Mem. Differential Equations Math. Phys. 6(1995), 109–112

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ON SOLVABILITY OF FUNCTIONAL EQUATIONS IN THE SPACE OF CONTINUOUS VECTOR FUNCTIONS

(Reported on September 18, 1995)

In the present note, we establish sufficient conditions for solvability of the functional equation

$$x(t) = p(x)(t) + q(x)(t), \tag{1}$$

where $p : C([a, b]; R^n) \to C([a, b]; R^n)$ and $q : C([a, b]; R^n) \to C([x, b]; R^n)$ are, respectively, linear and nonlinear operators.

Before passing to the statement of the basic result, we will give some notation and definitions necessary in the sequel.

R is the set of real numbers, $R_+ = [0, +\infty[$;

R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ $(i = 1, \dots, n)$ and the norm $\|x\| = \sum_{i=1}^n |x_i|$;

$R^{n\times n}$ is the space of $n \times n$-matrices $X = (x_{ik})_{i,k=1}^n$ with elements $x_{ik} \in R$ $(i, k = 1, \dots, n)$;

if $x = (x_i)_{i=1}^n \in R^n$ and $X = (x_{ik})_{i,k=1}^n \in R^{n\times n}$, then $|x| = (|x_i|)_{i=1}^n$ and $|X| = (|x_{ik}|)_{i,k=1}^n$;

$R_+^n = \{(x_i)_{i=1}^n : x_i \geq 0\,(i = 1, \dots, n)\}$, $R_+^{n\times n} = \{(x_{ik})_{i,k=1}^n : x_{ik} \geq 0\,(i, k = 1, \dots, n)\}$;

if $x = (x_i)_{i=1}^n$ and $y = (y_i)_{i=1}^n \in R^n$, then $x \leq y \Leftrightarrow x_i \leq y_i$ $(i = 1, \dots, n)$;

$r(X)$ is spectral radius of the matrix $X \in R^{n\times n}$;

$C([a, b]; R^n)$ is the space of continuous vector functions $x : I \to R^n$ with the norm

$$\|x\|_C = \max\Big\{\|x(t)\| : t \in [a, b]\Big\};$$

$$C([a, b]; R_+^n) = \Big\{x \in C([a, b]; R^n) : x(t) \in R_+^n \text{ for } t \in [a, b]\Big\};$$

An operator $g : C([a, b]; R^n) \to C([a, b]; R^n)$ is said to be **uniformly compact** if it is continuous and

$$\left\{\frac{1}{1 + \|x\|_C}\, g(x) : x \in C([a, b]; R^n)\right\}$$

is a relatively compact subset of $C([a, b]; R^n)$.

An operator $g : C([a, b]; R^n) \to C([a, b]; R^n)$ is said to be **positively homogeneous** if for every $x \in C([a, b]; R^n)$ and $\lambda \in R_+$ we have $g(\lambda x)(t) = \lambda g(x)(t)$ for $a \leq t \leq b$.

Along with (1), we have to consider the functional inequality

$$|x(t) - p(x)(t)| \leq g(x)(t). \tag{2}$$

Under solution of the functional equation (1) (functional inequality (2)) is meant a vector function $x \in C([a, b]; R^n)$ which for every $t \in [a, b]$ satisfies (1) (satisfies (2)).

1991 *Mathematics Subject Classification.* 47H10, 34K10.

Key words and phrases. Functional equation, functional differential equation, boundary value problem.

Theorem. *Let p and $q : C([a,b];R^n) \to C([a,b];R^n)$ be, respectively, a linear compact and a uniformly compact operators. Moreover, let there exist a positively homogeneous, continuous operator $g : C([a,b];R^n) \to C([a,b];R^n_+)$ and a vector $h \in R^n_+$ such that the functional inequality* (2) *has only trivial solution, and for every $x \in C([a,b];R^n)$ the inequality*

$$|q(x)(t)| \le g(x)(t) + h. \tag{3}$$

is fulfilled on $[a,b]$. Then the functional equation (1) *has at least one solution.*

To prove this theorem, we will need the following

Lemma. *Let a linear continuous operator $p : C([a,b];R^n) \to C([a,b];R^n)$ and a positively homogeneous continuous operator $g : C([a,b];R^n) \to C([a,b];R^n_+)$ be such that the functional inequality* (2) *has only trivial solution. Let, moreover, $h \in R^n_+$, C_0 be a non-empty subset of the space $C([a,b];R^n)$ such that the set*

$$\left\{ \frac{1}{1+\|x\|_C}\, x \, : \, x \in C_0 \right\} \tag{4}$$

is relatively compact. Then there exists a positive number ρ such that every vector function $x \in C_0$ satisfying on $[a,b]$ the functional inequality

$$|x(t) - p(x)(t)| \le g(|x|)(t) + h \tag{5}$$

admits the estimate

$$\|x\|_C \le \rho. \tag{6}$$

Proof. Suppose that the lemma is not true. Then for every natural k there exists $x_k \in C_0$ such that $\|x_k\|_C \ge k$, and the inequality

$$|x_k(t) - p(x_k)(t)| \le g(x_k)(t) + h$$

is fulfilled on $[a,b]$. Assume $\overline{x}_k(t) = (1+\|x_k\|_C)^{-1} x_k(t)$. Then

$$\lim_{k\to\infty} \|\overline{x}_k\|_C = 1 \tag{7}$$

and

$$\Big|\overline{x}_k(t) - p(\overline{x}_k(t))\Big| \le g(\overline{x}_k)(t) + \frac{1}{k+1}\, h. \tag{8}$$

Because (4) is relatively compact, without loss of generality we may regard the sequence $(\overline{x}_k)_{k=1}^{\infty}$ to be uniformly convergent on $[a,b]$. Suppose $x(t) = \lim_{k\to\infty} x_k(t)$. By (7) and (8), the vector function x is a solution of the functional inequality (2) satisfying $\|x\|_C = 1$. But this is impossible for (2) has only trivial solution. The obtained contradiction proves the lemma. $\square$

Proof of Theorem. First it should be noted that the linear homogeneous equation $(I - p)(x)(t) = 0$, where $I : C([a,b];R^n) \to C([a,b];R^n)$ is an identical operator, has only trivial solution. From this, by virtue of the Fredholm theorem ([1], Theorem 7.3.7) and the compactness of the operator p it follows that the operator $I - p$ has the bounded inverse $(I-p)^{-1}$.

Denote by C_0 the set of those $x \in C([a,b];R^n)$ for which there exists $\alpha(x) \in [0,1]$ such that

$$x(t) = p(x)(t) + \alpha(x) q(x)(t).$$

C_0 is non-empty, since $0 \in C_0$. On the other hand, because p is compact and q is uniformly compact, the set (4) is relatively compact.

Let ρ be the number appearing in the above proven lemma,

$$\sigma(s) = \begin{cases} 1 & \text{for} \quad 0 \leq s \leq \rho \\ 2 - \frac{s}{\rho} & \text{for} \quad \rho < s < 2\rho \\ 0 & \text{for} \quad s \geq 2\rho \end{cases}, \tag{9}$$

$$\widetilde{q}(x) = \big(\|x\|_C\big)(I - p)^{-1}(q(x)), \tag{10}$$

$$\begin{aligned} \rho_0 &= \sup\Big\{\|\widetilde{q}(x)\|_C \,:\, x \in C([a,b];R^n)\Big\}, \\ K &= \Big\{x \in C([a,b];R^n) : \|x\|_C \leq \rho_0\Big\}. \end{aligned} \tag{11}$$

Because the operator q is uniformly compact, it follows from the equalities (9)–(11) that $\widetilde{q}$ is a continuous compact operator transforming the ball K into itself. By Schauder's theorem, there exists a vector function $x \in K$ such that $x(t) = \widetilde{q}(x)(t)$ for $a \leq t \leq b$. By the definition of the set C_0 and owing to the equalities (9)–(11), it is clear that $x \in C_0$ and

$$x(t) = p(x)(t) + \sigma\big(\|x\|_C\big)q(x)(t). \tag{12}$$

From (3) and (12) we obtain the inequality (5). Therefore because of our choice of ρ, the vector function x admits the estimate (6). However, (6), (9) and (12) imply that x is the solution of the functional equation (1). $\square$

As an application, let us consider the functional differential equation

$$\frac{dx(t)}{dt} = f(t, x(\tau(t))), \tag{13}$$

with the boundary conditions

$$x(t) = 0 \quad \text{for} \quad t \notin [a,b] \quad \text{and} \quad x(a) = \sum_{k=1}^{m} A_k(x(b_k) - x(a)) + c, \tag{14}$$

where $f : [a,b] \times R^n \to R^n$ is a vector function satisfying the local Caratheodory conditions, $\tau : [a,b] \to R$ is a measurable function, $b_k \in [a,b]$, $A_k \in R^{n\times n}$ $(k = 1, \ldots, m)$, $c \in R^n$.

By χ we denote a characteristic function of the interval $[a,b]$.

Corollary. *Let the inequality*

$$\Big|f(t, \chi(\tau(t))y)\Big| \leq G_0(t)|y| + h_0(t), \tag{15}$$

be fulfilled on $[a,b] \times R^n$, *where* $G_0 : [a,b] \to R_+^{n\times n}$ *and* $h_0 : [a,b] \to R_+^n$ *are, respectively, a matrix and a vector functions with summable components, and*

$$r\Big(\sum_{k=1}^{m} |A_k| \int_a^{b_k} G_0(s)ds + \int_a^b G_0(s)ds\Big) < 1. \tag{16}$$

Then the boundary value problem (13), (14) *has at least one solution.*

Proof. The problem (13), (14) is equivalent to the functional equation (1), where $p(x)(t) = 0$,

$$q(x)(t) = c + \sum_{k=1}^{m} A_k \int_a^{b_k} f(s, \chi(\tau(s))x(\tau_0(s))ds + \int_a^t f\big(s, \chi(\tau(s))x(\tau_0(s))\big)ds, \tag{17}$$

$\tau_0(t) = \tau(a)$ for $\tau(t) \notin [a,b]$, $\tau_0(t) = \tau(t)$ for $\tau(t) \in [a,b]$.

For every $x = (x_i)_{i=1}^n \in C([a,b];R^n)$, suppose $|x|_C = (\|x_i\|_C)_{i=1}^n$. It is obvious from (16) and (17) that $q : C([a,b];R^n) \to C([a,b];R^n)$ is a uniformly compact operator satisfying the inequality (3), where $g(x)(t) \equiv G|x|_C$,

$$G = \sum_{k=1}^{n} |A_k| \int_a^{b_k} G_0(s)ds + \int_a^b G_0(s)ds, \quad h = |c| + \sum_{k=1}^{m} |A_k| \int_a^{b_k} h_0(s)ds + \int_a^b h(s)ds.$$

To prove the above Corollary, it suffices to determine by using the above proven theorem that the functional inequality

$$|x(t)| \le G|x|_C \tag{18}$$

has only trivial solution. Indeed, from (18) we have

$$(E - G)|x|_C \le 0, \tag{19}$$

where E is the identity $n \times n$ matrix. However, owing to (16), there exists $(E-G)^{-1} \in R_+^{n\times n}$. Multiplying both parts of (19) by $(E-G)^{-1}$, we obtain $|x|_C \le 0$, i.e., $x(t) \equiv 0$. $\square$

Example. Consider the problem

$$\frac{dx(t)}{dt} = G_0(t)|x(b)|, \tag{20}$$

$$x(a) = \sum_{k=1}^{m} A_k(x(b_k) - x(a)) + c, \tag{21}$$

where $G_0 : [a,b] \to R_+^{n\times n}$ is a matrix function with summable components, $A_k \in R_+^{n\times n}$, $b_k \in [a,b]$ $(k = 1,\dots,m)$, $c = (c_i)_{i=1}^n$, $c_i = 1$ $(i = 1,\dots,m)$. After direct checking we can easily see that the problem (20), (21) is solvable if and only if

$$r\Big(\sum_{k=1}^{m} A_k \int_a^{b_k} G_0(s)ds + \int_a^b G_0(s)ds\Big) < 1.$$

Consequently, the condition (16) in the above proven corollary is optimal and it cannot be weakened.

References

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