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THE EXISTENCE OF SOLUTIONS OF INTEGRAL EQUATIONS RELATED TO INVERSE PROBLEMS OF QUASILINEAR ORDINARY DIFFERENTIAL EQUATIONS

Dedicated to Professor T. Kusano
on the occasion of his 80-th birthday anniversary

Abstract. We consider nonlinear integral equations related to inverse problems for quasilinear ordinary differential equations. We establish a global existence of solutions for them by means of the method of successive approximations and fractional calculus. Our results give a generalization of the recent result in [3].

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რეზიუმე. განხილულია კვაზიწრფივ ჩვეულებრივ დიფერენციალურ განტოლებათა შებრუნებულ ამოცანებთან დაკავშირებული არაწრფივი ინტეგრალური განტოლებები. მიმდევრობითი მიახლოების მეთოდისა და წილადური აღრიცხვის გამოყენებით დამტკიცებულია მათი გლობალური ამოხსნადობა.

1. Introduction

In the present paper we will establish a global existence theorem of solutions to integral equations of the form

$$T(p) = 2\Big(\frac{m}{m+1}\Big)^{1/(m+1)} \int_0^p \frac{dv}{\big(\int_v^p f(u)\,du\big)^{1/(m+1)}}, \quad 0 \le p \le R, \tag{1.1}$$

where $m, R > 0$ are constants, $T(p)$ is a given positive function. We seek for a solution f, that is of the class $C[0,R]$ and $f(u) > 0$ on $(0,R]$. If we set $F(u) = \int_0^u f(\xi)d\xi$, then (1.1) is rewritten as

$$T(p) = 2\Big(\frac{m}{m+1}\Big)^{1/(m+1)} \int_0^p \big(F(p) - F(v)\big)^{-1/(m+1)}\,dv, \quad 0 \le p \le R. \tag{1.2}$$

Though equation (1.1) has a complicated appearance, it arises naturally from the following inverse problem for quasilinear ordinary differential equations:

Problem 1.1. Let $T(p)$ be a given positive function on $[0,R]$. Determine a nonlinearity $f(u)$ of an ordinary differential equation

$$\big(|u'|^{m-1}u'\big)' + f(u) = 0 \tag{1.3}$$

so that, for each $p \in (0,R]$, the solution $u(t) = u(t;p)$ of the equation with the stationary (maximal) value p has a half-period $T(p)$. (Note that when $f(0) = 0$ and $f(u)$ is extended to the interval $[-R,R]$ as an odd function, every solution of (1.3) oscillates and is periodic.)

In fact, we will explain how Problem 1.1 relates to equation (1.1). Let $p \in [0,R]$, and $u = u(t;p)$ be the solution of (1.3) satisfying the constraints in Problem 1.1, that is,

$$\begin{gathered}\big(|u'|^{m-1}u'\big)' + f(u) = 0 \ \text{ on } \ [0,T(p)],\\ u(0) = u(T(p)) = 0, \ \text{ and } \ u(t) > 0 \ \text{ in } \ (0,T(p)),\end{gathered}$$

and

$$\max_{[0,T(p)]} u = u(T(p)/2) = p \ \text{ and } \ u'(T(p)/2) = 0.$$

Here, the symmetry of u on $[0,T(p)]$ has been employed. It is easy to see that

$$T(p) = 2\int_0^p \Big(u'(0)^{m+1} - \frac{m+1}{m}F(v)\Big)^{-1/(m+1)}\,dv.$$

Since $u'(0)^{m+1} = (m+1)F(p)/m$, we can get

$$T(p) = B_0 \int_0^p \big(F(p) - F(v)\big)^{-1/(m+1)}\, dv, \tag{1.4}$$

where

$$B_0 = 2\Big(\frac{m}{m+1}\Big)^{1/(m+1)}.$$

Accordingly, (1.2) has been obtained.

We transform equation (1.4) further to the form which is easy to analyze. By the change of variables $s = F(v)$, $t = F(p)$, that is, $p = p(t) = F^{-1}(t)$, this equation is transformed to

$$T(p(t)) = B_0 \int_0^t \frac{p'(s)}{(t-s)^{1/(m+1)}}\, ds, \quad 0 \le t \le F(R).$$

By using the Riemann–Liouville integral operator, which will be defined later in the next section, this is rewritten as

$$T(p(t)) = B_0 \Gamma\Big(\frac{m}{m+1}\Big) I^{m/(m+1)} p'(t).$$

Here, Γ denotes the Gamma function. Applying the Riemann–Liouville integral operator $I^{1/(m+1)}$ to the both sides, we have

$$I^{1/(m+1)} T(p)(t) = B_0 \Gamma\Big(\frac{m}{m+1}\Big) p(t),$$

that is,

$$p(t) = \frac{1}{B_0 \Gamma(\frac{m}{m+1})}\, I^{1/(m+1)} T(p)(t), \tag{1.5}$$

or equivalently,

$$p(t) = \frac{\sin\big(\frac{\pi}{m+1}\big)}{\pi B_0} \int_0^t \frac{T(p(s))}{(t-s)^{1-1/(m+1)}}\, ds. \tag{1.6}$$

(Here we have employed the property (2.2) appearing in the next section.)

When $m = 1$ and T is Lipschitzian, it is shown conversely [1], [3] that a solution $p(t)$ of (1.5) (with $m = 1$) is necessarily differentiable and satisfies (1.1) (with $m = 1$). Thus solving of equation (1.1) (as well as of Problem 1.1) is equivalent to finding a solution of (1.5) if $m = 1$.

In the paper we will show that such a result still holds for equation (1.5) with $m > 0$. This is the main objective of the paper. In fact, we can establish the following result:

Theorem 1.2. *Let $T(r)$ be a Lipschitz continuous positive function defined on* $[0, R]$. *Then there exists a* (*unique*) *solution f of* (1.1) *that is continuous on* $[0, R]$ *and positive on* $(0, R]$.

When $m = 1$, this theorem reduces to [3, Theorem 1.2].

The paper is organized as follows. In Section 2 we construct a solution of equation (1.5) by the method of successive approximations as a preliminary result. The proof of Theorem 1.2 is given in Section 3. Other related results can be found in [2], [4], [6].

Though the arguments in the paper are based essentially on those in [3], the fact that $m \neq 1$ causes some difficulties, in particular, in the proof of Proposition 3.2.

2. Preliminary Results

As a first step, we must introduce the Riemann–Liouville integral operators. Let $\delta > 0$ be a constant. We define the integral operator I^δ by

$$I^\delta \phi(t) = \frac{1}{\Gamma(\delta)} \int_0^t \frac{\phi(s)}{(t-s)^{1-\delta}}\, ds \tag{2.1}$$

for $\phi \in C[0,R]$, where Γ is the Gamma function. We can show by interchange of the order of integration that

$$I^{\delta_1} I^{\delta_2} = I^{\delta_1+\delta_2} \text{ on } C[0,R] \tag{2.2}$$

for $\delta_1, \delta_2 > 0$. See, for example, [2], [5]. Note that this property has been already used in the Introduction.

Let us construct a continuous solution of integral equation (1.5), namely (1.6), by successive approximation.

Proposition 2.1. *Suppose that $T(r)$ is Lipschitz continuous on $[0,R]$, and $T(r) > 0$ there. Then there exists a positive number q and a continuous function $p(t)$ such that*

(i) *$p(t)$ satisfies equation (1.5) on $[0,q]$;*

(ii) *$p(0) = 0$ and $p(q) = R$;*

(iii) *$0 < p(t) < R$ for $t \in (0,q)$.*

Proof. Let L be a constant satisfying

$$\big|T(r_1) - T(r_2)\big| \le L|r_1 - r_2| \tag{2.3}$$

for $r_1, r_2 \in [0,R]$. Put

$$T^* = \max_{[0,R]} T(r), \quad T_* = \min_{[0,R]} T(r), \quad \text{and} \quad \widetilde{R} = T^* R / T_*.$$

We extend $T(r)$ (defined on $[0,R]$) to the continuous function on $[0,\widetilde{R}\,]$ so that $T(r) \equiv T(\widetilde{R})$ on $[R,\widetilde{R}\,]$. (In what follows, we may denote the extension by the same symbol T for simplicity.) Then T still satisfies (2.3) for $r_1, r_2 \in [0,\widetilde{R}\,]$, and $T_* \le T(r) \le T^*$ on $[0,\widetilde{R}\,]$. Furthermore, we set

$$A = \frac{(m+1)\sin\left(\frac{\pi}{m+1}\right)}{\pi B_0}, \quad \widetilde{t} = \left(\frac{R}{AT_*}\right)^{m+1},$$

and

$$\underline{p}(t) = AT_* t^{1/(m+1)}, \quad \overline{p}(t) = AT^* t^{1/(m+1)} \text{ on } [0, \widetilde{t}\,].$$

Let us define the sequence $\{p_n(t)\}_{n=0}^{\infty}$ inductively by $p_0(t) = \underline{p}(t)$ and

$$p_n(t) = \frac{1}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)} T(p_{n-1})(t), \quad n = 1, 2, \dots . \tag{2.4}$$

We will show that $p_n(t)$, $n = 1, 2, \dots$, are well-defined, and

$$\underline{p}(t) \le p_n(t) \le \overline{p}(t) \text{ on } [0, \widetilde{t}\,], \tag{2.5}$$

for $n = 0, 1, 2, \dots$, and hence $0 \le p_n(t) \le \widetilde{R}$.

For $p_0(t)$, inequalities (2.5) are obviously true. Let $p_{n-1}(t)$ satisfy them. Since $T(p_{n-1}(t)) \le T^*$, we have

$$\begin{aligned} p_n(t) &\le \frac{T^*}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)}(1) = \\ &= \frac{T^*}{B_0 \Gamma(\frac{m}{m+1}) \Gamma(\frac{1}{m+1})} \int_0^t \frac{ds}{(t-s)^{1-1/(m-1)}} = \\ &= \frac{(m+1)T^*}{B_0 \frac{\pi}{\sin(\pi/(m+1))}} t^{1/(m+1)} = AT^* t^{1/(m+1)} = \\ &= \overline{p}(t) \le \widetilde{R}. \end{aligned}$$

Thus $p_n(t)$ is well-defined and satisfies $p_n(t) \le \overline{p}(t)$. Similarly, we can show that $p_n(t) \ge \underline{p}(t)$. We therefore find that (2.5) is true for all $n = 0, 1, 2, \dots$.

It follows from (2.4) that

$$\begin{aligned} \big|p_{k+1}(t) - p_k(t)\big| &\le \frac{1}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)} \big|T(p_k) - T(p_{k-1})\big|(t) \le \\ &\le \frac{L}{B_0 \Gamma(\frac{m}{m+1})} I^{1/(m+1)} |p_k - p_{k-1}|(t) \le \\ &\le \Big(\frac{L}{B_0 \Gamma(\frac{m}{m+1})}\Big)^2 I^{2/(m+1)} |p_{k-1} - p_{k-2}|(t) \end{aligned}$$

for $k = 2, 3, \dots$. Repeating this procedure, we can get

$$\big|p_{k+1}(t) - p_k(t)\big| \le \Big(\frac{L}{B_0 \Gamma(\frac{m}{m+1})}\Big)^k I^{k/(m+1)} |p_1 - p_0|(t).$$

Putting $M = \max\limits_{[0,\widetilde{t}\,]} |p_1 - p_0|$, we find that

$$\max_{[0,\widetilde{t}\,]} |p_{k+1} - p_k| \le \frac{(m+1)M}{k\Gamma(\frac{k}{m+1})} \Big(\frac{L}{B_0 \Gamma(\frac{m}{m+1})}\Big)^k \widetilde{t}^{k/(m+1)} \equiv c_k.$$

By the Stirling's formula $\Gamma(z) = \sqrt{2\pi}\, e^{-z} z^{z-1/2}(1 + O(1/z))$, as $|z| \to \infty$, we find that $c_{k+1}/c_k \to 0$ as $k \to \infty$.

Consequently, the sequence $\{p_n(t)\}$ converges to a limit function $\widetilde{p}(t) \in C[0,\widetilde{t}\,]$ uniformly on $[0,\widetilde{t}\,]$. Moreover, by (2.5) we know that

$$\underline{p}(t) \le \widetilde{p}(t) \le \overline{p}(t) \ \text{ on } \ [0,\widetilde{t}\,].$$

In particular, $\widetilde{p}(\widetilde{t}) \ge \underline{p}(\widetilde{t}) = R$. So, there is a $q \in (0,\widetilde{t})$ such that $\widetilde{p}(t) < R$ on $[0,q)$ and $\widetilde{p}(q) = R$. We define a function $p(t)$ by the restriction of $\widetilde{p}(t)$ on $[0,q] : p(t) = \widetilde{p}\big|_{[0,q]}(t)$. Then $p(t)$ satisfies the desired properties (i)–(iii). This completes the proof. □

3. Proof of Theorem 1.2

To see Theorem 1.2, we first prove that the solution $p(t)$ constructed in Proposition 2.1 is differentiable and $p'(t) > 0$ on $(0,q]$. The discussion is based on the fractional calculus associated with the Riemann–Liouville integral operators introduced in the Introduction by (2.1) and corresponding differential operators D^δ defined by $D^\delta = (d/dt)I^{1-\delta} = DI^{1-\delta}$, $D = d/dt$.

Below, we introduce the weighted Hölder spaces. Let $0 < b < \infty$, $0 \le \alpha \le 1$, and $\eta \in \mathbf{R}$. We put for $\phi \in C(0,b]$

$$|\phi|_\eta = \sup_{t\in(0,b]} t^{-\eta}|\phi(t)|$$

and

$$|\phi|_{\alpha,\eta} = \sup_{t,s\in(0,b],\, t\neq s} \frac{|t^{\alpha-\eta}\phi(t) - s^{\alpha-\eta}\phi(s)|}{|t-s|^\alpha},$$

and define the Banach space $(C^\alpha(0,b]_\eta, \|\cdot\|_{\alpha,\eta})$ by

$$C^\alpha(0,b]_\eta = \Big\{\phi \in C(0,b] \ \Big| \ \|\phi\|_{\alpha,\eta} = |\phi|_\eta + |\phi|_{\alpha,\eta} < \infty\Big\}.$$

It is easy to prove that $C^{\alpha_1}[0,b)_{\eta_1} \supset C^{\alpha_2}[0,b)_{\eta_2}$ if $\alpha_1 \le \alpha_2$ and $\eta_1 \le \eta_2$. Note that if $\eta > 0$, then $\phi \in C^\alpha(0,b]_\eta$ is a continuous function and $\phi(0) = 0$.

Lemma 3.1. *Let $\eta > -1$.*

(i) *Let $0 \le \alpha < \alpha+\delta < 1$. Then $I^\delta : C^\alpha(0,b]_\eta \to C^{\alpha+\delta}(0,b]_{\eta+\delta}$ is a bounded operator.*

(ii) *Let $0 < \alpha < \alpha+\delta \le 1$. Then $D^\delta : C^{\alpha+\delta}(0,b]_{\eta+\delta} \to C^\alpha(0,b]_\eta$ is a bounded operator. For $\phi \in C^{\alpha+\delta}(0,b]_{\eta+\delta}$, the derivative $D^\delta\phi$ is expressed as*

$$D^\delta\phi(t) = \frac{1}{\Gamma(1-\delta)}\left(\frac{\phi(t)}{t^\delta} + \delta\int_0^t \frac{\phi(t)-\phi(s)}{(t-s)^{\delta+1}}\,ds\right).$$

The proof of this lemma can be found in [3]; and related results in [2].

Since equation (1.5) has somewhat complicated appearance, we will consider equation (3.1) below instead of equation (1.5) without loss of generality.

Proposition 3.2. *Let τ be a Lipschitz continuous function defined on an interval containing* 0 *and assume that* $\tau(0) > 0$*. Suppose, furthermore, that a continuous function $x(t)$ defined on $[0, b]$, $0 < b < \infty$, satisfies* $x(t) = I^{1/(m+1)}(\tau \circ x)(t)$, $0 \le t \le b$*, that is,*

$$x(t) = \frac{1}{\Gamma(\frac{1}{m+1})} \int_0^t \frac{\tau(x(s))}{(t-s)^{1-1/(m+1)}}\, ds, \quad 0 \le t \le b. \tag{3.1}$$

Then $x(t)$ is differentiable and $x'(t) > 0$ on $(0, b]$.

The following simple lemma is employed in proving Proposition 3.2:

Lemma 3.3. Let $k, l > 0$ be constants satisfying $k + l \le 1$. Then,

$$s^k(t^l - s^l) \le (t-s)^{k+l}, \quad t \ge s \ge 0.$$

Proof of Proposition 3.2. In the sequel, we denote a Lipschitz constant of τ by L. We may assume that $m > 1$, because the case where $0 < m \le 1$ can be treated similarly. The proof is divided into several steps.

Step 1. We show that

$$x \in C^{\beta+1/(m+1)}(0, b]_{1/(m+1)} \ \text{ for any } \beta, \ \ 0 \le \beta < 1/(m+1). \tag{3.2}$$

To see this we first note that $\tau \circ x \in C^0(0, b]_0$. So the fact that $x = I^{1/(m+1)}(\tau \circ x)$ and Lemma 3.1-(i) imply that $x \in C^{1/(m+1)}(0, b]_{1/(m+1)}$. Since the Lipschitz continuity implies that

$$\begin{aligned}\big|\tau(x(t)) - \tau(x(s))\big| &\le L|x(t) - x(s)| \le \\ &\le L|x|_{1/(m+1),1/(m+1)}|t-s|^{1/(m+1)} \le C_1|t-s|^{1/(m+1)}\end{aligned}$$

for some constant $C_1 > 0$, it follows that

$$\begin{aligned}&\Big|t^{1/(m+1)}\tau(x(t)) - s^{1/(m+1)}\tau(x(s))\Big| \le \\ &\quad \le t^{1/(m+1)}\big|\tau(x(t)) - \tau(x(s))\big| + |\tau(x(s))| \cdot \big|t^{1/(m+1)} - s^{1/(m+1)}\big| \le \\ &\le b^{1/(m+1)}C_1|t-s|^{1/(m+1)} + \big(\max_{[0,b]}|\tau \circ x|\big)|t-s|^{1/(m+1)} \le C_2|t-s|^{1/(m+1)}\end{aligned}$$

for some constant $C_2 > 0$. Thus, $\tau \circ x \in C^{1/(m+1)}(0, b]_0$, and hence, $\tau \circ x \in C^\beta(0, b]_0$ for any β, $0 \le \beta < 1/(m+1)$. Noting $x(t) = I^{1/(m+1)}(\tau \circ x)(t)$, we can show (3.2) by Lemma 3.1-(i).

Step 2. We show that

$$\tau(x(t)) - \tau(x(0)) \in C^{\beta+1/(m+1)}(0, b]_{1/(m+1)} \ \text{ for any } \beta, \ \ 0 \le \beta < 1/(m+1).$$

In fact, by Step 1, we know that for some $C_3 > 0$,

$$|x(t)| \le C_3 t^{1/(m+1)}, \tag{3.3}$$

and

$$\big|t^\beta x(t) - s^\beta x(s)\big| \le C_3|t-s|^{\beta+1/(m+1)} \tag{3.4}$$

for any β, $0 \le \beta < 1/(m+1)$. By the Lipschitz continuity of τ and (3.3), we find that

$$\bigl|\tau(x(t)) - \tau(x(0))\bigr| \le L|x(t) - x(0)| = L|x(t)| \le C_4 t^{1/(m+1)} \tag{3.5}$$

for some $C_4 > 0$. On the other hand, by the Lipschitz continuity of τ, (3.4), and (3.5), we find that

$$\begin{gathered}
\Bigl|t^\beta\bigl\{\tau(x(t)) - \tau(x(0))\bigr\} - s^\beta\bigl\{\tau(x(s)) - \tau(x(0))\bigr\}\Bigr| = \\
= \Bigl|t^\beta\bigl\{\tau(x(t)) - \tau(x(s))\bigr\} - (t^\beta - s^\beta)\bigl\{\tau(x(s)) - \tau(x(0))\bigr\}\Bigr| \le \\
\le Lt^\beta|x(t) - x(s)| + LC_3 s^{1/(m+1)}|t^\beta - s^\beta| = \\
= L\Bigl|\bigl\{t^\beta x(t) - s^\beta x(s)\bigr\} - (t^\beta - s^\beta)x(s)\Bigr| + LC_3 s^{1/(m+1)}|t^\beta - s^\beta| \le \\
\le L\Bigl(C_3|t-s|^{\beta+1/(m+1)} + C_3|t-s|^\beta s^{1/(m+1)}\Bigr) + LC_3 s^{1/(m+1)}|t^\beta - s^\beta| = \\
= 2LC_3|t-s|^{\beta+1/(m+1)} + LC_3 s^{1/(m+1)}|t^\beta - s^\beta|.
\end{gathered}$$

Employing Lemma 3.3, we can get

$$\Bigl|t^\beta\bigl\{\tau(x(t)) - \tau(x(0))\bigr\} - s^\beta\bigl\{\tau(x(s)) - \tau(x(0))\bigr\}\Bigr| \le 3LC_3|t-s|^{\beta+1/(m+1)}.$$

Step 3. We show that

$$x \in C^{\beta+1/(m+1)}(0,b]_{1/(m+1)} \quad \text{for any } \beta, \ 0 \le \beta < 1 - 1/(m+1). \tag{3.6}$$

Since the constant $\tau(x(0))$ is of the class $C^{\beta+1/(m+1)}(0,b]_0$ and $C^{\beta+1/(m+1)}(0,b]_{1/(m+1)} \subset C^{\beta+1/(m+1)}(0,b]_0$, we find by Step 2 that $\tau \circ x \in C^{\beta+1/(m+1)}(0,b]_0, 0 \le \beta < 1/(m+1)$. Thus, by Lemma 3.1-(i) again,

$$\begin{gathered}
x = I^{1/(m+1)}(\tau \circ x) \in C^{\beta_1+2/(m+1)}(0,b]_{1/(m+1)}, \\
0 \le \beta_1 < \min\Bigl\{\frac{m-1}{m+1}, \frac{1}{m+1}\Bigr\}.
\end{gathered} \tag{3.7}$$

So, if $1 < m \le 2$, then we have established (3.6).

Below, we suppose that $m > 2$. Then from (3.7), we get $x \in C^{\beta_2+1/(m+1)}(0,b]_{1/(m+1)}$, $0 \le \beta_2 < 2/(m+1)$. By the argument developed in Step 2, we find that $\tau(x(t)) - \tau(x(0)) \in C^{\beta_2+1/(m+1)}(0,b]_{1/(m+1)}$, $0 \le \beta_2 < 2/(m+1)$; and hence $\tau(x(t)) \in C^{\beta_2+1/(m+1)}(0,b]_0$, $0 \le \beta_2 < 2/(m+1)$. Again, applying Lemma 3-(i), we have

$$\begin{gathered}
x = I^{1/(m+1)}(\tau \circ x) \in C^{\beta_2+2/(m+1)}(0,b]_{1/(m+1)}, \\
0 \le \beta_2 < \min\Bigl\{\frac{m-1}{m+1}, \frac{2}{m+1}\Bigr\}.
\end{gathered}$$

So, if $2 < m \le 3$, then we have established (3.6). If $m > 3$, then $x \in C^{\beta_3+1/(m+1)}(0,b]_{1/(m+1)}$, $0 \le \beta_3 < 3/(m+1)$.

Continuing this procedure, we finally reach to the relation

$$x \in C^{\widetilde{\beta}+1/(m+1)}(0,b]_{1/(m+1)}, \quad 0 \le \widetilde{\beta} < \frac{[m]}{m+1}\,,$$

that is,

$$x \in C^{\widetilde{\beta}+1/(m+1)}(0,b]_{1/(m+1)}, \quad 0 \le \widetilde{\beta} < \frac{m-1}{m+1}\,,$$

where $[m]$ denotes the largest integer, not exceeding m, as usual. Then, one more application of the argument in Step 2 and Lemma 3.1-(i) show that (3.6) is valid.

Step 4. We show that $x(t)$ is differentiable on $(0,b]$, and

$$t^{m/(m+1)}x'(t) = \frac{\tau(0)}{\Gamma(\frac{1}{m+1})} + O\big(t^{m/(m+1)}\big) \text{ as } t \to +0. \tag{3.8}$$

Therefore, $x'(t) > 0$ near $+0$.

To see this, we notice by Step 3 and the observation in Step 2 that

$$\tau\circ x(t) - \tau\circ x(0) \in C^{\beta+1/(m+1)}(0,b]_{1/(m+1)}, \quad 0 \le \beta < 1 - 1/(m+1), \tag{3.9}$$

and accordingly, $\tau \circ x \in C^{\beta+1/(m+1)}(0,b]_0$. Then, by Lemma 3.1-(ii), $D^{m/(m+1)}(\tau \circ x) \equiv DI^{1-m/(m+1)}(\tau \circ x)$ is well-defined; and so, $x' = DI^{1/(m+1)}(\tau\circ x)$ is well-defined, and

$$\begin{aligned} x' &= DI^{1-m/(m+1)}(\tau\circ x) \equiv \\ &\equiv D^{m/(m+1)}(\tau\circ x) \in C^{\beta-(m-1)/(m+1)}(0,b]_{-m/(m+1)}. \end{aligned}$$

Therefore, we obtain

$$I^{m/(m+1)}x' = \tau\circ x \in C^{\beta+1/(m+1)}(0,b]_0, \tag{3.10}$$

and

$$\begin{aligned} x'(t) &= D^{m/(m+1)}\big((\tau\circ x)(t) - (\tau\circ x)(0)\big) + D^{m/(m+1)}((\tau\circ x)(0)) = \\ &= D^{m/(m+1)}\big((\tau\circ x)(t) - (\tau\circ x)(0)\big) + \frac{\tau(0)}{\Gamma(\frac{1}{m+1})}\, t^{-m/(m+1)}, \end{aligned}$$

by Lemma 3.1-(ii). Since

$$D^{m/(m+1)}\big((\tau\circ x)(t) - (\tau\circ x)(0)\big) \in C^{\beta-(m-1)/(m+1)}(0,b]_0$$

by (3.9), we have

$$D^{m/(m+1)}\big((\tau\circ x)(t) - (\tau\circ x)(0)\big) = O(1) \text{ as } t \to +0.$$

This implies the validity of (3.8).

Step 5. Finally, we show that $x'(t) > 0$ on $(0,b]$.

The proof of this step is essentially the same as that of [3, Step 2 of the proof of Proposition 3.2]. To see this, let $0 < \varepsilon < 1/(m+1)$ and choose

$\beta \in (0, 1-1/(m+1))$, so that $1-1/(m+1)-\beta < \varepsilon$. (For example, $\beta = 1-1/(m+1)-\varepsilon/2$.) We get from (3.10) that

$$D^{1/(m+1)-\varepsilon}x' = D^{1-\varepsilon}(\tau\circ x). \tag{3.11}$$

By Lemma 3.1-(ii), the left-hand side of (3.11) can be rewritten as

$$D^{1/(m+1)-\varepsilon}x'(t) =$$

$$= \frac{1}{\Gamma(1-\frac{1}{m+1}+\varepsilon)}\left(\frac{x'(t)}{t^{1/(m+1)-\varepsilon}} + \Big(\frac{1}{m+1}-\varepsilon\Big)\int_0^t \frac{x'(t)-x'(s)}{(t-s)^{1/(m+1)+1-\varepsilon}}\,ds\right).$$

To see $x'(t) > 0$ on $(0,b]$ by contradiction, we assume the contrary. Since $x'(t) > 0$ near the origin, there is an $a \in (0,b]$ such that $x'(t) > 0$ on $(0,a)$ and $x'(a) = 0$. Noting that

$$\int_0^a \frac{x'(s)}{(a-s)^{1+1/(m+1)-\varepsilon}}\,ds >$$

$$> a^{-1-1/(m+1)+\varepsilon}\int_0^a x'(s)\,ds = a^{-1-1/(m+1)}a^{\varepsilon}x(a),$$

we can find a constant $\rho > 0$ independent of ε such that

$$D^{1/(m+1)-\varepsilon}x'(t)\big|_{t=a} =$$

$$= -\frac{1/(m+1)-\varepsilon}{\Gamma(1-\frac{1}{m+1}+\varepsilon)}\int_0^a \frac{x'(s)}{(a-s)^{1+1/(m+1)-\varepsilon}}\,ds \le -\rho. \tag{3.12}$$

On the other hand, the right-hand side of (3.11) with $t = a$ can be rewritten as

$$D^{1-\varepsilon}(\tau\circ x)(a) = \frac{1}{\Gamma(\varepsilon)}\left\{\frac{\tau(x(a))}{a^{1-\varepsilon}} + (1-\varepsilon)\int_0^a \frac{\tau(x(a))-\tau(x(s))}{(a-s)^{2-\varepsilon}}\,ds\right\} \equiv$$

$$\equiv \frac{1}{\Gamma(\varepsilon)}\left\{\frac{\tau(x(a))}{a^{1-\varepsilon}} + (1-\varepsilon)\int_0^{a-\varepsilon} + (1-\varepsilon)\int_{a-\varepsilon}^a\right\}.$$

We observe that

$$(1-\varepsilon)\int_0^{a-\varepsilon} = (1-\varepsilon)\tau(x(a))\int_0^{a-\varepsilon}\frac{ds}{(a-s)^{2-\varepsilon}} - (1-\varepsilon)\int_0^{a-\varepsilon}\frac{\tau(x(s))}{(a-s)^{2-\varepsilon}}\,ds =$$

$$= \tau(x(a))(\varepsilon^{\varepsilon-1} - a^{\varepsilon-1}) - (1-\varepsilon)\int_0^{a-\varepsilon}\frac{\tau(x(s))}{(a-s)^{2-\varepsilon}}\,ds,$$

so we get

$$D^{1-\varepsilon}(\tau\circ x)(a)=\frac{\varepsilon^{\varepsilon-1}\tau(x(a))}{\Gamma(\varepsilon)}-\frac{1-\varepsilon}{\Gamma(\varepsilon)}\int\limits_0^{a-\varepsilon}\frac{\tau(x(s))}{(a-s)^{2-\varepsilon}}\,ds+$$
$$+\frac{1-\varepsilon}{\Gamma(\varepsilon)}\int\limits_{a-\varepsilon}^{a}\frac{\tau(x(a))-\tau(x(s))}{(a-s)^{2-\varepsilon}}\,ds\equiv$$
$$\equiv J_1(\varepsilon)-J_2(\varepsilon)+J_3(\varepsilon),$$

where $J_i(\varepsilon)$, $i=1,2,3$, are defined naturally by the last equality. Below we will estimate each $J_i(\varepsilon)$ separately.

It is easy to see that

$$J_1(\varepsilon)=\frac{\varepsilon^{\varepsilon}}{\Gamma(\varepsilon+1)}\,\tau(x(a))\longrightarrow\tau(x(a))\ \text{ as }\ \varepsilon\to+0.$$

By the change of variables, the term $J_2(\varepsilon)$ is expressed as

$$J_2(\varepsilon)=\frac{(1-\varepsilon)\varepsilon^{\varepsilon}}{\Gamma(\varepsilon+1)}\int\limits_1^{a/\varepsilon}\frac{\tau(x(a-\varepsilon v))}{v^{2-\varepsilon}}\,dv=\frac{(1-\varepsilon)\varepsilon^{\varepsilon}}{\Gamma(\varepsilon+1)}\int\limits_1^{\infty}h_\varepsilon(v)\,dv,$$

where

$$h_\varepsilon(v)=\begin{cases}\tau(x(a-\varepsilon v))/v^{2-\varepsilon} & \text{if }\ 1\le v\le a/\varepsilon,\\ 0 & \text{if }\ v\ge a/\varepsilon.\end{cases}$$

Since $|h_\varepsilon(v)|\le Cv^{-2}$ on $[1,\infty)$ for some constant $C>0$, and $\lim\limits_{\varepsilon\to+0}h_\varepsilon(v)=\tau(x(a))/v^2$, the dominated convergence theorem implies that

$$J_2(\varepsilon)\longrightarrow\int\limits_1^{\infty}\frac{\tau(x(a))}{v^2}\,dv=\tau(x(a))\ \text{ as }\ \varepsilon\to+0.$$

Finally, let us examine $J_3(\varepsilon)$. Recall that $x'\in C^{\beta-(m-1)/(m+1)}(0,b]_{-m/(m+1)}$ for any β, $0\le\beta<1-1/(m+1)$. Hence

$$t^{m/(m+1)}|x'(t)|\le C_4$$

and

$$\left|t^{\beta+1/(m+1)}x'(t)-s^{\beta+1/(m+1)}x'(s)\right|\le C_4|t-s|^{\beta-(m-1)/(m+1)}$$

for some constant $C_4>0$. Therefore, for $t_0>0$, we have

$$|x'(t)-x'(s)|\le t^{-\beta-1/(m+1)}\left|t^{\beta+1/(m+1)}x'(t)-s^{\beta+1/(m+1)}x'(s)\right|+$$
$$+\,t^{-\beta-1/(m+1)}|x'(s)|\cdot|t-s|^{\beta+1/(m+1)}\le C(t_0)|t-s|^{\beta},\ \ t,s\in[t_0,b],$$

where $C(t_0) > 0$ is a constant depending on t_0. Thus for $s \le a$ near a, we have

$$\begin{aligned}\big|\tau(x(a)) - \tau(x(s))\big| &\le L|x(a) - x(s)| \le \\ &\le L \int_s^a |x'(v) - x'(a)|\, dv \le LC_5 \int_s^a (a-v)^\beta\, dv = \frac{LC_5}{\beta+1}\,(a-s)^{\beta+1}\end{aligned}$$

for some $C_5 > 0$. Consequently,

$$\begin{aligned}|J_3(\varepsilon)| &\le \frac{(1-\varepsilon)LC_5}{(\beta+1)\Gamma(\varepsilon)} \int_{a-\varepsilon}^a (a-s)^{\beta+\varepsilon-1}\, ds = \\ &= \frac{(1-\varepsilon)LC_5}{(\beta+1)(\beta+\varepsilon)\Gamma(\varepsilon+1)}\,\varepsilon^{\beta+1+\varepsilon} \longrightarrow 0 \ \text{ as } \ \varepsilon \to +0.\end{aligned}$$

Hence $\lim\limits_{\varepsilon\to+0} D^{1-\varepsilon}(\tau\circ x)(t)\big|_{t=a} = 0$. By (3.11) this contradicts (3.12). So, $x'(t) > 0$ on $(0, b]$.

The proof of Proposition 3.2 is complete. □

We are now in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $p(t)$ be the solution of equation (1.5) constructed in Proposition 2.1. By Proposition 3.2 we know that $p(t)$ is differentiable, $p'(t) > 0$ on $(0, q]$, $p' \in C(0, q]$, and $p(t)$ satisfies the asymptotic formula

$$t^{m/(m+1)}p'(t) = C_0 + O(t^{m/(m+1)}) \ \text{ as } \ t \to +0$$

for some constant $C_0 > 0$. Applying $I^{m/(m+1)}$ to the both sides of (1.5), we get

$$I^1T(p)(t) = B_0\Gamma\Big(\frac{m}{m+1}\Big)I^{m/(m+1)}p(t) = B_0 \int_0^t (t-s)^{-1/(m+1)}p(s)\, ds.$$

By the integration by parts, we have

$$I^1T(p)(t) = \frac{(m+1)B_0}{m} \int_0^t (t-s)^{m/(m+1)}p'(s)\, ds$$

Differentiating this, we conclude that

$$T(p(t)) = B_0 \int_0^t \frac{p'(s)}{(t-s)^{1/(m+1)}}\, ds, \quad 0 \le t \le q \tag{3.13}$$

holds. Since $p'(t) > 0$ on $(0, q]$, $p = p(t)$ has the inverse function defined on $[0, R]$, which we denote by $t = F(p)$. Then F is differentiable on $[0, R]$ and satisfies $F'(u) = 1/p'(F(u))$. By putting $t = F(p)$ and $s = F(v)$ in (3.13), we have

$$T(p) = B_0 \int_0^p (F(p) - F(v))^{-1/(m+1)}\, dv, \quad 0 \le p \le R.$$

This means that F satisfies (1.2). So, the function f given by $f(u) = 1/p'(F(u))$ on $[0, R]$ gives the solution of integral equation (1.1). This completes the proof. □

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