{h}_1)$, $\frac{v(t)\xi_1'(t)}{v'(t)\xi_1^2(t)} \equiv -\frac{1}{\lambda_1 \ell}\,.$

Let us show that the conditions 1)–3) of Lemma 2 are satisfied for the equation (25). In this case we have:

$$\alpha(t) = \frac{v'(t)\xi_1}{v(t)} = \frac{\lambda_1 \ell v'(t)}{v(t)\ln v(t)}\,,$$

$$f(t,\widetilde{\xi}) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \Big(-\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell}\Big)\widetilde{\xi} + \frac{\widetilde{\eta}(t,\xi_1(1+\widetilde{\xi}))}{\xi_1^2}\,.$$

Using the properties of functions $v(t)$, $\widetilde{\eta}(t,\xi)$, $\xi_1(t)$, we obtain:

1) $0 \neq \alpha(t) \in \mathrm{C}(\Delta(t_1))$, $\int\limits_{t_1}^{+\infty} \alpha(t)\,dt = \lambda_1\ell \int\limits_{t_1}^{+\infty} \frac{v'(t)}{v(t)\ln v(t)}\,dt = \infty$;

2) $f(t,\widetilde{\xi}) \in \mathrm{C}^{0\,\infty}_{t\,\widetilde{\xi}}(D_4)$;

$$f(t,0) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \frac{\widetilde{\eta}(t,\xi_1)}{\xi_1^2}\,,$$

$$f'_{\widetilde{\xi}}(t,\widetilde{\xi}) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \frac{\widetilde{\eta}'_{\xi}(t,\xi_1(1+\widetilde{\xi}))}{\xi_1}\,,$$

$$f'_{\widetilde{\xi}}(t,0) = -\frac{1}{\xi_1} + \frac{1}{\lambda_1\ell} + \frac{\widetilde{\eta}'_{\xi}(t,\xi_1)}{\xi_1}\,.$$

Let us expand the functions $\widetilde{\eta}(t,\xi_1)$ and $\widetilde{\eta}'_{\xi}(t,\xi_1)$ with respect to the variable $\xi_1$ in $D_4$ using the Maclaurin's formula:

$$\widetilde{\eta}(t,\xi_1) = \widetilde{\eta}(t,0) + \widetilde{\eta}'_{\xi_1}(t,0)\xi_1 + \frac{1}{2}\,\widetilde{\eta}''_{\xi_1^2}(t,0)\xi_1^2 + O(\xi_1^3),$$

$$\widetilde{\eta}'_{\xi}(t,\xi_1) = \widetilde{\eta}'_{\xi}(t,0) + \widetilde{\eta}''_{\xi\xi_1}(t,0)\xi_1 + O(\xi_1^2).$$

Using Lemma 1, we obtain:

$$\widetilde{\eta}(t,0) \sim -\frac{\ell\psi_{00}(t)}{S_{01}+o(1)}\,,$$

$$\widetilde{\eta}'_{\xi_1}(t,0) = \widetilde{\eta}'_{\xi}(t,0) =$$

$$= -\frac{\sum\limits_{i=1}^{s} \alpha_i c_i^* \ell^{\alpha_i-1}(\ell+\widetilde{\eta}(t,0))^{\beta_i} + \sum\limits_{k=1}^{n} \alpha_k \varepsilon_k(t)\ell^{\alpha_k-1}(\ell+\widetilde{\eta}(t,0))^{\beta_k}}{\sum\limits_{i=1}^{s} \beta_i c_i^* \ell^{\alpha_i}(\ell+\widetilde{\eta}(t,0))^{\beta_i-1} + \sum\limits_{k=1}^{n} \beta_k \varepsilon_k(t)\ell^{\alpha_k}(\ell+\widetilde{\eta}(t,0))^{\beta_k-1}}\,,$$

$$\widetilde{\eta}'_{\xi_1}(+\infty,0) = \widetilde{\eta}'_{\xi}(+\infty,0) = -\frac{S_{10}}{S_{01}}\,,$$

$$\widetilde{\eta}''_{\xi_1^2}(+\infty,0) = \widetilde{\eta}''_{\xi\xi_1}(+\infty,0) = \widetilde{\eta}''_{\xi^2}(+\infty,0) = -\frac{2}{\lambda_1\ell}\,.$$

Then

$$f(t,0) = \frac{\widetilde{\eta}(t,0)}{\xi_1^2} + \frac{\widetilde{\eta}'_{\xi_1}(t,0)-1}{\xi_1} + \frac{1}{2}\,\widetilde{\eta}''_{\xi_1^2}(t,0) + \frac{1}{\lambda_1\ell} + O(\xi_1),$$

$$f'_{\widetilde{\xi}}(t,0) = \frac{\widetilde{\eta}'_{\xi}(t,0)-1}{\xi_1} + \widetilde{\eta}''_{\xi\xi_1}(t,0) + \frac{1}{\lambda_1\ell} + O(\xi_1).$$

From the conditions (22), (23) and $S_{10}+S_{01}=0$ it follows that

$$\lim_{t\to+\infty}\frac{\widetilde{\eta}(t,0)}{\xi_1^2}=-\lim_{t\to+\infty}\frac{\psi_{00}(t)\ln^2 v(t)}{\ell S_{01}\lambda_1^2}=0,$$
$$\lim_{t\to+\infty}\frac{\widetilde{\eta}'_{\xi_1}(t,0)-1}{\xi_1}=\lim_{t\to+\infty}\frac{\widetilde{\eta}'_{\xi}(t,0)-1}{\xi_1}=$$
$$=-\lim_{t\to+\infty}\frac{\ln v(t)}{\lambda_1 S_{01}}\biggl(\sum_{k=0}^{\infty}\frac{S_{1k}+S_{0k+1}}{k!\ell^{k+1}}\,\widetilde{\eta}^{\,k}(t,0)+$$
$$+\sum_{k=0}^{\infty}\frac{\psi_{1k}+\psi_{0k+1}}{k!\ell^{k+1}}\,\widetilde{\eta}^{\,k}(t,0)\biggr)=0,$$
$$\lim_{t\to+\infty}\Bigl(\frac{1}{2}\,\widetilde{\eta}''_{\xi_1^2}(t,0)+\frac{1}{\lambda_1\ell}\Bigr)=0,$$
$$\lim_{t\to+\infty}\Bigl(\frac{1}{2}\,\widetilde{\eta}''_{\xi\xi_1}(t,0)+\frac{1}{\lambda_1\ell}\Bigr)=-\frac{1}{\lambda_1\ell}\,.$$

As a result, we have found that $f(+\infty,0)=0$, $f'_{\widetilde{\xi}}(+\infty,0)=-\frac{1}{\lambda_1\ell}\neq 0$.

3) Since

$$f''_{\widetilde{\xi}^2}(t,\widetilde{\xi})=\widetilde{\eta}''_{\xi^2}(t,\xi_1(1+\widetilde{\xi})),\quad f''_{\widetilde{\xi}^2}(t,0)=\widetilde{\eta}''_{\xi^2}(t,\xi_1)=\widetilde{\eta}''_{\xi^2}(t,0)+O(\xi_1),$$
$$f''_{\widetilde{\xi}^2}(+\infty,0)=\widetilde{\eta}''_{\xi^2}(+\infty,0)=-\frac{2}{\lambda_1\ell}\neq 0,$$

the condition 3) of Lemma 2 is automatically satisfied.

Then the differential equation (25) satisfies the conditions of Lemma 2, where since $\operatorname{sign}\bigl(\frac{v'\xi_1}{\lambda_1\ell v}\bigr)=1$, there exists for the fixed $\ell$ a one-parameter set of $o$-solutions of the equation (25) in $\Delta(t_1)$.

Finally, having the dimension of the set of $o$-solutions of the equation (25), we have likewise obtained the dimension of a set of $R$-solutions of the equation (1) with the asymptotic properties (24) in $\Delta(t_1)$. $\square$

Consider now separately the exponential asymptotics $v_4=e^{\ell_0|I(a,t)|}$ (the values of the constants and functions we used, have been identified previously). We proceed from the assumption that of principal importance remain the first $s$ terms, and also the fact that

1) $\alpha_k+\beta_k=\alpha_1+\beta_1\neq 0$ $(k=\overline{2,s})$;
2) $\alpha_k+\beta_k=\alpha_1+\beta_1\neq 0$ $(k=\overline{s+1,s_1})$;
3) $\alpha_k+\beta_k\neq\alpha_1+\beta_1$ $(k=\overline{s_1+1,n})$.

The possibility that the summands with powers of type 2) or 3) are absent is not excluded.

The assumptions 1)–3) and the condition (18) imply that the condition (19) is not satisfied, as

$$S_{10}+S_{01}=\sum_{i=1}^{s}\alpha_i c_i^*\ell^{\alpha_i+\beta_i}+\sum_{i=1}^{s}\beta_i c_i^*\ell^{\alpha_i+\beta_i}=$$
$$=\sum_{i=1}^{s}(\alpha_i+\beta_i)c_i^*\ell^{\alpha_i+\beta_i}=(\alpha_1+\beta_1)\sum_{i=1}^{s}c_i^*\ell^{\alpha_i+\beta_i}=0.$$

Therefore, Theorem 1 cannot be applied to the given asymptotics. If Theorem 2 is likewise not satisfied, then under certain conditions we can achieve fulfilment of the conditions of Theorem 2 by defining the asymptotics $v_4(t)$ more exactly.

Consider the more precise asymptotics

$$v_{41}(t)=e^{\ell_0\int\limits_a^t I_t'(a,t)(1+z(t))\,dt}, \tag{26}$$

where

$$I_t'(a,t)=\Big|\frac{p_1(t)}{p_2(t)}\Big|^{\frac{1}{\beta_2-\beta_1}},$$
$$z(t)\in \mathrm{C}_{\Delta(a)},\quad z(+\infty)=0\Longrightarrow v_{41}(+\infty)=v_4(+\infty)=0\vee+\infty.$$

A solution will be sought in the form

$$y(t)=v_{41}(t)(\ell+\xi(t)), \tag{27}$$

where $\xi(t)\in \mathrm{C}^1_{\Delta(a)}$, $\xi(+\infty)=0$.

Differentiating the equation (27), we obtain:

$$y'(t)=v_{41}'(t)(\ell+\eta(t)), \tag{28}$$
$$\eta(t)=\xi(t)+\frac{v_{41}(t)}{v_{41}'(t)}\,\xi'(t),\quad \eta(t)\in \mathrm{C}_{\Delta(a)}.$$

The condition $y'(t)\sim \ell v_{41}'(t)$ requires the assumption that $\eta(+\infty)=0$.

Substituting (27) and (28) into the equation (1), we obtain the equality:

$$\sum_{k=1}^{n}p_k(t)(v_{41}(t))^{\alpha_k}(v_{41}'(t))^{\beta_k}(\ell+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k}=0. \tag{29}$$

In the equation (29) we put $\xi=0$, $\eta=0$ and get

$$\sum_{k=1}^{n}\ell^{\alpha_k+\beta_k}p_k(t)(v_{41}(t))^{\alpha_k}(v_{41}'(t))^{\beta_k}=0. \tag{30}$$

In accordance with the condition B), indicated in the statement of the problem, we consider the relations of the functions:

$$\frac{p_i(t)(v_{41}(t))^{\alpha_i}(v_{41}'(t))^{\beta_i}}{p_1(t)(v_{41}(t))^{\alpha_1}(v_{41}'(t))^{\beta_1}}=(c_i^*+\varepsilon_i(t))(1+z(t))^{\beta_i-\beta_1}=c_i^*+\varepsilon_{i1}(t), \tag{31}$$
$$\varepsilon_{i1}(+\infty)=0\ \ (i=\overline{1,s});$$

$$\frac{p_j(t)(v_{41}(t))^{\alpha_j}(v'_{41}(t))^{\beta_j}}{p_1(t)(v_{41}(t))^{\alpha_1}(v'_{41}(t))^{\beta_1}} = \varepsilon_j(t)(1+z(t))^{\beta_j-\beta_1} = \varepsilon_{j1}(t), \tag{32}$$
$$\varepsilon_{j1}(+\infty) = 0 \ \ (j = \overline{s+1, s_1});$$

$$\frac{p_k(t)(v_{41}(t))^{\alpha_k}(v'_{41}(t))^{\beta_k}}{p_1(t)(v_{41}(t))^{\alpha_1}(v'_{41}(t))^{\beta_1}} = \frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)(1+z(t))\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)(1+z(t))\,dt}} \times$$
$$\times (1+z(t))^{\beta_k-\beta_1} = \varepsilon_{k1}(t) \ \ (k = \overline{s_1+1, n}), \tag{33}$$

where

$$\lim_{t\to+\infty} \frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)\,dt}} = 0 \Longrightarrow \varepsilon_{k1}(+\infty) = 0 \ \ (k = \overline{s_1+1, n}).$$

Then, after the division by $p_1(t)(v_{41}(t))^{\alpha_1}(v'_{41}(t))^{\beta_1}$, the equation (30) takes the form:

$$\ell^{\alpha_1+\beta_1}\left(\sum_{i=1}^{s} c_i^*(1+z(t))^{\beta_i-\beta_1} + \sum_{j=1}^{s_1}\varepsilon_j(t)(1+z(t))^{\beta_j-\beta_1}\right) +$$
$$+ \sum_{k=s_1+1}^{n} \frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)(1+z(t))\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)(1+z(t))\,dt}}(1+z(t))^{\beta_k-\beta_1}\ell^{\alpha_k+\beta_k} = 0$$

or

$$F(t,z) = \ell^{\alpha_1+\beta_1}\left(\sum_{i=1}^{s} c_i^*(1+z)^{\beta_i} + \sum_{j=1}^{s_1}\varepsilon_j(t)(1+z)^{\beta_j}\right) +$$
$$+ \sum_{k=s_1+1}^{n} \frac{e^{\ell_0(\alpha_k+\beta_k)\int_a^t I'_t(a,t)(1+z(t))\,dt}}{e^{\ell_0(\alpha_1+\beta_1)\int_a^t I'_t(a,t)(1+z(t))\,dt}}\,(1+z)^{\beta_k}\ell^{\alpha_k+\beta_k} = 0. \tag{34}$$

We introduce into consideration the domain $\widetilde{D} = \Delta(a) \times [-h; h]$. The function $F(t,z) \in \mathrm{C}^{0\infty}_{tz}(\widetilde{D})$.

We consider in $\widetilde{D}$ a part of the function $F(t,z)$:

$$\widetilde{F}(t,z) = \ell^{\alpha_1+\beta_1}\left(\sum_{i=1}^{s} c_i^*(1+z)^{\beta_i} + \sum_{j=1}^{s_1}\varepsilon_j(t)(1+z)^{\beta_j}\right). \tag{35}$$

Taking into account the conditions (16), (18), we get:

$$\widetilde{F}(+\infty,0) = 0;$$
$$\widetilde{F}'_z(+\infty,0) = S_{01} \neq 0;$$
$$\widetilde{F}''_{z^2}(+\infty,0) = S_{02}.$$

Then, by Lemma 1, the equation (35) determines a unique function $z = \widetilde{z}(t,\xi)$, such that $\widetilde{z}(t) \in \mathrm{C}(\Delta(a_1))$ $(a_1 \geq a)$, $\widetilde{z}(+\infty) = 0$.

As $\widetilde{z}(t)$ we take an approximate solution of the equation (35):

$$\widetilde{z}(t) = -\frac{\ell^{\alpha_1+\beta_1} \sum\limits_{j=1}^{s_1} \varepsilon_j(t)}{S_{01} + \ell^{\alpha_1+\beta_1} \sum\limits_{j=1}^{s_1} \beta_j \varepsilon_j(t)}\,. \tag{36}$$

Next, we will need the following functions:

$$\widetilde{\psi}_{00}(t) = \sum_{k=1}^{n} \ell^{\alpha_k+\beta_k} \varepsilon_{k1}(t),$$
$$\widetilde{\psi}_{10}(t) = \sum_{k=1}^{n} \alpha_k \varepsilon_{k1}(t) \ell^{\alpha_k+\beta_k},$$
$$\widetilde{\psi}_{01}(t) = \sum_{k=1}^{n} \beta_k \varepsilon_{k1}(t) \ell^{\alpha_k+\beta_k}.$$

We express $\widetilde{\psi}_{00}(t)$, $\widetilde{\psi}_{10}(t) + \widetilde{\psi}_{01}(t)$ through the previously introduced functions:

$$\begin{aligned}
\widetilde{\psi}_{00}(t) &= \sum_{k=1}^{n} \ell^{\alpha_k+\beta_k} \varepsilon_{k1}(t) = \\
&= \frac{\widetilde{z}^{\,2}(t)}{(1+\widetilde{z}(t))^{\beta_1}} \big[S_{02} + \psi_{02}(t) + O(\widetilde{z})\big] = O(\psi_{00}^2(t)); \\
\widetilde{\psi}_{10}(t) + \widetilde{\psi}_{01}(t) &= \sum_{k=1}^{n} (\alpha_k + \beta_k) \varepsilon_{k1}(t) \ell^{\alpha_k+\beta_k} = \\
&= \frac{(\alpha_1+\beta_1)\widetilde{z}^{\,2}(t)}{(1+\widetilde{z}(t))^{\beta_1}} \big[S_{02}+\psi_{02}(t)+O(\widetilde{z})\big] = O(\psi_{00}^2(t)).
\end{aligned}$$

Thus, using Theorem 2, we formulate a theorem for the more precise asymptotics

$$v_{41} = e^{\ell_0 \int\limits_a^t I_t'(a,t)(1+\widetilde{z}(t))\,dt}. \tag{37}$$

**Theorem 3.** *Let for the function $v = v_{41}(t)$ of the form* (37) *the conditions of Theorem* 1, *except for* (19), *be fulfilled, and*

$$S \neq 0, \tag{21}$$
$$S_{02} \neq 0, \tag{38}$$
$$\psi_{00}(t) \ln v_{41}(t) = o(1). \tag{39}$$

*Then there exists a one-parameter set of $R$-solutions $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ of the differential equation* (1) *with the asymptotic properties*

$$y(t) = v_{41}(t)(\ell + \xi(t)), \quad y'(t) \sim \ell v_{41}'(t), \tag{40}$$

*where* $\xi(t) \sim \frac{\lambda_1 \ell}{\ln v_{41}(t)}$ .

## The Existence and Asymptotics of $R$-Solutions of the Equation (1) with the Condition $y(+\infty) = \gamma \in \mathbb{R}_+$

Since $y(+\infty) = \gamma \in \mathbb{R}_+$, a supposed asymptotics will be sought for the derivative of $n$-solutions $y'(t)$ to within a constant factor of the ratio of the first two summands. Taking into account $p_1(t),\ p_2(t) \neq 0$ $(t \in \Delta(a))$, we get:

$$y'(t) \dot{\sim} w(t) = \Big|\frac{p_1(t)}{p_2(t)}\Big|^{\frac{1}{\beta_2-\beta_1}} \quad (\beta_1 \neq \beta_2),$$

where $0 < w(t) \in \mathrm{C}_{\Delta(a)}$.

In the sequel, we will need the assumption that

$$\int\limits_a^{+\infty} w(t)\, dt < +\infty. \tag{41}$$

Let

$$y'(t) = w(t)(\ell + \eta(t)), \tag{42}$$

where $\ell, \ell^{\beta_k} \in \mathbb{R} \setminus \{0\}$ $(k = \overline{1,n})$; $\eta(t) \in \mathrm{C}_{\Delta(a)}$, $\eta(+\infty) = 0$.

Integrating (42), we obtain:

$$y(t) = \gamma - \int\limits_t^{+\infty} w(\tau)(\ell + \eta(\tau))\, d\tau,$$

where $\gamma \in \mathbb{R}_+$. Next, we show that the constants $\ell$ and $\gamma$ are related to each other by the equation (49).

Denoting

$$-\int\limits_t^{+\infty} w(\tau)(\ell + \eta(\tau))\, d\tau = \xi(t), \tag{43}$$

$\xi(t) \in \mathrm{C}^1_{\Delta(a)}$, $\xi(+\infty) = 0$, we obtain:

$$y(t) = \gamma + \xi(t). \tag{44}$$

We substitute (42) and (44) into the equation (1) and obtain the equality:

$$F\big(t, \gamma + \xi, w(\ell + \eta)\big) = \sum_{k=1}^{n} p_k(t)(\gamma + \xi)^{\alpha_k} w^{\beta_k} (\ell + \eta)^{\beta_k} = 0, \tag{45}$$

which is satisfied by the functions $\xi(t)$ and $\eta(t)$.

In accordance with the condition B), indicated in the statement of the problem, we assume that:

$$\frac{p_i(t)(w(t))^{\beta_i}}{p_1(t)(w(t))^{\beta_1}} = \widetilde{c}_i + \varepsilon_i(t), \quad \varepsilon_i(+\infty) = 0, \quad \widetilde{c}_i \in \mathbb{R} \setminus \{0\} \ \ (i = \overline{1,s}); \tag{46}$$

$$\frac{p_j(t)(w(t))^{\beta_j}}{p_1(t)(w(t))^{\beta_1}} = \varepsilon_j(t), \quad \varepsilon_j(+\infty) = 0 \ \ (j = \overline{s+1, n}). \tag{47}$$

Then, after the division by $p_1(t)(w(t))^{\beta_1}$, the equation (45) takes the form:

$$\widetilde{F}(t, \xi, \eta) = \sum_{i=1}^{s} (\widetilde{c}_i + \varepsilon_i(t))(\gamma + \xi)^{\alpha_i} (\ell + \eta)^{\beta_i} + \\ + \sum_{j=s+1}^{n} \varepsilon_j(t)(\gamma + \xi)^{\alpha_j} (\ell + \eta)^{\beta_j} = 0. \tag{48}$$

Obviously, the condition

$$\widetilde{F}(+\infty, 0, 0) = \sum_{i=1}^{s} \widetilde{c}_i \gamma^{\alpha_i} \ell^{\beta_i} = 0 \tag{49}$$

is necessary for the existence of a solution of the form (44) and of its derivative of the form (42).

**Theorem 4.** *Let a function $w(t)$ be a possible asymptotics of the derivative of R-solution of the equation* (1)*, which satisfies the conditions* (41)*,* (46)*,* (47)*. Moreover, let there exist $\gamma \in \mathbb{R}_+$, $\ell \in \mathbb{R} \setminus \{0\}$, satisfying the condition* (49)*.*

*Then for the existence of R-solution $y(t) \in \mathrm{C}^1_{\Delta(t_1)}$ of the differential equation* (1) *with the asymptotic properties*

$$y(t) \sim \gamma, \quad y'(t) \sim \ell w(t), \tag{50}$$

*it is sufficient that the condition*

$$\sum_{i=1}^{s} \beta_i \widetilde{c}_i \gamma^{\alpha_i} \ell^{\beta_i} \neq 0 \tag{51}$$

*be satisfied.*

*In this connection, for each pair $(\gamma, \ell)$ the differential equation* (1) *admits a unique R-solution $y(t)$ with the asymptotic properties* (50)*.*

*Proof.* Owing to its structure, the functions $\widetilde{F}(t, \xi, \eta) \in \mathrm{C}^{0\infty\infty}_{t\,\xi\,\eta}(D_1)$, $\frac{\partial^n \widetilde{F}}{\partial \xi^n}$, $\frac{\partial^m \widetilde{F}}{\partial \eta^m}$, $\frac{\partial^{n+m} \widetilde{F}}{\partial \xi^n \partial \eta^m}$ $(n = \overline{1, \infty},\ m = \overline{1, \infty})$ are bounded in $D_1$, where $D_1 = \Delta(a) \times [-h_1; h_1] \times [-h_2; h_2]$, $0 < h_1 < \gamma$, $0 < h_2 < |\ell|$.

To prove the above theorem, we will need expressions of the derivatives of the function $\widetilde{F}(t, \xi, \eta)$ of first and order with respect to the variables $\xi$, $\eta$ and also some of their properties:

$$\widetilde{F}'_{\xi}(t, \xi, \eta) = \sum_{i=1}^{s} \alpha_i \widetilde{c}_i (\gamma + \xi)^{\alpha_i - 1} (\ell + \eta)^{\beta_i} + \\ + \sum_{k=1}^{n} \alpha_k \varepsilon_k(t)(\gamma + \xi)^{\alpha_k - 1} (\ell + \eta)^{\beta_k},$$

$$\widetilde{F}'_{\xi}(+\infty,0,0)=\sum_{i=1}^{s}\alpha_i\widetilde{c}_i\gamma^{\alpha_i-1}\ell^{\beta_i}=\frac{1}{\gamma}\sum_{i=1}^{s}\alpha_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i};$$

$$\widetilde{F}'_{\eta}(t,\xi,\eta)=\sum_{i=1}^{s}\beta_i\widetilde{c}_i(\gamma+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-1}+ + \sum_{k=1}^{n}\beta_k\varepsilon_k(t)(\gamma+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-1},$$

$$\widetilde{F}'_{\eta}(+\infty,0,0)=\sum_{i=1}^{s}\beta_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i-1}=\frac{1}{\ell}\sum_{i=1}^{s}\beta_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i}\neq 0$$

by virtue of condition (51);

$$\widetilde{F}''_{\eta\eta}(t,\xi,\eta)=\sum_{i=1}^{s}\beta_i(\beta_i-1)\widetilde{c}_i(\gamma+\xi)^{\alpha_i}(\ell+\eta)^{\beta_i-2}+ + \sum_{k=1}^{n}\beta_k(\beta_k-1)\varepsilon_k(t)(\gamma+\xi)^{\alpha_k}(\ell+\eta)^{\beta_k-2}.$$

Owing to the conditions (49), (51) and the properties of the function $\widetilde{F}(t,\xi,\eta)$, in some domain $D_2\subset D_1$, $D_2=\Delta(t_0)\times[-\widetilde{h}_1;\widetilde{h}_1]\times[-\widetilde{h}_2;\widetilde{h}_2]$, $t_0\geq a$, $0<\widetilde{h}_1\leq h_1$, $0<\widetilde{h}_2<\min\Big\{h_2;\frac{\left|\sum_{i=1}^{s}\beta_i\widetilde{c}_i\gamma^{\alpha_i}\ell^{\beta_i}\right|}{4\ell\sup\limits_{D_1}\left|\widetilde{F}''_{\eta\eta}(t,\xi,\eta)\right|}\Big\}$, the equation (48) satisfies the conditions of Lemma 1. Consequently, there exists a unique function $\eta=\widetilde{\eta}(t,\xi)$, $\widetilde{\eta}(t,\xi)\in \mathrm{C}^{0\,\infty}_{t\,\xi}(D_3)$, $\sup\limits_{D_3}\left|\frac{\partial^n\widetilde{\eta}}{\partial\xi^n}\right|<+\infty$ $(n=\overline{1,\infty})$, such that $\widetilde{F}(t,\xi,\widetilde{\eta}(t,\xi))\equiv 0$, $\widetilde{\eta}(+\infty,0)=0$, $\|\widetilde{\eta}(t,\xi)\|\leq\widetilde{h}_2$. Moreover, we can write $\frac{\partial\widetilde{\eta}(t,\xi)}{\partial\xi}=-\frac{\widetilde{F}'_{\xi}(t,\xi,\widetilde{\eta})}{\widetilde{F}'_{\eta}(t,\xi,\widetilde{\eta})}$, $\sup\limits_{D_3}\left|\frac{\partial\widetilde{\eta}}{\partial\xi}\right|=M>0$.

In view of the replacement (43), we obtain the integral equation:

$$-\int_{t}^{+\infty}w(\tau)\big[\ell+\widetilde{\eta}(\tau,\xi(\tau))\big]\,d\tau=\xi(t). \tag{52}$$

The solution of the equation (52) will be sought in the class $\xi(t)\in\mathrm{C}^1_{\Delta(t_1)}$ $(t_1\geq t_0)$.

Next, we consider and estimate the difference $\widetilde{\eta}(t,\xi_2)-\widetilde{\eta}(t,\xi_1)$, $(t,\xi_i)\in D_3$ $(i=1,2)$, applying the Lagrange's theorem with respect to the variable $\xi$:

$$\widetilde{\eta}(t,\xi_2)-\widetilde{\eta}(t,\xi_1)=\widetilde{\eta}'_{\xi}(t,\xi^*)(\xi_2-\xi_1),\quad \xi^*\in]\xi_1;\xi_2[\,;$$

$$\left|\widetilde{\eta}(t,\xi_2)-\widetilde{\eta}(t,\xi_1)\right|\leq\sup_{D_3}|\widetilde{\eta}'_{\xi}(t,\xi)|\,|\xi_2-\xi_1|=M|\xi_2-\xi_1|.$$

Assuming $\xi_1=0$, $\xi_2=\xi$, we get:

$$|\widetilde{\eta}(t,\xi)|\leq M|\xi|.$$

To the equation (49) we out into the correspondence the operator

$$\xi(t) = T(t, \widetilde{\xi}(t)) \equiv -\int\limits_t^{+\infty} w(\tau)\big[\ell + \widetilde{\eta}(\tau, \widetilde{\xi}(\tau))\big]\, d\tau,$$

where $\widetilde{\xi}(t) \in B_1 \subset B$, $B = \big\{\widetilde{\xi}(t): \ \widetilde{\xi}(t) \in \mathrm{C}^1_{\Delta(t_1)}, \ \widetilde{\xi}(+\infty) = 0, \ \|\widetilde{\xi}(t)\| = \sup\limits_{\Delta(t_1)} |\widetilde{\xi}(t)|\big\}$ is the Banach space, $B_1 = \big\{\widetilde{\xi}(t): \widetilde{\xi}(t) \in B, \ \|\widetilde{\xi}(t)\| \le \widetilde{h}_1\big\}$ is a closed subset of the Banach space $B$.

Using the contraction mapping principle, we:

1) prove that if $\widetilde{\xi}(t) \in B_1$, then $\xi(t) = T(t, \widetilde{\xi}(t)) \in B_1$: $\widetilde{\xi}(t) \in \mathrm{C}^1_{\Delta(t_1)}$ and $\widetilde{\xi}(+\infty) = 0$, and by virtue of the structure of the operator, we get $\xi(t) \in \mathrm{C}^1_{\Delta(t_1)}$, $\xi(+\infty) = 0$;

$$\begin{aligned}\|\widetilde{\xi}(t)\| \le \widetilde{h}_1 \Longrightarrow \|\xi(t)\| = \big\|T(t, \widetilde{\xi}(t))\big\| = \\ = \bigg\| \int\limits_t^{+\infty} w(\tau)\big[\ell + \widetilde{\eta}(\tau, \widetilde{\xi}(\tau))\big]\, d\tau \bigg\| \le \int\limits_{t_1}^{+\infty} w(\tau)\big(|\ell| + \widetilde{h}_2\big)\, d\tau \le \widetilde{h}_1,\end{aligned}$$

if $t_1$ is sufficiently large.

2) check the condition of contraction:

$$\begin{aligned}\widetilde{\xi}_1(t), \widetilde{\xi}_2(t) \in B_1 \Longrightarrow \|\xi_2(t) - \xi_1(t)\| = \\ = \bigg\| \int\limits_t^{+\infty} w(\tau)\big[\widetilde{\eta}(\tau, \widetilde{\xi}_2(\tau)) - \widetilde{\eta}(\tau, \widetilde{\xi}_1(\tau))\big]\, d\tau \bigg\| \le \\ \le M \int\limits_{t_1}^{+\infty} w(\tau)\, d\tau\, \big\|\widetilde{\xi}_2(t) - \widetilde{\xi}_1(t)\big\| = \gamma \big\|\widetilde{\xi}_2(\tau) - \widetilde{\xi}_1(\tau)\big\|,\end{aligned}$$

where $\gamma = M \int\limits_{t_1}^{+\infty} w(\tau)\, d\tau < 1$, if $t_1$ is sufficiently large.

Thus, $t_1$ should necessarily be such that

$$\int\limits_{t_1}^{+\infty} w(\tau)\, d\tau < \min\bigg\{\frac{\widetilde{h}_1}{|\ell| + \widetilde{h}_2}, \frac{1}{M}\bigg\}.$$

As a result, we have found that by the contractive mapping principle the equation (52) admits a unique solution $\xi = \widetilde{\xi}(t) \in B_1$.

Thus, we have obtained that for each pair of constants $(\gamma, \ell)$, satisfying the condition (49), the differential equation (1) admits a unique $R$-solution $y(t)$ with the asymptotic properties (50) in $\Delta(t_1)$. Thus the Theorem is complete. □

## References

1. A. V. Kostin, The asymptotic properties of solutions of first order ordinary differential equations. (Russian). *Differencial'nye Uravnenija* **4** (1968), 1184–1195.
2. A. V. Kostin, On the question of the existence of bounded particular solutions and of particular solutions tending to zero as $t \to +\infty$ for a system of ordinary differential equations. (Russian) *Differencial'nye Uravnenija* **1** (1965), 585–604.
3. A. V. Kostin, Asymptotics of the regular solutions of nonlinear ordinary differential equations. (Russian) *Differentsial'nye Uravneniya* **23** (1987), No. 3, 524–526.
4. A. V. Kostin, Stability and asymptotics of quasilinear nonautonomous differential systems. (Russian) *Uchebnoe posobie, Izd-vo OGU, Odessa*, 1984.

(Received 13.07.2011)

**Authors' address:**

Odessa I. I. Mechnikov National University, 2 Dvoryanskaya St., Odessa 65082, Ukraine.

*E-mail:* a_kostin@ukr.net; koltsova.liliya@gmail.com